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Reduced models for the Poisson problem in perforated domains

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Reduced models for the Poisson problem in perforated domains

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Abstract

We develop a fictitious domain method to approximate a Dirichlet problem on a domain with small circular holes (simply called a perforated domain). To address the case of many small inclusions or exclusions, we propose a reduced model based on the projection of the homogeneous Dirichlet boundary constraint on a finite dimensional approximation space. We analyze the existence of the solution of this reduced problem and prove its convergence towards the limit problem without holes. We next obtain an estimate of the gap between the solution of the reduced model and the solution of the full initial model with small holes, the convergence rate depending on the size of the inclusion and on the number of modes of the finite dimensional space. The numerical discretization of the reduced problem is addressed by the finite element method, using a computational mesh that does not fit to the holes. The approximation properties of the finite element method are analyzed by a-priori estimates and confirmed by numerical experiments. elliptic differential equations, small inclusions, asymptotic analysis, approximated numerical method

1 Introduction

Many engineering problems involve domains with small holes, for example for the description of mechanical components with screws or bolts, for the modeling of heating or cooling systems by arrays of pipes, for the description of fluid-particle interaction, just to mention a few examples. Although the solution of partial differential equations (PDEs) on such domains is well understood, some challenges remain for the application of well known numerical discretization techniques, such as the finite element method, especially for the case of domains with many holes of small size. In such cases, a tradeoff between computational complexity (including the pre-processing phase where the CAD model and the computational mesh is generated) and accuracy of the results must be established. Several methods such as penalty methods, Nitsche's method, Lagrange multiplier methods have been proposed for addressing boundary conditions at the discrete level, see for example [1] for a review and [2, 3, 4, 5, 6] for a non exhaustive list of specific examples, which may represent a starting point for discretizing problems on domains with holes. We have been inspired in particular by the *fictitious domain* methods [7], where a distributed Lagrange multiplier is applied to impose the Dirichlet boundary conditions on the hole, while using a regular unfitted grid for the finite element discretization of the problem. Although this method finds

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its primary application in fluid-particle interaction problems, this technique has been studied also for Dirichlet problems governed by elliptic equations [8].

From the standpoint of the numerical analysis, fundamental questions arise about the stability [9], the error analysis [8] and the numerical solution [6, 10] of the proposed approach, which will be partially addressed here. From the standpoint of mathematical analysis, this work is also related to the ones on the analysis of the asymptotic extension of the Green's function around small perturbations [11, 12] which enables the computation of numerical solutions based on the problem without inclusion [13, 14, 15]. From the point of view of applications, this work is a first step towards the formulation of coupled three-dimensional (3D) and one-dimensional (1D) models, in the framework of 3D-1D mixed-dimensional PDEs. In this perspective, the present work addresses the simplified case of 2D-0D coupling. Several works by the authors and co-workers have already addressed these topics. For example, the present work can be regarded as an extension to the case of Dirichlet constraints of [16], where Robin boundary conditions on holes were considered. The Robin-Neumann interface conditions on small cylindrical inclusions has been later studied in [17] for 3D-1D mixed dimensional PDEs. Also Dirichlet-Neumann conditions were later addressed in [18] in a similar modelling context. The present study extends the previous works to a higher level of generality, as it will be explained later on.

In this work, we propose a new fictitious domain formulation that is particularly suited for modeling small circular holes (also generally called inclusions). We combine the fictitious domain technique with the idea of representing the holes as concentrated sources. It is well known that the latter approach gives rise to ill-posed problems in computational mechanics, see [19] for a thorough discussion on these issues, but it may still provide some answers at the level of numerical discretization. We look for a compromise approach where the treatment of boundary conditions on the holes is simplified, but the mathematical soundness of the problem is preserved. In this spirit, we name our approach as a *reduced model*.

Let Ω be a convex polygonal domain of $\mathbb{R}^2$ and ω_ε an inclusion of size ε defined for $\varepsilon > 0$ by

$$\omega_\varepsilon = \varepsilon\omega$$

with $\omega = B(0,1)$ the open ball of center 0 and radius 1. We denote the complementary of ω_ε in Ω by $\Omega_\varepsilon = \Omega\backslash\overline{\omega_\varepsilon}$. We assume that $\overline{\omega_\varepsilon} \subset \Omega$, this assumption is easily verified for ε small enough. Let $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$ be the boundary and volume data of the following Poisson problem:

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = \phi & \text{on } \partial\Omega, \\ u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{1.1}$$

We also require that $0 \notin \text{supp} f$ or equivalently that for ε small enough $\omega_\varepsilon \cap \text{supp} f = \emptyset$. This is a standard [12] but crucial assumption to ensure that the solution of (1.1) is harmonic in the neighborhood of the inclusion.

As previously pointed out, the numerical approximation of Problem (1.1) usually requires to use a computational mesh that conforms to the holes and this can be computationally expensive, especially in the case of many inclusions of small size. In our study, the model reduction approach mainly relies on the approximation of the homogeneous Dirichlet condition on the hole by a finite number of scalar constraints. As we will see, this operation will lead to the definition of a family of problems where, in the N^{th} problem, we impose $2N+1$ scalar constraints on $\partial\omega_\varepsilon$. For this family of problems, the implementation of the numerical approximation does not need to resolve ω_ε and the corresponding solutions represent a good approximations of u_ε when ε tends to 0. This setting also makes it possible to choose any balance between accuracy and model complexity, giving rise to a computational framework that is extremely flexible. These features represent a significant improvement with respect to the previously published works [16, 17, 18] in the context of the approximation of 3D-1D or 2D-0D interface conditions.

We now present the organization of our article and the main results obtained. After introducing the reduced approach in the next section, we focus in **Section 3** on the analysis of the reduced problem. We prove that the reduced problem is well posed, with a particular attention to the influence of the essential parameters that characterize our approach, such as the dependency with respect to the size of the hole or the number of approximation modes. We also study the limit case of vanishing inclusions, showing that both full and reduced problems converge to the problem without inclusions as $\varepsilon \to 0$. Most importantly, we prove that the difference between the solutions of the reduced problem and of the full problem converges to zero as ε goes to zero and this convergence is exponential with respect to the number of modes N.

In other words, we derive estimates of the reduced model error which show that the convergence rate to the full problem can be made arbitrarily fast, by suitably choosing the number of modes (the precise statement is given in **Theorem 3.6**). Finally in **Section 4**, we address the numerical discretization of the reduced problem by means of the finite element method. In the spirit of the fictitious domain approach, we privilege the discretization on grids that do not fit with the inclusion. In this case, we derive error estimates for the finite element method. As a result of the lack of regularity of solutions, the convergence rate of Lagrangian finite elements is sub-optimal. We also identify suitable conditions under which the expected optimal accuracy is restored. These results are illustrated by numerical experiments.

2 The Poisson problem in a perforated domain

2.1 Notation and first results

Throughout the paper, for a Lipschitz domain $\mathcal{D}$ in $\mathbb{R}^n$, we will use the classical notation $(\cdot,\cdot)_{\mathcal{D}}$ for the inner product on $L^2(\mathcal{D})$, more generally, for an Hilbert space X defined on $\mathcal{D}$, $(\cdot,\cdot)_X$ denotes the inner product on X. For a vector space V defined on $\mathcal{D}$, we denote by V' its dual space and $\langle\cdot,\cdot\rangle_{V'}$ is the pairing between V' and V. For $1 \leqslant p \leqslant \infty$ and $m \in \mathbb{N}$, the standard notation $W^{p,m}(\mathcal{D})$ is used to denote the Sobolev space of functions on $\mathcal{D}$ with all derivatives up to the order m in $L^p(\mathcal{D})$. In the specific case $p = 2$, we denote $W^{2,m}(\mathcal{D})$ by $H^m(\mathcal{D})$. For α a multi-index such that $|\alpha| \leqslant m$, the differential operator D^α is defined for $f \in H^m(\mathcal{D})$ by

$$D^\alpha f = \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \dots \partial x_n^{a_n}}.$$

If we equip $H^m(\mathcal{D})$ with the scalar product

$$(\cdot,\cdot)_{m,\mathcal{D}} = \sum_{|\alpha| \leqslant m} (D^\alpha \cdot, D^\alpha \cdot)_{\mathcal{D}}, \tag{2.1}$$

then $H^m(\mathcal{D})$ is an Hilbert space. We write $\|\cdot\|_{m,\mathcal{D}}$ the norm on $H^m(\mathcal{D})$ arising from the scalar product (2.1). We can generalize the definition of $H^m(\mathcal{D})$ to all $m \in \mathbb{R}$. To do so, for $\theta \in (0,1)$, we introduce the semi-norm $[\cdot]_{\theta,\mathcal{D}}$ defined for $f \in L^2(\mathcal{D})$ by

$$[f]_{\theta,\mathcal{D}} = \left(\int_{\mathcal{D}} \int_{\mathcal{D}} \frac{|f(x) - f(y)|^2}{|x-y|^{2\theta+n}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

Let $s > 0$, if we set $\theta = s - \lfloor s \rfloor$, the space $H^s(\mathcal{D})$ is then given by

$$H^s(\mathcal{D}) = \{f \in H^{\lfloor s \rfloor}(\mathcal{D}) \mid \sup_{|\alpha| = \lfloor s \rfloor} [D^\alpha f]_{\theta,\mathcal{D}} < \infty\}.$$

The space $H^s(\mathcal{D})$ is a reflexive Banach space for the norm

$$\|f\|^2_{s,\mathcal{D}} = \|f\|^2_{\lfloor s \rfloor,\mathcal{D}} + \sum_{|\alpha| = \lfloor s \rfloor} [D^\alpha f]^2_{\theta,\mathcal{D}}.$$

We have in particular for $s = \frac{1}{2}$,

$$\|f\|^2_{\frac{1}{2},\mathcal{D}} = \|f\|^2_{0,\mathcal{D}} + [f]^2_{\frac{1}{2},\mathcal{D}}. \tag{2.2}$$

For $s > 0$, the space $H^s_0(\mathcal{D})$ denotes the closure of $\mathcal{C}^\infty_0(\mathcal{D})$ functions in $H^s(\mathcal{D})$. In the particular case of the space $H^1_0(\mathcal{D})$, it is equal to the set $\{u \in H^1(\mathcal{D}) \mid \mathcal{T}_{\partial\mathcal{D}} u = 0\}$ where, for $\mathcal{S}$ a Lipschitz subset of $\overline{\mathcal{D}}$ of co-dimension one, $\mathcal{T}_{\mathcal{S}} : H^1(\mathcal{D}) \to H^{\frac{1}{2}}(\mathcal{S})$ is the trace operator such that $\mathcal{T}_{\mathcal{S}} v = v|_{\mathcal{S}}$ if v is regular enough. The space $H^s(\mathcal{D})$ for $s < 0$ is defined by $H^s = (H_0^{-s})'$.

In all the paper, C will denote the constant of a generic upper bound $a \leqslant Cb$ assumed to be independent of the variables of the inequality and of the mesh size h, the size of the hole ε and N which characterizes the number of scalar constraints considered to approximate the homogeneous Dirichlet condition on the hole. This generic upper bound being not necessarily the same from one occurrence to another. When it is necessary, its dependency on some parameters will be made precise, for example, if C depends on a domain $\mathcal{D}$, we will write $C(\mathcal{D})$.

We now introduce some preliminary lemmas useful in the following sections and a first result on the asymptotic behavior of u_ε when ε tends to 0.

The first lemma is a stability result on the behavior of the solution of the Poisson problem with a small inclusion of size $\varepsilon > 0$. As a reminder, the domain ω denotes the unit open ball centered at the origin.

Lemma 2.1. *For $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$ such that $\omega_\varepsilon \cap \operatorname{supp} f = \varnothing$, Problem (1.1) admits a unique weak solution in $H^1(\Omega_\varepsilon)$. Moreover there exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,*

$$\|u_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C independent of ε.

The second lemma describes the behavior of the solution of the Poisson problem with a constant Dirichlet boundary condition on $\partial\omega_\varepsilon$.

Lemma 2.2. *For any $L \in \mathbb{R}$, the problem*

$$\begin{cases} -\Delta v_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \\ v_\varepsilon = L & \text{on } \partial\omega_\varepsilon, \end{cases}$$

admits a unique weak solution in $H^1(\Omega_\varepsilon)$. Moreover, there exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,

$$\|v_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}|L|,$$

with C independent of ε.

The third lemma describes the behavior of the solution of the Poisson problem for a general Dirichlet boundary condition on $\partial\omega_\varepsilon$.

Lemma 2.3. *For any $\varphi \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, the problem*

$$\begin{cases} -\Delta v_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \\ v_\varepsilon = \varphi & \text{on } \partial\omega_\varepsilon, \end{cases}$$

admits a unique weak solution in $H^1(\Omega_\varepsilon)$. Moreover, there exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,

$$\|v_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega},$$

with C independent of ε.

For the convenience of the reader, the proofs of **Lemma 2.1**, **Lemma 2.2** and **Lemma 2.3** are given in the Appendix. They are based on the results presented in [[20], **Appendix A-D**] themselves described in [[21], **Chapter 3**] .

Let us now consider $u_0 \in H^1(\Omega)$ the unique solution of

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = \phi & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The function u_0 satisfies the following standard energy bound:

$$\|u_0\|_{1,\Omega} \leqslant C \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right). \tag{2.4}$$

Problem (2.3) represents the limit case of Problem (1.1) for $\varepsilon \to 0$. The following result gives the rate of convergence of u_ε towards u_0, namely an estimate of the difference $u_\varepsilon - u_0$ with respect to ε.

Theorem 2.1. *For $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$ such that $\omega_\varepsilon \cap \operatorname{supp} f = \varnothing$, there exist constants $C > 0$ and $\rho > 0$ such that, for all $0 < \varepsilon < \rho$, the solution u_ε of Problem (1.1) satisfies*

$$\|u_\varepsilon - u_0\|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C independent of ε.

The result presented in **Theorem 2.1** is a classical result which can be seen as a consequence of the fact that the H^1-capacity of ω_ε tends to 0 as $(-\log(\varepsilon))^{-1}$, see for example [22] for a discussion about capacity. For the sake of completeness, we also present a proof of **Theorem 2.1** in the Appendix.

Since the solution of Problem (1.1) tends to the solution of Problem (2.3) when $\varepsilon \to 0$, to approximate Problem (1.1), one may consider the limit problem without inclusion. In other words, one could just ignore the presence of the inclusion. However, **Theorem 2.1** shows that the convergence is very slow with respect to the size of the inclusion. To give an idea, $(-\log(\varepsilon))^{-\frac{1}{2}} \approx 0.201$ for $\varepsilon = 10^{-10}$. For this reason, we introduce and analyse a family of problems whose solutions can approximate u_ε better than u_0, with an arbitrarily high accuracy when $\varepsilon \to 0$.

Remark 2.1. *We consider in this paper a circular obstacle centered in 0 with homogeneous boundary conditions on $\partial\omega_\varepsilon$ but all the results can be generalized to an obstacle centered in $\mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^2$ with arbitrary constant boundary conditions on $\partial\omega_\varepsilon$. This construction is also easily generalized to multiple perforations, provided that they do not intersect and do not intersect the support of f.*

2.2 A reduced model for the boundary condition on the inclusion

To derive a family of reduced problems, we will first consider a weak formulation of Problem (1.1) where the constraint on the boundary $\partial\omega_\varepsilon$ is imposed by a Lagrange multiplier. Problem (1.1) can be written: find $u_\varepsilon \in H^1(\Omega_\varepsilon)$ and $\lambda_\varepsilon \in H^{-\frac{1}{2}}(\partial\omega_\varepsilon)$ such that

$$\begin{cases} (\nabla u_\varepsilon, \nabla v)_{\Omega_\varepsilon} + \langle \lambda_\varepsilon, v\rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = (f, v)_{\Omega_\varepsilon}, & \forall v \in H^1(\Omega_\varepsilon), \\ \langle \mu, u_\varepsilon\rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = 0, & \forall \mu \in H^{-\frac{1}{2}}(\partial\omega_\varepsilon), \\ u_\varepsilon - \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $\langle\cdot,\cdot\rangle_{-\frac{1}{2},\partial\omega_\varepsilon}$ denotes the pairing between $H^{-\frac{1}{2}}(\partial\omega_\varepsilon)$ and $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$. We now apply a model reduction method to Problem (2.5) based on the hypothesis that ε is very small.

The first assumption consists in identifying the domain Ω_ε with the entire domain Ω. As a consequence, we suppose that for $u, v \in H^1(\Omega)$,

$$(u, v)_{\Omega_\varepsilon} \simeq (u, v)_\Omega. \tag{2.6}$$

Let us note that this assumption alone is equivalent to extending the solution u_ε to the whole domain Ω by zero and this extension satisfies

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon. \end{cases}$$

In particular, the solution of Problem (2.5) with assumption (2.6) remains unchanged in Ω_ε.

For the second assumption, we replace the trace equality on $\partial\omega_\varepsilon$ by an approximation of the trace operator on $\partial\omega_\varepsilon$ thanks to a projection operator in a finite dimensional space. This space will approximate the trace space $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$. To describe this space, we take advantage of the circular geometry of the hole boundary and switch to cylindrical coordinates, that is for $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi[$, if u is a function defined in Ω and $\mathbf{x} = (\cos(\theta), \sin(\theta))$, we write $u(r\mathbf{x}) = u(r, \theta)$. In particular, if $\mathbf{x} = (\cos(\theta), \sin(\theta))$, for $\theta \in [0, 2\pi[$, we have $u(\mathbf{x}) = u(1, \theta)$. We then consider the space $\mathscr{M}^N$ of trigonometric polynomials of degree less than or equal to N given by

$$\mathscr{M}^N = \{v \in L^2(\partial\omega) \mid v(\mathbf{x}) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\theta) + b_n \sin(n\theta)), \text{ for a. e. } \mathbf{x} \in \partial\omega,$$
$$\text{with } (a_n)_{0 \leqslant n \leqslant N} \in \mathbb{R}^{N+1}, (b_n)_{1 \leqslant n \leqslant N} \in \mathbb{R}^N\}.$$

The space $\mathscr{M}^N$ is seen here as an approximation space of $L^2(\partial\omega)$ but by rescaling we will see that it can also be used to obtain an approximation space of $L^2(\partial\omega_\varepsilon)$ and $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$.

We then denote by $\Pi^N : L^2(\partial\omega) \to \mathscr{M}^N$ the L^2 projection on $\mathscr{M}^N$ given by

$$(\Pi^N u, v)_{\partial\omega} = (u, v)_{\partial\omega}, \forall u \in L^2(\partial\omega), \forall v \in \mathscr{M}^N. \tag{2.7}$$

We have for $u \in L^2(\partial\omega)$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta))$,

$$(\Pi^N u)(\mathbf{x}) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

where a_n and b_n are the n^{th} Fourier coefficients of u satisfying for $n \in \mathbb{N}^*$,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\mathrm{d}\theta, \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\cos(n\theta)\mathrm{d}\theta, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\sin(n\theta)\mathrm{d}\theta. \end{aligned}$$

Based on these notations, we introduce the following operators

$$\begin{aligned} \mathcal{A}^n &: L^2(\partial\omega) \to \mathbb{R} \quad \mathcal{A}^n(u) = a_n, \ \text{ for } n \in \mathbb{N}, \\ \mathcal{B}^n &: L^2(\partial\omega) \to \mathbb{R} \quad \mathcal{B}^n(u) = b_n, \ \text{ for } n \in \mathbb{N}^*. \end{aligned} \tag{2.8}$$

The next theorem justifies the good approximation properties of $\mathscr{M}^N$ on $L^2(\partial\omega)$ and allows the introduction of the Fourier series decomposition, see for example [23] for more details on this topic.

Theorem 2.2. *Let u be a function of $L^2(\partial\omega)$ and Π^N be the operator introduced in equation (2.7), we have*

$$\|u - \Pi^N u\|_{0,\partial\omega} \to 0 \text{ as } N \to \infty,$$

and

$$\lim_{N\to\infty} \|\Pi^N u\|^2_{0,\partial\omega} = 2\pi\left(a_0^2 + \sum_{n=1}^{\infty}\left(\frac{a_n^2}{2} + \frac{b_n^2}{2}\right)\right) = \|u\|^2_{0,\partial\omega}, \tag{2.9}$$

with for all $1 \leqslant n \leqslant N$, $a_0 = \mathcal{A}^0 u$, $a_n = \mathcal{A}^n u$, $b_n = \mathcal{B}^n u$.

We have in particular that any function $u \in L^2(\partial\omega)$ verifies, for $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta))$,

$$u(\mathbf{x}) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\theta) + b_n \sin(\theta)),$$

with for all $1 \leqslant n \leqslant N$, $a_0 = \mathcal{A}^0 u$, $a_n = \mathcal{A}^n u$, $b_n = \mathcal{B}^n u$. This decomposition is usually called the Fourier series decomposition of u and $(a_n \cos(n\theta), b_n \sin(n\theta))$ are the n^{th} Fourier modes of u. A consequence of **Theorem 3.3** is the stability of the operator Π^N on $L^2(\partial\omega)$.

Corollary 2.1. *Let $u \in L^2(\partial\omega)$,*

$$\|\Pi^N u\|^2_{0,\partial\omega} = 2\pi\left(a_0^2 + \sum_{n=1}^{N}\left(\frac{a_n^2}{2} + \frac{b_n^2}{2}\right)\right) \leqslant \|u\|^2_{0,\partial\omega}.$$

with for all $1 \leqslant n \leqslant N$, $a_0 = \mathcal{A}^0 u$, $a_n = \mathcal{A}^n u$, $b_n = \mathcal{B}^n u$.

In our case, the space we want to approximate is $H^{\frac{1}{2}}(\partial\omega)$. Since $H^{\frac{1}{2}}(\partial\omega) \subset L^2(\partial\omega)$, the elements of $H^{\frac{1}{2}}(\partial\omega)$ also admit a Fourier series decomposition. The norm usually employed on $H^{\frac{1}{2}}(\partial\omega)$ is $\|\cdot\|_{\frac{1}{2},\partial\omega}$ defined in (2.2), however here we introduce a more suitable auxiliary norm $\overline{\|\cdot\|}_{\frac{1}{2},\partial\omega}$ on $H^{\frac{1}{2}}(\partial\omega)$ depending on the Fourier coefficients and defined for all $v \in H^{\frac{1}{2}}(\partial\omega)$ by

$$\overline{\|v\|}_{\frac{1}{2},\partial\omega} = \left(a_0^2 + \sum_{n=1}^{\infty}(1+n)\left(a_n^2 + b_n^2\right)\right)^{\frac{1}{2}},$$

with for all $1 \leqslant n \leqslant N$, $a_0 = \mathcal{A}^0 u$, $a_n = \mathcal{A}^n u$, $b_n = \mathcal{B}^n u$. We have in particular

$$\overline{\|\Pi^N v\|}_{\frac{1}{2},\partial\omega} = \left(a_0^2 + \sum_{n=1}^{N} (1+n)\left(a_n^2 + b_n^2\right)\right)^{\frac{1}{2}}.$$

Both norms $\|\cdot\|_{\frac{1}{2},\partial\omega}$ and $\overline{\|\cdot\|}_{\frac{1}{2},\partial\omega}$ are equivalent on $H^{\frac{1}{2}}(\partial\omega)$ (see [[24], **Lemma 2.4.5**] for example), implying in particular that the norm $\overline{\|\cdot\|}_{\frac{1}{2},\partial\omega}$ is well defined. Using the Fourier norm on $H^{\frac{1}{2}}(\partial\omega)$ and the equivalence result on the norm $\|\cdot\|_{\frac{1}{2},\partial\omega}$ and $\overline{\|\cdot\|}_{\frac{1}{2},\partial\omega}$, we immediately obtain some stability properties on Π^N described in the following proposition.

Proposition 2.1. *For $u \in H^{\frac{1}{2}}(\partial\omega)$ we have,*

$$\overline{\|\Pi^N u\|}_{\frac{1}{2},\partial\omega} \leqslant \overline{\|u\|}_{\frac{1}{2},\partial\omega}.$$

There exists a constant $C > 0$ such that for $u \in H^{\frac{1}{2}}(\partial\omega)$,

$$\|\Pi^N u\|_{\frac{1}{2},\partial\omega} \leqslant C\|u\|_{\frac{1}{2},\partial\omega},$$

with C independent of N.

