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# On arbitrarily regular conforming virtual element methods for elliptic partial differential equations

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## Abstract

The Virtual Element Method (VEM) is a very effective framework to design numerical approximations with high global regularity to the solutions of elliptic partial differential equations. In this paper, we review the construction of such approximations for an elliptic problem of order  $p_1$  using conforming, finite dimensional subspaces of  $H^{p_2}(\Omega)$ , where  $p_1$  and  $p_2$  are two integer numbers such that  $p_2 \geq p_1 \geq 1$  and  $\Omega \in \mathbb{R}^2$  is the computational domain. An abstract convergence result is presented in a suitably defined energy norm. The space formulation and major aspects such as the choice and unisolvence of the degrees of freedom are discussed, also providing specific examples corresponding to various practical cases of high global regularity. Finally, the construction of the “enhanced” formulation of the virtual element spaces is also discussed in details with a proof that the dimension of the “regular” and “enhanced” spaces is the same and that the virtual element functions in both spaces can be described by the same choice of the degrees of freedom.

**Keywords:** polyharmonic problem, virtual element method, polytopal mesh, high-order methods, high-regular methods

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## 1. Introduction

The conforming finite element method is based on the construction of a finite dimensional approximation spaces that are typically only  $C^0$ -continuous [39] on the meshes covering the computational domain. The construction of such approximation spaces with higher regularity is normally deemed a difficult task because it requires a set of basis functions with such global regularity. Examples in this direction can be found all along the history of finite elements from the oldest works in the sixties of the last century, e.g., [12, 26, 40] to the most recent attempts in [44, 45, 63, 64]. Despite its intrinsic difficulty, designing approximations with global  $C^1$ - or higher regularity is still a major research topic. Such kind of approximations have indeed a natural application in the numerical treatment of problems involving high-order differential operators.

The Virtual Element Method (VEM) [16] does not require the explicit knowledge of the basis functions spanning the approximation spaces in its formulation and implementation. The crucial idea behind the VEM is that the elemental approximation spaces, which are globally “glued” in a highly regular conforming way, are defined elementwise as the solutions of a partial differential equation. The functions that belong to such approximation spaces are dubbed as “virtual” as they are never really computed, with the noteworthy exception of a subspace of polynomials that are indeed used in the formulation of the method. The virtual element functions are uniquely characterized by a set of values, the *degrees of freedom*, that are actually solved for in the method.

The virtual element ‘paradigm’ thus provides a major breakthrough in obtaining highly regular Galerkin methods as it allows the construction of numerical approximation of any order of accuracy and global conforming regularity that work on unstructured two-dimensional and three-dimensional meshes with very general polytopal elements.

Roughly speaking, the VEM is a Galerkin-type projection method that generalize the finite element method, which was originally designed for simplicial and quadrilateral/hexahedral meshes, to polytopal meshes. Other important families of methods that are suited to polytopal meshes are the polygonal/polyhedral finite element method [59] the mimetic finite difference method [21] the discontinuous Galerkin method on polygonal/polyhedral grids [7, 34]; the hybrid discontinuous Galerkin method [41]; and the hybrid high-order method [42].

The conforming VEM was first developed for second-order elliptic problems in primal formulation [16, 19], and then in mixed formulation [18, 32] and nonconforming formulation [15]. Despite its relative youthness (the first paper was published in 2013), the VEM has been very successful in a wide range of scientific and engineering applications. A non-exhaustive list of applications includes, for example, the works of References [2, 5, 6, 8, 13, 14, 27, 28, 35, 37, 50, 56–58, 62]. Virtual element spaces forming de Rham complexes for the Stokes, Navier-Stokes and Maxwell equations were proposed in [17, 22, 23]. A VEM for Helmholtz problems based on non-conforming approximation spaces of Trefftz functions, i.e., functions that belong to the kernel of the Helmholtz operator, is found [51].

The first works using a  $C^1$ -regular conforming VEM addressed the classical plate bending problems [33, 38], second-order elliptic problems [24, 25], and the nonlinear Cahn-Hilliard equation [3]. More recently, highly regular virtual element spaces were considered for the von Kármán equation modelling the deformation of very thin plates [49], geostrophic equations [53] and fourth-order subdiffusion equations [48], two-dimensional plate vibration problem of Kirchhoff plates [52], the transmission eigenvalue problems [54] the fourth-order plate buckling eigenvalue problem [55]. In [10], we proposed the highly-regular conforming VEM for the two-dimensional polyharmonic problem  $(-\Delta)^{p_1} u = f$ ,  $p_1 \geq 1$ . The VEM is based on an approximation space that locally contains polynomials of degree  $r \geq 2p_1 - 1$  and has a global  $H^{p_1}$  regularity. In [9], we extended this formulation to a virtual element space that can have arbitrary regularity  $p_2 \geq p_1 \geq 1$  and contains polynomials of degree  $r \geq p_2$ . This VEM is a generalization of the VEMs for second- and fourth-order problems since the approximation space for  $p_2 = p_1 = 1$  coincides with the conforming virtual element spaces for the Poisson equation of Reference [16] and the approximation space for  $p_2 = p_1 = 2$  coincides with the conforming virtual element spaces for the and the biharmonic equation of Reference [33]. VEMs for three-dimensional problems are also available for the fourth-order linear elliptic equation [20] (see also [31]), and highly-regular conforming VEM in any dimension has been proposed in [46].

In this paper, we review the detailed construction of the virtual element spaces with arbitrary order of accuracy and regularity for the numerical approximation of two-dimensional problems involving the polyharmonic operator of degree  $p_1$ . Such a construction follows the standard guidelines of the VEM, which we briefly summarize here. As the VEM is a conforming Galerkin variational method, its formulation requires the definition of a suitable finite dimensional approximation space, which is obtained by combining in a conforming way local (elemental) finite dimensional spaces. The local virtual element spaces are defined in every mesh element by all the solutions of a specific polyharmonic problem of degree  $p_2 \geq p_1$ . The loading terms of the partial differential equations defining the elemental virtual element spaces can be all the polynomials of degree (up to)  $r - 2p_1$ , where the integer number  $r \geq p_2$  is the order of the virtual element space. The traces of the virtual element functions and all its normal derivatives of order  $j$  from one to  $p_2 - 1$  on the elemental edges are univariate polynomials of degree at least  $r - j$  (in some cases the polynomial degree can be a little higher than  $r - j$ ). From the definition, it also follows the fundamental property that the polynomials of degree up to  $r$  inside all elements are a linear subspace of the virtual element space of degree  $r$ . Then, the elemental spaces are ‘‘glued’’ together to form a global space with  $H^{p_2}$ -regularity. A very careful choice of the degrees of freedom, which as usual are nodal

values associated with the mesh vertices or polynomial moments associated with edges and elements, makes the elliptic projection onto the polynomials of degree  $r$  computable. An  $L^2$ -orthogonal projection onto the polynomials of degree  $r - p_1$  in every mesh element is also computable in the “modified” (or “enhanced”) formulation of the virtual element method. In this work, we also present a detailed discussion of the enhanced formulation and its major properties. The enhanced formulation is obtained by extending the similar construction for the Poisson equation presenting in the pioneering paper [1] to our case. These polynomial projection operators are finally used to construct the discrete approximation of the bilinear form and the right-hand side that are used in the virtual element approximation. An abstract convergence result holds, that can be proved by assuming only a few fundamental properties of the virtual element formulation.

The remaining part of the manuscript is organized as follows. In Section 2 we introduce the continuous polyharmonic problem and its weak formulation. In Section 3 we introduce the virtual element discretization and recall the main abstract convergence result. In Section 4 we present the formulation of the conforming virtual element approximation with higher-order regularity. Finally, in Section 5 we draw our conclusions.

## 2. The continuous problem

Let  $\Omega \subset \mathbb{R}^2$  be an open, bounded, convex domain with polygonal boundary  $\Gamma$ . For any integer  $p_1 \geq 1$ , we consider the polyharmonic problem

$$(-\Delta)^{p_1} u = f \quad \text{in } \Omega, \quad (1a)$$

$$\partial_n^j u = 0 \quad \text{for } j = 0, \dots, p_1 - 1 \text{ on } \Gamma, \quad (1b)$$

where  $\partial_n u = \mathbf{n} \cdot \nabla u$  is the normal derivative of  $u$  and  $\partial_n^j u$  is the normal derivative applied  $j$  times to  $u$  with the useful convention that  $\partial_n^0 u = u$  for  $j = 0$ . Let

$$V := H_0^{p_1}(\Omega) = \{v \in H^{p_1}(\Omega) : \partial_n^j v = 0 \text{ on } \Gamma, j = 0, \dots, p_1 - 1\}.$$

Denoting the duality pairing between  $V$  and its dual  $V'$  by  $\langle \cdot, \cdot \rangle$ , the variational formulation of the polyharmonic problem (1) reads as

$$\text{Find } u \in V \text{ such that: } a_{p_1}(u, v) = \langle f, v \rangle \quad \forall v \in V, \quad (2)$$

where, for any nonnegative integer  $\ell$ , the bilinear form  $a_{p_1}(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  is given by

$$a_{p_1}(u, v) := \begin{cases} \int_{\Omega} \nabla \Delta^{\ell} u \cdot \nabla \Delta^{\ell} v \, d\mathbf{x} & \text{for } p_1 = 2\ell + 1, \ell \geq 0, \\ \int_{\Omega} \Delta^{\ell} u \Delta^{\ell} v \, d\mathbf{x} & \text{for } p_1 = 2\ell, \ell \geq 1. \end{cases} \quad (3)$$

If  $f \in L^2(\Omega)$  we have

$$\langle f, v \rangle := (f, v) = \int_{\Omega} f v \, d\mathbf{x}, \quad (4)$$

where  $(\cdot, \cdot)$  denotes the  $L^2$ -inner product. The bilinear form  $a_{p_1}(\cdot, \cdot)$  is coercive and continuous with respect to  $\|u\|_V := (a_{p_1}(u, u))^{1/2}$ , which is a norm on  $H_0^{p_1}(\Omega)$ . The coercivity and continuity constants are respectively denoted by  $\alpha$  and  $M$ , and their value depends on the regularity of  $\Omega$  and its boundary

Γ. Coercivity and continuity implies existence and uniqueness of the solution to (2) from an application of the Lax-Milgram theorem [30, Theorem 2.7.7]. About the regularity of the solution to (2), it is worth mentioning the result in [43, Corollary 2.21]. Accordingly, if the domain boundary  $\partial\Omega$  is  $C^k$ -regular for  $k \geq 2p_1$  and  $f \in H^{k-2p_1}(\Omega)$ , then  $u \in H^k(\Omega) \cap H_0^{p_1}(\Omega)$  and it holds that  $\|u\|_k \leq C\|f\|_{k-2p_1}$ . As pointed out in [9], the regularity of  $u$  for domains with irregular boundaries is still an open issue. However, we know that a similar result holds for the biharmonic problem, i.e.,  $p_1 = 2$ , if  $\Omega$  is a bounded, convex, polygonal domain see [29].

