Analysis of a Domain Decomposition Method for the Coupling of Stokes and Darcy Equations

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Summary

We introduce a differential system based on the coupling of (Navier) Stokes equations and Darcy equation for the modelling of the interaction between surface and subsurface flows. We formulate the problem as an interface problem and analyze the associated Steklov–Poincaré operator. We then propose a way to solve the coupled problem iteratively, based on a suitable splitting of the interface conditions, allowing at each step the solution of two subproblems.

Key words: *Stokes and Darcy equations, Domain Decomposition methods, Steklov–Poincaré operators, Finite Elements*

1 Introduction and Problem Setting

We consider a bounded domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$ composed of two subdomains Ω_f and Ω_p such that $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\Omega_f \cap \Omega_p = \emptyset$ and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. The hypersurface Γ (a line if $d = 2$, a surface if $d = 3$) is the interface separating the domain Ω_f, filled by a fluid, from the domain Ω_p formed by a porous medium. Unless otherwise specified, we understand that the fluid has a prescribed upper surface, to make a distinction with the more general case of a free surface fluid that was addressed in [1]. Let us denote by $\mathbf{n}_f$ the unit outward normal direction on $\partial\Omega_f$, and by $\mathbf{n}_p$ the normal direction on $\partial\Omega_p$, oriented outward. Then $\mathbf{n}_f = -\mathbf{n}_p$ on the interface Γ.

In the domain Ω_p, we describe the motion of the fluid through the porous medium using Darcy's equation, according to which the velocity field is proportional to the gradient of a potential φ called *piezometric head*. In particular, the piezometric head for an incompressible fluid is defined as:

$$\varphi := z + \frac{p_p}{\varrho_f g} ,$$

where z is the elevation from a reference level, representing the potential energy per unit weight of fluid, p_p is the pressure of the fluid, ϱ_f its density and g is the gravity acceleration. Darcy's law states precisely:

$$n\mathbf{u}_p = \mathbf{q} = -\mathsf{K}\nabla\varphi ,$$

where ∇ is the gradient operator $(\partial/\partial x_1, \ldots, \partial/\partial x_d)^T$ with respect to the space coordinate $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{u}_p$ is the fluid velocity vector in Ω_p, $\mathbf{q}$ the specific discharge vector, n the volumetric porosity and K the hydraulic conductivity tensor of the porous medium. In the following K will be assumed diagonal: $\mathsf{K} = diag(K_1, \ldots, K_d)$, with $K_i > 0$ and $K_i \in L^\infty(\Omega_p)$, $i = 1, \ldots, d$.

The motion of the fluid in Ω_p is therefore described by the following system of equations (see [2], [3]), $\forall t > 0$:

$$\begin{aligned} &S_0 \frac{\partial\varphi}{\partial t} + \mathrm{div}\mathbf{q} = 0 \qquad && \forall \mathbf{x} \in \Omega_p \\ &\mathbf{q} = -\mathsf{K}\nabla\varphi && \forall \mathbf{x} \in \Omega_p , \end{aligned} \tag{1}$$

where S_0 is the specific mass storativity coefficient.

Let us now consider the domain Ω_f. We suppose the fluid to be homogeneous and incompressible, thus the Navier–Stokes equations apply, $\forall t > 0$:

$$\begin{aligned} &\frac{\partial \mathbf{u}_f}{\partial t} - \mathbf{div}\mathsf{T}(\mathbf{u}_f, p_f) + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f = \mathbf{f} \qquad && \forall \mathbf{x} \in \Omega_f \\ &\mathrm{div}\mathbf{u}_f = 0 && \forall \mathbf{x} \in \Omega_f , \end{aligned} \tag{2}$$

where $\mathsf{T}(\mathbf{u}_f, p_f) = \nu(\nabla\mathbf{u}_f + \nabla^T\mathbf{u}_f) - p_f\mathbf{I}$ is the stress tensor; $\nu > 0$ is the kinematic viscosity coefficient, $\mathbf{f}$ represents external forces, while $\mathbf{u}_f$ and p_f denote the fluid velocity and pressure.

In Sect. 2 we will consider several kind of boundary conditions on $\partial\Omega_f \setminus \Gamma$ (for the Navier–Stokes equations) and on $\partial\Omega_p \setminus \Gamma$ (for the Darcy equation). Then we discuss the matching conditions to be satisfied by $\mathbf{u}_f$, p_f and φ at the interface Γ.

From Sect. 3 on, we will concentrate on the linear steady problem that is obtained by dropping time derivatives as well as the nonlinear convective term in (1) and (2). We formulate the global problem in a weak form and prove that there exists a unique solution. Then, in Sect. 4 we reformulate our problem as an interface problem $S\lambda = \chi$, whose unknown solution is the common value of

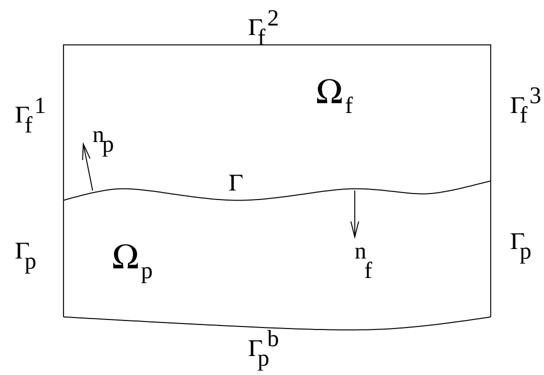

Figure 1: Schematic representation of a 2D vertical section of the computational domain

$\mathbf{u}_f \cdot \mathbf{n}_f$ and $\mathsf{K}\nabla\varphi \cdot \mathbf{n}_f$ of Γ, χ is a suitable term depending on boundary data and forcing terms, while S is the so-called Steklov–Poincaré operator. We prove that this interface problem has a unique solution, and that S can be split in two suboperators S_f and S_p, relative to Ω_f and Ω_p, respectively, where S_f is spectrally equivalent to S. This crucial property suggests the way to introduce suitable iterative methods whose preconditioner is S_f, see Sect. 5.

This is a preliminary, necessary step to devise, in the finite dimensional case after space discretization by finite elements, a domain decomposition substructuring method whose convergence rate is independent of the grid size. This method is briefly introduced in Sect. 6, its analysis however, together with a thorough investigation of the algorithmic aspects, will be carried out in a forthcoming paper.

2 Boundary and Interface Conditions

Let us split the boundaries $\partial\Omega_f$ and $\partial\Omega_p$ of Ω_f and Ω_p as $\partial\Omega_f = \Gamma\cup\Gamma_f^1\cup\Gamma_f^2\cup\Gamma_f^3$ and $\partial\Omega_p = \Gamma \cup \Gamma_p \cup \Gamma_p^b$, as shown in Fig. 1.

For the Darcy equation we assign the piezometric head $\varphi = \varphi_p$ on Γ_p; moreover, we require that the normal component of the velocity vanishes on the bottom surface, that is $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p^b.

For the Navier–Stokes problem several combinations of boundary conditions could be considered, representing different kind of flow problems; let us indicate some of them.