Next we rescale these spaces to further define appropriate norms and approximate spaces of $L^2(\partial\omega_\varepsilon)$ and $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$. Let $\Psi_\varepsilon : L^2(\partial\omega_\varepsilon) \longrightarrow L^2(\partial\omega)$ defined, for all $v \in L^2(\partial\omega_\varepsilon)$, by

$$\Psi_\varepsilon(v)(\mathbf{x}) = v(\varepsilon\mathbf{x}), \text{ for a. e. } \mathbf{x} \in \partial\omega, \tag{2.10}$$

in particular, Ψ_ε satisfies

$$\Psi_\varepsilon(L^2(\partial\omega_\varepsilon)) = L^2(\partial\omega),$$

and

$$\Psi_\varepsilon(H^{\frac{1}{2}}(\partial\omega_\varepsilon)) = H^{\frac{1}{2}}(\partial\omega).$$

We set, for $\varepsilon > 0$,

$$\mathscr{M}_\varepsilon^N = \{v \in L^2(\partial\omega_\varepsilon) \mid \Psi_\varepsilon(v) \in \mathscr{M}^N\},$$

and we define the operator $\Pi_\varepsilon^N : L^2(\partial\omega_\varepsilon) \to \mathscr{M}_\varepsilon^N$ (resp. $\mathcal{A}_\varepsilon^n : L^2(\partial\omega_\varepsilon) \to \mathbb{R}$ and $\mathcal{B}_\varepsilon^n : L^2(\partial\omega_\varepsilon) \to \mathbb{R}$) by

$$\Pi_\varepsilon^N = \Psi_\varepsilon^{-1} \circ \Pi^N \circ \Psi_\varepsilon \tag{2.11}$$

(resp. $\mathcal{A}_\varepsilon^n = \mathcal{A}^n \circ \Psi_\varepsilon$ and $\mathcal{B}_\varepsilon^n = \mathcal{B}^n \circ \Psi_\varepsilon$). Note that $\mathscr{M}_\varepsilon^N$ is equal to

$$\mathscr{M}_\varepsilon^N = \{v \in L^2(\partial\omega_\varepsilon) \mid \Psi_\varepsilon(v)(\mathbf{x}) = a_{\varepsilon,0} + \sum_{n=1}^{N} \left(a_{\varepsilon,n}\cos(n\theta) + b_{\varepsilon,n}\sin(n\theta)\right), \text{ for a. e. } \mathbf{x} \in \partial\omega,$$
$$\text{with } (a_{\varepsilon,n})_{0\leqslant n\leqslant N} \in \mathbb{R}^{N+1}, (b_{\varepsilon,n})_{1\leqslant n\leqslant N} \in \mathbb{R}^N\},$$

and for $v \in L^2(\partial\omega_\varepsilon)$, for $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,

$$(\Pi^N \circ \Psi_\varepsilon)(v)(\mathbf{x}) = a_{\varepsilon,0} + \sum_{n=1}^{N} \left(a_{\varepsilon,n}\cos(n\theta) + b_{\varepsilon,n}\sin(n\theta)\right),$$

with $a_{\varepsilon,0} = \mathcal{A}_\varepsilon^0 v$, $a_{\varepsilon,n} = \mathcal{A}_\varepsilon^n v$, $b_{\varepsilon,n} = \mathcal{B}_\varepsilon^n v$. We then consider the following norm on $L^2(\partial\omega_\varepsilon)$,

$$\|v\|_{0,\varepsilon} = \|\Psi_\varepsilon(v)\|_{0,\partial\omega},$$

and the associated scalar product $(\cdot,\cdot)_\varepsilon$ is defined by a rescaling of the scalar product $(\cdot,\cdot)_{\partial\omega}$ in $L^2(\partial\omega)$ as

$$(\mu, v)_\varepsilon = (\Psi_\varepsilon(\mu), \Psi_\varepsilon(v))_{\partial\omega}.$$

Due to the definitions of Π_ε^N and the scalar product on $L^2(\partial\omega_\varepsilon)$, we have the following proposition on Π_ε^N.

Proposition 2.2. *The operator Π_ε^N is auto-adjoint for the scalar product $(\cdot,\cdot)_\varepsilon$, that is, for any $v \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$ and $\mu \in \mathscr{M}_\varepsilon^N$,*

$$\left(\mu, \Pi_\varepsilon^N v\right)_\varepsilon = \left(\Pi_\varepsilon^N \mu, v\right)_\varepsilon = (\mu, v)_\varepsilon.$$

In a similar way, on the space $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, we consider the rescaled norm

$$\|v\|_{\frac{1}{2},\varepsilon} = \|\Psi_\varepsilon(v)\|_{\frac{1}{2},\partial\omega}.$$

We also set $\overline{\|\cdot\|}_{\frac{1}{2},\varepsilon}$ the norm on $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$ defined by

$$\overline{\|v\|}_{\frac{1}{2},\varepsilon} = \overline{\|\Psi_\varepsilon(v)\|}_{\frac{1}{2},\partial\omega}.$$

Let us note that for all $v \in L^2(\partial\omega_\varepsilon)$, we have

$$\|\Pi_\varepsilon^N v\|_{0,\varepsilon}^2 = 2\pi\left(a_{\varepsilon,0}^2 + \sum_{n=1}^{N}\left(\frac{a_{\varepsilon,n}^2}{2} + \frac{b_{\varepsilon,n}^2}{2}\right)\right) \text{ and } \|v\|_{0,\varepsilon}^2 = 2\pi\left(a_{\varepsilon,0}^2 + \sum_{n=1}^{\infty}\left(\frac{a_{\varepsilon,n}^2}{2} + \frac{b_{\varepsilon,n}^2}{2}\right)\right) \tag{2.12}$$

and if $v \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$,

$$\overline{\|\Pi_\varepsilon^N v\|}_{\frac{1}{2},\varepsilon}^2 = a_{\varepsilon,0}^2 + \sum_{n=1}^{N}(1+n)\left(a_{\varepsilon,n}^2 + b_{\varepsilon,n}^2\right), \text{ and } \overline{\|v\|}_{\frac{1}{2},\varepsilon}^2 = a_{\varepsilon,0}^2 + \sum_{n=1}^{\infty}(1+n)\left(a_{\varepsilon,n}^2 + b_{\varepsilon,n}^2\right) \tag{2.13}$$

with $a_{\varepsilon,0} = \mathcal{A}_\varepsilon^0 v$, $a_{\varepsilon,n} = \mathcal{A}_\varepsilon^n v$, $b_{\varepsilon,n} = \mathcal{B}_\varepsilon^n v$. Since the norms $\|\cdot\|_{\frac{1}{2},\partial\omega}$ and $\overline{\|\cdot\|}_{\frac{1}{2},\partial\omega}$ are equivalent, the norms $\|\cdot\|_{\frac{1}{2},\varepsilon}$ and $\overline{\|\cdot\|}_{\frac{1}{2},\varepsilon}$ are also equivalent. Moreover, since the constants appearing in the norm equivalence are the same as the ones appearing for $\|\cdot\|_{\frac{1}{2},\partial\omega}$ and $\overline{\|\cdot\|}_{\frac{1}{2},\partial\omega}$, they are independent of ε. Thanks to the last remark, equations (2.12) and equations (2.13), we can deduce some stability properties on Π_ε^N for the L^2 and $H^{\frac{1}{2}}$ rescaled norm described in the following proposition.

Proposition 2.3. *For $u \in L^2(\partial\omega_\varepsilon)$, we have*

$$\|\Pi_\varepsilon^N u\|_{0,\varepsilon} \leqslant \|u\|_{0,\varepsilon}.$$

For $u \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, we have

$$\overline{\|\Pi_\varepsilon^N u\|}_{\frac{1}{2},\varepsilon} \leqslant \overline{\|u\|}_{\frac{1}{2},\varepsilon}.$$

There exists a constant $C > 0$ such that for $u \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$,

$$\|\Pi_\varepsilon^N u\|_{\frac{1}{2},\varepsilon} \leqslant C\|u\|_{\frac{1}{2},\varepsilon},$$

with C independent of N and ε.

Now, taking into account the hypothesis that ε is small, for $u \in H^1(\Omega_\varepsilon)$, we will substitute the constraint $\mathcal{T}_{\partial\omega_\varepsilon} u = 0$ by

$$\Pi_\varepsilon^N \circ \mathcal{T}_{\partial\omega_\varepsilon} u = 0, \tag{2.14}$$

and look for a solution in

$$V^N = \{v \in H^1(\Omega) \mid \Pi_\varepsilon^N \circ \mathcal{T}_{\partial\omega_\varepsilon} u = 0\}.$$

In other words, instead of imposing u equal to 0 on $\partial\omega_\varepsilon$, we just impose this constraint to its first $(2N+1)$ Fourier coefficients. In particular, for $N = 0$, the space $\mathscr{M}_\varepsilon^N$ is the space of constant functions on $\partial\omega_\varepsilon$ and the constraint (2.14) is equivalent to substituting the trace constraint by a constraint on the average of u on $\partial\omega_\varepsilon$.

From now on, we set $\mathcal{T}_{\partial\omega_\varepsilon}^N : H^1(\Omega) \to \mathscr{M}_\varepsilon^N$ the operator defined by $\mathcal{T}_{\partial\omega_\varepsilon}^N = \Pi_\varepsilon^N \circ \mathcal{T}_{\partial\omega_\varepsilon}$ and we denote by u_ε^N the solution of the reduced problem obtained under the assumptions (2.6) and (2.14). The Lagrange multiplier λ_ε is also approximated in $\mathscr{M}_\varepsilon^N$ by a function λ_ε^N corresponding to the Lagrange multiplier associated to the constraint (2.14). The space $\mathscr{M}_\varepsilon^N$ is this time seen as an approximation space of $H^{-\frac{1}{2}}(\partial\omega_\varepsilon)$ equipped with the duality product $\langle\cdot,\cdot\rangle_{-\frac{1}{2},\varepsilon}$ and the norm $\|\cdot\|_{-\frac{1}{2},\varepsilon}$ defined for $\lambda \in H^{-\frac{1}{2}}(\partial\omega_\varepsilon)$ and $\underline{\mu} \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$ by

$$\langle\lambda,\underline{\mu}\rangle_{-\frac{1}{2},\varepsilon} = \varepsilon^{-1}\langle\lambda,\underline{\mu}\rangle_{-\frac{1}{2},\partial\omega_\varepsilon} \text{ and } \|\lambda\|_{-\frac{1}{2},\varepsilon} = \sup_{\underline{\mu}\in H^{\frac{1}{2}}(\partial\omega_\varepsilon)} \frac{\langle\lambda,\underline{\mu}\rangle_{-\frac{1}{2},\varepsilon}}{\|\underline{\mu}\|_{\frac{1}{2},\varepsilon}}. \tag{2.15}$$

Let us note that the duality product $\langle\cdot,\cdot\rangle_{-\frac{1}{2},\varepsilon}$ verifies in particular for $\lambda \in \mathscr{M}_\varepsilon^N \subset L^2(\partial\omega_\varepsilon)$ and $\underline{\mu} \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$,

$$\langle\lambda,\underline{\mu}\rangle_{-\frac{1}{2},\varepsilon} = \varepsilon^{-1}\left(\lambda,\underline{\mu}\right)_{\partial\omega_\varepsilon} = \left(\lambda,\underline{\mu}\right)_\varepsilon.$$

The resulting reduced problem writes: find $u_\varepsilon^N \in H^1(\Omega)$ and $\lambda_\varepsilon^N \in \mathscr{M}_\varepsilon^N$ such that

$$\begin{cases} \left(\nabla u_\varepsilon^N, \nabla v\right)_\Omega + \left(\lambda_\varepsilon^N, \mathcal{T}_{\partial\omega_\varepsilon}^N v\right)_\varepsilon = \left(f, \mathcal{T}_{\partial\omega_\varepsilon}^N v\right)_\Omega, & \forall v \in H_0^1(\Omega), \\ \left(\mu, \mathcal{T}_{\partial\omega_\varepsilon}^N u_\varepsilon^N\right)_\varepsilon = 0, & \forall \mu \in \mathscr{M}_\varepsilon^N, \\ u_\varepsilon^N - \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Let us note that the second equation of Problem (2.16) implies in particular $u_\varepsilon^N \in V^N$. Moreover according to **Proposition 2.2**, we have for $v \in H_0^1(\Omega)$ and $\mu \in \mathscr{M}_\varepsilon^N$,

$$\left(\lambda_\varepsilon^N, \mathcal{T}_{\partial\omega_\varepsilon}^N v\right)_\varepsilon = \left(\lambda_\varepsilon^N, v\right)_\varepsilon \text{ and } \left(\mu, \mathcal{T}_{\partial\omega_\varepsilon}^N u_\varepsilon^N\right)_\varepsilon = \left(\mu, u_\varepsilon^N\right)_\varepsilon,$$

as a consequence, we can omit the operator $\mathcal{T}_{\partial\omega_\varepsilon}^N$ in the writing of Problem (2.16), what we will do from now on. Eventually, notice that Problem (2.16) can be formally written in strong form: find $u_\varepsilon^N \in H^1(\Omega)$ and $\lambda_\varepsilon^N \in \mathscr{M}_\varepsilon^N$ such that

$$\begin{cases} -\Delta u_\varepsilon^N + \varepsilon^{-1}\lambda_\varepsilon^N \delta_{\partial\omega_\varepsilon} = f & \text{in } \Omega, \\ \mathcal{T}_{\partial\omega_\varepsilon}^N u_\varepsilon^N = 0 & \text{on } \partial\omega_\varepsilon, \\ u_\varepsilon^N = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.17}$$

where the distribution $v\delta_{\partial\omega_\varepsilon}$ is such that for all $\psi \in \mathcal{C}_0^\infty(\Omega)$, $\langle v\delta_{\partial\omega_\varepsilon}, \psi\rangle_\Omega = \langle v, \psi\rangle_{-\frac{1}{2},\partial\omega_\varepsilon}$.

3 Analysis of the reduced Poisson problem

3.1 Auxiliary results

Before studying the well-posedness of the reduced problem (2.16), we introduce some preliminary general results that will be useful in what follows. Let $\mathcal{D}$ be a domain in $\mathbb{R}^2$ and $\mathcal{S}$ be a Lipschitz subset of $\mathcal{D}$ of codimension 1.

Theorem 3.1 (Poincaré inequality). *There exists a positive constant $C_P(\mathcal{D})$ such that for any $v \in H_0^1(\mathcal{D})$,*

$$\|v\|_{0,\mathcal{D}} \leqslant C_P(\mathcal{D})\|\nabla v\|_{0,\mathcal{D}}.$$

Theorem 3.2 (Trace theorem). *For $\eta > 0$, there exists a positive constant $C_{T,\eta}(\mathcal{D},\mathcal{S})$ such that for any $v \in H^{\frac{1}{2}+\eta}(\mathcal{D})$,*

$$\|\mathcal{T}_\mathcal{S} v\|_{\eta,\mathcal{S}} \leqslant C_{T,\eta}(\mathcal{D},\mathcal{S})\|v\|_{\frac{1}{2}+\eta,\mathcal{D}}.$$

For the next theorem, we introduce X an Hilbert space and Q a reflexive Banach space, $a : X \times X \to \mathbb{R}$, $b : Q \times X \to \mathbb{R}$ two bounded bilinear forms and $c : X \to \mathbb{R}$, $d : Q \to \mathbb{R}$ two bounded linear forms. We consider the problem: find $u \in X$ and $\lambda \in Q$ such that

$$\begin{cases} a(u,v) + b(\lambda,v) = c(v) & \forall v \in X, \\ b(\mu,u) = d(\mu) & \forall \mu \in Q. \end{cases} \tag{3.1}$$

This problem has a so called saddle-point structure, the proof of its well-posedness relies on the following theorem, see [25].

Theorem 3.3 ("Inf-sup" condition)**.** *Under the following conditions*

$$\begin{cases} \exists\, \alpha > 0,\ \forall v \in X, & a(v,v) \geqslant \alpha \|v\|_X^2, \\ \exists\, \beta > 0, & \inf\limits_{\mu \in Q} \sup\limits_{v \in X} \dfrac{b(\mu,v)}{\|\mu\|_Q \|v\|_X} \geqslant \beta, \end{cases} \tag{3.2}$$

the saddle-point problem (3.1) *is well posed. Moreover we have the following estimates on u and λ:*

$$\|u\|_X \leqslant \alpha^{-1}\|c\| + \beta^{-1}(1 + \alpha^{-1}\|a\|)\|d\|, \tag{3.3}$$

and

$$\|\lambda\|_Q \leqslant \beta^{-1}(\|c\| + \|a\|\|u\|_X). \tag{3.4}$$

We now consider an extension of this theorem to the twofold saddle point problem discussed for example in [26]. For the convenience of the reader, the proof of the following theorem is given in the Appendix.

Theorem 3.4. *Let Q_1 and Q_2 be two reflexive Banach spaces, $a : X \times X \to \mathbb{R}$, $b_1 : Q_1 \times X \to \mathbb{R}$, $b_2 : Q_2 \times X \to \mathbb{R}$ three bilinear forms, $d_1 : Q_1 \to \mathbb{R}$, $d_2 : Q_2 \to \mathbb{R}$ two linear forms, we consider the twofold saddle point problem: find $(u, \lambda_1, \lambda_2) \in X \times Q_1 \times Q_2$ such that*

$$\begin{cases} a(u,v) + b_1(\lambda_1, v) + b_2(\lambda_2, v) = c(v), & \forall v \in X, \\ b_1(\mu_1, u) = d_1(\mu_1), & \forall \mu_1 \in Q_1, \\ b_2(\mu_2, u) = d_2(\mu_2), & \forall \mu_2 \in Q_2. \end{cases}$$

Let

$$Z_{b_i} := \{v \in X \mid b_i(\mu_i, v) = 0\ \forall \mu_i \in Q_i\} \subset X\ i = 1,2.$$

We suppose that conditions (3.2) *are satisfied with $Q = Q_1 \times Q_2$ and*

$$b : (Q_1 \times Q_2) \times X \to \mathbb{R} \quad b([\lambda_1, \lambda_2], u) = b_1(\lambda_1, u) + b_2(\lambda_2, u).$$

We also suppose that there exists $\beta_1 > 0$ such that for all $\lambda_1 \in Q_1$,

$$\sup_{v \in Z_{b_2}} \frac{b_1(\lambda_1, v)}{\|v\|_X} \geqslant \beta_1 \|\lambda_1\|_{Q_1}, \tag{3.5}$$

and that there exists $\beta_2 > 0$ such that for all $\lambda_2 \in Q_2$,

$$\sup_{v \in Z_{b_1}} \frac{b_2(\lambda_2, v)}{\|v\|_X} \geqslant \beta_2 \|\lambda_2\|_{Q_2}. \tag{3.6}$$

Then we have the following estimates on u, λ_1 and λ_2:

$$\|u\|_X \leqslant \alpha^{-1}\|c\| + \beta_1^{-1}(1 + \alpha^{-1}\|a\|)\|d_1\| + \beta_2^{-1}(1 + \alpha^{-1}\|a\|)\|d_2\|,$$

and

$$\|\lambda_1\|_{Q_1} \leqslant \beta_1^{-1}(\|c\| + \|a\|\|u\|_X), \|\lambda_2\|_{Q_2} \leqslant \beta_2^{-1}(\|c\| + \|a\|\|u\|_X).$$

3.2 Well-posedness of problem (2.16)

To study the reduced problem, we introduce $e_\varepsilon^N = u_\varepsilon^N - u_0$ where u_0 is the solution of (2.3). Then $e_\varepsilon^N \in H_0^1(\Omega)$ and $\lambda_\varepsilon^N \in \mathscr{M}_\varepsilon^N$ satisfy

$$\begin{cases} \left(\nabla e_\varepsilon^N, \nabla v\right)_\Omega + \left(\lambda_\varepsilon^N, v\right)_\varepsilon = 0, & \forall v \in H_0^1(\Omega), \\ \left(\mu, e_\varepsilon^N\right)_\varepsilon = -\left(\mu, u_0\right)_\varepsilon, & \forall \mu \in \mathscr{M}_\varepsilon^N. \end{cases} \tag{3.7}$$

We also introduce the space $\mathscr{M}^{N,*}$ defined by

$$\mathscr{M}^{N,*} = \{v \in L^2(\partial\omega) \mid v(\mathbf{x}) = \sum_{n=1}^{N} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right), \text{ for a. e } \mathbf{x} \in \partial\omega, \quad \text{with } (a_n)_{1\leqslant n\leqslant N} \in \mathbb{R}^N \text{ and } (b_n)_{1\leqslant n\leqslant N} \in \mathbb{R}^N\},$$

and

$$\mathscr{M}_\varepsilon^{N,*} = \{v \in L^2(\partial\omega_\varepsilon) \mid \Psi_\varepsilon(v) \in \mathscr{M}^{N,*}\},$$

where Ψ_ε is defined by (2.10). We further define the operators $\Pi_\varepsilon^{N,*} = \Psi_\varepsilon^{-1} \circ \Pi^{N,*} \circ \Psi_\varepsilon$ and $\mathcal{T}_{\partial\omega_\varepsilon}^{N,*} = \Pi_\varepsilon^{N,*} \circ \mathcal{T}_{\partial\omega_\varepsilon}$ where $\Pi^{N,*}$ is the L^2 projector on $\mathscr{M}^{N,*}$. For revealing the scaling of e_ε^N with respect to ε, we need to reformulate Problem (3.7). It is straightforward to show that there exists a unique pair $(\lambda_\varepsilon^0, \lambda_\varepsilon^{N,*}) \in \mathscr{M}_\varepsilon^0 \times \mathscr{M}_\varepsilon^{N,*}$ such that

$$\lambda_\varepsilon^N = \lambda_\varepsilon^0 + \lambda_\varepsilon^{N,*}.$$

We then deduce that Problem (3.7) is equivalent to: find $e_\varepsilon^N \in H_0^1(\Omega)$ and $(\lambda_\varepsilon^0, \lambda_\varepsilon^{N,*}) \in \mathscr{M}_\varepsilon^0 \times \mathscr{M}_\varepsilon^{N,*}$ such that

$$\begin{cases} \left(\nabla e_\varepsilon^N, \nabla v\right)_\Omega + \left(\lambda_\varepsilon^0, v\right)_\varepsilon + \left(\lambda_\varepsilon^{N,*}, v\right)_\varepsilon = 0, & \forall v \in H_0^1(\Omega), \\ \left(\mu_0, e_\varepsilon^N\right)_\varepsilon = -\left(\mu_0, u_0\right)_\varepsilon, & \forall \mu_0 \in \mathscr{M}_\varepsilon^0, \\ \left(\mu_{N,*}, e_\varepsilon^N\right)_\varepsilon = -\left(\mu_{N,*}, u_0\right)_\varepsilon, & \forall \mu_{N,*} \in \mathscr{M}_\varepsilon^{N,*}. \end{cases} \tag{3.8}$$

This is a two-fold saddle point problem that can be analyzed using **Theorem 3.4**.

This section is devoted to the proof of the well-posedness of Problem (3.7) which is established in the next theorem.

Theorem 3.5. *Problem* (3.7) *is well-posed in* $H_0^1(\Omega) \times \mathscr{M}_\varepsilon^N$. *Moreover there exist constants* $C > 0$ *and* $\rho > 0$ *such that for all* $0 < \varepsilon < \rho$,

$$\|e_\varepsilon^N\|_{1,\Omega} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \tag{3.9}$$

and

$$\begin{aligned} \|\lambda_\varepsilon^0\|_{-\frac{1}{2},\varepsilon} &\leqslant C(-\log(\varepsilon))^{-1} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \\ \|\lambda_\varepsilon^{N,*}\|_{-\frac{1}{2},\varepsilon} &\leqslant C(-\log(\varepsilon))^{-\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \end{aligned} \tag{3.10}$$

with C *independent of* N *and* ε.

This theorem shows that the proposed reduced problem converges towards the problem without holes and that the order of convergence with respect to ε is the same that the rate of convergence of the solution of the full problem, see **Theorem 2.1**. Moreover the norms of the two Lagrange multipliers, associated to the constraints on the average and on the higher moments, converges to zero as ε goes to zero. This is not surprising since the obstacle disappears in the limit. To prove this result, we first need the following three lemmas. The first lemma states a trace like estimate that gives an explicit dependency of the trace continuity constant according to the size ε of the hole.

Lemma 3.1. *There exists a constant* $C > 0$ *such that for all* $v \in H^1(\Omega_\varepsilon)$,

$$\|v\|_{\frac{1}{2},\varepsilon} \leqslant C\|v\|_{1,\Omega_\varepsilon},$$

with C *independent of* ε. *In particular if* $v \in H^1(\Omega)$, *we have*

$$\|v\|_{\frac{1}{2},\varepsilon} \leqslant C\|v\|_{1,\Omega},$$

with C *independent of* ε.

Proof. According to [[27], **Section 4.1.3**], if we consider the norm defined for $u \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$ by

$$\langle u \rangle_{\frac{1}{2},\partial\omega_\varepsilon} = \inf_{v \in H^1(\Omega_\varepsilon), \mathcal{T}_{\partial\omega_\varepsilon} v = u} \|v\|_{1,\Omega_\varepsilon}, \tag{3.11}$$

then the norm $\langle\cdot\rangle_{\frac{1}{2},\partial\omega_\varepsilon}$ and the norm $\|\cdot\|_{\frac{1}{2},\varepsilon}$ are equivalent independently of ε. By definition of the norm $\langle\cdot\rangle_{\frac{1}{2},\partial\omega_\varepsilon}$ given in (3.11), we then have, for all $v \in H^1(\Omega_\varepsilon)$,

$$\frac{1}{C}\|v\|_{\frac{1}{2},\varepsilon} \leqslant \langle v\rangle_{\frac{1}{2},\partial\omega_\varepsilon} \leqslant \|v\|_{1,\Omega_\varepsilon},$$

with C independent of ε. □

In the next lemma, we build a lifting of functions in $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$ over the whole domain Ω. Such lifting is endowed with a norm that only depends on $\|\cdot\|_{\frac{1}{2},\varepsilon}$.

Lemma 3.2. *There exist constants $C > 0$ and $\rho > 0$ such that for any $\underline{\mu} \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, there exists $\underline{v}_\varepsilon \in H^1_0(\Omega)$ satisfying $\underline{v}_\varepsilon = \underline{\mu}$ on $\partial\omega_\varepsilon$ and for all $0 < \varepsilon < \rho$,*

$$\|\underline{v}_\varepsilon\|_{1,\Omega} \leqslant C\|\underline{\mu}\|_{\frac{1}{2},\varepsilon}, \tag{3.12}$$

with C independent of ε. Moreover, if $\underline{\mu}$ is constant on $\partial\omega_\varepsilon$, then

$$\|\underline{v}_\varepsilon\|_{1,\Omega} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}\|\underline{\mu}\|_{0,\varepsilon}, \tag{3.13}$$

with C independent of ε.