### 3. The discrete problem and an abstract convergence result

Let  $r$  and  $p_2$  be two integer numbers such that  $r \geq p_2 \geq p_1 \geq 1$ . The virtual element approximation to the variational problem (2) reads as

$$\text{Find } u_h \in V_{h,r}^{p_2,p_1} \text{ such that: } a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_{h,r}^{p_2,p_1}, \quad (5)$$

where the virtual element space  $V_{h,r}^{p_2,p_1}$  is a finite-dimensional conforming subspace of  $V$ ;  $a_h(\cdot, \cdot) : V_{h,r}^{p_2,p_1} \times V_{h,r}^{p_2,p_1} \rightarrow \mathbb{R}$  is the virtual element bilinear form that approximates the bilinear form (3);  $\langle f_h, \cdot \rangle : V_{h,r}^{p_2,p_1} \rightarrow \mathbb{R}$  is the continuous linear functional that approximates (4) through an element  $f_h$  of the dual space  $(V_{h,r}^{p_2,p_1})^*$  of  $V_{h,r}^{p_2,p_1}$ . The formal definition and properties of  $V_{h,r}^{p_2,p_1}$ ,  $a_h(\cdot, \cdot)$  and  $f_h$  are discussed in the next section.

#### 3.1. Mesh notation, mesh regularity and some basic definitions

The virtual element method is formulated on the mesh family  $\{\Omega_h\}_h$ , where each mesh  $\Omega_h$  is a partition of the computational domain  $\Omega$  into nonoverlapping polygonal elements  $P$  and is labeled by the mesh size parameter  $h$  that is defined below. A polygonal element  $P$  is a compact subset of  $\mathbb{R}^2$  with boundary  $\partial P$ , area  $|P|$ , center  $\mathbf{x}_P$ , and diameter  $h_P = \sup_{\mathbf{x}, \mathbf{y} \in P} |\mathbf{x} - \mathbf{y}|$ . The mesh elements of  $\Omega_h$  form a finite cover of  $\Omega$  such that  $\bar{\Omega} = \cup_{P \in \Omega_h} P$  and the mesh size labeling each mesh  $\Omega_h$  is defined by  $h = \max_{P \in \Omega_h} h_P$ . A mesh edge  $e$  has center  $\mathbf{x}_e$  and length  $h_e$  and we denote the set of mesh edges by  $\mathcal{E}_h$ . A mesh vertex  $\mathbf{v}$  has position vector  $\mathbf{x}_v$  and we denote the set of mesh vertices by  $\mathcal{V}_h$ . Moreover, in the definition of the degrees of freedom of the next section, we associate every vertex  $\mathbf{v}$  with a characteristic length  $h_v$ . This characteristic length  $h_v$  can be the average of the diameters of the polygons sharing  $\mathbf{v}$ .

For any integer number  $\ell \geq 0$ , we let  $\mathbb{P}_\ell(P)$  and  $\mathbb{P}_\ell(e)$  denote the space of polynomials defined on  $P$  and  $e$ , respectively, and  $\mathbb{P}_\ell(\Omega_h)$  denotes the space of piecewise polynomials of degree  $\ell$  on the mesh  $\Omega_h$ . Accordingly, if  $q \in \mathbb{P}_\ell(\Omega_h)$  then it holds that  $q|_P \in \mathbb{P}_\ell(P)$  for all  $P \in \Omega_h$ . Finally, we define the (broken) seminorm of a function  $v \in \prod_{P \in \Omega_h} H^{p_1}(P)$  by

$$\|v\|_h^2 = \sum_{P \in \Omega_h} a_{p_1}^P(v, v).$$

Throughout the paper, we use the multi-index notation, so that  $\nu = (\nu_1, \nu_2)$  is a two-dimensional index defined by the two integer numbers  $\nu_1, \nu_2 \geq 0$ . Moreover,  $D^\nu w = \partial^{|\nu|} w / \partial x^{\nu_1} \partial y^{\nu_2}$  denotes the partial derivative of order  $|\nu| = \nu_1 + \nu_2 > 0$  of a given bivariate function  $w(x, y)$ , and we use the conventional notation that  $D^{(0,0)} w = w$  for  $\nu = (0, 0)$ . We denote the partial derivatives of  $w$  versus  $x$  and  $y$  by the shortcuts  $\partial_x w, \partial_y w, \partial_{xx} w, \partial_{xy} w, \partial_{yy} w$ , etc. We denote the normal and tangential derivatives with respect to a given edge and their mixed combination by  $\partial_n w, \partial_t w, \partial_{nt} w, \partial_{nn} w$ , etc, and use the shorter notation  $\partial_t^j w$  and  $\partial_n^j w$  for the tangential and normal derivatives of  $w$  of order  $j$ .

### 3.2. Abstract convergence theorem

For the mathematical formulation of the virtual element approximation (5), we require the two following assumptions on the virtual element space  $V_{h,r}^{p_2,p_1}$  and the bilinear form  $a_h(\cdot, \cdot)$ :

**(H1).** For all  $h > 0$ , the *global* virtual element space  $V_{h,r}^{p_2,p_1}$  is a conforming, finite-dimensional subspace of  $V = H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega)$  such that for all elements  $P$  of all mesh partitions  $\Omega_h$  it holds that

- $V_{h,r}^{p_2,p_1}(P)$ , the *local* (elemental) virtual element space that is defined as the restriction of  $V_{h,r}^{p_2,p_1}$  to the element  $P$  is a finite-dimensional subspace of  $H^{p_2}(P)$ ;
- $\mathbb{P}_r(P)$ , the space of polynomials of degree up to  $r$  defined on  $P$  is a subspace of  $V_{h,r}^{p_2,p_1}(P)$ .

**(H2).** The bilinear form  $a_h(\cdot, \cdot) : V_{h,r}^{p_2,p_1} \times V_{h,r}^{p_2,p_1} \rightarrow \mathbb{R}$  admits the elementwise decomposition

$$a_h(u_h, v_h) = \sum_{P \in \Omega_h} a_h^P(u_h, v_h) \quad \forall u_h, v_h \in V_{h,r}^{p_2,p_1},$$

where for all element  $P$  the local bilinear form  $a_h^P(\cdot, \cdot)$  is symmetric and such that

**(r-Consistency):** for every polynomial  $q \in \mathbb{P}_r(P)$  and every virtual element function  $v_h \in V_{h,r}^{p_2,p_1}(P)$  it holds that

$$a_h^P(v_h, q) = a_{p_1}^P(v_h, q); \quad (6)$$

**(Stability):** there exist two positive constants  $\alpha_*$ ,  $\alpha^*$  independent of  $h$  and  $P$  such that for every  $v_h \in V_{h,r}^{p_2,p_1}(P)$  it holds that

$$\alpha_* a_{p_1}^P(v_h, v_h) \leq a_h^P(v_h, v_h) \leq \alpha^* a_{p_1}^P(v_h, v_h). \quad (7)$$

The stability constant  $\alpha_*$  and  $\alpha^*$  may depend on the polynomial approximation degree  $r$ , see, e.g., [11] for the case  $p_1 = 1$ .

Assumption **(H2)** implies that the symmetric bilinear form  $a_h(\cdot, \cdot)$  is coercive and continuous, so that the existence and uniqueness of the solution  $u_h$  follows from an application of the Lax-Milgram theorem [30, Theorem 2.7.7]. Under these assumptions we can prove this abstract convergence result.

**Theorem 3.1.** Let  $u \in V$  be the solution to the variational problem (1) and  $u_h \in V_{h,r}^{p_2,p_1}$ ,  $r \geq p_2 \geq p_1 \geq 1$ , the solution to the virtual element approximation (5) under assumptions **(H1)**-**(H2)**. Then, there exists a constant  $C$  independent of  $h$  such that

$$\|u - u_h\|_V \leq C \left( \|u - u_I\|_V + \|u - u_\pi\|_h + \|f_h - f\|_{(V_{h,r}^{p_2,p_1})^*} \right), \quad (8)$$

for every virtual element approximation  $u_I$  in  $V_{h,r}^{p_2,p_1}$  and any piecewise polynomial approximation  $u_\pi \in \mathbb{P}_r(\Omega_h)$  of  $u$ . The constant  $C$  is proportional to  $(M/\alpha) (\alpha^*/\alpha_*)$ .

The proof of this theorem was first published in [10] for the case with  $p_2 = p_1$  and then extended to the case for  $p_2 \geq p_1$  in [9].

#### 4. The virtual element spaces of higher-order continuity

In this section, we present the formulation of the virtual element method of Eq. (5). To this end, we first introduce the local virtual element spaces, the degrees of freedom, and the global virtual element space, which is obtained by “gluing” in a conforming way the local spaces. Then, we discuss the computability of the elliptic projection operator, and the enhancement of the local spaces, which allows us to compute the orthogonal projection operators onto the subspace of polynomials of degree up to  $p_1 - 1$ . Finally, we discuss the construction of the bilinear form  $a_h(\cdot, \cdot)$  and the load term  $\langle f_h, \cdot \rangle$ .

##### 4.1. Local space definitions

Let  $P$  be a mesh element. For  $p_2 \leq r \leq 2p_2 - 2$ , we consider the local virtual element space defined as

$$V_{h,r}^{p_2,p_1}(P) = \left\{ v_h \in H^{p_2}(P) : \Delta^{p_2} v_h \in \mathbb{P}_{r-2p_1}(P), \partial_n^j v_h \in \mathbb{P}_{\alpha_j(p_2,r)}(e), \right. \\ \left. j = 0, \dots, p_2 - 1 \forall e \in \partial P \right\}, \quad (9)$$

where  $\alpha_j(p_2, r) = \max\{2(p_2 - j) - 1, r - j\}$ . For  $r \geq 2p_2 - 1$  it holds that  $\alpha_j = r - j$  for all  $j = 0, \dots, p_2 - 1$  and the definition of the local virtual element space on the element  $P$  takes the simpler form

$$V_{h,r}^{p_2,p_1}(P) = \left\{ v_h \in H^{p_2}(P) : \Delta^{p_2} v_h \in \mathbb{P}_{r-2p_1}(P), \partial_n^j v_h \in \mathbb{P}_{r-j}(e), \right. \\ \left. j = 0, \dots, p_2 - 1 \forall e \in \partial P \right\}. \quad (10)$$

In both definitions (9) and (10) we use the conventional notation that  $\mathbb{P}_r(P) = \{0\}$  if  $r < 0$ .