A first possibility is to assign the velocity vector $\mathbf{u}_f = \mathbf{0}$ on $\Gamma_f^1 \cup \Gamma_f^3$ and a natural boundary condition $\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f = \mathbf{g}$ on Γ_f^2 (a fictitious boundary), being $\mathbf{g}$ a given vector function, representing the flux across Γ_f^2 of the fluid column standing above.

Alternatively, we can prescribe a non-null inflow $\mathbf{u}_f = \mathbf{u}_{in}$ on the left-hand boundary Γ_f^1, a slip condition $\mathbf{u}_f \cdot \mathbf{n}_f = 0$ and $(\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f) \cdot \boldsymbol{\tau}_i = 0$ on Γ_f^2 ($\boldsymbol{\tau}_i$, $i = 1, \ldots, d-1$, are linear independent unit tangential vectors to the boundary) and an outflow $\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f = 0$ on the right-hand boundary Γ_f^3.

A third possibility consists of assigning again a non-null inflow $\mathbf{u}_f = \mathbf{u}_{in}$ on the left-hand boundary Γ_f^1 and a no-slip condition $\mathbf{u}_f = \mathbf{0}$ on the remaining boundary $\Gamma_f^2 \cup \Gamma_f^3$.

Our analysis shall consider the last choice we have indicated, but it can be modified to accommodate the other boundary conditions as well. From now on, we shall indicate Γ_f^1 as Γ_f^{in} (standing for Γ_f^{inflow}) and the remaining boundary $\Gamma_f^2 \cup \Gamma_f^3$ simply by Γ_f.

We propose the following set of interface conditions on Γ:

$$
\begin{aligned}
&\mathbf{u}_p \cdot \mathbf{n}_f = \mathbf{u}_f \cdot \mathbf{n}_f \ , \\
&-[(\mathsf{T}(\mathbf{u}_f, p_f)) \cdot \mathbf{n}_f] \cdot \boldsymbol{\tau}_i = \frac{\alpha_1}{\sqrt{k_i}}(\mathbf{u}_f - \mathbf{u}_p) \cdot \boldsymbol{\tau}_i \ , \quad i = 1, \ldots, d-1 \ , \\
&-[(\mathsf{T}(\mathbf{u}_f, p_f)) \cdot \mathbf{n}_f] \cdot \mathbf{n}_f = g\varphi \ ,
\end{aligned}
\tag{3}
$$

where $k_i = \boldsymbol{\tau_i} \cdot \mathsf{K} \cdot \boldsymbol{\tau_i}$, and α_1 is a positive dimensionless parameter which depends on the properties of the porous medium.

The above conditions impose the continuity of the normal velocity on Γ, as well as that of the normal component of the normal stress. The interface conditions (3) generalize those proposed by Payne and Straughan in [4], where the simplified case of a flat interface is accounted for. We notice that, in agreement with what advocated by Jäger and Mikelić (see [5, 6]), the pressure is allowed to be discontinuous across the interface.

Finally, since α_1 can be taken very small, we will assume it to be null from now on. In that case, note that the second condition in (3) does no longer couple $\mathbf{u}_f$ with $\mathbf{u}_p$, and will be recovered as "natural" boundary condition for the problem in Ω_f (see next Section).

3 Weak Formulation and Analysis

In our analysis we shall consider a linear coupled problem Stokes/Darcy, which corresponds to replacing the Navier–Stokes equations with the linear Stokes equations in Ω_f. This replacement is justified when the fluid velocity is small. Moreover, we will consider only the stationary case. Let us recall that a stationary Stokes problem can also be generated by a semi–implicit time advancement of the Navier–Stokes equations where all terms but the nonlinear convective one have dealt with implicitely. The differential formulation of the problem we are considering reads therefore as follows:

$$
\begin{array}{ll}
-\mathbf{div}\mathsf{T}(\mathbf{u}_f, p_f) = \mathbf{f} & \text{in } \Omega_f \\
\mathrm{div}\mathbf{u}_f = 0 & \text{in } \Omega_f \\
-\mathrm{div}(\mathsf{K}\nabla\varphi) = 0 & \text{in } \Omega_p \\
\mathbf{u}_f = \mathbf{u}_{in} & \text{on } \Gamma_f^{in} \\
\mathbf{u}_f = \mathbf{0} & \text{on } \Gamma_f \\
-\mathsf{K}\nabla\varphi \cdot \mathbf{n}_p = 0 & \text{on } \Gamma_p^b \\
\varphi = \varphi_p & \text{on } \Gamma_p \; ,
\end{array}
\tag{4}
$$

and it must be completed with the interface conditions (3) on Γ (with $\alpha_1 = 0$). We will assume that $\mathbf{u}_{in}$ is null in a neighborhood of the intersection $\overline{\Gamma} \cap \overline{\Gamma}_f^{in}$.

Let us introduce the following functional spaces:

$$
H_{\Gamma_f} := \{v \in H^1(\Omega_f) | v = 0 \text{ on } \Gamma_f\} \; , \tag{5}
$$

$$
H_{\Gamma_f \cup \Gamma_f^{in}} := \{v \in H_{\Gamma_f} | v = 0 \text{ on } \Gamma_f^{in}\} \; , \quad H_f := (H_{\Gamma_f \cup \Gamma_f^{in}})^d \; , \tag{6}
$$

$$
H_f^0 := \{\mathbf{v} \in H_f | \mathbf{v} \cdot \mathbf{n}_f = 0 \text{ on } \Gamma\} \; , \quad Q := L^2(\Omega_f) \; , \tag{7}
$$

$$
H_p := \{\psi \in H^1(\Omega_p) | \psi = 0 \text{ on } \Gamma_p\} \; , \quad H_p^0 := \{\psi \in H_p | \psi = 0 \text{ on } \Gamma\} \; . \tag{8}
$$

The space $W := H_f \times H_p$ is a Hilbert space with norm

$$
\|\underline{w}\|_W := \left(\|\mathbf{w}\|^2_{H^1(\Omega_f)} + \|\psi\|^2_{H^1(\Omega_p)} \right)^{1/2} \qquad \forall \underline{w} = (\mathbf{w}, \psi) \in W.
$$

Finally, we consider on Γ the trace space $\Lambda := H_{00}^{1/2}(\Gamma)$ and denote its norm by $\| \cdot \|_\Lambda$ (see [7]).

We introduce a continuous extension operator

$$
E_f \colon (H^{1/2}(\Gamma_f^{in}))^d \to (H_{\Gamma_f})^d \; . \tag{9}
$$

Then $\forall \mathbf{u}_{in} \in (H^{1/2}(\Gamma_f^{in}))^d$ we can construct a vector function $E_f \mathbf{u}_{in} \in (H_{\Gamma_f})^d$ such that $E_f \mathbf{u}_{in|_{\Gamma_f^{in}}} = \mathbf{u}_{in}$ and $\mathrm{div}(E_f \mathbf{u}_{in}) = 0$ in Ω_f (see e.g. [8], pp. 158–159). Moreover we define $\mathbf{u}_f^0 := \mathbf{u}_f - E_f \mathbf{u}_{in} \in H_f$.