Proof. Let $\underline{\mu} \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$ and let us consider the problem: find $\underline{v}_\varepsilon \in H^1_0(\Omega)$ and $\underline{\lambda}_\varepsilon \in H^{-\frac{1}{2}}(\partial\omega_\varepsilon)$ such that

$$\begin{cases} (\nabla\underline{v}_\varepsilon, \nabla v)_\Omega + \langle\underline{\lambda}_\varepsilon, \underline{v}_\varepsilon\rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = 0, & \forall v \in H^1_0(\Omega), \\ \langle\mu, \underline{v}_\varepsilon\rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = \langle\mu, \underline{\mu}\rangle_{-\frac{1}{2},\partial\omega_\varepsilon}, & \forall\mu \in H^{-\frac{1}{2}}(\partial\omega_\varepsilon), \\ \underline{v}_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

By applying the inf-sup theorem, we can prove that (3.14) is well-posed, see [28] for a discussion about this problem. Moreover $\underline{v}_\varepsilon$ satisfies

$$\begin{cases} -\Delta\underline{v}_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ -\Delta\underline{v}_\varepsilon = 0 & \text{in } \omega_\varepsilon, \\ \underline{v}_\varepsilon = \underline{\mu} & \text{on } \partial\omega_\varepsilon, \\ \underline{v}_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

We are now left with the estimates (3.12) and (3.13). To show them, we will consider $\underline{v}_\varepsilon$ separately on Ω_ε and on ω_ε. According to **Lemma 2.3**, for $\varepsilon > 0$ sufficiently small,

$$\|\underline{v}_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C\|\underline{\mu}\|_{\frac{1}{2},\varepsilon}.$$

If $\underline{\mu}$ is constant on $\partial\omega_\varepsilon$, then according to **Lemma 2.2**, for $\varepsilon > 0$ sufficiently small, we have

$$\|\underline{v}_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}|\underline{\mu}| = C(-\log(\varepsilon))^{-\frac{1}{2}}\|\underline{\mu}\|_{0,\varepsilon}.$$

Next we look at the estimate on ω_ε. Since $\underline{\mu} \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, $\underline{\mu}$ can be written for $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,

$$\underline{\mu}(\varepsilon\mathbf{x}) = a_{\varepsilon,0} + \sum_{n=1}^{\infty}\left(a_{\varepsilon,n}\cos(n\theta) + b_{\varepsilon,n}\sin(n\theta)\right).$$

As v_ε is harmonic in ω_ε, by the method of separation of variables, we have for $0 < r < \varepsilon$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,

$$\underline{v}_\varepsilon(r\mathbf{x}) = a_{\varepsilon,0} + \sum_{n=1}^{\infty}\left(\frac{r}{\varepsilon}\right)^n\left(a_{\varepsilon,n}\cos(n\theta) + b_{\varepsilon,n}\sin(n\theta)\right).$$

The proof of this expression can be found for example in [29]. We then deduce by orthogonality of the basis functions $\sin(n\theta)$ and $\cos(n\theta)$ that

$$\|\underline{v}_\varepsilon\|^2_{0,\omega_\varepsilon} = a^2_{\varepsilon,0}\pi\varepsilon^2 + \sum_{n=1}^{\infty}\pi\varepsilon^{-2n}\left(a^2_{\varepsilon,n} + b^2_{\varepsilon,n}\right)\int_0^\varepsilon r^{2n+1}\mathrm{d}r = a^2_{\varepsilon,0}\pi\varepsilon^2 + \varepsilon^2\sum_{n=1}^{\infty}\frac{\pi}{2n+2}\left(a^2_{\varepsilon,n} + b^2_{\varepsilon,n}\right),$$

and

$$\|\frac{1}{r}\frac{\partial \underline{v}_\varepsilon}{\partial \theta}\|^2_{0,\omega_\varepsilon} = \|\frac{\partial \underline{v}_\varepsilon}{\partial r}\|^2_{0,\omega_\varepsilon} = \sum_{n=1}^{\infty} \varepsilon^{-2n}\pi n^2 \left(a^2_{\varepsilon,n} + b^2_{\varepsilon,n}\right) \int_0^\varepsilon r^{2n-1}\mathrm{d}r = \sum_{n=1}^{\infty} \frac{n\pi}{2}\left(a^2_{\varepsilon,n} + b^2_{\varepsilon,n}\right),$$

so for $\varepsilon > 0$ sufficiently small,

$$\|\underline{v}_\varepsilon\|^2_{1,\omega_\varepsilon} = \|\underline{v}_\varepsilon\|^2_{0,\omega_\varepsilon} + \|\nabla v\|^2_{0,\omega_\varepsilon} \leqslant C\left(a^2_{\varepsilon,0} + \sum_{n=1}^{\infty}(1+n)\left(a^2_{\varepsilon,n} + b^2_{\varepsilon,n}\right)\right) = C\overline{\|\underline{\mu}\|}^2_{\frac{1}{2},\varepsilon} \leqslant C\|\underline{\mu}\|^2_{\frac{1}{2},\varepsilon},$$

with C independent of ε. Similarly, in the case where $\underline{\mu}$ is a constant, $a_{\varepsilon,n} = b_{\varepsilon,n} = 0$ for all $n \geqslant 1$, and we deduce a more accurate estimate,

$$\|\underline{v}_\varepsilon\|_{1,\omega_\varepsilon} \leqslant C\varepsilon\|\underline{\mu}\|_{0,\varepsilon},$$

with C independent of ε. We conclude the proof of the lemma noticing that

$$\|\underline{v}_\varepsilon\|_{1,\Omega} = \left(\|\underline{v}_\varepsilon\|^2_{1,\omega_\varepsilon} + \|\underline{v}_\varepsilon\|^2_{1,\Omega_\varepsilon}\right)^{\frac{1}{2}}.$$

□

The next lemma addresses the trace of the solution of the limit problem on the boundary $\partial\omega_\varepsilon$ and in particular it details its behavior with respect to the radius ε of the inclusion.

Lemma 3.3. *Let u_0 be the solution of Problem (2.3). There exist constants $C > 0$, $\rho > 0$ and $\Upsilon > 0$ such that for all $0 < \varepsilon < \rho < \Upsilon$, for all $n \in \mathbb{N}$,*

$$|\mathcal{A}^n_\varepsilon u_0| + |\mathcal{B}^n_\varepsilon u_0| \leqslant C\left(\frac{\varepsilon}{\Upsilon}\right)^n \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

and

$$\|\mathcal{T}^0_{\partial\omega_\varepsilon} u_0\|_{\frac{1}{2},\varepsilon} \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \|\mathcal{T}^{N,*}_{\partial\omega_\varepsilon} u_0\|_{\frac{1}{2},\varepsilon} \leqslant C\varepsilon^{\frac{1}{2}}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of n, N and ε.

Proof. Let us consider $\Upsilon > 0$ sufficiently small such that $\mathrm{supp} f \cap \omega_\Upsilon = \varnothing$. Such a Υ exists since we have assumed that $\mathrm{supp} f \cap \omega_\varepsilon = \varnothing$ for $\varepsilon > 0$ sufficiently small. The function u_0 is harmonic in ω_Υ, so by the method of separation of variables, we have for $0 < r < \Upsilon$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,

$$u_0(r\mathbf{x}) = A_{\Upsilon,0} + \sum_{n=0}^{\infty} r^n \left(A_{\Upsilon,n}\cos(n\theta) + B_{\Upsilon,n}\sin(n\theta)\right)$$

with

$$\begin{cases} A_{\Upsilon,0} = \mathcal{A}^0_\Upsilon u_0, \\ A_{\Upsilon,n} = \dfrac{1}{\Upsilon^n}\mathcal{A}^n_\Upsilon u_0, & \forall n \geqslant 1, \\ B_{\Upsilon,n} = \dfrac{1}{\Upsilon^n}\mathcal{B}^n_\Upsilon u_0, & \forall n \geqslant 1, \end{cases}$$

where $\mathcal{A}^n_\Upsilon$ and $\mathcal{B}^n_\Upsilon$ are defined on $H^1_0(\Omega)$ in a similar way as $\mathcal{A}^n_\varepsilon$ and $\mathcal{B}^n_\varepsilon$, see (2.8). We deduce that, for all $0 < \varepsilon < \Upsilon$,

$$|\mathcal{A}^n_\varepsilon u_0| = \varepsilon^n |A_{\Upsilon,n}| = \left(\frac{\varepsilon}{\Upsilon}\right)^n |\mathcal{A}^n_\Upsilon u_0|.$$

Using Cauchy-Schwarz inequality, we then have

$$|\mathcal{A}^n_\Upsilon u_0| \leqslant |\partial\omega_\Upsilon|^{-\frac{1}{2}}\|u_0\|_{0,\partial\omega_\Upsilon}.$$

Thus we get that, for all $0 < \varepsilon < \Upsilon$,

$$|\mathcal{A}^n_\varepsilon u_0| \leqslant |\partial\omega_\Upsilon|^{-\frac{1}{2}}\left(\frac{\varepsilon}{\Upsilon}\right)^n \|u_0\|_{0,\partial\omega_\Upsilon}.$$

Moreover, according to **Theorem 3.2** on trace inequality, we have

$$\begin{aligned} \|u_0\|_{0,\partial\omega_\Upsilon} &\leqslant \|u_0\|_{\frac{1}{2},\partial\omega_\Upsilon} \\ &\leqslant C_{T,\frac{1}{2}}(\omega_\Upsilon, \partial\omega_\Upsilon)\|u_0\|_{1,\omega_\Upsilon} \leqslant C_{T,\frac{1}{2}}(\omega_\Upsilon, \partial\omega_\Upsilon)\|u_0\|_{1,\Omega}. \end{aligned}$$

Eventually, thanks to energy estimate (2.4), we get

$$|\mathcal{A}_\varepsilon^n u_0| \leqslant C \left(\frac{\varepsilon}{\Upsilon}\right)^n \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

and similarly,

$$|\mathcal{B}_\varepsilon^n u_0| \leqslant C \left(\frac{\varepsilon}{\Upsilon}\right)^n \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of n, N and ε. This concludes the proof of the lemma. □

Proof of Theorem 3.5. We will verify the assumptions of **Theorem 3.3** to prove that Problem (3.7) is well-posed. First, we set $X = H_0^1(\Omega)$, $Q = \mathscr{M}_\varepsilon^N$ and also define the following linear and bilinear forms:

$$\begin{cases} a:\ H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R} & a(u,v) = (\nabla u, \nabla v)_\Omega\,, \\ b:\ \mathscr{M}_\varepsilon^N \times H_0^1(\Omega) \to \mathbb{R} & b(\mu,u) = (\mu,u)_\varepsilon\,, \\ c:\ H_0^1(\Omega) \to \mathbb{R} & c(v) = 0, \\ d:\ \mathscr{M}_\varepsilon^N \to \mathbb{R} & d(\mu) = -(\mu,u_0)_\varepsilon\,. \end{cases}$$

Let us note that for the sake of simplicity, we have deliberately omitted in the notation of b and d their dependency in N and ε. In addition, to verify the assumptions of **Theorem 3.3**, we will also need to specify the dependency of the coercivity and the inf-sup constants (which will be denoted respectively by α_ε^N and β_ε^N and which are defined by (3.2)), as well as the dependency of the norm of a and b with respect to ε, to obtain the desired estimates (3.9) and (3.10).

First, we notice that the forms a and b are bilinear and the function forms c and d are linear and bounded. The coercivity of a is a direct consequence of **Lemma 3.1** on Poincaré inequality. Indeed, for $u \in H_0^1(\Omega)$,

$$\|u\|_{1,\Omega}^2 (1 + C_P(\Omega))^{-2} \leqslant \|\nabla u\|_{0,\Omega}^2 = a(u,u).$$

As for the continuity property, it directly comes from Cauchy-Schwarz inequality: for all $u, v \in H_0^1(\Omega)$,

$$|a(u,v)| \leqslant \|u\|_{1,\Omega}\|v\|_{1,\Omega}.$$

So a satisfies the first condition of (3.2) with $\alpha_\varepsilon^N \geqslant (1 + C_P(\Omega))^{-2}$, and the continuity bound of a is $\|a\| \leqslant 1$.

Let us now prove that the bilinear form b is bounded. By definition of the norm $\|\cdot\|_{-\frac{1}{2},\varepsilon}$, for all $u \in H_0^1(\Omega)$ and $\mu \in \mathscr{M}_\varepsilon^N$,

$$|b(\mu,u)| = |\,(\mu,u)_\varepsilon\,| \leqslant \|\mu\|_{-\frac{1}{2},\varepsilon}\|u\|_{\frac{1}{2},\varepsilon}.$$

According to **Lemma 3.1**, we have

$$\|u\|_{\frac{1}{2},\varepsilon} \leqslant C\|u\|_{1,\Omega}.$$

So we get that b is bounded with a bound independent of N and ε. We are left to prove the inf-sup condition. It consists in proving that there exists $\beta_\varepsilon^N > 0$ such that for all $\mu \in \mathscr{M}_\varepsilon^N$,

$$\sup_{v \in H_0^1(\Omega)} \frac{(\mu,v)_\varepsilon}{\|v\|_{1,\Omega}} \geqslant \beta_\varepsilon^N \|\mu\|_{-\frac{1}{2},\varepsilon}.$$

According to **Lemma 3.2**, for all $\underline{\mu} \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, there exists $\underline{v}_\varepsilon \in H_0^1(\Omega)$ such that $\underline{v}_\varepsilon = \underline{\mu}$ on $\partial\omega_\varepsilon$ and for $\varepsilon > 0$ sufficiently small,

$$\|\underline{v}_\varepsilon\|_{1,\Omega} \leqslant C\|\underline{\mu}\|_{\frac{1}{2},\varepsilon}.$$

We deduce that, for all $\mu \in \mathscr{M}_\varepsilon^N$,

$$\sup_{v \in H_0^1(\Omega)} \frac{(\mu,v)_\varepsilon}{\|v\|_{1,\Omega}} \geqslant \sup_{\underline{\mu} \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)} \frac{1}{C}\frac{(\mu,\underline{\mu})_\varepsilon}{\|\underline{\mu}\|_{\frac{1}{2},\varepsilon}} \geqslant \frac{1}{C}\|\mu\|_{-\frac{1}{2},\varepsilon}, \tag{3.15}$$

with C independent of N and ε. We conclude on the well-posedness of Problem (3.7) using **Theorem 3.3** with $\alpha_\varepsilon^N = \alpha$ and $\beta_\varepsilon^N = \beta$ where α and β are independent of N and ε. Besides, a is bounded independently of ε so according to (3.3) and (3.4), for $\varepsilon > 0$ sufficiently small,

$$\|e_\varepsilon^N\|_{1,\Omega} \leqslant C\|\mathcal{T}_{\partial\omega_\varepsilon}^N u_0\|_{\frac{1}{2},\varepsilon},$$

and

$$\|\lambda_\varepsilon^N\|_{-\frac{1}{2},\varepsilon} \leqslant C\|e_\varepsilon^N\|_{1,\Omega},$$

with C independent of N and ε.

To prove estimates (3.9) and (3.10), we will apply **Theorem 3.4** to Problem (3.8). We set $Q_1 = \mathscr{M}_\varepsilon^0$, $Q_2 = \mathscr{M}_\varepsilon^{N,*}$ and

$$\begin{cases} b_1 : \mathscr{M}_\varepsilon^0 \times H_0^1(\Omega) \to \mathbb{R} & b_1(\mu_0, u) = (\mu_0, u)_\varepsilon\,, \\ b_2 : \mathscr{M}_\varepsilon^{N,*} \times H_0^1(\Omega) \to \mathbb{R} & b_2(\mu_{N,*}, u) = (\mu_{N,*}, u)_\varepsilon\,, \\ d_1 : \mathscr{M}_\varepsilon^0 \to \mathbb{R} & d(\mu_0) = -(\mu_0, u_0)_\varepsilon\,, \\ d_2 : \mathscr{M}_\varepsilon^{N,*} \to \mathbb{R} & d(\mu_{N,*}) = -(\mu_{N,*}, u_0)_\varepsilon\,. \end{cases}$$

The conditions (3.5) and (3.6) directly come from the inf-sup condition (3.15) which gives that: for $\varepsilon > 0$ sufficiently small and for all $\mu_{N,*} \in \mathscr{M}_\varepsilon^{N,*}$ and $\mu_0 \in \mathscr{M}_\varepsilon^0$, we have

$$\sup_{v \in H_0^1(\Omega), \mathcal{T}^0_{\partial\omega_\varepsilon} v = 0} \frac{(\mu_{N,*}, v)_\varepsilon}{\|v\|_{1,\Omega}} \geqslant \frac{1}{C}\|\mu_{N,*}\|_{-\frac{1}{2},\varepsilon},$$

and

$$\sup_{v \in H_0^1(\Omega), \mathcal{T}^{N,*}_{\partial\omega_\varepsilon} v = 0} \frac{(\mu_0, v)_\varepsilon}{\|v\|_{1,\Omega}} \geqslant \frac{1}{C}\|\mu_0\|_{-\frac{1}{2},\varepsilon},$$

where C is independent of N and ϵ. However we can improve in this last estimate the constant that appears in the right hand side. Indeed for $\mu_0 \in \mathscr{M}_\varepsilon^0$, according to **Lemma 3.2**, there exists $\underline{v}_\varepsilon \in H_0^1(\Omega)$ such that $\underline{v}_\varepsilon = \mu_0$ on $\partial\omega_\varepsilon$ and for $\varepsilon > 0$ sufficiently small,

$$\|\underline{v}_\varepsilon\|_{1,\Omega} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}\|\mu_0\|_{0,\varepsilon}.$$

We then get for $\mu_0 \in \mathscr{M}_\varepsilon^0$,

$$\begin{aligned} \sup_{v \in H_0^1(\Omega), \mathcal{T}^{N,*}_{\partial\omega_\varepsilon} v = 0} \frac{(\mu_0, v)_\varepsilon}{\|v\|_{1,\Omega}} &\geqslant \frac{1}{C(-\log(\varepsilon))^{-\frac{1}{2}}} \frac{(\mu_0, \mu_0)_\varepsilon}{\|\mu_0\|_{0,\varepsilon}} \\ &\geqslant \frac{1}{C(-\log(\varepsilon))^{-\frac{1}{2}}}\|\mu_0\|_{0,\varepsilon} \geqslant \frac{1}{C}(-\log(\varepsilon))^{\frac{1}{2}}\|\mu_0\|_{-\frac{1}{2},\varepsilon}, \end{aligned}$$

with C independent of ε. Then, according to **Theorem 3.4**, for $\varepsilon > 0$ small enough, we have

$$\|e_\varepsilon^N\|_{1,\Omega} \leqslant C\left((-\log(\varepsilon))^{-\frac{1}{2}}\|\mathcal{T}^0_{\partial\omega_\varepsilon} u_0\|_{\frac{1}{2},\varepsilon} + \|\mathcal{T}^{N,*}_{\partial\omega_\varepsilon} u_0\|_{\frac{1}{2},\varepsilon}\right),$$

and

$$\begin{cases} \|\lambda_\varepsilon^0\|_{-\frac{1}{2},\varepsilon} & \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}\|e_\varepsilon^N\|_{1,\Omega}, \\ \|\lambda_\varepsilon^{N,*}\|_{-\frac{1}{2},\varepsilon} & \leqslant C\|e_\varepsilon^N\|_{1,\Omega}, \end{cases}$$

with C independent of N and ε. According to **Lemma 3.3**, we obtain for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \|e_\varepsilon^N\|_{1,\Omega} &\leqslant C\left((-\log(\varepsilon))^{-\frac{1}{2}}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right) + \varepsilon^{\frac{1}{2}}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right)\right), \\ &\leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \end{aligned}$$

and

$$\begin{cases} \|\lambda_\varepsilon^0\|_{-\frac{1}{2},\varepsilon} & \leqslant C(-\log(\varepsilon))^{-1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \\ \|\lambda_\varepsilon^{N,*}\|_{-\frac{1}{2},\varepsilon} & \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \end{cases}$$

with C independent of N and ε. □

The estimates of **Theorem 3.5** show that the approximation by the reduced problem (2.16) is robust with respect to the size of the inclusion ε, in the sense that the solution u_ε^N converges to u_0 the solution of the limit problem (2.3) when $\varepsilon \to 0$. Using that $u_\varepsilon^N = e_\varepsilon^N + u_0$, we directly deduce from **Theorem 3.5** the following corollary that states that the solution u_ε^N is bounded independently of ε and N.

Corollary 3.1. *The reduced Poisson Problem* (2.16) *is well-posed in* $H^1(\Omega) \times \mathscr{M}_\varepsilon^N$ *and there exist constants* $C > 0$ *and* $\rho > 0$ *such that for all* $0 < \varepsilon < \rho$,

$$\|u_\varepsilon^N\|_{1,\Omega} \leqslant C \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C *independent of* N *and* ε. *Moreover* u_ε^N *belongs to* V^N.

3.3 Convergence of the reduced problem towards the full problem

We analyze the convergence rate between the solution u_ε of Problem (1.1) and the solution u_ε^N of Problem (2.17) in Ω_ε. To this purpose, we introduce the function $e_\varepsilon^{FN} = u_\varepsilon - u_\varepsilon^N$ which is solution of

$$\begin{cases} -\Delta e_\varepsilon^{FN} = 0 & \text{in } \Omega_\varepsilon, \\ e_\varepsilon^{FN} = -u_\varepsilon^N & \text{on } \partial\omega_\varepsilon, \\ e_\varepsilon^{FN} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

The convergence rate of e_ε^{FN} is specified in the following theorem.

Theorem 3.6. *There exist constants* $C > 0$, $\rho > 0$ *and* $\Upsilon > 0$ *such that for all* $0 < \varepsilon < \rho < \Upsilon$, *the solution* $e_\varepsilon^{FN} = u_\varepsilon - u_\varepsilon^N$ *of Problem* (3.16) *satisfies*

$$\|e_\varepsilon^{FN}\|_{1,\Omega_\varepsilon} \leqslant C(1+N)^{\frac{1}{2}} \left(\frac{\varepsilon}{\Upsilon}\right)^{N+1} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C *independent of* N *and* ε.

We see that the convergence rate in ε behaves as ε^{N+1} where (N+1) is the number of Fourier modes that can thus be adjusted to ensure a good approximation. Before proving this theorem, we will first state and prove two lemmas which will be useful in the proof. The first lemma gives an expression of λ_ε^N as a function of the gradient jump of u_ε^N at the interface $\partial\omega_\varepsilon$.

Lemma 3.4. *Let* $(u_\varepsilon^N, \lambda_\varepsilon^N) \in H_0^1(\Omega) \times \mathscr{M}_\varepsilon^N$ *be the solution of Problem* (2.16), *then for all* $\phi \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$,

$$\langle -\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ + \nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+, \phi \rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = -\left(\lambda_\varepsilon^N, \phi\right)_\varepsilon = -\varepsilon^{-1}\left(\lambda_\varepsilon^N, \phi\right)_{\partial\omega_\varepsilon},$$

where $\mathbf{n}^+$ *is the exterior normal on* $\partial\omega_\varepsilon$.

Proof. For all $\phi \in H_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega \nabla u_\varepsilon^N \nabla \phi \mathrm{d}x &= \int_{\Omega_\varepsilon} \nabla u_\varepsilon^N \nabla \phi \mathrm{d}x + \int_{\omega_\varepsilon} \nabla u_\varepsilon^N \nabla \phi \mathrm{d}x, \\ &= -\int_{\Omega_\varepsilon} \Delta u_\varepsilon^N \phi \mathrm{d}x - \int_{\omega_\varepsilon} \Delta u_\varepsilon^N \phi \mathrm{d}x + \langle -\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ + \nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+, \phi \rangle_{-\frac{1}{2},\partial\omega_\varepsilon}, \end{aligned}$$

and since $-\Delta u_\varepsilon^N = f$ in Ω_ε and $-\Delta u_\varepsilon^N = 0$ in ω_ε in a strong sense, we obtain the equality

$$\int_\Omega \nabla u_\varepsilon^N \nabla \phi \mathrm{d}x = \langle -\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ + \nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+, \phi \rangle_{-\frac{1}{2},\partial\omega_\varepsilon} + \int_\Omega f \phi \mathrm{d}x.$$

On the other hand, we have

$$\int_\Omega \nabla u_\varepsilon^N \nabla \phi \mathrm{d}x = -\left(\lambda_\varepsilon^N, \phi\right)_\varepsilon + \int_\Omega f \phi \mathrm{d}x.$$

Identifying the formulations, we get

$$\langle -\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ + \nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+, \phi \rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = -\left(\lambda_\varepsilon^N, \phi\right)_\varepsilon$$

for all $\phi \in H_0^1(\Omega)$. Thus, since the trace operator is surjective and continuous from $H_0^1(\Omega)$ on $H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, we get that

$$\langle -\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ + \nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+, \phi \rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = -\left(\lambda_\varepsilon^N, \phi\right)_\varepsilon$$

for all $\phi \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$. This concludes the proof of the lemma. □

The second lemma describes the behaviour of u_ε^N on $\partial\omega_\varepsilon$ which, according to **Lemma 2.3**, will allow to obtain a H^1-bound for e_ε^{FN} in Ω_ε.

Lemma 3.5. *There exists a constant $\Upsilon > 0$ such that for all $0 < \varepsilon < \Upsilon$, there exist $(\xi_{\varepsilon,n})_{n\geqslant N}$ and $(\zeta_{\varepsilon,n})_{n\geqslant N}$ such that for $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,*

$$u_\varepsilon^N(\varepsilon\mathbf{x}) = \sum_{n=N+1}^{\infty} \left(\frac{\varepsilon}{\Upsilon}\right)^n (\xi_{\varepsilon,n}\cos(n\theta) + \zeta_{\varepsilon,n}\sin(n\theta)).$$

Moreover, there exists a constant $C > 0$ and $\rho > 0$ such that for all $n \geqslant N+1$ and $0 < \varepsilon < \rho$,

$$|\xi_{\varepsilon,n}| + |\zeta_{\varepsilon,n}| \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of n, N and ε.