**Remark 4.1.** The space of polynomials  $\mathbb{P}_r(P)$  is a subspace of  $V_{h,r}^{p_2,p_1}(P)$  for both definitions (9) and (10).

**Remark 4.2.** Let  $\#(\mathcal{P})$  denote the cardinality of a (finite dimensional) space  $\mathcal{P}$  and  $N_P^E$  and  $N_P^V$  the number of edges and vertices of element  $P$ . The dimension of the local virtual element space (9) is given by

$$\dim V_{h,r}^{p_2,p_1}(P) = \#(\mathbb{P}_{r-2p_1}(P)) + \sum_{e \in \partial P} \sum_{j=0}^{p_2-1} \#(\mathbb{P}_{\alpha_j(p_2,r)}(e)) - N_P^V \frac{(p_2+1)p_2}{2} \\ = \frac{(r-2p_1+1)(r-2p_1+2)}{2} + N_P^E \sum_{j=0}^{p_2-1} (\alpha_j(p_2,r) + 1) \\ - N_P^V \frac{(p_2+1)p_2}{2}. \quad (11)$$

The dimension of the local virtual element space (10) is given by

$$\dim V_{h,r}^{p_2,p_1}(P) = \#(\mathbb{P}_{r-2p_1}(P)) + \sum_{e \in \partial P} \sum_{j=0}^{p_2-1} \#(\mathbb{P}_{r-j}(e)) - N_P^V \frac{(p_2+1)p_2}{2} \\ = \frac{(r-2p_1+1)(r-2p_1+2)}{2} + N_P^E \frac{p_2(2r+3-p_2)}{2}$$

$$- N_P^{\mathcal{V}} \frac{(p_2 + 1)p_2}{2}. \quad (12)$$

In both equations (11) and (12), the last term of the right-hand side, i.e.,  $N_P^{\mathcal{V}} p_2(p_2 + 1)/2$ , is subtracted to take into account the  $C^{p_2-1}$ -regularity of  $v_h$  at the elemental vertices.

#### 4.2. Local degrees of freedom

Let  $\beta_j = \alpha_j - \min\{2(p_2 - j) - 1, r - j\} - 1$ . For  $r = 2p_2 - 1 - k$  with  $k = 1, \dots, p_2 - 1$ , the virtual element functions in the elemental space (9) are uniquely identified by the following degrees of freedom

(D1)  $h_v^{|\nu|} D^{\nu} v_h(\mathbf{v})$ ,  $|\nu| \leq p_2 - 1$  for any vertex  $\mathbf{v}$  of  $\partial P$ ;

(D2)  $h_e^{-1+j} \int_e q \partial_n^j v_h ds$  for any  $q \in \mathbb{P}_{\beta_j}(e)$  and edge  $e$  of  $\partial P$ ,  $j = k + 1, \dots, p_2 - 1$ ;

(D3)  $h_P^{-2} \int_P q v_h d\mathbf{x}$  for any  $q \in \mathbb{P}_{r-2p_1}(P)$ .

For  $r \geq 2p_2 - 1$ , we note that  $\beta_j = r - (2p_2 - j)$  and we consider the polynomial edge moments in (D2) for  $j = 0, \dots, p_2 - 1$ .

**Remark 4.3.** The  $L^2$ -projection operator  $\Pi_{r-2p_1}^{0,P}$  onto the polynomial space  $\mathbb{P}_{r-2p_1}(P)$  is computable from the degrees of freedom (D3).

A virtual element function in  $v_h \in V_{h,r}^{p_2,p_1}(P)$  has the regularity property that  $(v_h)|_{\partial P} \in C^{p_2-1}(\partial P)$ . This regularity is reflected by the choice of the degrees of freedom, and is, indeed, provided by the vertex degrees of freedom of (D1). Furthermore, the traces of  $v_h$  ( $j = 0$ ) and the  $j$ -th normal derivatives (up to order  $j = p_2 - 1$ ) on every edge  $e \in \partial P$  are univariate polynomials of degree at least  $r - j$ . The information provided by (D1) makes it possible to build polynomial traces of degree higher than  $r - j$  if  $r$  is equal to  $p_2$  (or not “too bigger” than  $p_2$  as shown in the following examples). In such a case, we can compute the edge traces  $(v_h)|_e$  and  $(\partial_n^j v_h)|_e$  by solving the interpolation problem that uses the vertex values of  $v_h$  and its partial derivatives.

In the next three examples, we discuss the trace interpolation problem for  $r \geq p_2$  and  $j = 0, j = 1, j \geq 2$ . This process is also shown in Table 2 and Fig. 1

**Example 4.4** ( $j = 0$ ). We derive the higher-order tangential derivatives of  $v_h$  by repetitively applying the differential operator  $\mathbf{t} \cdot \nabla = (t_x \partial_x + t_y \partial_y)$  to the univariate polynomial trace of  $v_h$ , i.e.,  $\partial_t^\ell v_h$  along every elemental edge (recall that  $\partial_t^0 v_h = v_h$  for  $\ell = 0$ ). For example, for  $\ell = 1, 2, 3, 4$ , the tangential derivatives  $\partial_t^\ell v_h$  are given by

$$\begin{aligned} \partial_t v_h(\mathbf{v}_i) &= t_x \partial_x (v_h)(\mathbf{v}_i) + t_y \partial_y (v_h)(\mathbf{v}_i), \\ \partial_t^2 v_h(\mathbf{v}_i) &= (t_x \partial_x + t_y \partial_y) (\partial_t v_h)(\mathbf{v}_i) \\ &= t_x \partial_x (\partial_t v_h)(\mathbf{v}_i) + t_y \partial_y (\partial_t v_h)(\mathbf{v}_i) \\ &= t_x t_x \partial_{xx} v_h(\mathbf{v}_i) + 2t_x t_y \partial_{xy} v_h(\mathbf{v}_i) + t_y t_y \partial_{yy} v_h(\mathbf{v}_i), \\ \partial_t^3 v_h(\mathbf{v}_i) &= (t_x \partial_x + t_y \partial_y) (\partial_t^2 v_h)(\mathbf{v}_i) \\ &= t_x \partial_x (\partial_t^2 v_h)(\mathbf{v}_i) + t_y \partial_y (\partial_t^2 v_h)(\mathbf{v}_i) \end{aligned}$$



$$\begin{aligned}
&= t_x t_x t_x \partial_{xxx} v_h(v_i) + 3t_x t_x t_y \partial_{xxy} v_h(v_i) + 3t_x t_y t_y \partial_{xyy} v_h(v_i) \\
&\quad + t_y t_y t_y \partial_{yyy} v_h(v_i), \\
\partial_t^4 v_h(v_i) &= (t_x \partial_x + t_y \partial_y) (\partial_t^3 v_h)(v_i) \\
&= t_x \partial_x (\partial_t^3 v_h)(v_i) + t_y \partial_y (\partial_t^3 v_h)(v_i) \\
&= t_x t_x t_x t_x \partial_{xxxx} v_h(v_i) + 4t_x t_x t_x t_y \partial_{xxxy} v_h(v_i) + 6t_x t_x t_y t_y \partial_{xxyy} v_h(v_i) \\
&\quad + 4t_x t_y t_y t_y \partial_{xyyy} v_h(v_i) + t_y t_y t_y t_y \partial_{yyyy} v_h(v_i).
\end{aligned}$$

It is easy to recognize the pattern of the combinatorial coefficients in these expansions. According to the third column ( $j = 0$ ) of Table 1, the degrees of freedom **(D1)** at vertex  $v_i$  for a given regularity index  $p_2$  yield  $p_2$  pieces of information,  $v_h(v_i)$ ,  $\partial_t v_h(v_i)$ ,  $\partial_t^2 v_h(v_i)$ ,  $\dots$ ,  $\partial_t^{p_2-1} v_h(v_i)$ . Since each edge has two vertices, we have  $2p_2$  pieces of information and we can interpolate the edge trace of  $v_h$  as a univariate polynomial of degree  $2p_2 - 1$ . Such polynomial degree is clearly bigger than  $r$  if we choose  $r$  such that  $p_2 \leq r < 2p_2 - 1$ . This fact is not in conflict with the property that the virtual element space contains the subspace of polynomials of degree  $r$ .

The polynomial degrees of the edge trace of  $v_h$  that we can interpolate from the degrees of freedom **(D1)**-**(D2)** are illustrated in Table 2 by the rows for  $j = 0$  and different values of  $p_2$ . In this table, the values of  $r$  such that  $r = 2p_2 - 1$  are reported in bold, and the ones for  $r < 2p_2 - 1$  are those preceeding the bold ones on the same row. For these values of  $r$  the trace of  $v_h$  can be interpolated from the information provided by **(D1)**. However, if we increase the polynomial degree  $r$  so that  $r > 2p_2 - 1$ , the degrees of freedom **(D1)** are no longer enough to solve the interpolation problem. In such a case, we need the additional degrees of freedom of **(D2)**, i.e., the moments of  $v_h$  against a (basis of) polynomials of degree  $r - (2p_2 - 1)$  defined on  $e$ .  $\square$

**Example 4.5** ( $j = 1$ ). As for the case  $j = 0$ , we derive the higher-order tangential derivatives of  $\partial_n v_h = n_x \partial_x v_h + n_y \partial_y v_h$  by repetitively applying the differential operator  $\mathbf{t} \cdot \nabla = (t_x \partial_x + t_y \partial_y)$  to the univariate polynomial trace of  $\partial_n v_h$ , i.e.,  $\partial_t^\ell \partial_n v_h$  along every elemental edge (recall that  $\partial_t^0 \partial_n v_h = \partial_n v_h$  for  $\ell = 0$ ). For example, for  $\ell = 1, 2, 3$  we find that

$$\begin{aligned}
\partial_t \partial_n v_h(v_i) &= (t_x \partial_x + t_y \partial_y) (\partial_n v_h)(v_i) \\
&= t_x \partial_x (\partial_n v_h)(v_i) + t_y \partial_y (\partial_n v_h)(v_i) \\
&= t_x n_x \partial_{xx} v_h(v_i) + (t_x n_y + t_y n_x) \partial_{xy} v_h(v_i) + t_y t_y \partial_{yy} v_h(v_i), \\
\partial_t^2 \partial_n v_h(v_i) &= (t_x \partial_x + t_y \partial_y) (\partial_t \partial_n v_h)(v_i) \\
&= t_x \partial_x (\partial_t \partial_n v_h)(v_i) + t_y \partial_y (\partial_t \partial_n v_h)(v_i) \\
&= t_x t_x n_x \partial_{xxx} v_h(v_i) + (t_x (t_x n_y + t_y n_x) + t_y (t_x n_y + t_y n_x)) \partial_{xxy} v_h(v_i) \\
&\quad + t_x t_y (n_x + n_y) \partial_{xyy} v_h(v_i) + t_y t_y n_y \partial_{yyy} v_h(v_i), \\
\partial_t^3 \partial_n v_h(v_i) &= (t_x \partial_x + t_y \partial_y) (\partial_t^2 \partial_n v_h)(v_i) \\
&= (t_x \partial_x + t_y \partial_y) (\partial_t^2 v_h)(v_i) \\
&= t_x \partial_x (\partial_t^2 \partial_n v_h(v_i))(v_i) + t_y \partial_y (\partial_t^2 \partial_n v_h(v_i))(v_i), \\
&= \dots
\end{aligned}$$

According to the fourth column ( $j = 1$ ) of Table 1, the degrees of freedom **(D1)** at vertex  $v_i$  for a given  $p_2$  yield  $p_2 - 1$  pieces of information,  $\partial_n v_h(v_i)$ ,  $\partial_t \partial_n v_h(v_i)$ ,  $\partial_t^2 \partial_n v_h(v_i)$ ,  $\dots$ ,  $\partial_t^{p_2-2} \partial_n v_h(v_i)$ . Since each edge has two vertices, we have  $2(p_2 - 1)$  pieces of information that we can interpolate as a polynomial of degree  $2p_2 - 3$ . Such polynomial degree is clearly bigger than  $r - 1$  if we choose  $r$  such that  $p_2 \leq r < 2(p_2 - 1)$ .