We define the following bilinear forms: for all $\mathbf{v}, \mathbf{w} \in (H^1(\Omega_f))^d$,

$$
a_f(\mathbf{v}, \mathbf{w}) := \int_{\Omega_f} \frac{\nu}{2} \left(\nabla \mathbf{v} + \nabla^T \mathbf{v} \right) \cdot \left(\nabla \mathbf{w} + \nabla^T \mathbf{w} \right) ; \tag{10}
$$

$$
\begin{aligned}
\mathcal{A}(\underline{v}, \underline{w}) := \; & n \, a_f(\mathbf{v}, \mathbf{w}) + \int_{\Omega_p} g \nabla\psi \cdot \mathsf{K}\nabla\varphi \\
& + \int_\Gamma ng \, \varphi(\mathbf{w} \cdot \mathbf{n}_f) - \int_\Gamma ng \, \psi(\mathbf{v} \cdot \mathbf{n}_f)
\end{aligned}
\tag{11}
$$

for all $\underline{v} = (\mathbf{v}, \varphi)$ and $\underline{w} = (\mathbf{w}, \psi) \in W$;

$$
\mathcal{B}(\underline{w}, q) := -\int_{\Omega_f} n \, q \, \mathrm{div}\mathbf{w} \qquad \forall \underline{w} = (\mathbf{w}, \psi) \in W, \quad q \in Q. \tag{12}
$$

Let us introduce a continuous extension operator

$$E_p : H^{1/2}(\Gamma_p) \to H^1(\Omega_p) \ , \tag{13}$$

then we define the function $\varphi_0 \in H_p$ as $\varphi_0 := \varphi - E_p\varphi_p$, upon assuming that φ_p belongs to $H^{1/2}(\Gamma_p)$.
Finally, we can define the following linear functional for all $\underline{w} = (\mathbf{w}, \psi) \in W$:

$$\begin{aligned} < \mathcal{F}, \underline{w} > := & \int_{\Omega_f} n\mathbf{f} \cdot \mathbf{w} - n a_f(E_f \mathbf{u}_{in}, \mathbf{w}) \\ & + \int_{\Omega_p} g \nabla \psi \cdot \mathsf{K} \nabla (E_p \varphi_p) - \int_\Gamma n g \, (E_p \varphi_p) \mathbf{w} \cdot \mathbf{n}_f \\ & + \int_\Gamma n g \, (E_f \mathbf{u}_{in} \cdot \mathbf{n}_f) \psi \ . \end{aligned} \tag{14}$$

Using the above definitions, problem (4) can be written in the following weak form: find $\underline{u} = (\mathbf{u}_f^0, \varphi_0) \in W$, $p \in Q$:

$$\begin{array}{ll} \mathcal{A}(\underline{u}, \underline{v}) + \mathcal{B}(\underline{v}, p) = < \mathcal{F}, \underline{v} > & \forall \underline{v} = (\mathbf{v}, \psi) \in W \\ \mathcal{B}(\underline{u}, q) = 0 & \forall q \in Q. \end{array} \tag{15}$$

We remark that the interface conditions (3) have been incorporated in the above weak model: in fact, integrating by parts both Stokes and Darcy's equations, it can be seen that they are all natural conditions on Γ.

In order to prove existence and uniqueness for the solution of the coupled Stokes/Darcy problem, we introduce some preliminary results on the properties of the bilinear forms $\mathcal{A}$ and $\mathcal{B}$ and of the functional $\mathcal{F}$.

Lemma 3.1 *The following results hold:*

1. $\mathcal{A}(.,.)$ *is continuous and coercive on* W *and, in particular, is coercive on the space*
$$Z^0 := \{\underline{v} \in W \,|\, \mathcal{B}(\underline{v}, q) = 0 \quad \forall q \in Q\} \ ;$$
2. $\mathcal{B}(.,.)$ *is continuous on* $W \times Q$ *and satisfies the following Brezzi-Babuška condition: there exists a positive constant* $\beta > 0$ *such that* $\forall q \in Q$ $\exists \underline{w} \in W$:
$$\mathcal{B}(\underline{w}, q) \geq \beta \|\underline{w}\|_W \|q\|_{L^2(\Omega_f)} \ . \tag{16}$$
3. $\mathcal{F}$ *is a continuous linear functional on* W.

Proof. 1. The following trace inequalities hold (see [7]):

$$\exists C_f > 0 \ : \quad \|\mathbf{v}_{|\Gamma}\|_\Lambda \leq C_f \|\mathbf{v}\|_{H^1(\Omega_f)} \qquad \forall \mathbf{v} \in H_f \ ; \tag{17}$$

$$\exists C_p > 0 \ : \quad \|\psi_{|\Gamma}\|_\Lambda \leq C_p \|\psi\|_{H^1(\Omega_p)} \qquad \forall \psi \in H_p \ . \tag{18}$$

Thanks to the Cauchy–Schwarz inequality and the above trace inequalities the continuity of $\mathcal{A}(.,.)$ follows:

$$\mathcal{A}(\underline{v},\underline{w}) \leq \gamma \|\underline{v}\|_W \|\underline{w}\|_W \ , \qquad \forall \underline{v},\underline{w} \in W \ ,$$

with the following constant

$$\gamma := 2\max\{2n\nu, gM_K, C_f C_p ng\} \tag{19}$$

and $M_K := \max_{i=1,\dots,d} \|K_i\|_{L^\infty(\Omega_p)}$.
The coercivity is a consequence of the Korn inequality (see e.g. [8], p. 149): $\forall \mathbf{v} = (v_1,\dots,v_d) \in H_f$

$$\exists \kappa_f > 0 \ : \ \int_{\Omega_f} \sum_{j,l=1}^{d} \left(\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j} \right)^2 \geq \kappa_f \|\mathbf{v}\|^2_{H^1(\Omega_f)} \ , \tag{20}$$

and the Poincaré inequality (see [7] and [9], p. 11):

$$\exists C_{\Omega_p} > 0 : \quad \|\psi\|^2_{L^2(\Omega_p)} \leq C_{\Omega_p} \||\nabla\psi|\|^2_{L^2(\Omega_p)} \quad \forall \psi \in H_p \ . \tag{21}$$

In fact we have:

$$\mathcal{A}(\underline{v},\underline{v}) \geq \alpha \|\underline{v}\|^2_W, \qquad \forall \underline{v} = (\mathbf{v},\varphi) \in W,$$

where

$$\alpha := \frac{1}{2} \min\left\{ n\nu\kappa_f, gm_K \min\left(1, \frac{1}{C_{\Omega_p}}\right) \right\} \ , \tag{22}$$

$$m_K := \min_{i=1,\dots,d} \inf_{\mathbf{x}\in\Omega_p} K_i(\mathbf{x}), \quad (m_K > 0) \ . \tag{23}$$

2. For the continuity, thanks to the Cauchy–Schwarz inequality, we have

$$|\mathcal{B}(\underline{w},q)| \leq \|q\|_{L^2(\Omega_f)} \|\underline{w}\|_W, \qquad \text{for all} \quad \underline{w} \in W, \, q \in Q \ .$$