Proof. Let $\Upsilon > 0$ be sufficiently small such that $\overline{\omega_\Upsilon} \subset \Omega$ and $\mathrm{supp} f \cap \omega_\Upsilon = \varnothing$. For $0 < \varepsilon < \Upsilon$, the solution u_ε^N belongs to $H^1(\Omega)$ and is harmonic in ω_ε, so for $0 < r < \varepsilon$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$, we have

$$u_\varepsilon^N(r\mathbf{x}) = A_{\varepsilon,0} + \sum_{n=1}^{\infty} r^n (A_{\varepsilon,n}\cos(n\theta) + B_{\varepsilon,n}\sin(n\theta)), \tag{3.17}$$

with for $0 \leqslant n \leqslant N$,

$$\begin{cases} A_{\varepsilon,n} = 0, \\ B_{\varepsilon,n} = 0. \end{cases}$$

The solution u_ε^N is also harmonic in the annulus $\omega_\Upsilon\backslash\overline{\omega_\varepsilon}$, so for $\varepsilon < r < \Upsilon$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$, we have

$$u_\varepsilon^N(r\mathbf{x}) = C_{\varepsilon,0} + D_{\varepsilon,0}\log(r) + \sum_{n=1}^{\infty} \left(C_{\varepsilon,n}r^n + D_{\varepsilon,n}r^{-n}\right)\cos(n\theta) + \left(E_{\varepsilon,n}r^n + F_{\varepsilon,n}r^{-n}\right)\sin(n\theta). \tag{3.18}$$

with for $0 \leqslant n \leqslant N$,

$$\begin{cases} C_{\varepsilon,n}\varepsilon^n + D_{\varepsilon,n}\varepsilon^{-n} = 0, \\ E_{\varepsilon,n}\varepsilon^n + F_{\varepsilon,n}\varepsilon^{-n} = 0. \end{cases}$$

We refer to [29] for the derivation of formulas (3.17) and (3.18). By applying for all $n \geqslant N+1$ the operators $\mathcal{A}_\varepsilon^n$ and $\mathcal{B}_\varepsilon^n$ defined by (2.8) on equations (3.17) and (3.18), we obtain the following system satisfied by $A_{\varepsilon,n}$, $B_{\varepsilon,n}$, $C_{\varepsilon,n}$, $D_{\varepsilon,n}$, $E_{\varepsilon,n}$ and $F_{\varepsilon,n}$:

$$\begin{cases} A_{\varepsilon,n}\varepsilon^n = C_{\varepsilon,n}\varepsilon^n + D_{\varepsilon,n}\varepsilon^{-n}, & \forall n \geqslant N+1, \\ B_{\varepsilon,n}\varepsilon^n = E_{\varepsilon,n}\varepsilon^n + F_{\varepsilon,n}\varepsilon^{-n}, & \forall n \geqslant N+1. \end{cases} \tag{3.19}$$

Moreover, according to **Lemma 3.4**, we have

$$\langle -\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ + \nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+, \phi\rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = -\left(\lambda_\varepsilon^N, \phi\right)_\varepsilon = -\varepsilon^{-1}\left(\lambda_\varepsilon^N, \phi\right)_{\partial\omega_\varepsilon} \tag{3.20}$$

for all $\phi \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$ with

$$\nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+ = \frac{\partial u_{\varepsilon,int}^N}{\partial r} = \sum_{n=1}^{\infty} nr^{n-1}(A_{\varepsilon,n}\cos(n\theta) + B_{\varepsilon,n}\sin(n\theta)),$$

and

$$\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ = \frac{\partial u_{\varepsilon,ext}^N}{\partial r} = \frac{D_{\varepsilon,0}}{r} + \sum_{n=1}^{\infty} n(C_{\varepsilon,n}r^{n-1} - D_{\varepsilon,n}r^{-n-1})\cos(n\theta) + n(E_{\varepsilon,n}r^{n-1} - F_{\varepsilon,n}r^{-n-1})\sin(n\theta).$$

By applying for all $n \geqslant N+1$ operators $\mathcal{A}_\Upsilon^n$ and $\mathcal{B}_\Upsilon^n$ on equation (3.20), we obtain the following system satisfied by $A_{\varepsilon,n}$, $B_{\varepsilon,n}$, $C_{\varepsilon,n}$, $D_{\varepsilon,n}$, $E_{\varepsilon,n}$ and $F_{\varepsilon,n}$:

$$\begin{aligned} A_{\varepsilon,n}\varepsilon^n &= C_{\varepsilon,n}\varepsilon^n - D_{\varepsilon,n}\varepsilon^{-n}, \\ B_{\varepsilon,n}\varepsilon^n &= E_{\varepsilon,n}\varepsilon^n - F_{\varepsilon,n}\varepsilon^{-n}. \end{aligned} \tag{3.21}$$

From equations (3.19) and (3.21), we deduce that for all $n \geqslant N+1$,

$$D_{\varepsilon,n} = F_{\varepsilon,n} = 0 \text{ and } \left\{ \begin{array}{lcl} A_{\varepsilon,n} & = & C_{\varepsilon,n}, \\ B_{\varepsilon,n} & = & E_{\varepsilon,n}. \end{array} \right.$$

By applying now for all $n \geqslant N+1$ the operators $\mathcal{A}_\Upsilon^n$ and $\mathcal{B}_\Upsilon^n$ on equation (3.18), we obtain for $n \geqslant N+1$,

$$C_{\varepsilon,n}\Upsilon^n + D_{\varepsilon,n}\Upsilon^{-n} = \mathcal{A}_\Upsilon^n u_\varepsilon^N,$$
$$E_{\varepsilon,n}\Upsilon^n + F_{\varepsilon,n}\Upsilon^{-n} = \mathcal{B}_\Upsilon^n u_\varepsilon^N,$$

and

$$\begin{array}{lclcl} A_{\varepsilon,n} & = & C_{\varepsilon,n} & = & \dfrac{1}{\Upsilon^n}\mathcal{A}_\Upsilon^n u_\varepsilon^N, \\ B_{\varepsilon,n} & = & E_{\varepsilon,n} & = & \dfrac{1}{\Upsilon^n}\mathcal{B}_\Upsilon^n u_\varepsilon^N. \end{array}$$

Even if this result is not directly useful for the proof of the lemma, let us note that we can obtain an expression of λ_ε^N as a function of $\mathcal{A}_\Upsilon^n u_\varepsilon^N$ and $\mathcal{B}_\Upsilon^n u_\varepsilon^N$. Indeed, according to **Lemma 3.4**, we also have

$$-\varepsilon^{-1}\lambda_\varepsilon^N = -\nabla u_{\varepsilon,ext}^N \cdot \mathbf{n}^+ + \nabla u_{\varepsilon,int}^N \cdot \mathbf{n}^+ \tag{3.22}$$

in $H^{-\frac{1}{2}}(\partial\omega_\varepsilon)$, that is

$$\lambda_\varepsilon^N = a_{\varepsilon,0} + \sum_{n=1}^{N} (a_{\varepsilon,n}\cos(n\theta) + b_{\varepsilon,n}\sin(n\theta)),$$

with for $0 \leqslant n \leqslant N$,

$$a_{\varepsilon,0} = \frac{\mathcal{A}_\Upsilon^0 u_\varepsilon^N}{(\log(\Upsilon) - \log(\varepsilon))},$$
$$a_{\varepsilon,n} = 2n\frac{\varepsilon^n \mathcal{A}_\Upsilon^n u_\varepsilon^N}{\Upsilon^n - \varepsilon^{2n}\Upsilon^{-n}},$$
$$b_{\varepsilon,n} = 2n\frac{\varepsilon^n \mathcal{B}_\Upsilon^n u_\varepsilon^N}{\Upsilon^n - \varepsilon^{2n}\Upsilon^{-n}}.$$

Let us now set $\xi_{\varepsilon,n} = \Upsilon^n A_{\varepsilon,n}$ and $\zeta_{\varepsilon,n} = \Upsilon^n B_{\varepsilon,n}$. Using the same argument as we used for u_0 in the proof of the **Lemma 3.3**, we have for all $n \geqslant 0$,

$$\begin{aligned} |\mathcal{A}_\Upsilon^n u_\varepsilon^N| &\leqslant |\partial\omega_\Upsilon|^{-\frac{1}{2}}\|u_\varepsilon^N\|_{0,\partial\omega_\Upsilon}, \\ &\leqslant |\partial\omega_\Upsilon|^{-\frac{1}{2}} C_{T,\frac{1}{2}}(\omega_\Upsilon, \partial\omega_\Upsilon)\|u_\varepsilon^N\|_{1,\omega_\Upsilon} \leqslant C\|u_\varepsilon^N\|_{1,\Omega}. \end{aligned} \tag{3.23}$$

Eventually, **Corollary 3.1** gives the existence of $C > 0$ independent of n, N and ε such that for all $n \geqslant N+1$ and for $\varepsilon > 0$ sufficiently small,

$$|\xi_{\varepsilon,n}| + |\zeta_{\varepsilon,n}| \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right).$$

Let us note that, by identification, we also have for all $n \geqslant 1$, for $\varepsilon > 0$ sufficiently small,

$$|\mathcal{A}_\varepsilon^0 \lambda_\varepsilon^N| \leqslant C\log(\varepsilon)^{-1}$$
$$|\mathcal{A}_\varepsilon^n \lambda_\varepsilon^N| + |\mathcal{B}_\varepsilon^n \lambda_\varepsilon^N| \leqslant Cn\left(\frac{\varepsilon}{\Upsilon}\right)^n \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of ε. This concludes the proof of the lemma. □

We are now ready to prove **Theorem 3.6** that gives the convergence rate with respect to ε of the reduced problem solution towards the solution of the full problem.

Proof of Theorem 3.6. Let us first note that according to **Lemma 2.3**, for $\varepsilon > 0$ sufficiently small,

$$\|e_\varepsilon^{FN}\|_{1,\Omega_\varepsilon} \leqslant C\|u_\varepsilon^N\|_{\frac{1}{2},\varepsilon} \leqslant C\overline{\|u_\varepsilon^N\|}_{\frac{1}{2},\varepsilon}.$$

Then, according to **Lemma 3.5**, there exist $\Upsilon > 0$ and $\rho > 0$ such that for $0 < \varepsilon < \rho < \Upsilon$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$, we have

$$u_\varepsilon^N(\varepsilon\mathbf{x}) = \sum_{n=N+1}^{\infty} \left(\frac{\varepsilon}{\Upsilon}\right)^n (\xi_{\varepsilon,n}\cos(n\theta) + \zeta_{\varepsilon,n}\sin(n\theta))$$

with $|\xi_{\varepsilon,n}| + |\zeta_{\varepsilon,n}| \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right)$ for all $n \geqslant N+1$. It follows that

$$\begin{aligned}\|e_\varepsilon^{FN}\|_{1,\Omega_\varepsilon} &\leqslant C\overline{\|u_\varepsilon^N\|}_{\frac{1}{2},\varepsilon},\\ &\leqslant C\left(\sum_{n=N+1}^{\infty}(1+n)\left(\frac{\varepsilon}{\Upsilon}\right)^{2(n-N-1)}\right)^{\frac{1}{2}}\left(\frac{\varepsilon}{\Upsilon}\right)^{N+1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right).\end{aligned}$$

Since

$$\sum_{n=N+1}^{\infty}(1+n)\left(\frac{\varepsilon}{\Upsilon}\right)^{2(n-N-1)} = \sum_{n=0}^{\infty}(1+n+N+1)\left(\frac{\varepsilon}{\Upsilon}\right)^{2n}$$

and $0 < \varepsilon < \rho$, we finally get

$$\begin{aligned}\|e_\varepsilon^{FN}\|_{1,\Omega_\varepsilon} &\leqslant C\left(\sum_{n=0}^{\infty}(1+n)\left(\frac{\rho}{\Upsilon}\right)^{2n}\right)^{\frac{1}{2}}(1+N)^{\frac{1}{2}}\left(\frac{\varepsilon}{\Upsilon}\right)^{N+1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right)\\ &\leqslant C(1+N)^{\frac{1}{2}}\left(\frac{\varepsilon}{\Upsilon}\right)^{N+1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),\end{aligned}$$

with C independent of N and ε. □

Remark 3.1. *The parameter Υ plays an important role in the extension of the method to several obstacles. Indeed, it is bounded by the minimal distance of the inclusion to the boundary or to the nearest inclusion.* ***Theorem 3.6*** *shows that the closer the obstacles are, the worse the convergence in ε will be observed. The advantage of our approach is that the loss of precision in ε can be compensated by increasing the number of moments.*

Remark 3.2. *Let us also note that the minimal global regularity H^1 is sufficient to obtain the estimates in ε and N of* ***Theorem 3.6****.*

Remark 3.3. *If we consider the weak form associated with the Problem* (3.16) *where the constraint on $\partial\omega_\varepsilon$ is imposed by Lagrange multipliers, as we did when we wrote the weak form of Problem* (1.1) *as Problem* (2.5)*, we can notice that the Lagrange multiplier associated to this Problem is $\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N$ and verifies*

$$\left(\nabla e_\varepsilon^{FN}, \nabla v\right)_\Omega + \langle\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N, v\rangle_{-\frac{1}{2},\partial\omega_\varepsilon} = 0,\ \forall v \in H_0^1(\Omega),$$

or equivalently

$$\left(\nabla e_\varepsilon^{FN}, \nabla v\right)_\Omega + \varepsilon\langle\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N, v\rangle_{-\frac{1}{2},\varepsilon} = 0,\ \forall v \in H_0^1(\Omega).$$

We write $\|\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N\|_{-\frac{1}{2},\varepsilon}$ as

$$\|\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N\|_{-\frac{1}{2},\varepsilon} = \sup_{\mu\in H^{\frac{1}{2}}(\partial\omega_\varepsilon)}\frac{\langle\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N, \mu\rangle_{-\frac{1}{2},\varepsilon}}{\|\mu\|_{\frac{1}{2},\varepsilon}}.$$

For all $\mu \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, if we take v defined in ***Lemma 3.2*** *such that $v = \mu$ on $\partial\omega_\varepsilon$, we have for $\varepsilon > 0$ sufficiently small,*

$$\|\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N\|_{-\frac{1}{2},\varepsilon} \leqslant \frac{1}{C}\sup_{v\in H_0^1(\Omega)}\frac{\langle\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N, v\rangle_{-\frac{1}{2},\varepsilon}}{\|v\|_{1,\Omega}},$$

with C independent of ε. We deduce that

$$\|\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N\|_{-\frac{1}{2},\varepsilon} \leqslant \frac{1}{\varepsilon C}\sup_{v\in H_0^1(\Omega)}\frac{\left(\nabla e_\varepsilon^{FN}, \nabla v\right)}{\|v\|_{1,\Omega}} \leqslant \frac{1}{\varepsilon C}\|e_\varepsilon^{FN}\|_{1,\Omega},$$

and conclude

$$\|\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N\|_{-\frac{1}{2},\varepsilon} \leqslant C(1+N)^{\frac{1}{2}} \left(\frac{\varepsilon}{\Upsilon}\right)^N \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \tag{3.24}$$

*with C independent of N and ε. Note that for $N=0$, the inequality (3.24) does not imply the convergence of $\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N$ in the $H^{-\frac{1}{2}}$ rescaled norm. However, by separating $\lambda^{N,*}$ and λ_ε^0 as we did for **Theorem 3.5**, it is possible to prove a convergence in $\log(\varepsilon)^{-\frac{1}{2}}$. Moreover, using definition (2.15) of the $H^{-\frac{1}{2}}$ rescaled norm, we can prove that for $v \in H^{-\frac{1}{2}}(\partial\omega_\varepsilon)$, for $\varepsilon > 0$ sufficiently small,*

$$\|v\|_{-\frac{1}{2},\partial\omega_\varepsilon} \leqslant C\varepsilon^{\frac{1}{2}}\|v\|_{-\frac{1}{2},\varepsilon},$$

with C independent of ε, so, if we consider the standard norm $\|\cdot\|_{-\frac{1}{2},\partial\omega_\varepsilon}$ instead of the norm $\|\cdot\|_{-\frac{1}{2},\varepsilon}$, we obtain

$$\|\lambda_\varepsilon - \varepsilon^{-1}\lambda_\varepsilon^N\|_{-\frac{1}{2},\partial\omega_\varepsilon} \leqslant C(1+N)^{\frac{1}{2}} \left(\frac{\varepsilon}{\Upsilon}\right)^{N+\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of N and ε.

Before considering the finite element approximation of the reduced model we next derive some additional regularity estimates of the reduced model and investigate their dependency with respect to the size of the hole.

3.4 Additional regularity of the solution

Due to the presence of the Dirac source $\lambda_\varepsilon^N \delta_{\partial\omega_\varepsilon}$ in Problem (2.17), the global H^2 regularity for e_ε^N and u_ε^N cannot be recovered. However, we can prove a better regularity than H^1 for e_ε^N. To do so, let us first note that if we consider the Lagrange multiplier associated to the solution of Problem (3.7), namely λ_ε^N, as a datum depending both on f and ϕ, then e_ε^N satisfies

$$\begin{cases} -\Delta e_\varepsilon^N = -\varepsilon^{-1}\lambda_\varepsilon^N \delta_{\partial\omega_\varepsilon} & \text{in } \Omega, \\ e_\varepsilon^N = 0 & \text{on } \partial\Omega. \end{cases}$$

To analyze this problem we introduce an auxiliary lemma presented in [30].

Lemma 3.6. *Let $\mathcal{D}$ be a generic bounded, convex domain in $\mathbb{R}^2$. Let $\gamma \subset \mathcal{D}$ be a $\mathcal{C}^2$-surface such that the distance between γ and $\partial\mathcal{D}$ is positive. Consider the following problem*

$$\begin{cases} -\Delta y = \zeta\delta_\gamma & \text{in } \mathcal{D}, \\ y = 0 \ \text{on} & \partial\mathcal{D}, \end{cases} \tag{3.25}$$

where $\zeta \in L^2(\gamma)$. Problem (3.25) admits a unique solution which belongs to $H^{\frac{3}{2}-\eta}(\mathcal{D})$ for any $\eta > 0$. Furthermore there exists a constant C such that

$$\|y\|_{\frac{3}{2}-\eta,\mathcal{D}} \leqslant C\|\zeta\delta_\gamma\|_{-\frac{1}{2}-\eta,\mathcal{D}}$$

with C independent of ζ and γ.

We will apply this lemma in order to prove the following theorem.

Theorem 3.7. *For any $0 < \eta < \frac{1}{2}$, the solution of Problem (3.7) satisfies the additional regularity $e_\varepsilon^N \in H^{\frac{3}{2}-\eta}(\Omega)$ and the following estimate holds true: there exists a constant $C(\varepsilon,\eta) > 0$ such that*

$$\|e_\varepsilon^N\|_{\frac{3}{2}-\eta,\Omega} \leqslant C(\varepsilon,\eta)(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}),$$

with $C(\varepsilon,\eta)$ independent of N.

Proof. Since Ω a polygonal domain and $\varepsilon^{-1}\lambda_\varepsilon^N \in L^2(\partial\omega_\varepsilon)$, we can apply **Lemma 3.6** and get that $e_\varepsilon^N \in H^{\frac{3}{2}-\eta}(\Omega)$ and that there exists a constant $C > 0$ such that

$$\|e_\varepsilon^N\|_{\frac{3}{2}-\eta,\Omega} \leqslant C\varepsilon^{-1}\|\lambda_\varepsilon^N\delta_{\partial\omega_\varepsilon}\|_{-\frac{1}{2}-\eta,\Omega}. \tag{3.26}$$

Moreover, we have

$$\|\lambda_\varepsilon^N \delta_{\partial\omega_\varepsilon}\|_{-\frac{1}{2}-\eta,\Omega} = \sup_{v \in H_0^{\frac{1}{2}+\eta}(\Omega)} \frac{(\lambda_\varepsilon^N, v)_{\partial\omega_\varepsilon}}{\|v\|_{\frac{1}{2}+\eta,\Omega}}. \tag{3.27}$$

Note that here the considered scalar product on $L^2(\partial\omega_\varepsilon)$ is the standard one and not the rescaled one. Using Cauchy-Schwarz inequality we have, for all $v \in H_0^{\frac{1}{2}+\eta}(\Omega)$,

$$\left(\lambda_\varepsilon^N, v\right)_{\partial\omega_\varepsilon} \leqslant \|\lambda_\varepsilon^N\|_{0,\partial\omega_\varepsilon} \|v\|_{0,\partial\omega_\varepsilon},$$

and according to equality (3.22) and estimate (3.23), there exist constants $\rho > 0$ and $\Upsilon > 0$ such that for all $0 < \varepsilon < \rho < \Upsilon$, we have

$$\|\lambda_\varepsilon^N\|_{0,\partial\omega_\varepsilon} = (2\pi\varepsilon)^{\frac{1}{2}} \left(a_{\varepsilon,0}^2 + \sum_{n=1}^N \left(\frac{a_{\varepsilon,n}^2}{2} + \frac{b_{\varepsilon,n}^2}{2} \right) \right)^{\frac{1}{2}},$$

with

$$|a_{\varepsilon,0}| \leqslant C(-\log(\varepsilon))^{-1} \left(\|f\|_{0,\Omega} + \|\phi\|_{\frac{1}{2},\partial\Omega} \right),$$

and for $n \geqslant 1$,

$$|a_{\varepsilon,n}| + |b_{\varepsilon,n}| \leqslant Cn \left(\frac{\varepsilon}{\Upsilon} \right)^n \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C independent of n and ε. We then deduce that for $0 < \varepsilon < \rho < \Upsilon$,

$$\begin{aligned} \|\lambda_\varepsilon^N\|_{0,\partial\omega_\varepsilon} &\leqslant C\varepsilon^{\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right) \left(\log(\rho)^{-2} + \sum_{n=1}^\infty \frac{n^2}{2} \left(\frac{\rho}{\Upsilon} \right)^{2n} \right)^{\frac{1}{2}} \\ &\leqslant C\varepsilon^{\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right), \end{aligned} \tag{3.28}$$

with C independent of N and ε. We obtain, using **Theorem 3.2** on trace inequality, the following upper bound for any $v \in H^{\frac{1}{2}+\eta}(\Omega)$:

$$\|v\|_{0,\partial\omega_\varepsilon} \leqslant C_{T,\eta}(\Omega, \partial\omega_\varepsilon) \|v\|_{\frac{1}{2}+\eta,\Omega}. \tag{3.29}$$

Gathering (3.26), (3.27), (3.28) and (3.29), we deduce that

$$\|e_\varepsilon^N\|_{\frac{3}{2}-\eta,\Omega} \leqslant C\varepsilon^{-\frac{1}{2}} C_{T,\eta}(\Omega, \partial\omega_\varepsilon) \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C independent of N and ε. Note that the dependency with respect to ε is not explicit here because the behavior of the constant appearing in (3.29) coming from a trace inequality is not. □

Let $0 < \eta < \frac{1}{2}$, we then have the following corollary on u_ε^N.

Corollary 3.2. *If $\phi \in H^{1-\eta}(\partial\Omega)$ and $f \in L^2(\Omega)$, then the solution u_ε^N of Problem* (2.16) *satisfies $u_\varepsilon^N \in H^{\frac{3}{2}-\eta}(\Omega)$ and*

$$\|u_\varepsilon^N\|_{\frac{3}{2}-\eta,\Omega} \leqslant C(\varepsilon,\eta) \left(\|\phi\|_{1-\eta,\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with $C(\varepsilon,\eta)$ independent of N.

Proof. This result is just a consequence of the fact that $u_\varepsilon^N = e_\varepsilon^N + u_0$ and if $\phi \in H^{1-\eta}(\partial\Omega)$ then $u_0 \in H^{\frac{3}{2}-\eta}(\Omega)$ and

$$\|u_0\|_{\frac{3}{2}-\eta,\Omega} \leqslant C(\eta) \left(\|\phi\|_{1-\eta,\partial\Omega} + \|f\|_{0,\Omega} \right)$$

with $C(\eta)$ independent of ε. □

Finally, using and explicit representation of e_ε^N inside and outside ω_ε separately, we obtain the following result.

Theorem 3.8. *Let e_ε^N solution of Problem* (3.7)*, there exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,*

$$\|e_\varepsilon^N\|_{2,\Omega_\varepsilon} + \|e_\varepsilon^N\|_{2,\omega_\varepsilon} \leqslant C\varepsilon^{-1} \left(\|f\|_{0,\Omega} + \|\phi\|_{\frac{1}{2},\partial\Omega} \right),$$

with C independent of N and ε.

The proof of this theorem can be found in the Appendix.

Remark 3.4. *The statement of* ***Theorem 3.7*** *does not give an estimate with respect to* ε *of the constant* $C(\varepsilon,\eta)$. *However, proceeding in the same way as for the proof of* ***Theorem 3.8****, we can obtain some estimates for the* $H^{\frac{3}{2}-\eta}$*-norm on each subdomain. Indeed, if* e_ε^N *is solution of* (3.7)*, we can prove the existence of constants* $C(\eta)>0$ *and* $\rho>0$ *such that for all* $0<\varepsilon<\rho$,

$$\|e_\varepsilon^N\|_{\frac{3}{2}-\eta,\Omega_\varepsilon} + \|e_\varepsilon^N\|_{\frac{3}{2}-\eta,\omega_\varepsilon} \leqslant C(\eta)\varepsilon^{-\frac{1}{2}+\eta}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

where $C(\eta)$ *is independent of* N *and* ε.