The polynomial degrees of the edge trace of  $\partial_n v_h$  that we can interpolate from the degrees of freedom **(D1)**-**(D2)** are illustrated in Table 2 by the rows for  $j = 1$  and different values of  $p_2$ . In this table, the values of  $r$  such that  $r - 1 = 2(p_2 - 1) - 1$  are reported in bold, and the ones for  $r - 1 < 2(p_2 - 1) - 1$  are those preceeding the bold ones on the same row. For these values of  $r$  the trace of  $\partial_n v_h$  can be interpolated from the information provided by **(D1)**. However, if we increase the polynomial degree  $r$  so that  $r - 1 > 2(p_2 - 1) - 1$ , the degrees of freedom **(D1)** are no longer enough to solve the interpolation problem. In such a case, we need the additional degrees of freedom of **(D2)**, i.e., the moments of  $\partial_n v_h$  against a (basis of) polynomials of degree  $r - 2(p_2 - 1)$  defined on  $e$ .  $\square$

**Example 4.6** ( $j \geq 2$ ). As for the cases  $j = 0$  and  $j = 1$ , we derive the higher-order tangential derivatives of  $\partial_n^j v_h$  by repetitively applying the differential operator  $\mathbf{t} \cdot \nabla = (t_x \partial_x + t_y \partial_y)$  to the univariate polynomial trace of  $\partial_n^j v_h$ , i.e.,  $\partial_t^\ell \partial_n^j v_h$  along every elemental edge. For example, for  $j = 2$ , since  $\partial_n^2 v_h = n_x n_x \partial_{xx} v_h + 2n_x n_y \partial_{xy} v_h + t_y t_y \partial_{yy} v_h$ , we find that

$$\begin{aligned} \partial_t \partial_n^2 v_h(v_i) &= (t_x \partial_x + t_y \partial_y)(\partial_n^2 v_h)(v_i) \\ &= t_x \partial_x (\partial_n^2 v_h)(v_i) + t_y \partial_y (\partial_n^2 v_h)(v_i) \\ &= t_x n_x n_x \partial_{xxx} v_h + (2t_x t_x n_x + t_y n_x n_x) \partial_{xxy} v_h \\ &\quad + (t_x n_y n_y + 2t_y n_x n_y) \partial_{xyy} v_h + t_y n_y n_y \partial_{yyy} v_h. \end{aligned}$$

For  $j \geq 0$ , each edge vertex  $v_i$ ,  $i = 1, 2$ , provides the values of  $\partial_n^j v_h$  and its first  $p_2 - j - 1$  tangential derivatives. Hence, there are  $2(p_2 - j)$  pieces of information available on each edge and we can interpolate the edge trace of  $\partial_n^j v_h$  as a univariate polynomial of degree  $2(p_2 - j) - 1$ . Such polynomial degree is clearly bigger than  $r - j$  if we choose  $r$  such that  $p_2 \leq r < 2p_2 - j - 1$ .

The polynomial degrees of the edge trace of  $\partial_n^j v_h$  that we can interpolate from the degrees of freedom **(D1)**-**(D2)** are illustrated in Table 2 by the rows for  $j \geq 0$  and different values of  $p_2$ . In this table, the values of  $r$  such that  $r = 2p_2 - j - 1$  are reported in bold, and the ones for  $r < 2p_2 - j - 1$  are those preceeding the bold ones on the same row. For these values of  $r$  the trace of  $\partial_n^j v_h$  can be interpolated from the information provided by **(D1)**. However, if we increase the polynomial degree  $r$  so that  $r > 2p_2 - j - 1$ , the degrees of freedom **(D1)** are no longer enough to solve the interpolation problem. In such a case, we need the additional degrees of freedom of **(D2)**, i.e., the moments of  $\partial_n^j v_h$  against a (basis of) polynomials of degree  $r - (2p_2 - j - 1)$  defined on  $e$ . It is worth noting that for increasing values of  $r$ , we need to supplement this information starting from the higher-order normal derivatives (see Fig. 1).  $\square$

**Lemma 4.7.** The degrees of freedom **(D1)**-**(D3)** are unisolvent in the virtual element space  $V_{h,r}^{p_2,p_1}(\mathcal{P})$ .

*Proof.* Let  $\mathcal{P}$  be a polygonal element. First, a counting argument shows that the number of degrees of freedom **(D1)**-**(D3)** is equal to the dimension of  $V_{h,r}^{p_2,p_1}(\mathcal{P})$  (see Remark 4.2). Then, we prove that a virtual element function  $v_h$  is necessarily zero if all its degrees of freedom **(D1)**-**(D3)** are zero. In particular, assuming that the degrees of freedom **(D1)** and **(D2)** are zero implies that the polynomial traces of  $v_h$  and its normal derivatives of order up to  $p_2 - 1$  are identically zero on all edges of  $\partial\mathcal{P}$ , and so are their tangential derivatives of any order. Likewise, assuming that the degrees of freedom **(D3)** are zero implies that the elemental moments of  $v_h$  against the polynomials of degree up to  $r - 2p_1$  (for  $r \geq 2p_1$ ) are zero.

$p_2 - 1$	Degrees of freedom ( <b>D1</b> )	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
0	$v_h(\mathbf{v}_i)$	$v_h$	--	--	--	--
1	$\partial_x v_h(\mathbf{v}_i), \partial_y v_h(\mathbf{v}_i)$	$\partial_t v_h$	$\partial_n v_h$	--	--	--
2	$\partial_{xx} v_h(\mathbf{v}_i), \partial_{xy} v_h(\mathbf{v}_i), \partial_{yy} v_h(\mathbf{v}_i)$	$\partial_t^2 v_h$	$\partial_t \partial_n v_h$	$\partial_n^2 v_h$	--	--
3	$\partial_{xxx} v_h(\mathbf{v}_i), \partial_{xxy} v_h(\mathbf{v}_i), \partial_{xyy} v_h(\mathbf{v}_i), \partial_{yyy} v_h(\mathbf{v}_i)$	$\partial_t^3 v_h$	$\partial_t^2 \partial_n v_h$	$\partial_t \partial_n^2 v_h$	$\partial_n^3 v_h$	--
4	$\partial_{xxxx} v_h(\mathbf{v}_i), \partial_{xxxy} v_h(\mathbf{v}_i), \partial_{xxyy} v_h(\mathbf{v}_i), \partial_{xyyy} v_h(\mathbf{v}_i), \partial_{yyyy} v_h(\mathbf{v}_i)$	$\partial_t^4 v_h$	$\partial_t^3 \partial_n v_h$	$\partial_t^2 \partial_n^2 v_h$	$\partial_t \partial_n^3 v_h$	$\partial_n^4 v_h$

Table 1: Vertex degrees of freedom for the trace interpolation process on the elemental edges. The first column shows the value of  $\max\{|\nu|\} = p_2 - 1$  that we use to define the degrees of freedom (**D1**). The second column shows the degrees of freedom at vertex  $\mathbf{v}_i$  corresponding to  $p_2 - 1$  in the first column. The remaining columns show the quantities that we can compute using the degrees of freedom listed in the second column. Recalling that  $|\nu| \leq p_2 - 1$  and  $0 \leq j \leq p_2 - 1$ , on the columns for  $j = 0, \dots, 4$  we read the pieces of information that are available for the interpolation of  $\partial_n^j v_h$  (with  $\partial_n^0 v_h = v_h$  for  $j = 0$ ). For example, if  $p_2 = 3$ , we can use only the objects of the first three table rows (i.e.,  $|\nu| = 0, 1, 2$ ), and each column for  $j = 0, 1, 2$  lists the pieces of information that are available to construct the polynomial trace of  $v_h$ ,  $\partial_n v_h$  and  $\partial_n^2 v_h$  on each edge. In such a case, the vertex degrees of freedom allows us to interpolate the trace of  $v_h$  as a polynomial of degree 5, the trace of  $\partial_n v_h$  as a polynomial of degree 3, and the trace of  $\partial_n^2 v_h$  as a polynomial of degree 1. These trace interpolations are consistent with  $r = p_2$  and a global virtual element space with  $C^{p_2-1}$ -regularity on  $\Omega$ .

Consider separately the case of odd and even values of  $p_2$ . If  $p_2 = 2\ell + 1$  with  $\ell \geq 0$ , a repeated application of the integration by parts formula yields

$$\begin{aligned}
\int_{\mathbf{P}} |\nabla \Delta^\ell v_h|^2 d\mathbf{x} &= - \int_{\mathbf{P}} (\Delta^{p_2} v_h) v_h d\mathbf{x} + \int_{\partial\mathbf{P}} (\partial_n \Delta^\ell v_h) \Delta^\ell v_h ds \\
&\quad + \sum_{i=1}^{\ell} \left( \int_{\partial\mathbf{P}} (\partial_n \Delta^{p_2-i} v_h) \Delta^{i-1} v_h ds - \int_{\partial\mathbf{P}} (\Delta^{p_2-i} v_h) \partial_n \Delta^{i-1} v_h ds \right). \quad (13)
\end{aligned}$$

Similarly, if  $p_2 = 2\ell$  with  $\ell \geq 1$ , we find that

$$\begin{aligned}
\int_{\mathbf{P}} |\Delta^\ell v_h|^2 d\mathbf{x} &= \int_{\mathbf{P}} (\Delta^{p_2} v_h) v_h d\mathbf{x} \\
&\quad - \sum_{i=1}^{\ell} \left( \int_{\partial\mathbf{P}} (\partial_n \Delta^{p_2-i} v_h) \Delta^{i-1} v_h ds - \int_{\partial\mathbf{P}} (\Delta^{p_2-i} v_h) \partial_n \Delta^{i-1} v_h ds \right). \quad (14)
\end{aligned}$$

Since  $\Delta^{p_2} v_h$  is a polynomial of degree  $r - 2p_1$  according to the definition of the virtual element space, the volume integral in the right-hand sides of (13) and (14) is an elemental moment of  $v_h$ . This integral must be zero since we assumed that the degrees of freedom (**D3**) of  $v_h$  are zero.