Now it can be shown that the following compatibility condition holds (see e.g. [8], Proposition 5.3.2, or [10]): there exists a constant $\beta^0 > 0$, such that $\forall q \in Q \quad \exists \mathbf{w} \in H_f, \, \mathbf{w} \neq 0 :$

$$-\int_{\Omega_f} q \,\mathrm{div}\mathbf{w} \geq \beta^0 \|\mathbf{w}\|_{H^1(\Omega_f)} \|q\|_{L^2(\Omega_f)} \ . \tag{24}$$

Then, considering $\underline{w} = (\mathbf{w}, 0) \in H_f \times H_p$ the thesis follows with $\beta = n\beta^0$.
3. Thanks to the Cauchy–Schwarz inequality and the trace inequality (17), we have:

$$| < \mathcal{F}, \underline{w} > | \leq \sqrt{2} C_{\mathcal{F}} \|\underline{w}\|_W \ ,$$

where

$$\begin{aligned} C_{\mathcal{F}} := \max\{ & (n\|\mathbf{f}\|_{L^2(\Omega_f)} + 2n\nu \|E_f \mathbf{u}_{in}\|_{H^1(\Omega_f)} \\ & + ngC_f\kappa_p \|\varphi_p\|_{H^{1/2}(\Gamma_p)}), \ gM_K\kappa_p\|\varphi_p\|_{H^{1/2}(\Gamma_p)}\} \ , \end{aligned} \tag{25}$$

being κ_p the continuity coefficient of the continuous extension operator E_p. □

We can now prove the main result of this Section:

Proposition 3.1 *The Stokes/Darcy coupled problem (15) admits a unique solution* $(\mathbf{u}_f, p, \varphi_0) \in H_f \times Q \times H_p$*, satisfying the following a-priori estimates:*

$$\|(\mathbf{u}_f, \varphi_0)\|_W \le \frac{\sqrt{2}C_{\mathcal{F}}}{\alpha} ,$$

$$\|p\|_{L^2(\Omega_f)} \le \frac{\sqrt{2}C_{\mathcal{F}}}{\beta}\left(1 + \frac{\gamma}{\alpha}\right) ,$$

where β, γ, α *and* $C_{\mathcal{F}}$ *are the constants defined in (16), (19), (22) and (25), respectively.*

Proof. It's a straightforward consequence of the existence and uniqueness theorem by Brezzi (see [11]), whose hypotheses are satisfied thanks to Lemma 3.1. □

Remark 3.1 *From Proposition 3.1 it follows in particular that* $-\mathsf{K}\nabla\varphi \cdot \mathbf{n}_{f|_\Gamma} \in \Lambda$*, since* $\mathbf{u}_f \cdot \mathbf{n}_{f|_\Gamma} \in \Lambda$*. Then, on* Γ*,* φ *has a higher regularity than one might have expected.*

4 Steklov–Poincaré Operators Associated to the Coupled Problem

In this Section we will apply a Domain Decomposition technique (at the differential level) to study the Stokes/Darcy coupled problem. In particular, we shall introduce and analyze the Steklov–Poincaré interface equation associated to our problem, in order to reformulate it in terms of interface unknowns solely. This re-interpretation is crucial to allow the set up on an iterative procedure between the subdomains Ω_f and Ω_p.

First of all let us state the following result, whose proof is omitted since it is similiar to that of Lemma 5.3.5 of [8].

Proposition 4.1 *Problem (15) can be reformulated in an equivalent way as follows: find* $\mathbf{u}_f^0 \in H_f$*,* $p \in Q$*,* $\varphi_0 \in H_p$ *such that*

$$\begin{aligned}
&a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, \mathbf{w}) - \int_{\Omega_f} p\, div\mathbf{w} = \int_{\Omega_f} \mathbf{f}\cdot\mathbf{w} \qquad \forall \mathbf{w} \in H_f^0 ,\\
&\int_{\Omega_f} q\, div\mathbf{u}_f^0 = 0 \qquad \forall q \in Q ,\\
&\int_\Gamma (\mathbf{u}_f^0 + E_f\mathbf{u}_{in})\cdot\mathbf{n}_f\, \mu = -\frac{1}{n}\int_\Gamma [\mathsf{K}\nabla(\varphi_0 + E_p\varphi_p)\cdot\mathbf{n}_f]\mu \qquad \forall \mu \in \Lambda ,\\
&\int_{\Omega_p} \nabla\psi\cdot\mathsf{K}\nabla(\varphi_0 + E_p\varphi_p) = 0 \qquad \forall \psi \in H_p^0 ,\\
&\int_\Gamma g(\varphi_0 + E_p\varphi_p)\mu = \int_{\Omega_f} \mathbf{f}\cdot(R_1\mu)\\
&\quad - a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, R_1\mu) + \int_{\Omega_f} p\, div(R_1\mu) \quad \forall \mu \in \Lambda ,
\end{aligned} \tag{26}$$

where R_1 is any possible extension operator from Λ to H_f, i.e. *a continuous operator from Λ to H_f such that $(R_1\mu)\cdot\mathbf{n}_f = \mu$ on Γ, for all $\mu \in \Lambda$.*

Now, let us choose as governing variable on the interface Γ the normal component of the velocity field:

$$\lambda := \mathbf{u}_f \cdot \mathbf{n}_f = -\frac{1}{n}\mathsf{K}\nabla\varphi \cdot \mathbf{n}_f \ . \qquad (27)$$

Should we know a priori the value of λ on Γ, from (27) we would obtain a Dirichlet boundary condition for the Stokes system in Ω_f ($\mathbf{u}_f\cdot\mathbf{n}_f = \lambda$ on Γ) and a Neumann boundary condition for the Darcy equation in Ω_p ($-(\mathsf{K}\nabla\varphi\cdot\mathbf{n}_f)/n = \lambda$ on Γ).

Joint with $(3)_2$ (with $\alpha_1 = 0$), these conditions allow us to recover (independently) the solutions $(\mathbf{u}_f^0, p)$ of the Stokes problem and φ_0 of the Darcy problem.

We introduce two auxiliary problems whose solutions (which depend on the problem data) are related to that of the global problem (26), as we will see later on:

find $\boldsymbol{\omega}_0^* \in H_f^0, \pi^* \in L_0^2(\Omega_f)$ s.t. $\forall \mathbf{v} \in H_f^0$, $\forall q \in L_0^2(\Omega_f)$

$$\begin{aligned} & a_f(\boldsymbol{\omega}_0^* + E_f\mathbf{u}_{in}, \mathbf{v}) - \int_{\Omega_f} \pi^* \,\mathrm{div}\mathbf{v} = \int_{\Omega_f} \mathbf{f}\cdot\mathbf{v} \\ & \int_{\Omega_f} q\,\mathrm{div}\boldsymbol{\omega}_0^* = 0 \ ; \end{aligned} \qquad (28)$$

find $\varphi_0^* \in H_p$ s.t. $\forall \psi \in H_p$

$$\int_{\Omega_p} \nabla\psi\cdot\mathsf{K}\nabla\varphi_0^* = -\int_{\Omega_p} \nabla\psi\cdot\mathsf{K}\nabla(E_p\varphi_p) \ . \qquad (29)$$