4 Numerical approximation

4.1 Finite Element discretization

In this section, we study the convergence behavior of a standard finite element method applied to the variational Problem (2.16). We assume that the data f and ϕ are smooth as well as the solution u_0 of the limit problem. We introduce a shape regular triangulation $\mathcal{T}_h^\Omega$ of Ω, where h is the characteristic size of the mesh. We set for $k \geqslant 1$,

$$X_h^k(\Omega) = \{v_h \in \mathcal{C}(\Omega) \mid v_h|\tau \in \mathbb{P}^k \ \forall \tau \in \mathcal{T}_h^\Omega\},$$

where $\mathbb{P}^k$ is the set of polynomials of degree less or equal than k. We look for $u_{\varepsilon,h}^N$ solution of the discrete version of Problem (2.16): find $u_{\varepsilon,h}^N \in X_h^k(\Omega)$ and $\lambda_{\varepsilon,h}^N \in \mathscr{M}_\varepsilon^N$ such that

$$\begin{cases} \left(\nabla u_{\varepsilon,h}^N, \nabla v_h\right)_\Omega + \left(\lambda_{\varepsilon,h}^N, v_h\right)_\varepsilon = (f, v_h)_\Omega, & \forall v_h \in X_h^k(\Omega) \cap H_0^1(\Omega), \\ \left(\mu, u_{\varepsilon,h}^N\right)_\varepsilon = 0, & \forall \mu \in \mathscr{M}_\varepsilon^N, \\ u_{\varepsilon,h}^N = \mathcal{I}_h^\partial \phi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\mathcal{I}_h^\partial$ is the Lagrange interpolant on $\partial\Omega$. We then write $u_{\varepsilon,h}^N = e_{\varepsilon,h}^N + u_{0,h}$ where $e_{\varepsilon,h}^N$ is solution of the following discrete problem: find $e_{\varepsilon,h}^N \in X_h^k(\Omega) \cap H_0^1(\Omega)$ and $\lambda_{\varepsilon,h}^N \in \mathscr{M}_\varepsilon^N$ such that

$$\begin{cases} \left(\nabla e_{\varepsilon,h}^N, \nabla v_h\right)_\Omega + \left(\lambda_{\varepsilon,h}^N, v_h\right)_\varepsilon = 0, & \forall v_h \in X_h^k(\Omega) \cap H_0^1(\Omega), \\ \left(\mu, e_{\varepsilon,h}^N\right)_\varepsilon = -\left(\mu, u_{0,h}\right)_\varepsilon, & \forall \mu \in \mathscr{M}_\varepsilon^N, \end{cases} \tag{4.2}$$

and $u_{0,h} \in X_h^k(\Omega)$ satisfies

$$\begin{cases} \left(\nabla u_{0,h}, \nabla v_h\right)_\Omega = (f, v_h)_\Omega, & \forall v_h \in X_h^k(\Omega) \cap H_0^1(\Omega), \\ u_{0,h} = \mathcal{I}_h^\partial \phi & \text{on } \partial\Omega. \end{cases}$$

We then have

$$u_\varepsilon^N - u_{\varepsilon,h}^N = e_\varepsilon^N - e_{\varepsilon,h}^N + u_0 - u_{0,h}. \tag{4.3}$$

Let us note that for $\mathcal{I}_h^\partial \phi$ to exist, ϕ needs to be at least continuous, actually, in all this section, we will suppose that ϕ is regular enough to have a lifting in $\mathcal{C}^0(\overline{\Omega}) \cap H^1(\Omega)$, we then know from classical argument (see [[31], **Corollary 3.29**]) that if $\phi \in H^{k+\frac{1}{2}}(\partial\Omega)$ and $f \in L^{k-1}(\Omega)$, then $u_0 \in H^{k+1}(\Omega)$ and

$$\|u_0 - u_{0,h}\|_{1,\Omega} \leqslant Ch^k \|u_0\|_{k+1,\Omega} \leqslant C\left(\|\phi\|_{k+\frac{1}{2},\partial\Omega} + \|f\|_{k-1,\Omega}\right), \tag{4.4}$$

with C independent of h. As a consequence, we will focus on the well-posedness of Problem (4.2) and the convergence in h of the difference $e_{\varepsilon,h}^N - e_\varepsilon^N$, admitting that the term $u_0 - u_{0,h}$ does not deteriorate the convergence in h.

We first introduce the discretized version of the standard existence **Theorem 3.3** and some other preliminary results useful in what follows. The discrete equivalent of Problem (3.1) consists in finding $u_h \in X_h \subset X$ and $\lambda_h \in Q_h \subset Q$ such that

$$\begin{cases} a(u_h, v_h) + b(\lambda_h, v_h) = c(v_h), & \forall v_h \in X_h, \\ b(\mu_h, u_h) = d(\mu_h), & \forall \mu_h \in Q_h, \end{cases} \tag{4.5}$$

with $a: X_h \times X_h \to \mathbb{R}$ and $b: Q_h \times X_h \to \mathbb{R}$ two bounded bilinear forms, $c: X_h \to \mathbb{R}$ and $d: Q_h \to \mathbb{R}$ two bounded linear forms. The discrete version of **Theorem 3.3** reads as follows (see [[31], **Proposition 4.14**]) :

Theorem 4.1. *Under the following conditions*

$$\begin{cases} \exists\, \alpha_h > 0,\, \forall\, v_h \in X_h, & a(v_h, v_h) \geqslant \alpha_h \|v_h\|_X^2, \\ \exists\, \beta_h > 0, & \inf_{\mu_h \in Q_h} \sup_{v_h \in X_h} \dfrac{b(\mu_h, v_h)}{\|\mu_h\|_X \|v_h\|_Q} \geqslant \beta_h, \end{cases} \tag{4.6}$$

the discrete saddle point problem (4.5) *is well-posed. Moreover for* u *solution of* (3.1) *and* u_h *solution of* (4.5)*, we have*

$$\|u - u_h\|_X \leqslant \left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \frac{\|b\|}{\alpha_h} \inf_{\mu_h \in Q_h} \|\lambda - \mu_h\|_Q,$$

and

$$\|\lambda - \lambda_h\|_Q \leqslant \frac{\|a\|}{\beta_h}\left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \left(1 + \frac{\|b\|}{\beta_h}\left(1 + \frac{\|a\|}{\alpha_h}\right)\right) \inf_{\mu_h \in Q_h} \|\lambda - \mu_h\|_Q.$$

If we look at Problem (4.2), we see that this problem actually writes: find $\tilde{u}_h \in X_h$ and $\tilde{\lambda}_h \in Q$ such that

$$\begin{cases} a(\tilde{u}_h, v_h) + b(\tilde{\lambda}_h, v_h) = c(v_h), & \forall v_h \in X_h, \\ b(\mu, \tilde{u}_h) = d_h(\mu), & \forall \mu \in Q. \end{cases} \tag{4.7}$$

where $d_h : Q \to \mathbb{R}$ is a linear form on Q. We notice in particular that $Q_h = Q$ and $d_h \neq d$. To prove the convergence of $e^N_{\varepsilon,h} - e^N_\varepsilon$, we introduce a corollary of **Theorem 4.5** more suited to the problem at hand.

Corollary 4.1. *Under the conditions* (4.6)*, the discrete saddle point problem* (4.7) *is well-posed. Moreover, for* u *solution of* (3.1) *and* $\tilde{u}_h$ *solution of* (4.7)*,*

$$\|u - \tilde{u}_h\|_X \leqslant \left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \frac{1}{\beta_h}\left(1 + \frac{\|a\|}{\alpha_h}\right) \|d - d_h\|,$$

and

$$\|\lambda - \tilde{\lambda}_h\|_Q \leqslant \frac{\|a\|}{\beta_h}\left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \frac{\|a\|}{\beta_h}\frac{1}{\beta_h}\left(1 + \frac{\|a\|}{\alpha_h}\right) \|d - d_h\|.$$

Proof. The well-posedness of Problem (4.7) is a direct consequence of **Theorem 4.1**. Moreover, if we set $\tilde{e}_h = u_h - \tilde{u}_h$ we see that $\tilde{e}_h$ satisfies

$$\begin{cases} a(\tilde{e}_h, v_h) + b(\lambda_h - \tilde{\lambda}_h, v_h) = 0, & \forall v_h \in X_h, \\ b(\mu, \tilde{e}_h) = (d - d_h)(\mu), & \forall \mu \in Q, \end{cases} \tag{4.8}$$

so according to **Theorem 3.3**,

$$\|\tilde{e}_h\|_X \leqslant \frac{1}{\beta_h}\left(1 + \frac{\|\alpha\|}{\alpha_h}\right) \|d - d_h\|$$

and

$$\|\lambda - \tilde{\lambda}_h\|_Q \leqslant \frac{1}{\beta_h}\frac{\|a\|}{\alpha_h}\left(1 + \frac{\|\alpha\|}{\alpha_h}\right) \|d - d_h\|.$$

We conclude by noticing that $u - \tilde{u}_h = u - u_h + \tilde{e}_h$ and $\lambda - \tilde{\lambda}_h = \lambda - \lambda_h + \lambda_h - \tilde{\lambda}_h$. □

Moreover, if the inf-sup condition is verified in the continuous case, then the discrete inf-sup condition can be directly derived by the definition of a Fortin operator by the following lemma.

Lemma 4.1 (Fortin's trick)**.** *If the continuous inf-sup condition holds with the constant* β *for the operator* b *and if there exists a linear operator* $\Pi_h : X \to X_h$ *such that*

$$b(q, u - \Pi_h u) = 0,$$
$$\|\Pi_h u\|_{X_h} \leqslant C\|u\|_X, \forall u \in X, \forall q \in Q,$$

then the discrete inf-sup condition holds and $\beta_h \geqslant \beta |||\Pi_h|||^{-1}$ *where* $|||\cdot|||$ *denotes the operator norm.*

4.2 Well-posedness of Problem (4.2)

Note that here the discretization space associated to the Lagrange multiplier is not a finite element space but a Fourier space and does not depend on h. In particular, it is equal to the continuous space associated to the Lagrange multiplier of the reduced model. However, any function of $\mathscr{M}_\varepsilon^N$ is uniquely defined by the values it takes on a finite dimensional space, more specifically, the computation of $\left(\lambda_{\varepsilon,h}^N, \phi\right)_\varepsilon$ for $\phi \in \{1, \cos(n\theta), \sin(n\theta)\}_{1\leqslant n\leqslant N}$ is sufficient to determine $\lambda_{\varepsilon,h}^N$. The inf-sup condition of Problem (4.2) reads as follows: there exists $\beta_{\varepsilon,h}^N > 0$ such that

$$\inf_{\mu_h\in\mathscr{M}_\varepsilon^N} \sup_{v_h\in X_h^k(\Omega)\cap H_0^1(\Omega)} \frac{(\mu_h, v_h)_\varepsilon}{\|\mu_h\|_{\mathscr{M}_\varepsilon^N}\|v_h\|_{1,\Omega}} \geqslant \beta_{\varepsilon,h}^N. \tag{4.9}$$

The next theorem gives sufficient conditions such that this inf-sup constant can be bounded from below by a strictly positive constant independent of the parameter N, h and ε.

Theorem 4.2. *For all $N \in \mathbb{N}$, there exist constants $\rho_1 > 0$, $\rho_2(N) > 0$ and $C > 0$ such that for $0 < \varepsilon < \rho_1$ and $0 < \frac{h}{\varepsilon} < \rho_2(N)$, the inf-sup condition (4.1) is satisfied and*

$$\beta_{\varepsilon,h}^N \geqslant C, \tag{4.10}$$

with C independent of N, h and ε. In the specific case $N \leqslant k$, if we set $r^* = \sup\{r > 0,\ \overline{\omega_r} \subset \Omega\}$ then if

$$h \leqslant r^* - \varepsilon, \tag{4.11}$$

the inf-sup condition is satisfied with no further condition on h and ε

Note that in the latter theorem, we have considered two cases depending on the degree of the polynomial approximation and the number of Fourier modes. In the case where the degree is greater than the number of Fourier modes, then the assumption (4.11) is sufficient for the inf-sup condition to be satisfied. Note that this assumption is not very restrictive as it is equivalent to imposing the presence of at least one tile in the region between the boundary of the hole and the boundary of the domain. In the general case, which includes the case $N \leqslant k$, the condition on h and ε is more restrictive and requires that the mesh size should be small compared to the size of the hole.

We now introduce some tools and lemmas used to prove **Theorem 4.2**. We denote by $R_h^k : H_0^1(\Omega) \to X_h^k(\Omega) \cap H_0^1(\Omega)$ the Scott-Zhang operator (see [32]), R_h^k satisfies the following lemmas.

Lemma 4.2. *There exists a constant $C > 0$ such that for $v \in H_0^1(\Omega)$,*

$$\|R_h^k v\|_{1,\Omega} \leqslant C\|v\|_{1,\Omega},$$

with C independent of h.

Lemma 4.3. *For all $v \in X_h^k(\Omega) \cap H_0^1(\Omega)$,*

$$R_h^k v = v.$$

Lemma 4.4. *Let $k \in \mathbb{N}^*$ and $s \geqslant 1$, there exists a constant $C > 0$ independent of h such that for $\tau \in \mathcal{T}_h^\Omega$ and $v \in H^s(S_\tau)$,*

$$h^{-\frac{1}{2}}\|R_h^k v - v\|_{0,\tau} + h^{\frac{1}{2}}\|\nabla(R_h^k v - v)\|_{0,\tau} \leqslant Ch^{l+\frac{1}{2}}\|v\|_{s,S_\tau},\ l = \min\{k, s-1\},$$

where S_τ is a domain made of the elements neighboring τ. Moreover, we have for $v \in H_0^s(\Omega)$,

$$\|v - R_h^k v\|_{1,\Omega} \leqslant Ch^{l+\frac{1}{2}}|v|_{s,\Omega},\ l = \min\{k, s-1\}.$$

The next lemma ensures the existence of a linear extension operator from $H^k(\omega_\varepsilon)$ to $H^k(\Omega)$ with a norm independent of ε. The proof of this lemma can be found in [[33], **Theorem 6**].

Lemma 4.5. *Let Ω_ε a domain with a hole ω_ε and $\Omega = \Omega_\varepsilon \cup \overline{\omega_\varepsilon}$. Then there exists a linear extension operator $\mathcal{E}_\Omega$ which maps the space $H^k(\omega_\varepsilon)$ onto the space $H^k(\Omega)$ for all $k \geqslant 0$ and satisfying for all $v \in H^k(\omega_\varepsilon)$,*

$$\|\mathcal{E}_\Omega v\|_{k,\Omega} \leqslant C(\Omega, k)\|v\|_{k,\omega_\varepsilon},$$

with $C(\Omega, k)$ independent of ε.

Now we are going to prove **Theorem 4.2**.

Proof of Theorem 4.2. To prove the discrete inf-sup condition (4.9), we use **Lemma 4.1** which asserts that, since the inf-sup condition has been proved in the continuous case, it is sufficient to provide a Fortin operator $\Pi^N_{\varepsilon,h} : H^1_0(\Omega) \to X^k_h(\Omega) \cap H^1_0(\Omega)$ such that

$$\|\Pi^N_{\varepsilon,h} v\|_{1,\Omega} \leqslant C\|v\|_{1,\Omega}, \ \forall v \in H^1_0(\Omega),$$

with C independent of h and

$$(\mu_h, v)_\varepsilon = \left(\mu_h, \Pi^N_{\varepsilon,h} v\right)_\varepsilon, \ \forall \mu_h \in \mathscr{M}^N_\varepsilon. \tag{4.12}$$

We first consider the general case where k can be smaller or greater than N. Let Υ_1 such that $\overline{\omega_{\Upsilon_1}} \subset \Omega$ and $0 < \varepsilon < \Upsilon_1$, we denote by c_n and s_n the harmonic lifting of the (N + 1) first Fourier modes on $\partial\omega_\varepsilon$ into ω_ε. If we follow the same computations as we did for proving **Lemma 3.2** and **Theorem 3.8**, we can prove that, for $\varepsilon > 0$ sufficiently small,

$$\|c_n\|_{1,\omega_\varepsilon} + \|s_n\|_{1,\omega_\varepsilon} \leqslant C(1+n)^{\frac{1}{2}} \text{ and } \|c_n\|_{2,\omega_\varepsilon} + \|s_n\|_{2,\omega_\varepsilon} \leqslant C(1+n)^{\frac{3}{2}}\varepsilon^{-1}, \tag{4.13}$$

with C independent of n and ε. Then, we extend c_n and s_n to ω_{Υ_1} thanks to the extension operator defined in **Lemma 4.5** such that, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \|\mathcal{E}_{\omega_{\Upsilon_1}} c_n\|_{1,\omega_{\Upsilon_1}} + \|\mathcal{E}_{\omega_{\Upsilon_1}} s_n\|_{1,\omega_{\Upsilon_1}} &\leqslant C(1+n)^{\frac{1}{2}} \\ \text{and} \quad \|\mathcal{E}_{\omega_{\Upsilon_1}} c_n\|_{2,\omega_{\Upsilon_1}} + \|\mathcal{E}_{\omega_{\Upsilon_1}} s_n\|_{2,\omega_{\Upsilon_1}} &\leqslant C(1+n)^{\frac{3}{2}}\varepsilon^{-1}, \end{aligned} \tag{4.14}$$

with C independent of n and ε. Eventually, we extend $\mathcal{E}_{\omega_{\Upsilon_1}} c_n$ and $\mathcal{E}_{\omega_{\Upsilon_1}} s_n$ to all Ω with the harmonic extension such that $\mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} c_n) = \mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} s_n) = 0$ on $\partial\Omega$, and $\mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} c_n) = \mathcal{E}_{\omega_{\Upsilon_1}} c_n$ and $\mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} s_n) = \mathcal{E}_{\omega_{\Upsilon_1}} s_n$ on $\partial\omega_{\Upsilon_1}$. Identifying c_n with $\mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} c_n)$ and s_n with $\mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} s_n)$, we have $c_n, s_n \in H^1_0(\Omega)$ and for $\varepsilon > 0$ sufficiently small,

$$\|c_n\|_{1,\Omega} + \|s_n\|_{1,\Omega} \leqslant C(1+n)^{\frac{1}{2}}.$$

For any $v \in H^1(\Omega)$, we search $\Pi^N_{\varepsilon,h} v$ in the form

$$\Pi^N_{\varepsilon,h} v = a^0_{\varepsilon,h}(v) R^k_h c_0 + \sum_{n=1}^{N} a^n_{\varepsilon,h}(v) R^k_h c_n + b^n_{\varepsilon,h}(v) R^k_h s_n, \tag{4.15}$$

such that $\Pi^N_{\varepsilon,h}$ verifies (4.12). When h tends to 0, we expect $R^k_h c_n$ and $R^k_h s_n$ to tend to the first (N + 1) Fourier modes on $\partial\omega_\varepsilon$ and consequently $\Pi^N_{\varepsilon,h}$ to tend to Π^N_ε for which we have the existence and the uniqueness of the coefficients $a^n_{\varepsilon,h}(v)$ and $b^n_{\varepsilon,h}(v)$. In order to satisfy (4.12), the coefficients $a^n_{\varepsilon,h}(v)$ and $b^n_{\varepsilon,h}(v)$ defined by (4.15) have to verify, for all $v \in H^1(\Omega)$, for all $1 \leqslant n \leqslant N$,

$$\begin{cases} \mathcal{A}^0_\varepsilon[\Pi^N_{\varepsilon,h} v] &= \mathcal{A}^0_\varepsilon v, \\ \mathcal{A}^n_\varepsilon[\Pi^N_{\varepsilon,h} v] &= \mathcal{A}^n_\varepsilon v, \\ \mathcal{B}^n_\varepsilon[\Pi^N_{\varepsilon,h} v] &= \mathcal{B}^n_\varepsilon v. \end{cases}$$

We can write these equations as a linear system of size $2N + 1$ given by

$$\mathbf{P}^N_{\varepsilon,h} \mathbf{X}^N_{\varepsilon,h}(v) = \mathbf{Q}^N_{\varepsilon,h},$$

where for $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N$, $\mathbf{P}^N_{\varepsilon,h} \in \mathcal{M}_{2N+1}(\mathbb{R})$ is given by

$$\begin{cases} (\mathbf{P}^N_{\varepsilon,h})_{1,j} = \mathcal{A}^0_\varepsilon[R^k_h c_j], & (\mathbf{P}^N_{\varepsilon,h})_{i,1} = \mathcal{A}^i_\varepsilon[R^k_h c_0] \quad (4.16a) \\ (\mathbf{P}^N_{\varepsilon,h})_{2i,2j} = \mathcal{A}^i_\varepsilon[R^k_h c_j], & (\mathbf{P}^N_{\varepsilon,h})_{2i,2j+1} = \mathcal{A}^i_\varepsilon[R^k_h s_j], \quad (4.16b) \\ (\mathbf{P}^N_{\varepsilon,h})_{2i+1,2j} = \mathcal{B}^i_\varepsilon[R^k_h c_j], & (\mathbf{P}^N_{\varepsilon,h})_{2i+1,2j+1} = \mathcal{B}^i_\varepsilon[R^k_h s_j], \quad (4.16c) \end{cases}$$

$\mathbf{X}_{\varepsilon,h}^N \in \mathbb{R}^{2N+1}$ is given by

$$(\mathbf{X}_{\varepsilon,h}^N(v))_1 = a_{\varepsilon,h}^0(v),\ (\mathbf{X}_{\varepsilon,h}^N(v))_{2i+1} = a_{\varepsilon,h}^i(v),\ (\mathbf{X}_{\varepsilon,h}^N(v))_{2i+2} = b_{\varepsilon,h}^i(v),$$

and $\mathbf{Q}_{\varepsilon,h}^N \in \mathbb{R}^{2N+1}$ is given by

$$(\mathbf{Q}_{\varepsilon,h}^N)_1 = \mathcal{A}_\varepsilon^0 v, (\mathbf{Q}_{\varepsilon,h}^N)_{2i} = \mathcal{A}_\varepsilon^i v, (\mathbf{Q}_{\varepsilon,h}^N)_{2i+1} = \mathcal{B}_\varepsilon^i v.$$

Using the property of the Scott-Zhang projector, we next show that $\mathbf{P}_{\varepsilon,h}^N$ tends to the identity matrix $\mathbf{I}_{2N+1}$ when $\frac{h}{\varepsilon}$ tends to 0. For $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N$, the difference between the coefficient $(2i, 2j)$ of $\mathbf{P}_{\varepsilon,h}^N$ and the identity matrix writes

$$|(\mathbf{P}_{\varepsilon,h}^N)_{2i,2j} - \delta_{2i,2j}| = |\mathcal{A}_\varepsilon^i(R_h^k c_j - c_j)|.$$

Using Cauchy-Schwarz inequality on the last term, we obtain an estimate which only depends of the value of $R_h^k c_j - c_j$ on $\partial\omega_\varepsilon$,

$$|\mathcal{A}_\varepsilon^i(R_h^k c_j - c_j)| \leqslant \varepsilon^{-\frac{1}{2}}|\partial\omega|^{-\frac{1}{2}}\|R_h^k c_j - c_j\|_{0,\partial\omega_\varepsilon}.$$

Let $0 < \Upsilon_2 < \Upsilon_1$ such that $\omega_{\Upsilon_2} \subset \omega_{\Upsilon_1}$ and $2h \leqslant \text{dist}(\partial\omega_{\Upsilon_1}, \partial\omega_{\Upsilon_2})$, according to **Lemma 3.1**, if we consider$(\omega_{\Upsilon_2}\backslash\overline{\omega_\varepsilon})$ instead of Ω_ε as exterior domain, we have

$$\|R_h^k c_j - c_j\|_{\frac{1}{2},\varepsilon} \leqslant C\|R_h^k c_j - c_j\|_{1,(\omega_{\Upsilon_2}\backslash\overline{\omega_\varepsilon})},$$

so noticing that for $v \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$,

$$\|v\|_{\frac{1}{2},\varepsilon}^2 = \varepsilon^{-1}\|v\|_{0,\partial\omega_\varepsilon}^2 + [v]_{\frac{1}{2},\partial\omega_\varepsilon}^2,$$

we deduce that

$$\|v\|_{0,\partial\omega_\varepsilon} \leqslant \varepsilon^{\frac{1}{2}}\|v\|_{\frac{1}{2},\varepsilon},$$

in particular

$$\|R_h^k c_j - c_j\|_{0,\varepsilon} \leqslant \varepsilon^{\frac{1}{2}}\|R_h^k c_j - c_j\|_{\frac{1}{2},\varepsilon},$$

and we then obtain

$$|(\mathbf{P}_{\varepsilon,h}^N)_{2i,2j} - \delta_{2i,2j}| \leqslant C\|R_h^k c_j - c_j\|_{1,(\omega_{\Upsilon_2}\backslash\overline{\omega_\varepsilon})}, \tag{4.17}$$

with C independent of i, j and ε. For $r > 0$, we denote by $(\tau_{r,l}^h)_0^{N_{r,h}}$ the tiles intersecting $\partial\omega_r$ and by T_r^h the domain

$$T_r^h = \bigcup_{l=1}^{N_{r,h}} S_{\tau_{r,l}^h},$$

where S_τ denotes the domain made of the elements neighboring τ (see **Figure 1** for an explicit description of $\bigcup_{l=1}^{N_{\Upsilon_2,h}} S_{\tau_{\Upsilon_2,l}^h}$ and $\bigcup_{l=1}^{N_{\Upsilon_2,h}} \tau_{\Upsilon_2,l}^h$). According to **Lemma 4.4**,

$$\|R_h^k c_j - c_j\|_{1,(\omega_{\Upsilon_2}\backslash\overline{\omega_\varepsilon})} \leqslant Ch\|c_j\|_{2,(\omega_{\Upsilon_2}\backslash\overline{\omega_\varepsilon})\cup T_{\Upsilon_2}^h}. \tag{4.18}$$

Moreover, since $2h \leqslant \text{dist}(\partial\omega_{\Upsilon_1}, \partial\omega_{\Upsilon_2})$, then $(\omega_{\Upsilon_2}\backslash\overline{\omega_\varepsilon}) \cup T_{\Upsilon_2}^h \subset \omega_{\Upsilon_1}$, and according to equation (4.14), for $\varepsilon > 0$ sufficiently small,

$$\|c_j\|_{2,(\omega_{\Upsilon_2}\backslash\overline{\omega_\varepsilon})\cup T_{\Upsilon_2}^h} \leqslant \|c_j\|_{2,\omega_{\Upsilon_1}} \leqslant C(1+j)^{\frac{3}{2}}\varepsilon^{-1}. \tag{4.19}$$

Gathering equations (4.17), (4.18) and (4.19), we obtain

$$|(\mathbf{P}_{\varepsilon,h}^N)_{2i,2j} - \delta_{2i,2j}| \leqslant C(1+j)^{\frac{3}{2}}\left(\frac{h}{\varepsilon}\right),$$

with C independent of i, j, h and ε. We can prove in a similar way that for all $0 \leqslant k, l \leqslant 2N+1$, $(\mathbf{P}_{\varepsilon,h}^N)_{k,l}$ tends to δ_{kl} when $(1+N)^{\frac{3}{2}}\left(\frac{h}{\varepsilon}\right) \to 0$. Besides, the open ball of center $\mathbf{I}_{2N+1}$ and radius $2N+1$ for the matricial infinity-norm belongs to the space of invertible matrix of dimension $2N+1$. So we deduce that there exists $\rho(N) > 0$ proportional to $(2N+1)^{-1}(1+N)^{-\frac{3}{2}}$ such that for $\varepsilon > 0$ sufficiently small and $0 < \frac{h}{\varepsilon} < \rho(N)$, $\mathbf{P}^N_{\varepsilon,h}$ is invertible and that at N fixed, $\lim\limits_{\frac{h}{\varepsilon}\to 0} (\mathbf{P}^N_{\varepsilon,h})^{-1} = \mathbf{I}_{2N+1}$. In particular, we have for $0 \leqslant n \leqslant N$,

$$\lim_{\frac{h}{\varepsilon}\to 0} a^n_{\varepsilon,h}(v) = \mathcal{A}^n_\varepsilon v \text{ and } \lim_{\frac{h}{\varepsilon}\to 0} b^n_{\varepsilon,h}(v) = \mathcal{B}^n_\varepsilon v.$$