		$r - j$						
$p_2 = 1$	$2(p_2 - j) - 1$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7 \dots$
$j = 0$	<b>1</b>	<b>1</b>	2	3	4	5	6	7 $\dots$
$p_2 = 2$	$2(p_2 - j) - 1$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7 \dots$
$j = 0$	<b>3</b>	–	2	<b>3</b>	4	5	6	7 $\dots$
$j = 1$	<b>1</b>	–	<b>1</b>	2	3	4	5	6 $\dots$
$p_2 = 3$	$2(p_2 - j) - 1$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7 \dots$
$j = 0$	<b>5</b>	–	–	3	4	<b>5</b>	6	7 $\dots$
$j = 1$	<b>3</b>	–	–	2	<b>3</b>	4	5	6 $\dots$
$j = 2$	<b>1</b>	–	–	<b>1</b>	2	3	4	5 $\dots$
$p_2 = 4$	$2(p_2 - j) - 1$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7 \dots$
$j = 0$	<b>7</b>	–	–	–	4	5	6	<b>7</b> $\dots$
$j = 1$	<b>5</b>	–	–	–	3	4	<b>5</b>	6 $\dots$
$j = 2$	<b>3</b>	–	–	–	2	<b>3</b>	4	5 $\dots$
$j = 3$	<b>1</b>	–	–	–	<b>1</b>	2	3	4 $\dots$

Table 2: Polynomial orders of the edge traces of  $v_h$  and its normal derivatives  $\partial_n^j v_h$  for  $p_2 = 1, 2, 3, 4$  and  $r = p_2, \dots, 7$  (we recall that  $r \geq p_2$ ). The first column on the left reports the value of  $p_2$  and  $j = 0, \dots, p_2 - 1$ ; the second column reports the value of  $2(p_2 - j) - 1$ , which is a threshold value, and the remaining columns the possible values of  $r - j$  (remember that on each edge  $\partial_n^j v_h \in \mathbb{P}_{\alpha_j}$  with  $\alpha_j = \max\{2(p_2 - j) - 1, r - j\}$ ). The values of the polynomial degree  $r - 1$  such that  $r - j = 2(p_2 - j) - 1$  (or, equivalently, that  $r = 2p_2 - j - 1$ ) are reported in bold font. The polynomial traces with degree equal or higher than this bold value, which are above it in every column and correspond to the smaller order  $j$  of derivation, can be interpolated using only the vertex degrees of freedom **(D1)**. To interpolate the remaining edge traces we need the additional information provided by **(D2)**.

To prove that the edge integrals in (13) and (14) are zero, we first note that such integrals contain the edge trace of  $\Delta^\mu v_h$  for  $\mu = 0, \dots, p_2 - 1$  and its normal and tangential derivatives. Since  $\Delta v_h|_e = \partial_t^2 v_h + \partial_n^2 v_h$ , it holds that

$$\Delta^\mu v_h = (\partial_t^2 + \partial_n^2)^\mu v_h = \sum_{\nu=0}^{\mu} C_{\mu,\nu} \partial_t^{2(\mu-\nu)} \partial_n^{2\nu} v_h, \quad (15)$$

where  $C_{\mu,\nu}$  denote the  $\nu$ -th combinatorial coefficient of the  $\mu$ -th power expansion. Therefore, all the edge integrals either contain the normal derivatives  $\partial_n^\ell v_h$  for some integer  $\ell = 0, \dots, p_2 - 1$ , or the tangential derivatives of these quantities. As noted at the beginning of this proof, all these quantities are zero since we assumed that the degrees of freedom **(D1)**-(**D2**) of  $v_h$  are zero.

Finally, we note that a function  $v_h \in V_{h,r}^{p_2,p_1}(\mathbf{P})$  with all zero degrees of freedom also belongs to  $H_0^{p_2}(\mathbf{P}) = \{v \in H^{p_2}(\mathbf{P}) : \partial^j v|_{\partial\mathbf{P}} = 0 \forall j = 0, \dots, p_2 - 1\}$ . Since both left-hand sides of (13) and (14) are a norm on  $H_0^{p_2}(\mathbf{P})$ , it follows that  $v_h = 0$ .  $\square$   $\square$

#### 4.3. Global virtual element spaces

Building upon the local virtual element spaces, the *global* conforming virtual element space  $V_{h,r}^{p_2,p_1}$  is defined on  $\Omega$  as

$$V_{h,r}^{p_2,p_1} = \left\{ v_h \in H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega) : v_h|_{\mathbf{P}} \in V_{h,r}^{p_2,p_1}(\mathbf{P}) \forall \mathbf{P} \in \Omega_h \right\}, \quad (16)$$

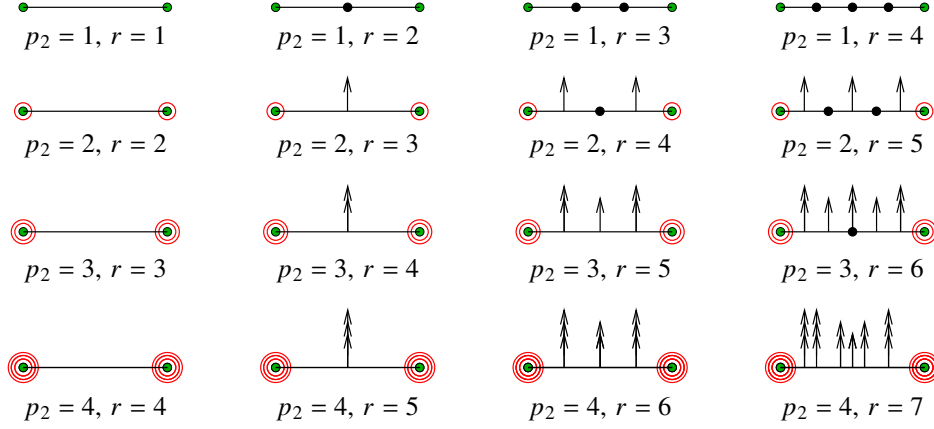


Figure 1: Edge degrees of freedom of the virtual element space  $V_{h,r}^{p_2, p_1}$  with regularity index  $p_2 = 1$  (Laplace operator),  $p_2 = 2$  (bi-harmonic operator),  $p_2 = 3$  (tri-harmonic operator),  $p_2 = 4$ , and polynomial degree  $r$  such that  $p_2 \leq r \leq p_2 + 4$ . The (green) dots at the vertices represent the vertex values and each (red) vertex circle represents an order of derivation. The (black) dots on the edge represent the polynomial moments of the trace  $v_h|_e$ ; the arrows represent the polynomial moments of  $\partial_n v_h|_e$ ; the double arrows represent the polynomial moments of  $\partial_n^2 v_h|_e$ .

where  $V_{h,r}^{p_2, p_1}(\mathbf{P})$  is the local space defined in (9) if  $p_2 \leq r \leq 2p_2 - 2$  and the local space defined in (10) if  $r \geq 2p_2 - 1$ .

**Remark 4.8.** Let  $N^{\mathcal{P}}$ ,  $N^{\mathcal{E}}$  and  $N^{\mathcal{V}}$  denote the number of element, edges and vertices of  $\Omega_h$ . The dimension of the global virtual element space built upon (9) is given by

$$\dim V_{h,r}^{p_2, p_1} = N^{\mathcal{P}} \frac{(r - 2p_1 + 1)(r - 2p_1 + 2)}{2} + N^{\mathcal{E}} \sum_{j=0}^{p_2-1} (\alpha_j(p_2, r) + 1) - N^{\mathcal{V}} \frac{(p_2 + 1)p_2}{2}.$$

The dimension of the global virtual element space built upon (10) is given by

$$\dim V_{h,r}^{p_2, p_1} = N^{\mathcal{P}} \frac{(r - 2p_1 + 1)(r - 2p_1 + 2)}{2} + N^{\mathcal{E}} \frac{(p_2 - 1)(2(r + 1) - (p_2 - 1))}{2} - N^{\mathcal{V}} \frac{(p_2 + 1)p_2}{2}.$$

The set of global degrees of freedom are inherited from the local degrees of freedom of section 4.2. Therefore, we consider

(D1)  $h_v^{|v|} D^v v_h(\mathbf{v})$ ,  $|v| \leq p_2 - 1$  for every vertex  $\mathbf{v}$  of  $\mathcal{V}_h$ ;

(D2)  $h_e^{-1+j} \int_e q \partial_n^j v_h ds$  for any  $q \in \mathbb{P}_{\beta_j}(e)$  and every edge  $e$  of  $\mathcal{E}_h$ ,  $j = k + 1, \dots, p_2 - 1$ ;

(D3)  $h_{\mathbf{P}}^{-2} \int_{\mathbf{P}} q v_h d\mathbf{x}$  for any  $q \in \mathbb{P}_{r-2p_1}(\mathbf{P})$  and every element  $\mathbf{P}$  of  $\Omega_h$ ,

where, again,  $\beta_j = \alpha_j - \min\{2(p_2 - j) - 1, r - j\} - 1$  and  $\alpha_j(p_2, r) = \max\{2(p_2 - j) - 1, r - j\}$ ,  $j = 0, \dots, p_2 - 1$ . For  $r \geq 2p_2 - 1$ , these degrees of freedom become

(D1)  $h_v^{|v|} D^v v_h(v)$ ,  $|v| \leq p_2 - 1$  for every interior vertex  $v$  of  $\mathcal{V}_h$ ;

(D2)  $h_e^{-1+j} \int_e q \partial_n^j v_h ds$  for any  $q \in \mathbb{P}_{r-2p_2+j}(e)$   $j = 0, \dots, p_2 - 1$  and every interior edge  $e$  of  $\mathcal{E}_h$ ;

(D3)  $h_P^{-2} \int_P q v_h d\mathbf{x}$  for any  $q \in \mathbb{P}_{r-2p_1}(P)$  and every element  $P$  of  $\Omega_h$ .