Now we define the following extension operators:

$$R_f\colon \Lambda_0 \to H_f \times L_0^2(\Omega_f), \quad \eta \to R_f\eta := (R_f^1\eta, R_f^2\eta)$$

such that $(R_f^1\eta)\cdot\mathbf{n}_f = \eta$ on Γ and

$$\begin{aligned} & a_f(R_f^1\eta, \mathbf{v}) - \int_{\Omega_f} (R_f^2\eta)\mathrm{div}\mathbf{v} = 0 && \forall \mathbf{v} \in H_f^0 \\ & \int_{\Omega_f} q\,\mathrm{div}(R_f^1\eta) = 0 && \forall q \in L_0^2(\Omega_f) \ , \end{aligned} \qquad (30)$$

where

$$\Lambda_0 := \left\{ \mu \in \Lambda \,\middle|\, \int_\Gamma \mu = 0 \right\} \ ; \qquad (31)$$

$$R_p\colon \Lambda \to H_p \ , \quad \eta \to R_p\eta$$

such that

$$\int_{\Omega_p} \nabla\psi \cdot \mathsf{K}\nabla(R_p\eta) = \int_\Gamma n\eta\psi \qquad \forall\psi \in H_p\ . \tag{32}$$

Let us define the *Steklov–Poincaré* operator S as follows:

$$\begin{aligned} < S\eta, \mu > \;\; &:= \;\; a_f(R_f^1\eta, R_1\mu) - \int_{\Omega_f} (R_f^2\eta)\,\mathrm{div}(R_1\mu) \\ &\quad + \int_\Gamma g(R_p\eta)\mu \qquad \forall\eta \in \Lambda_0,\ \forall\mu \in \Lambda\ , \end{aligned} \tag{33}$$

which can be split as the sum of two suboperators $S = S_f + S_p$:

$$< S_f\eta, \mu > := a_f(R_f^1\eta, R_1\mu) - \int_{\Omega_f} (R_f^2\eta)\,\mathrm{div}(R_1\mu)\ , \tag{34}$$

$$< S_p\eta, \mu > := \int_\Gamma g\,(R_p\eta)\mu \tag{35}$$

for all $\eta \in \Lambda_0$ and $\mu \in \Lambda$, and the functional $\chi : \Lambda_0 \to \mathbb{R}$,

$$\begin{aligned} < \chi, \mu > \;\; &:= \;\; \int_{\Omega_f} \mathbf{f} \cdot (R_1\mu) - a_f(\boldsymbol{\omega}_0^* + E_f\mathbf{u}_{in}, R_1\mu) \\ &\quad + \int_{\Omega_f} \pi^*\,\mathrm{div}(R_1\mu) - \int_\Gamma g\,(\varphi_0^* + E_p\varphi_p + R_p\lambda_{in})\mu\ , \end{aligned} \tag{36}$$

for all $\mu \in \Lambda$. We have set $\lambda_{in} := E_f\mathbf{u}_{in} \cdot \mathbf{n}_{f\,|\Gamma}$. Thus

$$\lambda = \lambda_0 + \lambda_{in}\ , \tag{37}$$

where $\lambda_0 = \mathbf{u}_f^0 \cdot \mathbf{n}_{f\,|\Gamma}$.

Theorem 4.1 *The solution to (26) can be characterized as follows:*

$$\mathbf{u}_f^0 = \boldsymbol{\omega}_0^* + R_f^1\lambda_0, \quad p = \pi^* + R_f^2\lambda_0 + \hat{p}_f, \quad \varphi_0 = \varphi_0^* + R_p(\lambda_0 + \lambda_{in})\ , \tag{38}$$

where $\hat{p}_f = (meas(\Omega_f))^{-1}\int_{\Omega_f} p$ *, and* $\lambda_0 \in \Lambda_0$ *is the solution of the following Steklov–Poincaré problem:*

$$< S\lambda_0, \mu_0 > = < \chi, \mu_0 > \qquad \forall\mu_0 \in \Lambda_0\ . \tag{39}$$

Moreover, $\hat{p}_f$ *can be obtained from* λ_0 *by solving the algebraic equation*

$$\hat{p}_f = \frac{1}{meas(\Gamma)} < S\lambda_0 - \chi, \varepsilon >\ , \tag{40}$$

where $\varepsilon \in \Lambda$ *is a fixed function such that*

$$\frac{1}{meas(\Gamma)} \int_\Gamma \varepsilon = 1\ . \tag{41}$$

Proof. By direct inspection, the functions defined in (38) satisfy $(26)_1$, $(26)_3$ and $(26)_4$. Moreover $(26)_2$ is satisfied too. Indeed, $\forall q \in Q$

$$\int_{\Omega_f} q \mathrm{div}(\boldsymbol{\omega}_0^* + R_f^1\lambda_0) = \int_{\Omega_f} (q - \overline{q})\mathrm{div}(\boldsymbol{\omega}_0^* + R_f^1\lambda_0) + \int_{\Omega_f} \overline{q}\mathrm{div}(\boldsymbol{\omega}_0^* + R_f^1\lambda_0),$$

$\overline{q}$ being the constant $\overline{q} := (meas(\Omega_f))^{-1}\int_{\Omega_f} q$. Thanks to $(28)_2$, $(30)_2$ and applying the divergence theorem to the last term on the right-hand side, we obtain $(26)_2$.

Let us now consider $(26)_5$. We substitute (38) in it and we obtain, $\forall \mu \in \Lambda$,

$$\begin{aligned} &\int_\Gamma g(R_p\lambda_0)\mu + a_f(R_f^1\lambda_0, R_1\mu) - \int_{\Omega_f}(R_f^2\lambda_0)\,\mathrm{div}(R_1\mu) \\ &= \int_{\Omega_f} \mathbf{f}\cdot(R_1\mu) - \int_\Gamma g(\varphi_0^* + E_p\varphi_p + R_p\lambda_{in})\mu \\ &\quad - a_f(\boldsymbol{\omega}_0^* + E_f\mathbf{u}_{in}, R_1\mu) + \int_{\Omega_f}\pi^*\,\mathrm{div}(R_1\mu) + \hat{p}_f\int_{\Omega_f}\mathrm{div}(R_1\mu)\ , \end{aligned}$$

that is

$$< S\lambda_0, \mu > = < \chi, \mu > + \hat{p}_f \int_{\Omega_f} \mathrm{div}(R_1\mu) \qquad \forall \mu \in \Lambda\ . \tag{42}$$

In particular, we can invoke the divergence theorem and conclude that λ_0 is the solution to the Steklov–Poincaré equation (39).

Now any $\mu \in \Lambda$ can be decomposed as $\mu = \mu_0 + \mu_\Gamma\varepsilon$, with $\mu_\Gamma := (meas(\Gamma))^{-1}\int_\Gamma \mu$, so that $\mu_0 \in \Lambda_0$.