From now on, for the sake of simplicity, the dependence in N of the constant $\rho(N)$ will be implied in the formulation *for* $\frac{h}{\varepsilon} > 0$ *sufficiently small.* Let us now prove that $\Pi^N_{\varepsilon,h}$ is bounded independently of h, N and ε. Let $v \in H^1_0(\Omega)$, for $\varepsilon > 0$ and $\frac{h}{\varepsilon} > 0$ sufficiently small, we have

$$|a^n_{\varepsilon,h}(v)| \leqslant 2|\mathcal{A}^n_\varepsilon v| \text{ and } |b^n_{\varepsilon,h}(v)| \leqslant 2|\mathcal{B}^n_\varepsilon v|, \ \forall 0 \leqslant n \leqslant N. \tag{4.20}$$

By linearity of the operator R^k_h, we have

$$\Pi^N_{\varepsilon,h} v = R^k_h \left(a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n \right),$$

so we deduce from **Lemma 4.2** that

$$\|\Pi^N_{\varepsilon,h} v\|_{1,\Omega} \leqslant C \|a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n\|_{1,\Omega}.$$

As well, the extension operator $\mathcal{H}_\Omega \circ \mathcal{E}_{\omega_{\Upsilon_1}}$ from ω_ε into Ω is linear, so we have in Ω

$$\begin{aligned}
&a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n \\
&= a^0_{\varepsilon,h}(v) \mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} c_0) + \sum_{n=1}^N a^n_{\varepsilon,h}(v) \mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} c_n) + b^n_{\varepsilon,h}(v) \mathcal{H}_\Omega(\mathcal{E}_{\omega_{\Upsilon_1}} s_n) \\
&= \mathcal{H}_\Omega \left(\mathcal{E}_{\omega_{\Upsilon_1}} \left(a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n \right) \right),
\end{aligned}$$

and we deduce from the continuity of the harmonic extension and **Lemma 4.5** that

$$\begin{aligned}
&\|\mathcal{H}_\Omega \left(\mathcal{E}_{\omega_{\Upsilon_1}} \left(a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n \right) \right) \|_{1,\Omega} \\
&\leqslant C \|\mathcal{E}_{\omega_{\Upsilon_1}} \left(a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n \right) \|_{1,\omega_{\Upsilon_1}}, \\
&\leqslant C \|a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n\|_{1,\omega_\varepsilon}.
\end{aligned}$$

In the same way, the harmonic lifting from $\partial\omega_\varepsilon$ into ω_ε is linear, so following the same computations as we did for proving **Lemma 3.2**, we have for $\varepsilon > 0$ sufficiently small,

$$\|a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n\|_{1,\omega_\varepsilon} \leqslant C \|a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n\|_{\frac{1}{2},\varepsilon},$$

and we deduce that

$$\begin{aligned}
\|\Pi^N_{\varepsilon,h} v\|_{1,\Omega} &\leqslant C \overline{\|a^0_{\varepsilon,h}(v) c_0 + \sum_{n=1}^N a^n_{\varepsilon,h}(v) c_n + b^n_{\varepsilon,h}(v) s_n\|}_{\frac{1}{2},\varepsilon}, \\
&\leqslant C \left(a^0_{\varepsilon,h}(v)^2 + \sum_{n=1}^N (1+n) \left(a^n_{\varepsilon,h}(v)^2 + b^n_{\varepsilon,h}(v)^2 \right) \right)^{\frac{1}{2}},
\end{aligned}$$

with C independent of h, N and ε. Eventually, according to inequality (4.20) and **Proposition 2.3**, for $\varepsilon > 0$ and $\frac{h}{\varepsilon} > 0$ sufficiently small, we have

$$\begin{aligned}\left(a^0_{\varepsilon,h}(v)^2 + \sum_{n=1}^{N}(1+n)\left(a^n_{\varepsilon,h}(v)^2 + b^n_{\varepsilon,h}(v)^2\right)\right)^{\frac{1}{2}} &\leqslant C\left(\mathcal{A}^0_\varepsilon(v)^2 + \sum_{n=1}^{N}(1+n)\left(\mathcal{A}^n_\varepsilon(v)^2 + \mathcal{B}^n_\varepsilon(v)^2\right)\right)^{\frac{1}{2}},\\ &\leqslant C\overline{\|v\|}_{\frac{1}{2},\varepsilon} \leqslant C\|v\|_{\frac{1}{2},\varepsilon},\end{aligned}$$

so **Lemma 3.1** allows to conclude

$$\|\Pi^N_{\varepsilon,h}v\|_{1,\Omega} \leqslant C\|v\|_{1,\Omega},$$

with C independent of h, N and ε. Let us now consider the case $N \leqslant k$. For $\theta \in [0, 2\pi[$ and $0 < r < \varepsilon$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$, since c_n and s_n are harmonic in ω_ε, we have $c_n(r\mathbf{x}) = \left(\frac{r}{\varepsilon}\right)^n\cos(n\theta)$ and $s_n(r\mathbf{x}) = \left(\frac{r}{\varepsilon}\right)^n\sin(n\theta)$ in ω_ε. Moreover, if $r\mathbf{x} = (x, y)$, then c_n and s_n can be written as polynomials of x and y of degree smaller than n. Indeed using Chebyshev polynomials, for $0 < r < \varepsilon$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta))$, then

$$c_n(r\mathbf{x}) = \left(\frac{r}{\varepsilon}\right)^n\cos(n\theta) = \sum_{0\leqslant 2j\leqslant n}\left(\frac{1}{\varepsilon}\right)^n\binom{n}{2j}(-1)^j y^{2j}x^{n-2j}, \tag{4.21}$$

and

$$s_n(r\mathbf{x}) = \left(\frac{r}{\varepsilon}\right)^n\sin(n\theta) = \sum_{0\leqslant 2j+1\leqslant n}\left(\frac{1}{\varepsilon}\right)^n\binom{n}{2j+1}(-1)^j y^{2j+1}x^{n-2j-1}. \tag{4.22}$$

To get rid of the invertibility condition on $\mathbf{P}^N_{\varepsilon,h}$, we would like to have for all $0 \leqslant n \leqslant N$,

$$R^k_h c_n = c_n \text{ and } R^k_h s_n = s_n \text{ on } \partial\omega_\varepsilon. \tag{4.23}$$

The harmonic or even the H^k-extension defined in **Lemma 4.5** of c_n and s_n on Ω_ε are not sufficient this time because they do not ensure that c_n and s_n are polynomials on all tiles spanning $\partial\omega_\varepsilon$. Let r^* as defined in **Theorem 4.2** and let suppose $h < r^* - \varepsilon$ such that $\overline{\omega_{\varepsilon+h}} \subset \Omega$, we extend c_n and s_n in $\omega_{\varepsilon+h}$ such that for $\theta \in [0, 2\pi[$ and $0 < r < \varepsilon + h$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$, $c_n(r\mathbf{x}) = \left(\frac{r}{\varepsilon}\right)^n\cos(n\theta)$ and $s_n(r\mathbf{x}) = \left(\frac{r}{\varepsilon}\right)^n\sin(n\theta)$. We also extend c_n and s_n outside $\omega_{h+\varepsilon}$ such that c_n and s_n are harmonic in $\Omega_{h+\varepsilon} = \Omega\backslash\overline{\omega_{\varepsilon+h}}$ and $c_n = s_n = 0$ on $\partial\Omega$. Consequently, for $0 \leqslant l \leqslant N_{\varepsilon,h}$, $c_n|_{\tau^l_{\varepsilon,h}}$ and $s_n|_{\tau^l_{\varepsilon,h}}$ are respectively equal to (4.21) and (4.22) and condition (4.23) is satisfied. The proof of the continuity of the operator $\Pi^N_{\varepsilon,h}$ with this definition of c_n and s_n is then similar to the case $N > k$. This concludes the proof of the existence of the Fortin operator and also the proof of the existence of the inf-sup condition with, for $\varepsilon > 0$ and $\frac{h}{\varepsilon} > 0$ sufficiently small,

$$\beta^N_{\varepsilon,h} \geqslant \beta^N_\varepsilon \left|\!\left|\!\left|\Pi^N_{\varepsilon,h}\right|\!\right|\!\right|^{-1} = C.$$

where $\beta^N_{\varepsilon,h}$ is the inf-sup constant of the continuous bilinear form b and C is independent of h, N and ε. □

Remark 4.1. *Let us note that the dependence in N of $\rho_2(N)$ in **Theorem 4.1** implies that the more modes there are, the larger the number of points on $\partial\omega_\varepsilon$ must be, which is consistent with the fact that the number of constraints on the solution increases.*

Theorem 4.3. *For all $N \in \mathbb{N}$, there exists a constant $\rho_1 > 0$, $\rho_2(N) > 0$ such that, for $0 < \varepsilon < \rho_1$ and $0 < \frac{h}{\varepsilon} < \rho_2(N)$, Problem (4.1) is well-posed in $X^k_h(\Omega) \times \mathscr{M}^N_\varepsilon$. Moreover, if u^N_ε is solution of Problem (2.16) and $u^N_{\varepsilon,h}$ is solution of Problem (4.1), then for $k = 1$ and $1 < s < \frac{3}{2}$, if $\phi \in H^{s-\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$, there exists a constant $C(s,\varepsilon) > 0$ such that*

$$\|u^N_\varepsilon - u^N_{\varepsilon,h}\|_{1,\Omega} \leqslant C(s,\varepsilon)h^{s-1}\left(\|\phi\|_{s-\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with $C(s,\varepsilon)$ independent of N.

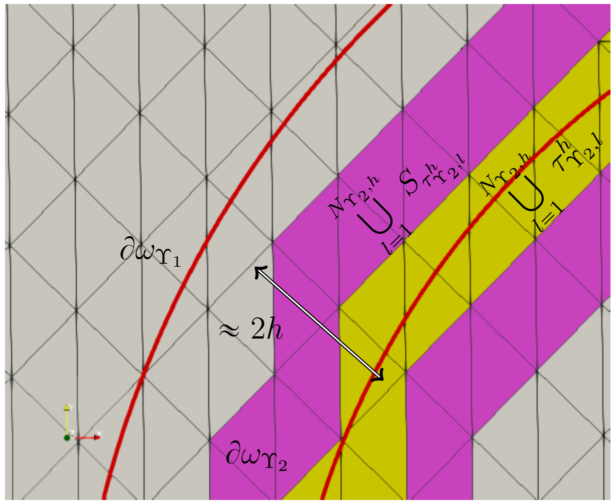


Figure 1: Geometric configuration of domain ω_{Υ_2} and sets $\{\bigcup_{l=1}^{N_{\Upsilon_2,h}} S_{\tau^h_{\Upsilon_2,l}}, \bigcup_{l=1}^{N_{\Upsilon_2,h}} \tau^h_{\Upsilon_2,l}\}$.

Proof. First, let us verify the assumptions of **Theorem 4.1** in order to prove that Problem (4.2) is well-posed. The continuity of the bilinear forms a and b and of the linear forms c and d, as well as the coercivity constraint on a, are a direct consequence of the fact that $X_h^k(\Omega) \cap H_0^1(\Omega) \subset H_0^1(\Omega)$ as well as the estimates

$$\|a\| \leqslant C, \ \|b\| \leqslant C, \ \alpha^N_{\varepsilon,h} \geqslant C,$$

where $\alpha^N_{\varepsilon,h}$ is the coercivity constant of a and C is a constant independent of h, N and ε. Moreover if we set

$$\begin{cases} d_h: & \mathscr{M}^N_\varepsilon \to \mathbb{R}, \\ & \mu \mapsto (\mu, u_{0,h})_\varepsilon, \end{cases}$$

it is straightforward to show that

$$\|d - d_h\| \leqslant \|b\| \|u_0 - u_{0,h}\|_{1,\Omega},$$

where $\|b\|$ is bounded independently of N and ε. According to **Theorem 4.2**, for $\varepsilon > 0$ and $\frac{h}{\varepsilon} > 0$ sufficiently small, the inf-sup condition is satisfied so we can conclude on the well-posedness of Problem (4.2) and, according to **Corollary 4.1**, we have

$$\|e^N_\varepsilon - e^N_{\varepsilon,h}\|_{1,\Omega} \leqslant C \left(\inf_{v_h \in X_h^k(\Omega)\cap H_0^1(\Omega)} \|e^N_\varepsilon - v_h\|_{1,\Omega} + \|u_0 - u_{0,h}\|_{1,\Omega} \right), \tag{4.24}$$

with C independent of h, N and ε. Then according to **Lemma 4.4**,

$$\inf_{v_h \in X_h^k(\Omega)\cap H_0^1(\Omega)} \|e^N_\varepsilon - v_h\|_{1,\Omega} \leqslant \|e^N_\varepsilon - R^k_h e^N_\varepsilon\|_{1,\Omega} \leqslant C h^{s-1} \|e^N_\varepsilon\|_{s,\Omega}, \ \forall\, 1 < s < \frac{3}{2}.$$

We deduce thanks to equations (4.3) and (4.4) that, for $\varepsilon > 0$ and $\frac{h}{\varepsilon} > 0$ sufficiently small,

$$\|u^N_\varepsilon - u^N_{\varepsilon,h}\|_{1,\partial\Omega} \leqslant C h^{s-1} \left(\|e^N_\varepsilon\|_{s,\Omega} + \|\phi\|_{s-\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C independent of h, N and ε. We conclude using **Theorem 3.7**. □

Remark 4.2. *Let us note that due to the low regularity of the solution u^N_ε described in **Subsection 3.4**, the convergence rate in h in **Theorem 4.3** is less than the maximum convergence rate reachable with P_1 elements. This result is similar to the one obtained for the discretization of the full problem* (2.5) *with classical finite element methods (see [28] for more details on this topic).*

4.3 δ mesh

We can prove a convergence result better than the one stated in the previous corollary under some additional restrictions of the mesh. Let $\mathcal{T}_h^{\Omega}$ be a δ-resolving with respect to the interface $\partial\omega_\varepsilon$, that is the boundaries of the submesh corresponding to ω_ε and Ω_ε have a maximum distance of δ to the interface. This condition is in particular fulfilled with $\delta = \mathcal{O}(h^2)$ when the nodes of the mesh $\mathcal{T}_h^{\Omega}$ fall on the interface $\partial\omega_\varepsilon$. With little abuse of notation, in the numerical simulations we call this case the *conforming mesh* configuration. In the case of conforming meshes we obtain the following theorem.

Theorem 4.4. *Let u_ε^N be the solution of Problem (2.16) and $u_{\varepsilon,h}^N$ be the solution of Problem (4.1) with $k = 1$. If $\phi \in H^{\frac{3}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$, for all $N \in \mathbb{N}$, there exist constants $\rho_1 > 0$, $\rho_2(N) > 0$ and $C > 0$ such that for $0 < \varepsilon < \rho_1$ and $0 < \frac{h}{\varepsilon} < \rho_2(N)$,*

$$\|u_\varepsilon^N - u_{\varepsilon,h}^N\|_{1,\Omega} \leqslant C\left(\frac{h}{\varepsilon}\right)\left(\|\phi\|_{\frac{3}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of h, N and ε.

Proof. By using a modified Clément operator S_{ch} presented in [34] we get

$$\|e_\varepsilon^N - S_{ch}e_\varepsilon^N\|_{1,\Omega}^2 \leqslant C\left(h^2\left(\|e_\varepsilon^N\|_{2,\Omega_\varepsilon}^2 + \|e_\varepsilon^N\|_{2,\omega_\varepsilon}^2\right) + \delta\left(\|e_\varepsilon^N\|_{2,\omega_\varepsilon}^2 + \|e_\varepsilon^N\|_{2,\Omega_\varepsilon}^2\right)\right),$$

with C independent of h. We deduce thanks to equations (4.3), (4.4) and (4.24), and the hypothesis $\delta = \mathcal{O}(h^2)$ that, for $\varepsilon > 0$ and $\frac{h}{\varepsilon} > 0$ sufficiently small,

$$\|u_\varepsilon^N - u_{\varepsilon,h}^N\|_{1,\Omega} \leqslant Ch\left(\|e_\varepsilon^N\|_{2,\omega_\varepsilon} + \|e_\varepsilon^N\|_{2,\Omega_\varepsilon} + \|\phi\|_{\frac{3}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of h, N and ε. Then, using **Theorem 3.8**, we can conclude on the proof of the theorem. ☐

Remark 4.3. *We can this way obtain an estimate of the difference between the discrete solution $u_{\varepsilon,h}^N$ obtained with a δ-mesh and the continuous solution u_ε in Ω_ε with respect to h and ε: there exists $\Upsilon > 0$ such that for $\varepsilon > 0$ and $\frac{h}{\varepsilon} > 0$ sufficiently small,*

$$\begin{aligned}\|u_\varepsilon - u_{\varepsilon,h}^N\|_{1,\Omega_\varepsilon} &\leqslant \|u_\varepsilon - u_\varepsilon^N\|_{1,\Omega_\varepsilon} + \|u_\varepsilon^N - u_{\varepsilon,h}^N\|_{1,\Omega},\\ &\leqslant C(1+N)^{\frac{1}{2}}\left(\frac{\varepsilon}{\Upsilon}\right)^{N+1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right) + C\left(\frac{h}{\varepsilon}\right)\left(\|\phi\|_{\frac{3}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),\end{aligned}$$

with C independent of h, N and ε.

4.4 Numerical experiments

4.4.1 Convergence in ε

In this section, we consider a rectangular domain Ω of width L and and height l with an hole located in (x_0, y_0). For simplicity we set $L = l = 1$, $(x_0, y_0) = (0.2, 0.1)$ and we make ε vary in the set $\{0.05, 0.04, 0.03, 0.02, 0.01\}$. The boundary conditions of the problem are defined in **Figure 2**. We compute the convergence rate with respect to ε of the reduced problem towards the full problem using increasing values of N in the definition of $\mathscr{M}_\varepsilon^N$ for the enforcement of the internal boundary condition. In the general case, these solutions are not known a priori. Therefore, we verify **Theorem 3.6** using the linear finite element approximation of the reduced and full problem. The convergence rates of the different solutions are given in **Figure 3** where $e_{\varepsilon,h}^{F0}$, $e_{\varepsilon,h}^{F1}$ and $e_{\varepsilon,h}^{F2}$ correspond to the discrete differences between the full Poisson problem and the reduced Poisson problems where the internal boundary conditions are respectively approximated by $N = 0$, $N = 1$ and $N = 2$ moments. The numerical solution of the reduced Poisson problem and the discrete model errors $e_{\varepsilon,h}^{F0}$, $e_{\varepsilon,h}^{F1}$ and $e_{\varepsilon,h}^{F2}$ are reported in **Figure 4**.

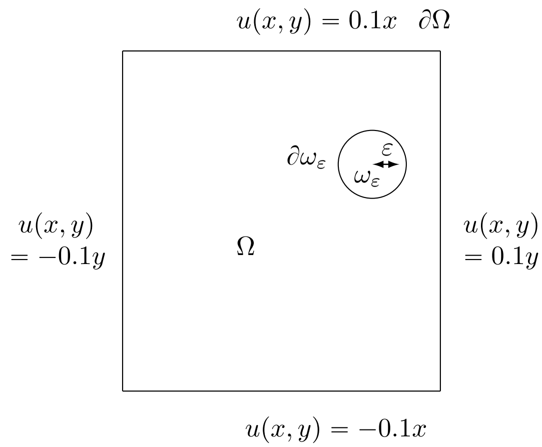


Figure 2: Boundary conditions for the inclusion in a square exterior domain.

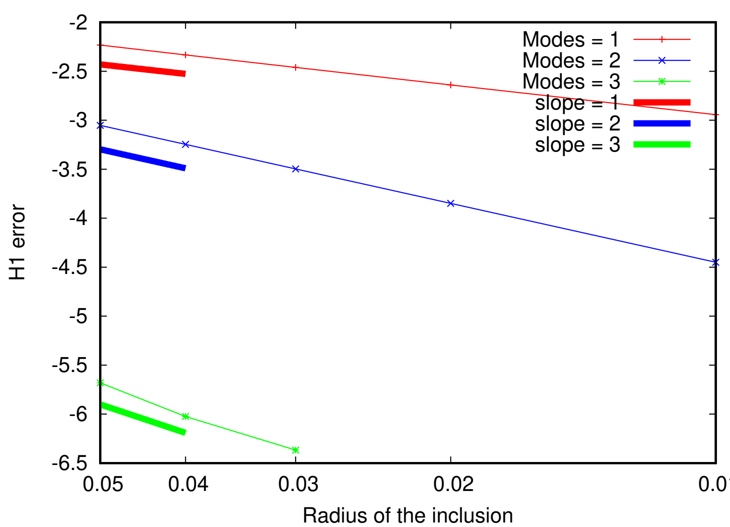
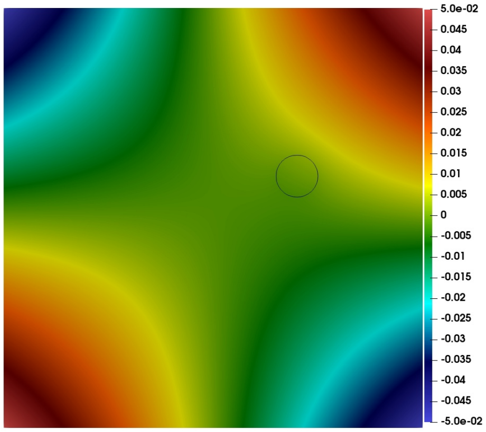
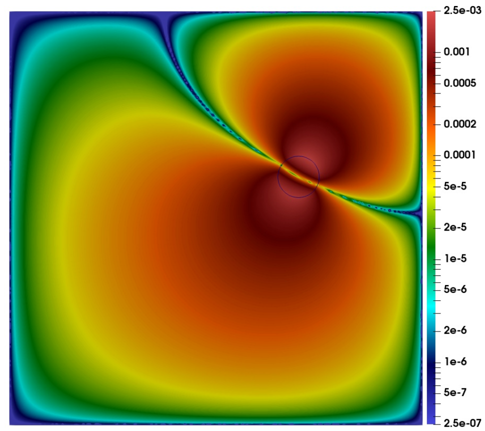


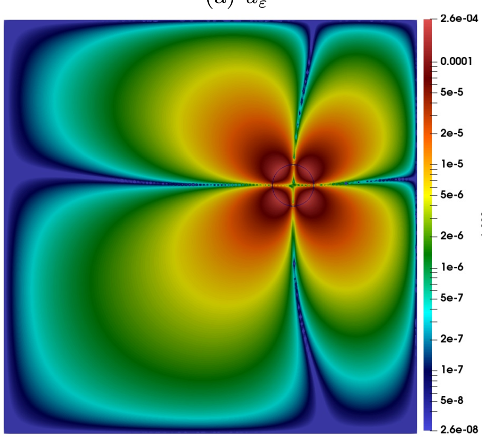
Figure 3: Numerical modelling errors for different radii and different numbers of moments for a single circular inclusion.



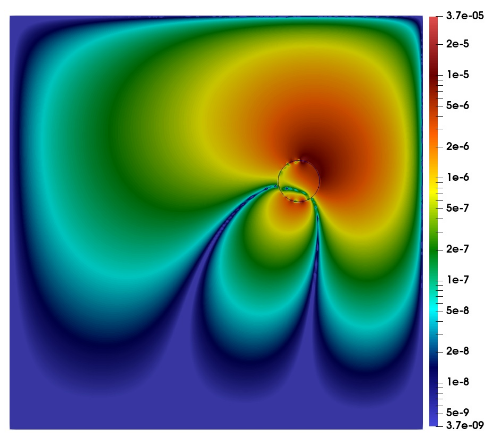
(a) u_ε^N



(b) $e_{\varepsilon,h}^{F0}$



(c) $e_{\varepsilon,h}^{F1}$



(d) $e_{\varepsilon,h}^{F2}$

Figure 4: Contour plots of the solution u_ε and of the discrete model errors $e_{\varepsilon,h}^{F0}$, $e_{\varepsilon,h}^{F1}$ and e_{ε}^{F2} on a log-scale axis for a circule of radius $\varepsilon = 0.05$.

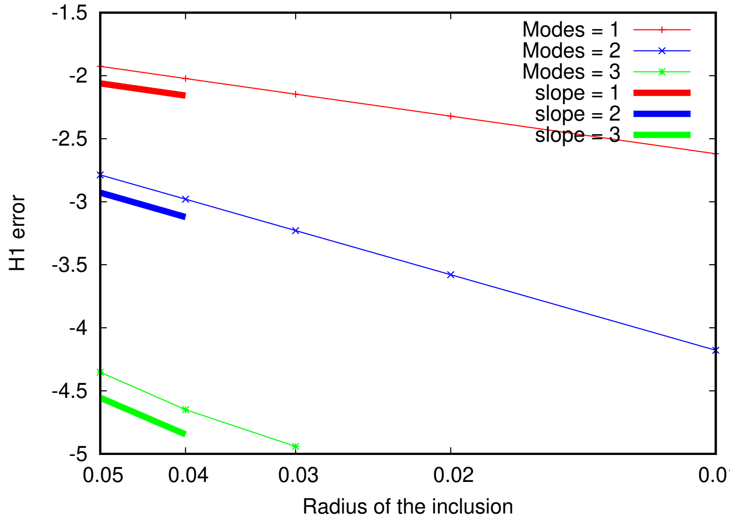


Figure 5: Numerical modelling errors for different radii and different numbers of modes for multiple circular inclusions.