We remark that the associated global space is made of  $H^{p_2}(\Omega)$  functions. Indeed, the restriction of a virtual element function  $v_h$  to each element  $P$  belongs to  $H^{p_2}(P)$  and glues with  $C^{p_2-1}$ -regularity across the internal mesh faces.

Finally, the unisolvence of these degrees of freedom is an immediate consequence of the unisolvence of the elementwise degrees of freedom in any elemental space  $V_{h,r}^{p_2,p_1}(P)$ , cf. Lemma 4.7.

#### 4.4. Elliptic projection operator

The elliptic projection operator  $\Pi_r^{p_1,P} : V_{h,r}^{p_2,p_1}(P) \rightarrow \mathbb{P}_r(P)$  is such that for all  $v_h \in V_{h,r}^{p_2,p_1}(P)$ , the projection  $\Pi_r^{p_1,P} v_h$  is the solution of the finite dimensional variational problem

$$a_{p_1}^P(\Pi_r^{p_1,P} v_h - v_h, q) = 0 \quad \forall q \in \mathbb{P}_r(P), \quad (17)$$

$$\int_{\partial P} (\Pi_r^{p_1,P} v_h - v_h) q ds = 0 \quad \forall q \in \mathbb{P}_{p_1-1}(P). \quad (18)$$

Condition (18) allows us to fix the nontrivial kernel of  $a_{p_1}^P(\cdot, \cdot)$ , which is the subspace of polynomials of degree (up to)  $p_1 - 1$ .

**Remark 4.9.** Instead of (18), we can consider the alternative condition [10]

$$\widehat{\Pi}^P D^v \Pi_r^{p_1,P} v_h = \widehat{\Pi}^P D^v v_h \quad \text{with } |v| \leq p_1 - 1,$$

by using the vertex average projection  $\widehat{\Pi}^P : C(P) \rightarrow \mathbb{P}_0(P)$ , which is such that

$$\widehat{\Pi}^P \psi = \frac{1}{N^P} \sum_{v \in \partial P} \psi(v), \quad (19)$$

for all continuous function  $\psi$ .

**Lemma 4.10.** The elliptic projection operator  $\Pi_r^{p_1,P}$  is polynomial preserving in the sense that  $\Pi_r^{p_1,P} q = q$  for every  $q \in \mathbb{P}_r(P)$ .

*Proof.* Let  $\mathbb{P}_r(P) \setminus \mathbb{P}_{p_1-1}(P)$  denote the linear space of polynomials of degrees  $s$  such that  $p_1 \leq s \leq r$ , and consider the decomposition

$$\mathbb{P}_r(P) = \mathbb{P}_r(P) \setminus \mathbb{P}_{p_1-1}(P) \oplus \mathbb{P}_{p_1-1}(P). \quad (20)$$

We expand the polynomial  $q \in \mathbb{P}_r(\mathbf{P})$  and its projection  $\Pi_r^{p_1, \mathbf{P}} q$  as follows

$$q = \sum_{\ell'} c_{\ell'}(q) \mu_{\ell'} + \sum_{\ell'} \tilde{c}_{\ell'}(q) \tilde{\mu}_{\ell'}, \quad (21)$$

$$\Pi_r^{p_1, \mathbf{P}} q = \sum_{\ell'} c_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q) \mu_{\ell'} + \sum_{\ell'} \tilde{c}_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q) \tilde{\mu}_{\ell'}, \quad (22)$$

where  $\{\mu_{\ell'}\}$  is a basis of  $\mathbb{P}_r(\mathbf{P}) \setminus \mathbb{P}_{p_1-1}(\mathbf{P})$ ,  $\{\tilde{\mu}_{\ell'}\}$  is a basis of  $\mathbb{P}_{p_1-1}(\mathbf{P})$ , and  $c_{\ell'}(q)$ ,  $\tilde{c}_{\ell'}(q)$ ,  $c_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q)$ , and  $\tilde{c}_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q)$  are the coefficients of such expansions. The range of the summation index  $\ell'$ , which is not explicitly indicated in (21) and (22), is consistent with the dimensions of  $\mathbb{P}_r(\mathbf{P}) \setminus \mathbb{P}_{p_1-1}(\mathbf{P})$  and  $\mathbb{P}_{p_1-1}(\mathbf{P})$ . We assume that the polynomials  $\mu_{\ell'}$  are orthogonal with respect to the semi-inner product  $a_{p_1}^{\mathbf{P}}(\cdot, \cdot)$ , which is the restriction of  $a_{p_1}^{\mathbf{P}}(\cdot, \cdot)$  to a polygonal element  $\mathbf{P}$ , so that  $a_{p_1}^{\mathbf{P}}(\mu_{\ell'}, \mu_{\ell}) = |\mathbf{P}| \delta_{\ell', \ell}$ . Since the polynomials  $\tilde{\mu}_{\ell'}$  belong to the kernel of  $a_{p_1}^{\mathbf{P}}(\cdot, \cdot)$ , we substitute the expansions (21) and (22) in (17) (with  $v_h = q$  and  $q = \mu_{\ell}$ ) and we find that

$$\begin{aligned} 0 &= a_{p_1}^{\mathbf{P}}(\Pi_r^{p_1, \mathbf{P}} q - q, \mu_{\ell}) = \sum_{\ell'} (c_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q) - c_{\ell'}(q)) a_{p_1}^{\mathbf{P}}(\mu_{\ell}, \mu_{\ell'}) \\ &= |\mathbf{P}| \sum_{\ell'} (c_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q) - c_{\ell'}(q)) \delta_{\ell, \ell'} = |\mathbf{P}| (c_{\ell}(\Pi_r^{p_1, \mathbf{P}} q) - c_{\ell}(q)), \end{aligned}$$

which holds for all possible integers  $\ell$ . This relation implies that

$$\Pi_r^{p_1, \mathbf{P}} q - q = \sum_{\ell'} (\tilde{c}_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q) - \tilde{c}_{\ell'}(q)) \tilde{\mu}_{\ell'} \in \mathbb{P}_{p_1-1}(\mathbf{P}). \quad (23)$$

Then, we assume that the polynomials  $\tilde{\mu}_{\ell'}$  are orthogonal with respect to the inner product  $(v, u)_{\partial \mathbf{P}} = \int_{\partial \mathbf{P}} v u \, ds$ , so that  $(\tilde{\mu}_{\ell'}, \tilde{\mu}_{\ell})_{\partial \mathbf{P}} = |\partial \mathbf{P}| \delta_{\ell', \ell}$ , where  $|\partial \mathbf{P}|$  is the perymeter of  $\partial \mathbf{P}$ . We substitute (23) in (18) (with  $v_h = q$  and  $q = \tilde{\mu}_{\ell}$ ) and we find that

$$\begin{aligned} 0 &= \int_{\partial \mathbf{P}} (\Pi_r^{p_1, \mathbf{P}} q - q) \tilde{\mu}_{\ell} \, ds = \sum_{\ell'} (\tilde{c}_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q) - \tilde{c}_{\ell'}(q)) \int_{\partial \mathbf{P}} \tilde{\mu}_{\ell'} \tilde{\mu}_{\ell} \, ds \\ &= |\partial \mathbf{P}| \sum_{\ell'} (\tilde{c}_{\ell'}(\Pi_r^{p_1, \mathbf{P}} q) - \tilde{c}_{\ell'}(q)) \delta_{\ell', \ell} = |\partial \mathbf{P}| (\tilde{c}_{\ell}(\Pi_r^{p_1, \mathbf{P}} q) - \tilde{c}_{\ell}(q)). \end{aligned}$$

which holds for all possible integers  $\ell$ . This implies that  $\Pi_r^{p_1, \mathbf{P}} q - q = 0$ , which is the assertion of the lemma.  $\square$

**Lemma 4.11.** *The polynomial projection  $\Pi_r^{p_1, \mathbf{P}} v_h$  is computable using only the degrees of freedom (D1)-(D3) of  $v_h \in V_{h,r}^{p_2, p_1}(\mathbf{P})$ .*

*Proof.* To prove the assertion of the lemma, we only need to prove that  $a_{p_1}^{\mathbf{P}}(v_h, q)$  and  $(v, u)_{\partial \mathbf{P}} = \int_{\partial \mathbf{P}} v_h q \, ds$  are computable for all  $v_h \in V_{h,r}^{p_2, p_1}(\mathbf{P})$  and scalar polynomial  $q \in \mathbb{P}_r(\mathbf{P})$ . To this end, we integrate by parts  $a_{p_1}^{\mathbf{P}}(v_h, q)$ . For an odd  $p_1$ , i.e.,  $p_1 = 2\ell + 1$ , we find that

$$a_{p_1}^{\mathbf{P}}(v_h, q) = - \int_{\mathbf{P}} (\Delta^{p_1} v_h) q \, d\mathbf{x} + \int_{\partial \mathbf{P}} (\partial_n \Delta^{\ell} v_h) \Delta^{\ell} q \, ds$$

$$+ \sum_{i=1}^{\ell} \left( \int_{\partial P} (\partial_n \Delta^{p_1-i} v_h) \Delta^{i-1} q \, ds - \int_{\partial P} (\Delta^{p_1-i} v_h) \partial_n \Delta^{i-1} q \, ds \right). \quad (24)$$

For an even  $p_1$ , i.e.,  $p_1 = 2\ell$ , we find that

$$\begin{aligned} a_{p_1}^P(v_h, q) &= \int_P (\Delta^{p_1} v_h) q \, d\mathbf{x} \\ &\quad - \sum_{i=1}^{\ell} \left( \int_{\partial P} (\partial_n \Delta^{p_1-i} v_h) \Delta^{i-1} q \, ds - \int_{\partial P} (\Delta^{p_1-i} v_h) \partial_n \Delta^{i-1} q \, ds \right). \end{aligned} \quad (25)$$

The first integral of the right-hand side of both formulas (24) and (25) is computable from the degrees of freedom **(D3)**. In turn, all the edge integrals are computable since we can expand the trace of  $\Delta^\mu v_h$  in terms of  $\partial_t^{2(\mu-\nu)} \partial_n^{2\mu} v_h$  and use the same argument of the proof of Lemma 4.7. Since the edge traces of  $v_h$  and its normal derivatives (and all their tangential derivatives) are computable from the degrees of freedom of **(D1)**-**(D2)** through a polynomial interpolation, we deduce that all the edge integrals for both odd and even  $p_1$  and the boundary integral  $(v, u)_{\partial P}$  are computable.  $\square$   $\square$