From (42) we obtain

$$< S\lambda_0, \mu_0 > + < S\lambda_0, \mu_\Gamma\varepsilon > = < \chi, \mu_0 > + < \chi, \mu_\Gamma\varepsilon > + \hat{p}_f \int_\Gamma \mu \qquad \forall \mu \in \Lambda\ .$$

Therefore, thanks to (39), we have

$$\mu_\Gamma < S\lambda_0 - \chi, \varepsilon > = \hat{p}_f \int_\Gamma \mu \qquad \forall \mu \in \Lambda\ .$$

Since $\int_\Gamma \mu = \mu_\Gamma meas(\Gamma)$, we conclude that (40) holds.□

In next Section we will prove that (39) has a unique solution.

4.1 Analysis of the Steklov–Poincaré Operators

We shall now prove some properties of the Steklov–Poincaré operators S_f, S_p and S.

Lemma 4.1 *The Steklov–Poincaré operators enjoy the following properties:*

1. *S_f and S_p are linear continuous operators on Λ_0 (*i.e., *$S_f\eta \in \Lambda_0'$, $S_p\eta \in \Lambda_0'$, $\forall \eta \in \Lambda_0$);*

2. S_f is symmetric and coercive;

3. S_p is symmetric and positive.

Proof. 1. S_f and S_p are obviously linear. Then, we notice that for every $\mu \in \Lambda_0$ we can make the special choice $R_1\mu = R_f^1\mu$. Consequently, from (34) and (30) it follows that S_f can be characterized as:

$$< S_f\eta, \mu >= a_f(R_f^1\eta, R_f^1\mu) \qquad \forall \eta, \mu \in \Lambda_0 . \tag{43}$$

To prove its continuity, we introduce the vector operator $\mathcal{H}\colon \Lambda_0 \to H_f$, $\mu \to \mathcal{H}\mu$, s.t.

$$\begin{array}{ll} \displaystyle\int_{\Omega_f} \nabla(\mathcal{H}\mu)\cdot\nabla\mathbf{v} = 0 & \forall \mathbf{v} \in [H_0^1(\Omega_f)]^d \\ (\mathcal{H}\mu)\cdot\mathbf{n}_f = \mu \quad \text{on } \Gamma & \\ (\mathcal{H}\mu)\cdot\boldsymbol{\tau}_i = 0 \quad \text{on } \Gamma \ , & i = 1,\dots,d-1 \\ \mathcal{H}\mu = 0 \quad \text{on } \partial\Omega_f \setminus \Gamma \ , & \end{array} \tag{44}$$

where $H_0^1(\Omega_f) = \{v \in H^1(\Omega_f) | \, v = 0 \quad \text{on } \partial\Omega_f\}$.
By comparison with the operator R_f^1 introduced in (30), we see that for all $\mu \in \Lambda_0$, the vector function

$$\mathbf{z}(\mu) := R_f^1\mu - \mathcal{H}\mu \tag{45}$$

satisfies $\mathbf{z}(\mu)\cdot\mathbf{n}_f = 0$ on Γ , therefore $\mathbf{z}(\mu) \in H_f^0$. By taking $\mathbf{v} = \mathbf{z}(\mu)$ in $(30)_1$, in view of the definition (45) we have:

$$\begin{aligned} |a_f(R_f^1\mu, \mathbf{z}(\mu))| \quad &= \quad \left| -\int_{\Omega_f} (R_f^2\mu)\,\mathrm{div}(\mathcal{H}\mu) \right| \\ &\leq \quad \|R_f^2\mu\|_{L^2(\Omega_f)} \|\mathcal{H}\mu\|_{H^1(\Omega_f)} \ . \end{aligned} \tag{46}$$

Let us consider now the function $R_f^2\mu$. Since it belongs to $L_0^2(\Omega_f)$, there exists $\mathbf{w} \in (H_0^1(\Omega_f))^d$, $\mathbf{w} \neq 0$, such that

$$\beta^0 \|R_f^2\mu\|_{L^2(\Omega_f)} \|\mathbf{w}\|_{H^1(\Omega_f)} \leq -\int_{\Omega_f} (R_f^2\mu)\,\mathrm{div}\mathbf{w} \ ,$$

where $\beta_0 > 0$ is the inf-sup constant (independent of μ) (see e.g. [10]). Since $\mathbf{w} \in (H_0^1(\Omega_f))^d \subset H_f^0$, we can use $(30)_1$ and obtain:

$$\beta^0 \|R_f^2\mu\|_{L^2(\Omega_f)} \|\mathbf{w}\|_{H^1(\Omega_f)} \leq |a_f(R_f^1\mu, \mathbf{w})| \leq 2\nu \|\nabla R_f^1\mu\|_{L^2(\Omega_f)} \|\mathbf{w}\|_{H^1(\Omega_f)} \ .$$

The last inequality follows from the Cauchy–Schwarz inequality. Therefore

$$\|R_f^2\mu\|_{L^2(\Omega_f)} \leq \frac{2\nu}{\beta^0} \|R_f^1\mu\|_{H^1(\Omega_f)} \ , \qquad \forall \mu \in \Lambda_0 \ . \tag{47}$$

Now, using the Korn inequality (20) and relations (45), (46), (47), we obtain:

$$\begin{aligned}
\|R_f^1\mu\|^2_{H^1(\Omega_f)} &\leq C a_f(R_f^1\mu, R_f^1\mu) \\
&= C\left[a_f(R_f^1\mu, \mathbf{z}(\mu)) + a_f(R_f^1\mu, \mathcal{H}\mu)\right] \\
&\leq C\left[\|R_f^2\mu\|_{L^2(\Omega_f)}\|\mathcal{H}\mu\|_{H^1(\Omega_f)} + 2\nu\|R_f^1\mu\|_{H^1(\Omega_f)}\|\mathcal{H}\mu\|_{H^1(\Omega_f)}\right] \\
&\leq 2\nu C\left(1 + \frac{1}{\beta^0}\right)\|R_f^1\mu\|_{H^1(\Omega_f)}\|\mathcal{H}\mu\|_{H^1(\Omega_f)} ,
\end{aligned}$$

for all $\mu \in \Lambda_0$, where $C := 2/(n\nu\kappa_f)$. Therefore

$$\begin{aligned}
\|R_f^1\mu\|_{H^1(\Omega_f)} &\leq 2\nu C\left(1 + \frac{1}{\beta^0}\right)\|\mathcal{H}\mu\|_{H^1(\Omega_f)} \\
&\leq 2\nu\alpha^* C\left(1 + \frac{1}{\beta^0}\right)\|\mu\|_\Lambda . \qquad (48)
\end{aligned}$$

The last inequality follows from the observation that $\mathcal{H}\mu$ is a harmonic extension of μ, then there exists a positive constant $\alpha^* > 0$ (independent of μ) such that

$$\|\mathcal{H}\mu\|_{H^1(\Omega_f)} \leq \alpha^*\|\mathcal{H}\mu_{|\Gamma}\|_\Lambda = \alpha^*\|\mu\|_\Lambda ,$$

(see e.g. [8]).
Thanks to (48) we can now prove the continuity of S_f; in fact, for all $\mu, \eta \in \Lambda_0$, we have:

$$| < S_f\mu, \eta > | = |a_f(R_f^1\mu, R_f^1\eta)| \leq \beta_f\|\mu\|_\Lambda\|\eta\|_\Lambda ,$$

where β_f is the positive continuity constant

$$\beta_f := \frac{4\nu}{n^2}\left[\frac{\alpha^*}{\kappa_f}\left(1 + \frac{1}{\beta^0}\right)\right]^2 . \qquad (49)$$