4.4.2 Convergence in ε for multiple obstacles

As mentioned in the introduction, the proposed approach can be easily extended to multiple obstacles with different sizes and not necessarily centered in $(0, 0)$. To test this assumption, we consider a rectangular computational domain $\Omega = [-0.5, 0.5] \times [-0.5, 0.5]$ with four inclusions positioned respectively in $(0.2, 0.1)$, $(-0.3, 0.1)$, $(-0.2, 0.2)$, $(0.1, -0.2)$ and of respective initial sizes 0.05, 0.04, 0.06 and 0.03. Boundary conditions on $\partial\Omega$ are the same as those of **Figure 2**. The results obtained for $N = \{1, 2, 3\}$ are reported in **Figure 5**. There, the parameter ε corresponds then to the size of the first inclusion and all the other holes are scaled proportionally. The numerical solution of the reduced Poisson problem and of the discrete corresponding model errors $e^{F0}_{\varepsilon,h}$, $e^{F1}_{\varepsilon,h}$ and $e^{F2}_{\varepsilon,h}$ are reported in **Figure 6**.

4.4.3 Convergence in h

We now test the convergence results obtained in **Theorem 4.3** and **Theorem 4.4**. For the convergence test in h, we consider a single source and a single mode $N = 1$, the obstacle has radius $\varepsilon = 0.2$ and center $\mathbf{z} = (0, 0)$. As computational domain, we choose the rectangle $\Omega = [-0.5, 0.5] \times [-0.5, 0.5]$. Setting the appropriate Dirichlet boundary conditions, the exact solution $u_{e,\varepsilon}$ of this problem is the following

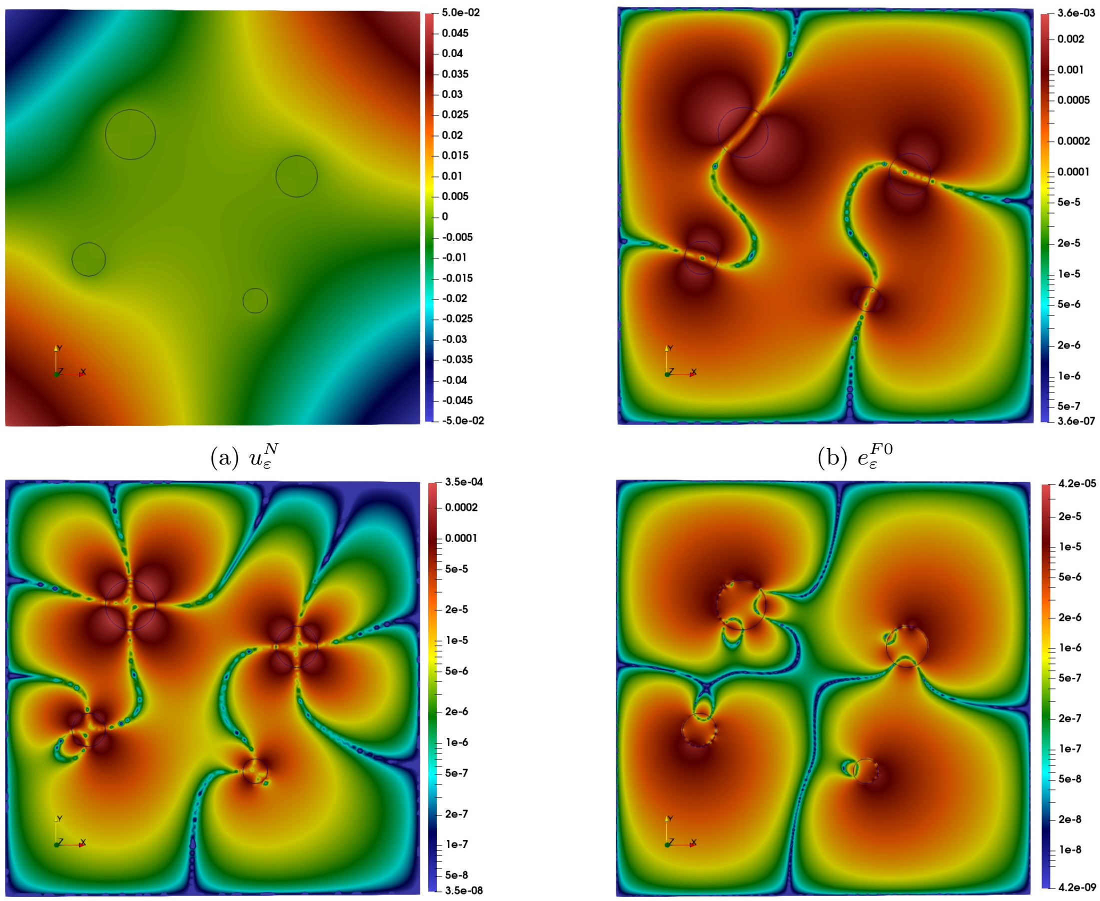
$$u_{\varepsilon,e}(r,\theta) = \begin{cases} r\cos(\theta) + r\sin(\theta) & \text{if } r \leqslant R, \\ r\cos(\theta) + r\sin(\theta) - 0.5\frac{\log(r)-\log(\varepsilon)}{\log(\varepsilon)} & \text{if } r > R. \end{cases}$$

The computed errors and the corresponding h-convergence rates are given in **Figure 7**.

4.4.4 Behavior of the model error for close obstacles

We consider here a last test case to illustrate **Remark 3.1**. The exterior domain is the same as in **Figure 2** with uniform Dirichlet boundary conditions equal to 1 and two holes of radius 0.05 initially centered in $(-0.1, 0)$ and $(0.1, 0)$. The values imposed on each inclusion are respectively 0.5 and 1.5 and are enforced using 3 modes. The numerical solution of such problem is displayed in **Figure 8** panel (a). Then, the distance between the two inclusions is gradually reduced. The numerical solution with inclusions separated by a gap of 0.01 is shown in panel (b). The comparison of the two top panels shows that the solution in the circle deviates from the constant as the two inclusions get closer. More quantitative results are given in **Figure 8** panel (c).

We see that the model error increases as the distance between the two obstacles decreases, as predicted by the inverse dependence on Υ of **Theorem 3.6**. We also notice that this effect becomes more severe as the number of modes increases.



(a) u_ε^N (b) e_ε^{F0}

(c) e_ε^{F1} (d) e_ε^{F2}

Figure 6: Contour plots of the solution u_ε and of the discrete model errors $e_{\varepsilon,h}^{F0}$, $e_{\varepsilon,h}^{F1}$ and $e_{\varepsilon,h}^{F2}$ on a log-scale axis for multiple obstacles of radius $\varepsilon = 0.05$.

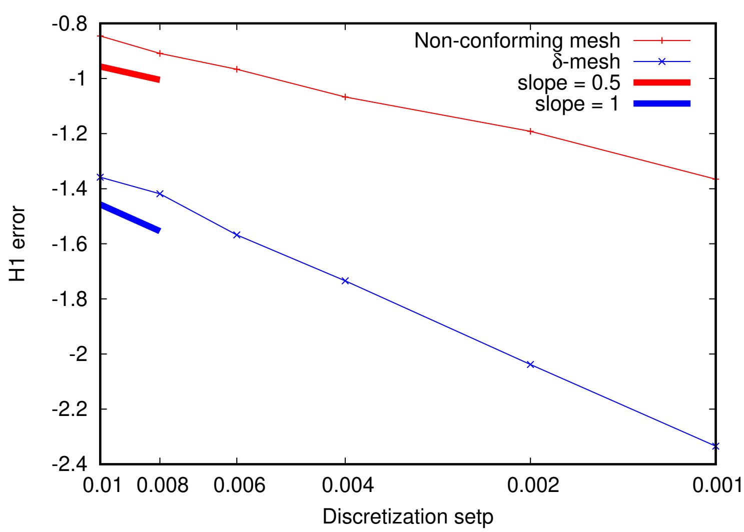
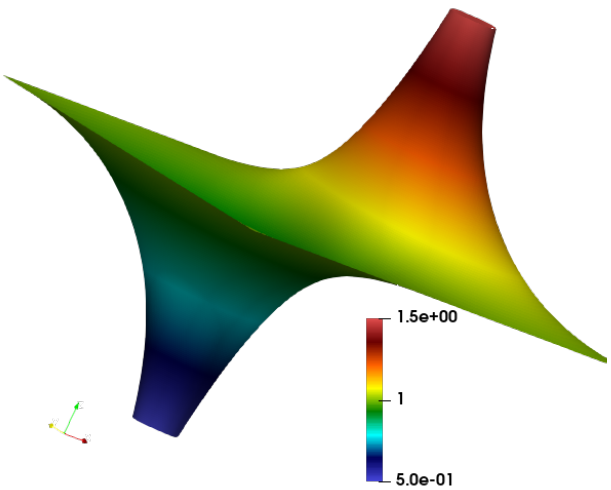
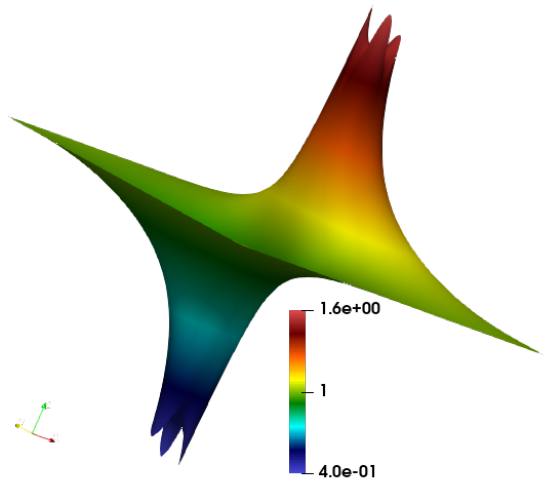


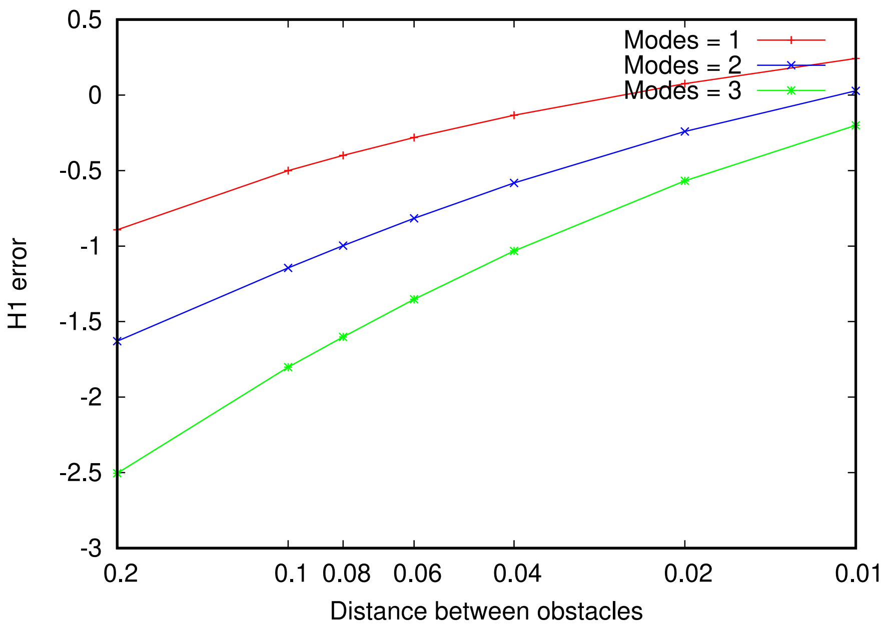
Figure 7: H^1-norms of $u_\varepsilon^N - u_{\varepsilon,h}^N$ and convergence rates for two different meshes (uniform, δ-resolving).



(a) Distance between the obstacles equal to 0.2.



(b) Distance between the obstacles equal to 0.01.



(c) Evolution of the error with respect to the distance between the obstacles.

Figure 8: Effect of the inter-inclusions distance on the model error.

Conclusions

In this work we have considered a second order PDE defined on a domain with small circular holes subject to Dirichlet boundary conditions. This problem can be seen as a template for different families of applications. It may represent solid mechanics problem where the holes play the role of small supports of a mechanical part, but it may also be used as a prototype problem for fluid-structure interaction applications where the small inclusions represent particles or fibers immersed into a fluid. To address these challenging applications using computational models, a thorough mathematical understanding of the fundamental mathematical aspects of the problem is extremely useful. As highlighted in [19], a mathematically-informed approach is a prerequisite for safe and reliable computations.

For these reasons, we focused on the fundamental aspects of the approximation of the problem. On one hand, we addressed the approximation of the Dirichlet boundary conditions on the inclusion by means of a reduced modeling approach based on the projection on Fourier modes. On the other hand we have studied the properties of the finite element method used for the approximation of the reduced model.

A particularly important question to be addressed is the robustness of this approach with respect to the size of the holes, which may become arbitrarily small with respect to the domain. To this purpose, we have studied three relevant problems: i) the full problem, corresponding to the standard enforcement of Dirichlet boundary conditions on the holes by means of Lagrange multipliers; ii) the reduced problem, characterized by the approximate weak enforcement of Dirichlet boundary conditions by projection on Fourier modes; iii) the limit problem obtained when the diameter of the holes vanishes. Understanding the mutual interaction of these problems characterizes what we call the modeling error in terms of the size of the holes. By means of suitable a priori estimates of the modeling error and of the finite element approximation error, we provide guidelines to optimally balance the approximation parameters of the proposed reduced modeling approach. These theoretical results will be particularly useful in view of forthcoming applications of this methodology to fluid-structure interaction problems.

Acknowledgments

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A Appendix

In this appendix, we prove several results presented in the previous sections. We recall the statements of these results to facilitate the reading of this part.

A.1 Proofs of Lemmas 2.1, 2.2 and 2.3

Lemma 2.1. *For $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$ such that $\omega_\varepsilon \cap \operatorname{supp} f = \emptyset$, the following problem*

$$\begin{cases} -\Delta v_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = \phi & \text{on } \partial\Omega, \\ v_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon, \end{cases} \tag{A.1}$$

admits a unique weak solution in $H^1(\Omega_\varepsilon)$. Moreover there exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,

$$\|v_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of ε.

Proof. The proof is similar to the proof of [[20], **Lemma C.1**] for the Stokes equations which is itself based on the results of [[21], **Chapter 3**]. In this proof, we will consider two cases, the first where $f = 0$ and the second where $\phi = 0$ and conclude by linearity.
Let us first suppose that $f = 0$. For $\varepsilon_0 > 0$, we consider v_{ε_0} the solution of (A.1) for $\varepsilon = \varepsilon_0$. It satisfies:

$$|v_{\varepsilon_0}|_{1,\Omega_{\varepsilon_0}} = \left(\int_{\Omega_{\varepsilon_0}} |\nabla v_{\varepsilon_0}|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leqslant C(\varepsilon_0)\|\phi\|_{\frac{1}{2},\partial\Omega}. \tag{A.2}$$

Now consider $\tilde{v}_{\varepsilon_0} \in H^1(\Omega)$ the extension by 0 of v_{ε_0} to all Ω. Notice that if $\varepsilon < \varepsilon_0$ then $\varepsilon\omega \subset \varepsilon_0\omega$ and $\Omega_{\varepsilon_0} \subset \Omega_\varepsilon$, so for all $\varepsilon \in (0, \varepsilon_0)$, by minimization of energy, we have

$$|v_\varepsilon|_{1,\Omega_\varepsilon} \leqslant |\tilde{v}_{\varepsilon_0}|_{1,\Omega_\varepsilon} = |v_{\varepsilon_0}|_{1,\Omega_{\varepsilon_0}},$$

and thanks to equation (A.2),

$$|v_\varepsilon|_{1,\Omega_\varepsilon} \leqslant C(\varepsilon_0)\|\phi\|_{\frac{1}{2},\partial\Omega}. \tag{A.3}$$

Let u_0 be the solution of Problem (2.3). Since $\tilde{v}_\varepsilon - u_0 \in H^1_0(\Omega)$, we can apply **Theorem 3.1** on Poincaré inequality, we get

$$\|v_\varepsilon\|_{0,\Omega_\varepsilon} = \|\tilde{v}_\varepsilon\|_{0,\Omega} \leqslant \|\tilde{v}_\varepsilon - u_0\|_{0,\Omega} + \|u_0\|_{0,\Omega} \leqslant C_P(\Omega)\left(|v_\varepsilon|_{1,\Omega_\varepsilon} + |u_0|_{1,\Omega}\right) + \|u_0\|_{0,\Omega}.$$

The well-posedness of Problem (2.3) also gives the existence of a constant $C > 0$ independent of ε such that

$$\|u_0\|_{1,\Omega} \leqslant C(\Omega)\|\phi\|_{\frac{1}{2},\partial\Omega}. \tag{A.4}$$

Combining equations (A.3) and (A.4) finally gives us

$$\|v_\varepsilon\|_{0,\Omega_\varepsilon} \leqslant C(\varepsilon_0, \Omega)\|\phi\|_{\frac{1}{2},\partial\Omega}.$$

Now, let us suppose that $\phi = 0$. Let us note that for all $v \in H^1_0(\Omega_\varepsilon)$, denoting by $\tilde{v}$ the extension by zero of v to Ω, we have

$$\|v\|_{0,\Omega_\varepsilon} = \|\tilde{v}\|_{0,\Omega} \leqslant C_P(\Omega)\|\nabla\tilde{v}\|_{0,\Omega} = C_P(\Omega)\|\nabla v\|_{0,\Omega_\varepsilon}.$$

Using this inequality and Lax-Milgram theorem, we get the existence of a constant $C > 0$ independent of ε such that

$$\|v_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C\|f\|_{0,\Omega}.$$

This conclude the proof of the lemma. □

Lemma 2.2. *For any $L \in \mathbb{R}$, the problem*

$$\begin{cases} -\Delta v_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \\ v_\varepsilon = L & \text{on } \partial\omega_\varepsilon, \end{cases}$$

admits a unique weak solution in $H^1(\Omega_\varepsilon)$. Morevover, there exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,

$$\|v_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}|L|,$$

with C independent of ε.

Proof. As for the previous lemma, the proof is adapted from [[20], **Lemma C.3**] for the Stokes equations itself presented in [[21], **Chapter 3**]. Let us consider the following quantity:

$$\rho = \sup\{r > 0 \mid \overline{\omega_r} \subset \Omega\},$$

and, for all $0 < \varepsilon < \rho$, let us define w_ε the unique solution of the system

$$\begin{cases} -\Delta w_\varepsilon = 0 & \text{in } \omega_{\rho/\varepsilon} \backslash \overline{\omega}, \\ w_\varepsilon = 0 & \text{on } \partial\omega_{\rho/\varepsilon}, \\ w_\varepsilon = L & \text{on } \partial\omega. \end{cases}$$

We also consider the function $\hat{v}_\varepsilon$ defined on $\frac{\Omega}{\varepsilon}\backslash\overline{\omega}$ by $\hat{v}_\varepsilon(\mathbf{x}) = v_\varepsilon(\varepsilon\mathbf{x})$ for all $\mathbf{x} \in \frac{\Omega}{\varepsilon}\backslash\overline{\omega}$. The function $\hat{v}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \hat{v}_\varepsilon = 0 & \text{in } \frac{\Omega}{\varepsilon}\backslash\overline{\omega}, \\ \hat{v}_\varepsilon = 0 & \text{on } \frac{1}{\varepsilon}\partial\Omega, \\ \hat{v}_\varepsilon = L & \text{on } \partial\omega. \end{cases} \tag{A.5}$$

Notice that we have $\overline{\omega} \subset \omega_{\rho/\varepsilon} \subset \frac{\Omega}{\varepsilon}$. Now we consider $\tilde{w}_\varepsilon$ the extension of w_ε to $\frac{\Omega}{\varepsilon}\backslash\overline{\omega}$ by zero in the outer part of the extended domain. Therefore, by the principle of minimization of energy, we have

$$|v_\varepsilon|_{1,\Omega_\varepsilon} = |\hat{v}_\varepsilon|_{1,\frac{\Omega}{\varepsilon}\backslash\overline{\omega}} \leqslant |\tilde{w}_\varepsilon|_{1,\frac{\Omega}{\varepsilon}\backslash\overline{\omega}} = |w_\varepsilon|_{1,\omega_{\rho/\varepsilon}\backslash\overline{\omega}}. \tag{A.6}$$

A computation provides for all $\mathbf{x} \in \omega_{\rho/\varepsilon}\backslash\overline{\omega}$,

$$w_\varepsilon(\mathbf{x}) = L\frac{\log(\rho/\varepsilon) - \log(|\mathbf{x}|)}{\log(\rho/\varepsilon)}$$

and for $\varepsilon > 0$ sufficiently small,

$$|w_\varepsilon|_{1,\omega_{\rho/\varepsilon}\backslash\overline{\omega}} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}|L|,$$

with C independent of ε. So we get that for $\varepsilon > 0$ sufficiently small,

$$|v_\varepsilon|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}|L|, \tag{A.7}$$

with C independent of ε. Finally, we consider $\tilde{v}_\varepsilon$ the extension of v_ε to Ω by L. Since this extension is in $H^1_0(\Omega)$, we can use the Poincaré inequality given by **Theorem 3.1**:

$$\begin{aligned} \|v_\varepsilon\|_{0,\Omega_\varepsilon} &\leqslant \|\tilde{v}_\varepsilon\|_{0,\Omega} \\ &\leqslant C_P(\Omega)|\tilde{v}_\varepsilon|_{1,\Omega} \leqslant C_P(\Omega)|v_\varepsilon|_{1,\Omega_\varepsilon}. \end{aligned}$$

Using equation (A.7), we get the result. □

Lemma 2.3. *For any $\varphi \in H^{\frac{1}{2}}(\partial\omega_\varepsilon)$, the problem*

$$\begin{cases} -\Delta v_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \\ v_\varepsilon = \varphi & \text{on } \partial\omega_\varepsilon, \end{cases}$$

admits a unique weak solution in $H^1(\Omega_\varepsilon)$. Moreover, there exist constant $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,

$$\|v_\varepsilon\|_{1,\Omega_\varepsilon} \leqslant C\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega},$$

with C independent of ε.

Proof. Once again, this proof is adapted from a similar proof conducted in [[20], **Lemma B.2** and **Lemma 4.2**] for the Stokes problem which is itself inspired from a proof described in [[21], **Chapter 3**]. Lax-Milgram theorem allows to prove that the problem

$$\begin{cases} -\Delta V = 0 & \text{in } \mathbb{R}^2 \backslash \overline{\omega}, \\ V = \varphi(\varepsilon \mathbf{x}) & \text{in } \partial\omega, \end{cases} \tag{A.8}$$

is well posed and has a unique solution in

$$W_0^{1,2}(\mathbb{R}^2 \backslash \overline{\omega}) = \{u \in \mathcal{D}'(\mathbb{R}^2 \backslash \overline{\omega}) \mid \log(\rho)^{-1} u \in L^2_{-1}(\mathbb{R}^2 \backslash \overline{\omega}), \nabla u \in L^2(\mathbb{R}^2 \backslash \overline{\omega})\}$$

where

$$\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\frac{1}{2}}$$

and

$$L^2_{-1}(\mathbb{R}^2 \backslash \overline{\omega}) = \{u \in \mathcal{D}'(\mathbb{R}^2 \backslash \overline{\omega}) \mid \rho^{-1} u \in L^2(\mathbb{R}^2 \backslash \overline{\omega})\},$$

(see [35] for example). We will try to give an explicit representation of V. By setting $-\Delta V = 0$ in ω, Problem (A.8) has a unique solution in $\mathbb{R}^2$ and we have that

$$-\Delta V = \nabla V \cdot \mathbf{n}^+ \delta_{\partial\omega}$$

in $\mathcal{D}'(\mathbb{R}^2)$ where $\mathbf{n}^+$ is the exterior normal on $\partial\omega$. Now let us define

$$W = E * \left(\nabla V \cdot \mathbf{n}^+ \delta_{\partial\omega}\right)$$

where E is the fundamental solution of the Laplace equation given for $\mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ by

$$E(\mathbf{x}) = -\frac{\log(|\mathbf{x}|)}{2\pi},$$

and $*$ is the convolution product. We have

$$-\Delta W = \nabla V \cdot \mathbf{n}^+ \delta_{\partial\omega}$$

in $\mathcal{D}'(\mathbb{R}^2)$. Then $V - W$ is a harmonic tempered distribution. A classical result of Fourier analysis states that harmonic tempered distribution are polynomials (see [36] for example). Then $V = L + W$ with L a polynomial and for $\mathbf{y} \in \mathbb{R}^2 \backslash \omega$,

$$W(\mathbf{y}) = \int_{\partial\omega} t(\mathbf{x}) E(\mathbf{y} - \mathbf{x}) \mathrm{d}s(\mathbf{x}),$$

with $t(\mathbf{x}) = \nabla V \cdot \mathbf{n}^+$. Using a Taylor development for E, we get

$$E(\mathbf{y} - \mathbf{x}) = E(\mathbf{y}) - \nabla E(\mathbf{y} - \alpha\mathbf{x}) \cdot \mathbf{x}$$

for some $\alpha \in (0, 1)$. We then have

$$W(\mathbf{y}) = E(\mathbf{y}) \int_{\partial\omega} t(\mathbf{x}) \mathrm{d}s(\mathbf{x}) - \int_{\partial\omega} t(\mathbf{x}) \nabla E(\mathbf{y} - \alpha\mathbf{x}) \cdot \mathbf{x} \mathrm{d}s(\mathbf{x}).$$

Let us denote

$$U(\mathbf{y}) = \int_{\partial\omega} t(\mathbf{x}) \nabla E(\mathbf{y} - \alpha\mathbf{x}) \cdot \mathbf{x} \mathrm{d}s(\mathbf{x}).$$

By computation, we get that $U(\mathbf{y}) = o(1/|\mathbf{y}|)$ when $|\mathbf{y}| \to \infty$ so $(\log(\rho))^{-1} U \in L^2_{-1}(\mathbb{R}^2 \backslash \overline{\omega})$. As $(\log(\rho))^{-1} V \in L^2_{-1}(\mathbb{R}^2 \backslash \overline{\omega})$ and $(\log(\rho))^{-1} \notin L^2_{-1}(\mathbb{R}^2 \backslash \overline{\omega})$, we necessarily have that

$$\int_{\partial\omega} t(\mathbf{x}) \mathrm{d}s(\mathbf{x}) = 0$$

and that L is a constant. By computation, we have that for $\|\mathbf{y}\|$ sufficiently large,

$$|W(\mathbf{y})| \leqslant C \|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega} \frac{1}{\|\mathbf{y}\|} \text{ and } |\nabla W(\mathbf{y})| \leqslant C \|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega} \frac{1}{\|\mathbf{y}\|^2}.$$

Let $A > 0$ such that the previous inequality is satisfied for $\|\mathbf{y}\| > A$. We have for $\|\mathbf{y}\| > A$,

$$|L| \leqslant |V(\mathbf{y})| + C\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega}\frac{1}{\|\mathbf{y}\|}$$

Integrating for $\|\mathbf{y}\| > A$, we get

$$|L|\left(\int_{\|\mathbf{y}\|>A}\frac{1}{\log(\|\mathbf{y}\|)^2\|\mathbf{y}\|^2}\right)^{\frac{1}{2}} \leqslant \left(\int_{\|\mathbf{y}\|>A}\frac{|V(\mathbf{y})|^2}{\log(\|\mathbf{y}\|)^2\|\mathbf{y}\|^2}\right)^{\frac{1}{2}} + C\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega}\left(\int_{\|\mathbf{y}\|>A}\frac{1}{\log(\|\mathbf{y}\|)^2\|\mathbf{y}\|^4}\right)^{\frac{1}{2}}$$

The fact that A is independent of ϕ and the well-posedness of the problem (A.8) give

$$|L| \leqslant C\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega}.$$

Using similar computations as in [[37], **Lemma 7.1**] for $\varepsilon > 0$ small enough, we also have

$$\|W(\frac{\mathbf{x}}{\varepsilon})\|_{1,\Omega_\varepsilon} \leqslant C\varepsilon\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega}.$$

We then define $z_\varepsilon := v_\varepsilon - W(\frac{\mathbf{x}}{\varepsilon})$, z_ε satisfies

$$\begin{cases} -\Delta z_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ z_\varepsilon = -W(\frac{\mathbf{x}}{\varepsilon}) & \text{on } \partial\Omega, \\ z_\varepsilon = L & \text{on } \partial\omega_\varepsilon. \end{cases}$$

Using **Lemma 2.1** and **Lemma 2.2** we get that

$$\begin{aligned} \|z_\varepsilon\|_{1,\Omega_\varepsilon} &\leqslant C(\|W(\frac{\mathbf{x}}{\varepsilon})\|_{\frac{1}{2},\partial\Omega} + (-\log(\varepsilon))^{-\frac{1}{2}}|L|), \\ &\leqslant C\left(\varepsilon\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega} + (-\log(\varepsilon))^{-\frac{1}{2}}\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega}\right). \end{aligned}$$

So finally, for $\varepsilon > 0$ small enough, we get

$$\begin{aligned} \|v_\varepsilon\|_{1,\Omega_\varepsilon} &\leqslant \|z_\varepsilon\|_{1,\Omega_\varepsilon} + \|W(\frac{\mathbf{x}}{\varepsilon})\|_{1,\Omega_\varepsilon}, \\ &\leqslant C\left(\varepsilon\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega} + (-\log(\varepsilon))^{-\frac{1}{2}}\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega} + \|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega}\right), \\ \|v_\varepsilon\|_{1,\Omega_\varepsilon} &\leqslant C\|\varphi(\varepsilon\mathbf{x})\|_{\frac{1}{2},\partial\omega}, \end{aligned}$$

with C independent of ε. □

A.2 Proof of Theorem 2.1

Theorem 2.1. *For $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$ such that $\omega_\varepsilon \cap \text{supp} f = \varnothing$, there exist constants $C > 0$ and $\rho > 0$ such that, for all $0 < \varepsilon < \rho$, the solution u_ε of the problem (1.1) satisfies*

$$\|u_\varepsilon - u_0\|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of ε.