#### 4.5. Enhancement

As noted in Remark 4.3, the  $L^2$ -projection operator  $\Pi_{r-2p_1}^{0,P} : V_{h,r}^{p_2,p_1}(P) \rightarrow \mathbb{P}_r(P)$  is computable from the degrees of freedom **(D3)**. Instead, to compute the orthogonal projection onto the polynomial subspace  $\mathbb{P}_{r-p_1}(P)$ , we need to modify the space definition as follows thus obtaining the so called “enhanced” virtual element space. Our construction follows the guidelines in [1]. First, we consider the mesh element  $P$  and the “extended” virtual element space for  $r \geq 2p_2 - 1$  (recall that  $p_2 \geq p_1$ ) defined as

$$\begin{aligned} \widetilde{V}_{h,r}^{p_2,p_1}(P) &:= \left\{ v_h \in H^{p_2}(P) : \Delta^{p_2} v_h \in \mathbb{P}_{r-p_1}(P), \partial_n^j v_h \in \mathbb{P}_{r-j}(e), \right. \\ &\quad \left. j = 0, \dots, p_2 - 1 \, \forall e \in \partial P \right\}. \end{aligned} \quad (26)$$

Then, we define the enhanced virtual element space as

$$\begin{aligned} W_{h,r}^{p_2,p_1}(P) &:= \left\{ v_h \in \widetilde{V}_{h,r}^{p_2,p_1}(P) : \int_P v_h q \, d\mathbf{x} = \int_P \Pi_{r-p_1}^{p_1,P} v_h q \, d\mathbf{x} \right. \\ &\quad \left. \forall q \in \mathbb{P}_{r-p_1} \setminus \mathbb{P}_{r-2p_1}(P) \right\}. \end{aligned} \quad (27)$$

The polynomial space  $\mathbb{P}_r(P)$  is a subspace of  $W_{h,r}^{p_2,p_1}(P)$  and, thus, of  $\widetilde{V}_{h,r}^{p_2,p_1}(P)$ , and the elliptic projection  $\Pi_{r-p_1}^{p_1,P} : \widetilde{V}_{h,r}^{p_2,p_1}(P) \rightarrow \mathbb{P}_{r-p_1}(P)$  that is defined in (17)-(18) is still computable and only depends on the degrees of freedom **(D1)**, **(D2)** and **(D3)**. This assertion can easily be proved by repeating the argument of Lemma 4.10.

The virtual element functions of the space  $\widetilde{V}_{h,r}^{p_2,p_1}(P)$  are uniquely characterized by the set of degrees of freedom **(D1)**, **(D2)**, **(D3)** and the set of additional degrees of freedom **(D3)**

$$\widetilde{\mathbf{(D3)}} \quad h_P^{-2} \int_P q v_h \, d\mathbf{x} \text{ for any } q \in \mathbb{P}_{r-p_1}(P) \setminus \mathbb{P}_{r-2p_1}(P).$$

We state the unisolvence of these degrees of freedom in the following lemma. The proof is equal to the proof of Lemma 4.7 (consider the degrees of freedom **(D3)**, **(D3)** instead of **(D3)**) and is omitted.



**Lemma 4.12.** *The degrees of freedom (D1), (D2), (D3), ( $\widetilde{\text{D3}}$ ) are unisolvent in the virtual element space  $\widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P})$ .*

**Remark 4.13.** *According to Lemma 4.12, the dimension of  $\widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P})$  must be equal to the cardinality of the set of the degrees of freedom (D1), (D2), (D3), ( $\widetilde{\text{D3}}$ ). This statement can also be proved by a counting argument.*

Next, we want to prove that the degrees of freedom (D1), (D2) and (D3) are unisolvent in the enhanced space  $W_{h,r}^{p_2,p_1}(\mathbf{P})$ . To this end, we first need to establish a technical result. Consider the set of linear, bounded functionals  $\lambda_{\ell_1}^{(\text{D1})}, \lambda_{\ell_2}^{(\text{D2})}, \lambda_{\ell_3}^{(\text{D3})} : \widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P}) \rightarrow \mathbb{R}$ , which respectively return the degrees of freedom (D1), (D2) and (D3) when applied to a virtual element function  $v_h \in \widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P})$ . The indices  $\ell_1, \ell_2$ , and  $\ell_3$  run from 1 to  $\#(\text{D1})$ ,  $\#(\text{D2})$  and  $\#(\text{D3})$ , respectively, where  $\#(\mathcal{D})$  denotes the cardinality of the discrete set  $\mathcal{D}$ . Renumbering  $\ell_2$  and  $\ell_3$  may require the introduction of suitable sets of basis functions for the polynomial spaces  $\mathbb{P}_{r-j}(e)$  and  $\mathbb{P}_{r-2p_1}(\mathbf{P})$  in (D2) and (D3), respectively. We left this aspect undefined as this technicality is not crucial in this presentation, although important in the practical implementations of the method. We introduce the additional set of linear functionals  $\lambda_{\tilde{\ell}}^{(\widetilde{\text{D3}})}$  that are such that

$$\lambda_{\tilde{\ell}}^{(\widetilde{\text{D3}})}(v_h) = h_{\mathbf{P}}^{-2} \int_{\mathbf{P}} q_{\tilde{\ell}} (\Pi_{r-p_1}^{p_1,\mathbf{P}} - \Pi_{r-p_1}^{0,\mathbf{P}}) v_h \, d\mathbf{x} \quad \tilde{\ell} = 1, \dots, \#(\widetilde{\text{D3}})$$

where  $\{q_{\tilde{\ell}}\}_{\tilde{\ell}}^{\#(\widetilde{\text{D3}})}$  is a basis of  $\mathbb{P}_{r-p_1}(\mathbf{P}) \setminus \mathbb{P}_{r-2p_1}(\mathbf{P})$ , and the index  $\tilde{\ell}$  runs from 1 to  $\#(\widetilde{\text{D3}})$ , the number of degrees of freedom of ( $\widetilde{\text{D3}}$ ).

Then, we collect these different types of functionals in the functional set

$$\Lambda = (\lambda_{\ell}) = (\lambda_{\ell_1}^{(\text{D1})}, \lambda_{\ell_2}^{(\text{D2})}, \lambda_{\ell_3}^{(\text{D3})}, \lambda_{\tilde{\ell}}^{(\widetilde{\text{D3}})}). \quad (28)$$

We assume that the integer index  $\ell$  is consistent with a suitable renumbering of such degrees of freedom; so that  $\ell$  runs from 1 to  $m' = m + \#(\widetilde{\text{D3}}) = \dim \widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P})$  and  $m = \#(\text{D1}) + \#(\text{D2}) + \#(\text{D3})$ . These functionals satisfy the property stated in the following lemma.

**Lemma 4.14.** *The linear functionals  $\Lambda$  are linearly independent.*

*Proof.* Let  $v_h \in \widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P})$  such that  $\lambda_{\ell}(v_h) = 0$  for all  $\ell = 1, \dots, m'$ . Now, the degrees of freedom (D1), (D2) and (D3) of  $v_h$  are (obviously) zero as they are the values of the functionals  $\lambda_{\ell_1}^{(\text{D1})}(v_h)$ ,  $\lambda_{\ell_2}^{(\text{D2})}(v_h)$  and  $\lambda_{\ell_3}^{(\text{D3})}(v_h)$ . Moreover, it holds that  $\Pi_r^{p_1,\mathbf{P}} v_h = 0$ , and, hence,  $\Pi_{r-p_1}^{p_1,\mathbf{P}} v_h = 0$ , as these projections only depend on the degrees of freedom (D1), (D2) and (D3), cf. Lemma 4.10. Then, we observe that the definition of the orthogonal projection  $\Pi_{r-p_1}^{0,\mathbf{P}} v_h$  and the facts that  $\lambda_{\tilde{\ell}}^{(\widetilde{\text{D3}})}(v_h) = 0$  and  $\Pi_{r-p_1}^{p_1,\mathbf{P}} v_h = 0$  imply that

$$\int_{\mathbf{P}} q v_h \, d\mathbf{x} = \int_{\mathbf{P}} q \Pi_{r-p_1}^{0,\mathbf{P}} v_h \, d\mathbf{x} = \int_{\mathbf{P}} q \Pi_{r-p_1}^{p_1,\mathbf{P}} v_h \, d\mathbf{x} = 0 \quad (29)$$

for all  $q \in \mathbb{P}_{r-p_1}(\mathbf{P}) \setminus \mathbb{P}_{r-2p_1}(\mathbf{P})$ . Therefore, the degrees of freedom ( $\widetilde{\text{D3}}$ ) of  $v_h$  are equal to zero and finally  $v_h = 0$  because the degrees of freedom (D1), (D2), (D3) and ( $\widetilde{\text{D3}}$ ) are unisolvent in  $\widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P})$ . This argument proves that the intersection of the kernels of all the linear functionals  $\lambda_{\ell}$  contains only the virtual element function that is identically zero over  $\mathbf{P}$ , so that these linear functionals are necessarily linearly independent.  $\square$   $\square$

Using the linear functionals  $\Lambda$ , we reformulate the definition of space  $W_{h,r}^{p_2,p_1}(\mathbf{P})$  in the following equivalent way:

$$W_{h,r}^{p_2,p_1}(\mathbf{P}) := \left\{ v_h \in \widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P}) : \lambda_{\widetilde{\ell}_3}^{(\widetilde{\mathbf{D3}})}(v_h) = 0 \quad \forall \widetilde{\ell}_3 = m+1, \dots, m' \right\}. \quad (30)$$

In other words,  $W_{h,r}^{p_2,p_1}(\mathbf{P})$  belongs to the intersection of the kernels of all the additional linear functionals  $\lambda_\ell$  with  $\ell = m+1, \dots, m'$ . The space  $W_{h,r}^{p_2,p_1}(\mathbf{P})$  has the two important properties that are stated in the following lemma.