Let us now consider the issue of continuity of S_p. Let m_K be the positive constant introduced in (23). Thanks to the Poincaré inequality (21) and (32) we have:

$$\begin{aligned}
\|R_p\mu\|^2_{H^1(\Omega_p)} &\leq (1 + C_{\Omega_p})\|\nabla R_p\mu\|^2_{L^2(\Omega_p)} \\
&\leq \frac{1 + C_{\Omega_p}}{m_K}\int_{\Omega_p}\nabla(R_p\mu)\cdot\mathsf{K}\nabla(R_p\mu) \\
&= \frac{1 + C_{\Omega_p}}{m_K}\int_\Gamma (R_p\mu)_{|\Gamma}\,\mu .
\end{aligned}$$

Finally, the Cauchy–Schwarz inequality and the trace inequality (18) allow us to deduce

$$\|R_p\mu\|_{H^1(\Omega_p)} \leq \frac{C_p}{m_K}(1 + C_{\Omega_p})\|\mu\|_\Lambda , \qquad \forall\mu \in \Lambda_0 .$$

Then, $\forall\mu, \eta \in \Lambda_0$,

$$\begin{aligned}
| < S_p\mu, \eta > | &\leq g\|R_p\mu_{|\Gamma}\|_{L^2(\Gamma)}\|\eta\|_{L^2(\Gamma)} \\
&\leq gC_p\|R_p\mu\|_{H^1(\Omega_p)}\|\eta\|_\Lambda \leq \frac{gC_p^2(1 + C_{\Omega_p})}{m_K}\|\mu\|_\Lambda\|\eta\|_\Lambda ,
\end{aligned}$$

Thus S_p is continuous, with continuity constant $\beta_p := [gC_p^2(1+C_{\Omega_p})]m_K^{-1}$.

2. S_f is symmetric thanks to (43). Using again the Korn inequality (20) and the trace inequality (17), for all $\mu \in \Lambda_0$ it holds:

$$
\begin{aligned}
< S_f\mu, \mu > \quad &\geq \quad \frac{n\nu\kappa_f}{2}\|R_f^1\mu\|^2_{H^1(\Omega_f)} \\
&\geq \quad \frac{n\nu\kappa_f}{2C_f}\|(R_f^1\mu\cdot\mathbf{n}_f)_{|\Gamma}\|^2_\Lambda \;=\alpha_f\|\mu\|^2_\Lambda ,
\end{aligned}
$$

thus S_f is coercive, with a coercivity constant given by

$$
\alpha_f := \frac{n\nu\kappa_f}{2C_f} . \tag{50}
$$

3. S_p is symmetric by definition. Moreover, thanks to (32), $\forall\mu\in\Lambda_0$,

$$
< S_p\mu, \mu >= \int_\Gamma g\,(R_p\mu)\mu = \frac{1}{n}\int_{\Omega_p} g\,\nabla(R_p\mu)\cdot \mathsf{K}\nabla(R_p\mu) .
$$

On the other hand, $\exists C_0 > 0$:

$$
\begin{aligned}
\|\mu\|_{\Lambda'} \quad &= \quad \sup_{\eta\in\Lambda_0}\frac{< \mathsf{K}\nabla(R_p\mu)\cdot\mathbf{n}_p, \eta >}{\|\mathcal{H}_p\eta\|_{H^1(\Omega_p)}} \\
&= \quad \sup_{\eta\in\Lambda_0}\frac{\int_{\Omega_p}\nabla(\mathcal{H}_p\eta)\cdot\mathsf{K}\nabla(R_p\mu)}{\|\mathcal{H}_p\eta\|_{H^1(\Omega_p)}} \leq C_0\|R_p\mu\|_{H^1(\Omega_p)} .
\end{aligned}
$$

We have denoted by Λ' the dual space of Λ_0 , and by $<\cdot,\cdot>$ the duality pairing between Λ' and Λ_0 . Moreover , we have denoted by $\mathcal{H}_p\eta$ the harmonic extension of η to $H^1(\Omega_p)$, i.e. the (weak) solution of the problem:

$$
\begin{array}{ll}
\mathrm{div}(\mathsf{K}\nabla\mathcal{H}_p\eta) = 0 & \text{in } \Omega_p \\
\mathsf{K}\nabla(\mathcal{H}_p\eta)\cdot\mathbf{n}_p = \mu & \text{on } \Gamma_p \\
\mathcal{H}_p\eta = 0 & \text{on } \Gamma_p^b .
\end{array}
$$

We conclude that $< S_p\mu, \mu > \geq \|\mu\|^2_{\Lambda'}$, for a suitable constant $C > 0$. □

The following result is a straightforward consequence of Lemma 4.1.

Corollary 4.1 *The global Steklov–Poincaré operator S is symmetric, continuous and coercive. Moreover S and S_f are spectrally equivalent;* i.e. *there exist two positive constants k_1 and k_2 such that*

$$
k_1 < S_f\eta, \eta > \leq < S\eta, \eta > \leq\; k_2 < S_f\eta, \eta > \qquad \forall\eta\in\Lambda.
$$

5 Subdomain Iterative Method for the Coupled Problem Stokes/Darcy

Our aim is to solve the Stokes/Darcy problem by an appropriate numerical scheme based on domain decomposition methods and, in particular, inspired by the Dirichlet–Neumann method in heterogeneous domain decomposition theory (see [8]).

The method we advocate, and that we will illustrate more precisely in next Section, computes the solution of the coupled problem through the independent solution of Darcy's equation in Ω_p and of the Stokes problem in Ω_f, and comprises three steps:

1. solve Darcy's equation in Ω_p using $(3)_1$ as Neumann boundary condition on the interface Γ;

2. solve the Stokes problem in Ω_f using $(3)_3$, and recovering $(3)_2$ (with $\alpha_1 = 0$) as natural boundary condition on Γ ;

3. use a suitable relaxation depending on a positive parameter θ to enforce the continuity of the normal velocity $\mathbf{u}_p \cdot \mathbf{n}_f = \mathbf{u}_f \cdot \mathbf{n}_f$ on the interface Γ at the following iterative step.

This procedure will be iterated till the fulfillment of a suitable convergence test.

The iterative method we propose can be improved by a suitable re-interpretation. In fact, one step of the iterative procedure can be regarded as a preconditioned Richardson iteration for the Steklov–Poincaré problem (39) on the interface Γ, the preconditioner being S_f. Precisely,

$$\begin{cases} \lambda_0 \in \Lambda_0 \\ \lambda^{k+1} = \lambda^k + \theta S_f^{-1}(\chi - S\lambda^k), \quad k \geq 0 \, , \end{cases} \tag{51}$$

λ_0 being an initial guess in Λ_0 and the equality (51) being valid in Λ_0.

The sequence $\{((\mathbf{u}_f^0)^k, p^k, \varphi_0^k)\}_{k\geq 0}$, generated by the iterative method, converges, for $k \to \infty$, to the solution of the Stokes/Darcy coupled problem for values of the relaxation parameter θ in a bounded interval $(0, \theta_{max})$, where $\theta_{max} < 1$ depends on the coecivity and continuity constants of the Steklov–Poincaré operators S_f and S_p. This result can be obtained by applying an abstract convergence theorem (see [8], Theorem 4.2.2).

Obviously, more effective iterative solvers (e.g. the Conjugate Gradient) with the same preconditioner can be used, which yield a convergence rate independent of the grid-size of the numerical approximation and select the acceleration parameter dynamically.

6 Finite Element Approximation of the Stokes/Darcy Problem

We consider a regular triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, or tetrahedra in the

3–dimensional case, such that the triangulations $\mathcal{T}_{fh}$ and $\mathcal{T}_{ph}$ induced on the subdomains Ω_f and Ω_p are compatible on Γ, and the triangulation $\mathcal{M}_h$ induced on Γ is quasi-uniform (see e.g. [9]).
Several choices of finite element spaces can be made in order to approximate the functional spaces H_f, Q and H_p, however the spaces H_{fh} and Q_h, that approximate H_f and Q, respectively, have to satisfy the discrete inf–sup condition: $\exists\beta^* > 0$, independent of h, such that $\forall q_h \in Q_h$

$$\exists \mathbf{v}_h \in H_{fh},\, \mathbf{v}_h \neq 0 : \quad \int_{\Omega_f} q_h \,\mathrm{div}\mathbf{v}_h \, d\Omega_f \geq \beta^* \|\mathbf{v}_h\|_{H^1(\Omega_f)} \|q_h\|_{L^2(\Omega_f)} \,. \tag{52}$$

One possible choice is

$$X^2_{fh} := \{v_h \in C^0(\overline{\Omega}_f) |\, v_h = 0 \text{ on } \Gamma_f \cup \Gamma_f^{in},\; v_{h|K} \in \mathbb{P}_2(K),\, \forall K \in \mathcal{T}_{fh}\}\,; \tag{53}$$

$$Q_h := \{q_h \in C^0(\overline{\Omega}_f) |\, q_{h|K} \in \mathbb{P}_1(K),\, \forall K \in \mathcal{T}_{fh}\}\,; \tag{54}$$

$$H_{ph} := \{\psi_h \in C^0(\overline{\Omega}_p) |\, \psi_h = 0 \text{ on } \Gamma_p,\; \psi_{h|K} \in \mathbb{P}_1(K),\, \forall K \in \mathcal{T}_{ph}\}\,; \tag{55}$$

$$H_{fh} := (X^2_{fh})^d,\; d = 2, 3\,, \qquad W_h := H_{fh} \times H_{ph}\,. \tag{56}$$

Finally we consider the space $\Lambda_h := \{v_{h|_\Gamma} | v_h \in X^2_{fh}\}$ to approximate the trace space Λ on Γ.

The Galerkin approximation of (15) reads: find $\underline{u}_h = (\mathbf{u}_{fh}, \varphi_{0h}) \in W_h$ and $p_h \in Q_h$:

$$\begin{array}{ll} \mathcal{A}(\underline{u}_h, \underline{v}_h) + \mathcal{B}(\underline{v}_h, p_h) = < \mathcal{F}^*, \underline{v}_h > & \forall \underline{v}_h \in W_h \\ \mathcal{B}(\underline{u}_h, q_h) = 0 & \forall q_h \in Q_h\,, \end{array} \tag{57}$$

where $\mathcal{F}^*$ is a linear functional accounting for a suitable discrete extension of an approximation of the boundary data $\mathbf{u}_{in}$ and φ_p, assigned on Γ_f^{in} and Γ_p, respectively. In the following we shall indicate these discrete extensions by $E_{fh}\mathbf{u}_{in\,h}$ and $E_{ph}\varphi_{ph}$.

Remark 6.1 *In the discrete case the coupling condition (3)$_3$ has to be intended in the sense of the $L^2(\Gamma)$–projection on the finite element space H_{ph} on Γ. In fact in the weak form (57) we are imposing*

$$\int_\Gamma \left(\frac{1}{n}\nabla\varphi_h \cdot \mathbf{n}_f - \mathbf{u}_{fh}\cdot\mathbf{n}_f\right)\psi_{h|_\Gamma} = 0 \qquad \forall \psi_h \in H_{ph}\,,$$

that is $\Pi(\mathbf{u}_{fh}\cdot\mathbf{n}_f) = (1/n)\nabla\varphi_h\cdot\mathbf{n}_f$, where Π is the projection operator on $H_{ph_{|\Gamma}}$ with respect to the scalar product of $L^2(\Gamma)$.

We remark that the existence, uniqueness and stability of the discrete solution of problem (57) can be proved following the same approach of the continuous case, using the theory developed by Brezzi (see [11]).

6.1 Iterative Method for the Numerical Solution of the Coupled Problem

The iterative method we propose to compute the solution of the Stokes/Darcy problem reads as follows:

let λ_h^0 be an initial guess; solve for $k \geq 0$:

$$\begin{aligned}&\text{find}\, \varphi_{0h}^{k+1} \in H_{ph} :\\ &\int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla\varphi_{0h}^{k+1} - \int_\Gamma n\,\psi_h\,\lambda_h^k = -\int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla(E_{ph}\varphi_{ph})\\ &\forall\psi \in H_{ph}\ ;\end{aligned} \tag{58}$$

$$\begin{aligned}&\text{find}\, (\mathbf{u}_{fh}^0)^{k+1} \in H_{fh},\, p_h^{k+1} \in Q_h :\\ &a_f((\mathbf{u}_{fh}^0)^{k+1}, \mathbf{w}_h) - \int_{\Omega_f} p_h^{k+1}\,\mathrm{div}\mathbf{w}_h + \int_\Gamma g\varphi_h^{k+1}\mathbf{w}_h \cdot \mathbf{n}_f\\ &= \int_{\Omega_f} \mathbf{f}\cdot\mathbf{w}_h - a_f(E_{fh}\mathbf{u}_{in\,h}, \mathbf{w}_h) \qquad \forall\mathbf{w}_h \in H_{fh}\\ &\int_{\Omega_f} q_h\,\mathrm{div}(\mathbf{u}_{fh}^0)^{k+1} = 0 \qquad \forall q_h \in Q_h,\end{aligned} \tag{59}$$

with $\varphi_h^{k+1} = \varphi_{0h}^{k+1} + E_{ph}\varphi_{ph}$;

$$\lambda_h^{k+1} := \theta_h[((\mathbf{u}_{fh}^0)^{k+1} + E_{fh}\mathbf{u}_{in})\cdot\mathbf{n}_f|_\Gamma] + (1-\theta_h)\lambda_h^k\ , \tag{60}$$

being θ_h a positive relaxation parameter.

The above iterative scheme can be reinterpreted using Galerkin approximation of the Steklov-Poincaré operators introduced in Sect. 4. This interpretation is useful to carry out the convergence analysis of the scheme.

Due to space constraints we don't develop here a complete convergence analysis; together with all the details concerning the finite element approximation of the Stokes/Darcy problem, this will make the subject of a future paper, where we also present numerical results on several test problems.

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