**Lemma 4.15.** *The virtual element space  $W_{h,r}^{p_2,p_1}(\mathbf{P})$  has the same dimension of the “regular” space  $V_{h,r}^{p_2,p_1}(\mathbf{P})$  and the set of degrees of freedom **(D1)**, **(D2)** and **(D3)** are unisolvent in  $W_{h,r}^{p_2,p_1}(\mathbf{P})$ .*

*Proof.* In view of Lemma 4.14 the linear functionals in  $\Lambda$  are linearly independent and the cardinality of  $\Lambda$ , i.e.,  $m' = \#(\Lambda)$ , is equal to the dimension of  $\widetilde{V}_{h,r}^{p_2,p_1}(\mathbf{P})$ . Therefore,  $(\mathbf{P}, \mathbb{P}_r(\mathbf{P}), \Lambda)$  is a *finite element* in the sense of Ciarlet, cf. [39]. So, there exists a set of  $m'$  dual basis functions  $\psi_{\ell'}$  such that

$$\lambda_{\ell'}(\psi_{\ell'}) = \delta_{\ell,\ell'} \quad \ell, \ell' = 1, \dots, m'.$$

Now, it holds that  $\lambda_{\widetilde{\ell}_3}^{(\widetilde{\mathbf{D3}})}(\psi_{\ell'}) = \lambda_{\ell'}(\psi_{\ell'}) = 0$  for  $\ell' = 1, \dots, m$ ,  $\ell = m+1, \dots, m'$  (and the corresponding values of the index  $\widetilde{\ell}_3$ ). This fact has the following consequences. The first  $m$  linearly independent functions  $\psi_{\ell'}$ ,  $\ell' = 1, \dots, m$ , belong to  $W_{h,r}^{p_2,p_1}(\mathbf{P})$ , cf. formulation (30), since  $\lambda_{\widetilde{\ell}_3}^{(\widetilde{\mathbf{D3}})}(\psi_{\ell'}) = \lambda_{\ell'}(\psi_{\ell'}) = 0$  for  $\ell = m+1, \dots, m'$  (and corresponding indices  $\widetilde{\ell}_3$ ). This implies that  $\dim W_{h,r}^{p_2,p_1}(\mathbf{P}) \geq m$ . Furthermore, according to the space definition (30) all virtual element functions  $w_h \in W_{h,r}^{p_2,p_1}(\mathbf{P})$  are such that  $\lambda_{\widetilde{\ell}_3}^{(\widetilde{\mathbf{D3}})}(w_h) = \lambda_{\ell'}(w_h) = 0$  for  $\ell' = m+1, \dots, m'$ . Therefore, such functions can be written as a linear combination of *only* the first  $m$  basis functions  $\psi_{\ell'}$ ,  $\ell' = 1, \dots, m$ , are thus identified by the values of the linear functionals  $\lambda_{\ell_1}^{(\mathbf{D1})}v_h$ ,  $\lambda_{\ell_2}^{(\mathbf{D2})}v_h$  and  $\lambda_{\ell_3}^{(\mathbf{D3})}v_h$ . Consequently, all virtual element functions of  $W_{h,r}^{p_2,p_1}(\mathbf{P})$  are uniquely identified by the degrees of freedom **(D1)**, **(D2)**, **(D3)** and, consequently,  $\dim W_{h,r}^{p_2,p_1}(\mathbf{P}) = m$ .  $\square$

In view of Lemma 4.15, the orthogonal projection operator  $\Pi_{r-p_1}^{0,\mathbf{P}} : W_{h,r}^{p_2,p_1}(\mathbf{P}) \rightarrow \mathbb{P}_{r-p_1}(\mathbf{P})$  is computable from the degrees of freedom **(D1)**, **(D2)** and **(D3)**.

Finally, we collect the local virtual element spaces into a *global* conforming virtual element space  $W_{h,r}^{p_2,p_1}$  defined on  $\Omega$  as

$$W_{h,r}^{p_2,p_1} = \left\{ w_h \in H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega) : w_h|_{\mathbf{P}} \in W_{h,r}^{p_2,p_1}(\mathbf{P}) \quad \forall \mathbf{P} \in \Omega_h \right\}, \quad (31)$$

where  $W_{h,r}^{p_2,p_1}(\mathbf{P})$  is the local space defined above.

#### 4.6. The virtual element bilinear form $a_h(\cdot, \cdot)$

We discuss the definition of the bilinear form  $a_h(\cdot, \cdot)$  that approximates the bilinear form  $a(\cdot, \cdot)$  in the virtual element discretization (5). This construction is the same for the “regular” virtual element spaces (9) and (10) and the “enhanced” space (27). In this section we use the symbol  $V_{h,r}^{p_2,p_1}(\mathbf{P})$  to denote both choices of the spaces. However, such construction holds also for the enhanced virtual element space (31). The symmetric bilinear form  $a_h : V_{h,r}^{p_2,p_1} \times V_{h,r}^{p_2,p_1} \rightarrow \mathbb{R}$ , is written as the sum of local terms

$$a_h(u_h, v_h) = \sum_{\mathbf{P} \in \Omega_h} a_{\mathbf{P}}^h(u_h, v_h), \quad (32)$$

where each local term  $a_h^P : V_{h,r}^{p_2,p_1}(P) \times V_{h,r}^{p_2,p_1}(P) \rightarrow \mathbb{R}$  is a symmetric bilinear form. We set

$$a_h^P(u_h, v_h) = a_{p_1}^P(\Pi_r^{p_1,P} u_h, \Pi_r^{p_1,P} v_h) + S^P(u_h - \Pi_r^{p_1,P} u_h, v_h - \Pi_r^{p_1,P} v_h), \quad (33)$$

where  $S^P : V_{h,r}^{p_2,p_1}(P) \times V_{h,r}^{p_2,p_1}(P) \rightarrow \mathbb{R}$  provides the stabilization term. The stabilization form  $S^P(\cdot, \cdot)$  is a symmetric, positive definite bilinear form for which there exist two positive constants  $\sigma_*$  and  $\sigma^*$  such that

$$\sigma_* a_{p_1}^P(v_h, v_h) \leq S^P(v_h, v_h) \leq \sigma^* a_{p_1}^P(v_h, v_h) \quad \forall v_h \in V_{h,r}^{p_2,p_1}(P) \text{ with } \Pi_r^{p_1,P} v_h = 0. \quad (34)$$

The constants  $\sigma_*$ ,  $\sigma^*$  are independent of  $h$  (and  $P$ ). A possible proof of the validity of (34) for the so called “dof-dof” stabilization in the context of arbitrarily regular conforming VEM can be found in [46] (for the case  $p_1 = 2$  see also [36, 61]). This construction has the  $r$ -consistency and stability properties stated in (6) and (7),

#### 4.7. The virtual element approximation of the load term

To approximate the right-hand side term of (5) we first assume the elemental decomposition

$$\langle f_h, v_h \rangle = \sum_{P \in \Omega_h} \int_P f_h v_h \, d\mathbf{x}. \quad (35)$$

In Eq. (35), the elemental term  $f_h|_P$  is defined as

$$f_h|_P = \begin{cases} \Pi_{r-2p_1}^{0,P} f, & (a) \text{ if } p_2 + 2p_1 - 1 \leq r, \\ \Pi_{r-p_1}^{0,P} f, & (b) \text{ if } p_2 \leq r \leq p_2 + 2p_1 - 2. \end{cases} \quad (36)$$

We discuss the two definitions of  $f_h$  given above separately.

**Remark 4.16.** The right-hand side of (35) is fully computable by using only the degrees of freedom **(D3)** if  $r \geq 2p_1$  and we choose  $f_h$  as the piecewise polynomial approximation of  $f$  on  $\Omega_h$  in accordance with (a). In such a case, we do not need to resort to the enhanced virtual element space defined in (27).

Now, consider decomposition (35) and definition (a). Since  $p_2 \geq p_1$ , it holds that  $r - 2p_1 \geq p_2 - 1$  (equivalently,  $r \geq 3p_1 - 1$ ). Thus, using the definition of the  $L^2$ -orthogonal projection, from (35), we find that

$$\langle f_h, v_h \rangle = \sum_{P \in \Omega_h} \int_P \Pi_{r-2p_1}^{0,P} f v_h \, d\mathbf{x} = \sum_{P \in \Omega_h} \int_P \Pi_{r-2p_1}^{0,P} f \Pi_{p_1-1}^{0,P} v_h \, d\mathbf{x}. \quad (37)$$

Applying standard approximation results to (37) and recalling that  $v_h \in V_{h,r}^{p_2,p_1} \subset H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega)$  yield the following estimate

$$\langle f - f_h, v_h \rangle \leq C h^{r-p_1+1} |v_h|_{p_1} |f|_{r-2p_1+1},$$

for some positive constant  $C$  that is independent of  $h$ . In particular, for  $p_1 = p_2 \geq 2$  it is enough to choose  $r \geq 2p_2 + 1$  (the case  $p_1 = p_2 = 2$  and  $r \geq 5$  has been originally treated in [33]). Note that for fixed values of  $p_1$ , larger values of the regularity parameter  $p_2$  ensure higher convergence rate for the approximation of the right-hand side. This is a specific attractive feature of arbitrarily regular conforming VEM (which can not be exploited, e.g., in the nonconforming setting).

Now, consider decomposition (35) and definition (b). Similarly to the previous case, using the definition of the  $L^2$ -orthogonal projection yields

$$\langle f_h, v_h \rangle = \sum_{P \in \Omega_h} \int_P \Pi_{r-p_1}^{0,P} f v_h \, d\mathbf{x} = \sum_{P \in \Omega_h} \int_P \Pi_{r-p_1}^{0,P} f \Pi_0^{0,P} v_h \, d\mathbf{x}. \quad (38)$$

Applying again standard approximation results to (38) we find that

$$\langle f - f_h, v_h \rangle \leq Ch^{r-p_1+2} |v_h|_{p_1} |f|_{r-p_1+1}.$$

For arbitrary values of  $p_1$  and  $p_2$ , the use of the enhancement approach might be avoided using arguments similar to those employed in [47].

#### 4.8. Error analysis

In this section, we briefly recall a convergence result in the energy norm [10] (see also [4, 46]) for the approximation of (1a)-(1b). In particular, employing Theorem 3.1 together with standard results of approximation (see, e.g., Reference [19, 46, 60]) and the approximation properties of the right-hand side contained in Section 4.7.

**Theorem 4.17.** *Let  $u \in H_0^{p_1}(\Omega) \cap H^{r+1}(\Omega)$  be the solution of the polyharmonic problem (1a)-(1b) and let  $u_h \in V_{h,r}^{p_2,p_1}$  be the solution of the discrete problem (5). Assume that  $f$  is sufficiently regular. Then, there exists a positive constant  $C$  independent of  $h$  such that*

$$\|u - u_h\|_V \leq Ch^{r-(p_1-1)}. \quad (39)$$

**Remark 4.18.** *Convergence estimates in lower order norms can be established provided that classical duality arguments can be used and that the polynomial approximation order  $r$  is sufficient large [4, 10, 38].*

## 5. Conclusion

We reviewed the construction of highly regular virtual element spaces for the conforming approximations in two spatial dimensions of elliptic problems of order  $p_1 \geq 1$ . The resulting finite dimensional virtual spaces are subspaces of  $H^{p_2}(\Omega)$ ,  $p_2 \geq p_1$ . We presented an abstract convergence result in a suitably defined energy norm. Moreover, after discussing the construction of the approximation spaces and major aspects such as the choice and unisolvence of the degrees of freedom, we provided specific examples of highly regular virtual spaces, corresponding to various practical cases. Finally, a detailed discussion of the properties of the “enhanced” formulation of the virtual element spaces is provided.

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