Proof. We first define $e^0_\varepsilon = u_\varepsilon - u_0$. It is solution of

$$\begin{cases} -\Delta e^0_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ e^0_\varepsilon = -u_0 & \text{on } \partial\omega_\varepsilon, \\ e^0_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The function $u_0(\varepsilon\mathbf{x})$ belongs to $H^{\frac{1}{2}}(\partial\omega)$, so if we set $\mathbf{x} = (\cos(\theta), \sin(\theta))$, we can write its Fourier series decomposition on $\partial\omega$ as follows:

$$u_0(\varepsilon\mathbf{x}) = a_{\varepsilon,0} + \sum_{n=1}^{\infty}\left(a_{\varepsilon,n}\cos(n\theta) + b_{\varepsilon,n}\sin(n\theta)\right),$$

with $a_{\varepsilon,0} = \mathcal{A}_\varepsilon^0 u_0$ and for all $n \geqslant 1$, $a_{\varepsilon,n} = \mathcal{A}_\varepsilon^n u_0, b_{\varepsilon,n} = \mathcal{B}_\varepsilon^n u_0$ where $\mathcal{A}_\varepsilon^n$ and $\mathcal{B}_\varepsilon^n$ are defined in (2.11). Then by linearity, we can decompose e_ε^0 into $e_\varepsilon^0 = \overline{e}_\varepsilon^0 + \underline{e}_\varepsilon^0$ where $\overline{e}_\varepsilon^0$ and $\underline{e}_\varepsilon^0$ are respectively solution of

$$\begin{cases} -\Delta \underline{e}_\varepsilon^0 = 0 & \text{in } \Omega_\varepsilon, \\ \underline{e}_\varepsilon^0 = -a_{\varepsilon,0} & \text{on } \partial\omega_\varepsilon, \\ \underline{e}_\varepsilon^0 = 0 & \text{on } \partial\Omega. \end{cases} \text{ and } \begin{cases} -\Delta \overline{e}_\varepsilon^0 = 0 & \text{in } \Omega_\varepsilon, \\ \overline{e}_\varepsilon^0 = a_{\varepsilon,0} - u_0 & \text{on } \partial\omega_\varepsilon, \\ \overline{e}_\varepsilon^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

According to **Lemma 2.2**, there exists a constant $C > 0$ such that, for $\varepsilon > 0$ sufficiently small,

$$\|\underline{e}_\varepsilon^0\|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}} |a_{\varepsilon,0}|.$$

According to **Lemma 2.3**, there exists a constant $C > 0$ such that, for $\varepsilon > 0$ sufficiently small,

$$\|\overline{e}_\varepsilon^0\|_{1,\Omega_\varepsilon} \leqslant C \|u_0(\varepsilon \mathbf{x}) - a_{\varepsilon,0}\|_{\frac{1}{2},\partial\omega} \leqslant C \left(\sum_{n=1}^{\infty} (1+n)((a_{\varepsilon,n})^2 + (b_{\varepsilon,n})^2) \right)^{\frac{1}{2}}.$$

Moreover, following the proof of **Lemma 3.3** which does not interact with this proof, we get the existence of constants $C > 0$, $\rho > 0$ and $\Upsilon > 0$ such that for all $0 < \varepsilon < \rho < \Upsilon$ and $n \geqslant 1$,

$$|a_{\varepsilon,0}| \leqslant C \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right) \text{ and } |a_{\varepsilon,n}| + |b_{\varepsilon,n}| \leqslant C \left(\frac{\varepsilon}{\Upsilon} \right)^n \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right).$$

with C independent of n, N and ε. We deduce that

$$\|\underline{e}_\varepsilon^0\|_{1,\Omega_\varepsilon} \leqslant C(-\log(\varepsilon))^{-\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

and

$$\|\overline{e}_\varepsilon^0\|_{1,\Omega_\varepsilon} \leqslant C\varepsilon \left(\sum_{n=1}^{\infty} (1+n) \left(\frac{\rho}{\Upsilon} \right)^{2(n-1)} \right)^{\frac{1}{2}} \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right) \leqslant C\varepsilon \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega} \right),$$

with C independent of ε. □

A.3 Proof of Theorem 3.4

Theorem 3.4. *Let Q_1 and Q_2 be two reflexive Banach spaces, $a : X \times X \to \mathbb{R}$, $b_1 : Q_1 \times X \to \mathbb{R}$, $b_2 : Q_2 \times X \to \mathbb{R}$ three bilinear forms, $d_1 : Q_1 \to \mathbb{R}$, $d_2 : Q_2 \to \mathbb{R}$ two linear forms, we consider the twofold saddle point problem: find $(u, \lambda_1, \lambda_2) \in X \times Q_1 \times Q_2$ such that*

$$\begin{cases} a(u,v) + b_1(\lambda_1, v) + b_2(\lambda_2, v) = c(v), & \forall v \in X, \\ b_1(\mu_1, u) = d_1(\mu_1), & \forall \mu_1 \in Q_1, \\ b_2(\mu_2, u) = d_2(\mu_2), & \forall \mu_2 \in Q_2. \end{cases} \tag{A.9}$$

Let

$$Z_{b_i} := \{v \in X \mid b_i(\mu_i, v) = 0 \; \forall \mu_i \in Q_i\} \subset X \; i = 1,2. \tag{A.10}$$

We suppose that conditions (3.2) *are satisfied with $Q = Q_1 \times Q_2$ and*

$$b : (Q_1 \times Q_2) \times X \to \mathbb{R} \quad b([\lambda_1, \lambda_2], u) = b_1(\lambda_1, u) + b_2(\lambda_2, u).$$

We also suppose that there exists $\beta_1 > 0$ such that for all $\lambda_1 \in Q_1$,

$$\sup_{v \in Z_{b_2}} \frac{b_1(\lambda_1, v)}{\|v\|_X} \geqslant \beta_1 \|\lambda_1\|_{Q_1}, \tag{A.11}$$

and that there exists $\beta_2 > 0$ such that for all $\lambda_2 \in Q_2$,

$$\sup_{v \in Z_{b_1}} \frac{b_2(\lambda_2, v)}{\|v\|_X} \geqslant \beta_2 \|\lambda_2\|_{Q_2}. \tag{A.12}$$

Then we have the following estimates on u, λ_1 and λ_2:

$$\|u\|_X \leqslant \alpha^{-1}\|c\| + \beta_1^{-1}(1 + \alpha^{-1}\|a\|)\|d_1\| + \beta_2^{-1}(1 + \alpha^{-1}\|a\|)\|d_2\|, \tag{A.13}$$

and

$$\|\lambda_1\|_{Q_1} \leqslant \beta_1^{-1}(\|c\| + \|a\|\|u\|_X), \|\lambda_2\|_{Q_2} \leqslant \beta_2^{-1}(\|c\| + \|a\|\|u\|_X).$$

Proof. We begin by noticing that condition (A.11) implies that there exists $w_1 \in Z_{b_2}$ such that

$$b_1(\mu_1, w_1) = d_1(\mu_1),\ \forall \mu_1 \in Q_1,$$

and

$$\|w_1\|_X \leqslant \beta_1^{-1}\|d_1\|.$$

Similarly, we deduce from (A.12) that there exists $w_2 \in Z_{b_1}$ such that

$$b_2(\mu_2, w_2) = d_2(\mu_2),\ \forall \mu_2 \in Q_2,$$

and

$$\|w_2\|_X \leqslant \beta_2^{-1}\|d_2\|.$$

Setting now

$$\kappa = u - w \tag{A.14}$$

with $w = w_1 + w_2$, we have for all $\mu_1 \in Q_1$, $\mu_2 \in Q_2$,

$$\begin{aligned} b_1(\mu_1, \kappa) &= b_1(\mu_1, u) - b_1(\mu_1, w_1) - b_1(\mu_1, w_2) = d_1(\mu_1) - d_1(\mu_1) = 0, \\ b_2(\mu_2, \kappa) &= b_2(\mu_2, u) - b_2(\mu_2, w_1) - b_2(\mu_2, w_2) = d_2(\mu_2) - d_2(\mu_2) = 0. \end{aligned}$$

We then deduce that $\kappa \in Z_{b_1} \cap Z_{b_2}$. Besides we have for all $v \in Z_{b_1} \cap Z_{b_2}$,

$$a(\kappa, v) = a(u, v) - a(w, v) = c(v) - a(w, v).$$

The continuity assumption on a and c, as well as the coercivity assumption on a, imply

$$\alpha\|\kappa\|_X \leqslant \sup_{v \in Z_{b_1} \cap Z_{b_2}, v \neq 0} \frac{a(\kappa, v)}{\|v\|_X} = \sup_{v \in Z_{b_1} \cap Z_{b_2}, v \neq 0} \frac{c(v) - a(w, v)}{\|v\|_X} \leqslant \|c\| + \|a\|\|w\|_X.$$

We get

$$\|\kappa\|_X \leqslant \alpha^{-1}(\|c\| + \|a\|\|w\|_X).$$

Applying triangular inequality to equation (A.14), we eventually obtain

$$\|u\|_X \leqslant \|\kappa\|_X + \|w\|_X \leqslant \alpha^{-1}\|c\| + \beta_1^{-1}(\alpha^{-1}\|a\| + 1)\|d_1\| + \beta_2^{-1}(\alpha^{-1}\|a\| + 1)\|d_2\|,$$

which corresponds to (A.13). Taking $v \in Z_{b_1}$ in the first equation of system (A.9), we have

$$b_2(\lambda_2, v) = c(v) - a(u, v).$$

By continuity of a and c, and the inf-sup condition (A.10) on b_2, we get

$$\beta_2\|\lambda_2\|_{Q_2} \leqslant \sup_{v \in Z_{b_1}, v \neq 0} \frac{b_2(\lambda_2, v)}{\|v\|_X} = \sup_{v \in Z_{b_1}, v \neq 0} \frac{c(v) - a(u, v)}{\|v\|_X} \leqslant \|c\| + \|a\|\|u\|_X.$$

Thus, we eventually obtain

$$\|\lambda_2\|_{Q_2} \leqslant \beta_2^{-1}(\|c\| + \|a\|\|u\|_X),$$

and we have in a similar way

$$\|\lambda_1\|_{Q_1} \leqslant \beta_1^{-1}(\|c\| + \|a\|\|u\|_X).$$

This concludes the proof of the lemma. □

A.4 Proof of Theorem 3.8

Theorem 3.8. *Let e_ε^N solution of Problem (3.7), there exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,*

$$\|e_\varepsilon^N\|_{2,\Omega_\varepsilon} + \|e_\varepsilon^N\|_{2,\omega_\varepsilon} \leqslant C\varepsilon^{-1}\left(\|f\|_{0,\Omega} + \|\phi\|_{\frac{1}{2},\partial\Omega}\right),$$

with C independent of N and ε.

In order to prove **Theorem 3.8**, we will consider separately the regularity of e_ε^N in Ω_ε and in ω_ε, e_ε^N is solution of

$$\begin{cases} -\Delta e_\varepsilon^N = 0 & \text{in } \Omega_\varepsilon, \\ -\Delta e_\varepsilon^N = 0 & \text{in } \omega_\varepsilon, \\ \mathcal{T}^N_{\partial\omega_\varepsilon} e_\varepsilon^N = -\mathcal{T}^N_{\partial\omega_\varepsilon} u_0 & \text{on } \partial\omega_\varepsilon, \\ e_\varepsilon^N = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, e_ε^N is harmonic in ω_ε and Ω_ε and in particular $e_\varepsilon^N \in H^2(\Omega_\varepsilon)$ and $e_\varepsilon^N \in H^2(\omega_\varepsilon)$. The estimates on the H^2-norm of e_ε^N on Ω_ε and ω_ε will then depend on the value of u_0 on $\partial\omega_\varepsilon$ and in particular on f and ϕ. Let $0 < \varepsilon < \Upsilon_1 < \Upsilon_2$ such that $\omega_\varepsilon \subset \omega_{\Upsilon_1} \subset \omega_{\Upsilon_2} \subset \Omega$. We will consider separately the regularity of e_ε^N in the disk ω_ε, in the annulus $\omega_{\Upsilon_1}\backslash\overline{\omega_\varepsilon}$ and in the exterior domain Ω_{Υ_1}, the space ω_{Υ_2} being introduced only for the proof. We will then introduce two lemmas describing the behavior of e_ε^N on ω_{Υ_1} and Ω_{Υ_1}.

Lemma AA1. *There exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,*

$$\|e_\varepsilon^N\|_{2,\omega_\varepsilon} + \|e_\varepsilon^N\|_{2,\omega_{\Upsilon_1}\backslash\overline{\omega_\varepsilon}} \leqslant C\varepsilon^{-1}(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}),$$

with C independent of N and ε.

Proof. In the beginning of the proof, we proceed as in **Lemma 3.5**. Since e_ε^N is harmonic in ω_ε, we have for $0 < r < \varepsilon$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,

$$e_\varepsilon^N(r\mathbf{x}) = A_{\varepsilon,0} + \sum_{n=1}^{\infty} r^n(A_{\varepsilon,n}\cos(n\theta) + B_{\varepsilon,n}\sin(\theta)), \tag{A.15}$$

with for $1 \leqslant n \leqslant N$,

$$\begin{cases} A_{\varepsilon,0} &= -\mathcal{A}^0_\varepsilon u_0, \\ A_{\varepsilon,n} &= -\dfrac{1}{\varepsilon^n}\mathcal{A}^n_\varepsilon u_0, \\ B_{\varepsilon,n} &= -\dfrac{1}{\varepsilon^n}\mathcal{B}^n_\varepsilon u_0. \end{cases}$$

The solution being also harmonic in $\omega_{\Upsilon_1}\backslash\overline{\omega_\varepsilon}$, for $\varepsilon < r < \Upsilon_1$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$, we have

$$e_\varepsilon^N(r\mathbf{x}) = C_{\varepsilon,0} + D_{\varepsilon,0}\log(r) + \sum_{n=1}^{\infty}(C_{\varepsilon,n}r^n + D_{\varepsilon,n}r^{-n})\cos(n\theta) + (E_{\varepsilon,n}r^n + F_{\varepsilon,n}r^{-n})\sin(n\theta). \tag{A.16}$$

In the same way as we did in the proof of **Lemma 3.5**, we use the continuity of the Fourier coefficients of e_ε^N on $\partial\omega_\varepsilon$ and the gradient jump of ∇e_ε^N given in equation (3.20) to obtain for $1 \leqslant n \leqslant N$,

$$\begin{cases} C_{\varepsilon,0} + D_{\varepsilon,0}\log(\varepsilon) = -\mathcal{A}^0_\varepsilon u_0, \\ C_{\varepsilon,n}\varepsilon^n + D_{\varepsilon,n}\varepsilon^{-n} = -\mathcal{A}^n_\varepsilon u_0, \\ E_{\varepsilon,n}\varepsilon^n + F_{\varepsilon,n}\varepsilon^{-n} = -\mathcal{B}^n_\varepsilon u_0, \end{cases}$$

and for $n \geqslant N + 1$,

$$D_{\varepsilon,n} = F_{\varepsilon,n} = 0 \text{ and } \begin{cases} A_{\varepsilon,n} &= C_{\varepsilon,n}, \\ B_{\varepsilon,n} &= E_{\varepsilon,n}. \end{cases}$$

Applying for all $n \geqslant 0$ the operators $\mathcal{A}^n_{\Upsilon_1}$ and $\mathcal{B}^n_{\Upsilon_1}$ defined in (2.8) on equation (A.16), we get for $1 \leqslant n \leqslant N$,

$$\begin{cases} C_{\varepsilon,0} + D_{\varepsilon,0}\log(\Upsilon_1) = \mathcal{A}^0_{\Upsilon_1} e_\varepsilon^N, \\ C_{\varepsilon,n}\Upsilon_1^n + D_{\varepsilon,n}\Upsilon_1^{-n} = \mathcal{A}^n_{\Upsilon_1} e_\varepsilon^N, \\ E_{\varepsilon,n}\Upsilon_1^n + F_{\varepsilon,n}\Upsilon_1^{-n} = \mathcal{B}^n_{\Upsilon_1} e_\varepsilon^N, \end{cases}$$

and for $n \geqslant N + 1$,

$$\begin{cases} A_{\varepsilon,n} &= C_{\varepsilon,n} = \dfrac{1}{\Upsilon_1^n}\mathcal{A}^n_{\Upsilon_1} e_\varepsilon^N, \\ B_{\varepsilon,n} &= E_{\varepsilon,n} = \dfrac{1}{\Upsilon_1^n}\mathcal{B}^n_{\Upsilon_1} e_\varepsilon^N. \end{cases}$$

We deduce an expression of e_ε^N in ω_ε and in $\omega_{\Upsilon_1}\backslash\overline{\omega_\varepsilon}$. For $0<r<\varepsilon$ and $\theta\in[0,2\pi[$, if $\mathbf{x}=(\cos(\theta),\sin(\theta))\in\partial\omega$,

$$e_\varepsilon^N(r\mathbf{x}) = -\sum_{n=1}^{N}\left(\frac{r}{\varepsilon}\right)^n\left(\mathcal{A}_\varepsilon^0 u_0\cos(n\theta)+\mathcal{B}_\varepsilon^0 u_0\sin(n\theta)\right) \\ +\sum_{n=N+1}^{\infty}\left(\frac{r}{\Upsilon_1}\right)^n\left(\mathcal{A}_{\Upsilon_1}^n e_\varepsilon^N\cos(n\theta)+\mathcal{B}_{\Upsilon_1}^n e_\varepsilon^N\sin(n\theta)\right), \quad (A.17)$$

and for $\varepsilon<r<\Upsilon_1$ and $\theta\in[0,2\pi[$, if $\mathbf{x}=(\cos(\theta),\sin(\theta))\in\partial\omega$,

$$e_\varepsilon^N(r\mathbf{x}) = C_{\varepsilon,0}+D_{\varepsilon,0}\log(r)+\sum_{n=1}^{N}\left(C_{\varepsilon,n}r^n+E_{\varepsilon,n}r^{-n}\right)\cos(n\theta)+\left(D_{\varepsilon,n}r^n+F_{\varepsilon,n}r^{-n}\right)\sin(n\theta) \\ +\sum_{n=N+1}^{\infty} r^n(C_{\varepsilon,n}\cos(n\theta)+D_{\varepsilon,n}\sin(n\theta)), \quad (A.18)$$

with

$$\begin{cases} C_{\varepsilon,0} &= \dfrac{\log(\varepsilon)\mathcal{A}_{\Upsilon_1}^0 e_\varepsilon^N+\log(\Upsilon_1)\mathcal{A}_\varepsilon^0 u_0}{\log(\varepsilon)-\log(\Upsilon_1)}, \\ D_{\varepsilon,0} &= \dfrac{\mathcal{A}_{\Upsilon_1}^0 e_\varepsilon^N+\mathcal{A}_\varepsilon^0 u_0}{\log(\Upsilon_1)-\log(\varepsilon)}, \end{cases} \quad (A.19)$$

for $1\leqslant n\leqslant N$,

$$\begin{cases} C_{\varepsilon,n} &= \dfrac{\varepsilon^{-n}\mathcal{A}_{\Upsilon_1}^n e_\varepsilon^N+\Upsilon_1^{-n}\mathcal{A}_\varepsilon^n u_0}{\Upsilon_1^n\varepsilon^{-n}-\Upsilon_1^{-n}\varepsilon^n}, \\ D_{\varepsilon,n} &= \dfrac{\varepsilon^{-n}\mathcal{B}_{\Upsilon_1}^n e_\varepsilon^N+\Upsilon_1^{-n}\mathcal{B}_\varepsilon^n u_0}{\Upsilon_1^n\varepsilon^{-n}-\Upsilon_1^{-n}\varepsilon^n}, \\ E_{\varepsilon,n} &= \dfrac{\varepsilon^{n}\mathcal{A}_{\Upsilon_1}^n e_\varepsilon^N+\Upsilon_1^{n}\mathcal{A}_\varepsilon^n u_0}{\varepsilon^n\Upsilon_1^{-n}-\Upsilon_1^{n}\varepsilon^{-n}}, \\ F_{\varepsilon,n} &= \dfrac{\varepsilon^{n}\mathcal{B}_{\Upsilon_1}^n e_\varepsilon^N+\Upsilon_1^{n}\mathcal{B}_\varepsilon^n u_0}{\varepsilon^n\Upsilon_1^{-n}-\Upsilon_1^{n}\varepsilon^{-n}}, \end{cases} \quad (A.20)$$

and for $n\geqslant N+1$,

$$\begin{cases} C_{\varepsilon,n} &= \dfrac{1}{\Upsilon_1^n}\mathcal{A}_{\Upsilon_1}^n e_\varepsilon^N, \\ D_{\varepsilon,n} &= \dfrac{1}{\Upsilon_1^n}\mathcal{B}_{\Upsilon_1}^n e_\varepsilon^N. \end{cases} \quad (A.21)$$

Now we look more closely to regularity of e_ε^N on the domain ω_ε, where e_ε^N is given by (A.17). We see that we cannot compute the successive derivatives according to $\mathbf{e}_\theta$ and to $\mathbf{e}_r$ because there are not defined in 0, then we will write e_ε^N in Cartesian coordinates thanks to Chebyshev polynomials which write for $n\geqslant 1$,

$$\begin{cases} \cos(n\theta) &= \displaystyle\sum_{0\leqslant 2k\leqslant n}\binom{n}{2k}(-1)^k\cos^{n-2k}(\theta)\sin^{2k}(\theta), \\ \sin(n\theta) &= \displaystyle\sum_{0\leqslant 2k+1\leqslant n}\binom{n}{2k+1}(-1)^k\cos^{n-2k-1}(\theta)\sin^{2k+1}(\theta). \end{cases}$$

If we write (x,y) the Cartesian coordinates in the map centered in 0, for $\theta\in[0,2\pi[$, if $\mathbf{x}=(\cos(\theta),\sin(\theta))$, we have $r\mathbf{x}=(x,y)$ and

$$e_\varepsilon^N(x,y) = -\sum_{n=1}^{N}\left(\frac{1}{\varepsilon}\right)^n\left((\mathcal{A}_\varepsilon^n u_0)\sum_{0\leqslant 2k\leqslant n}\binom{n}{2k}(-1)^k x^{2k}y^{n-2k}+(\mathcal{B}_\varepsilon^n u_0)\sum_{0\leqslant 2k+1\leqslant n}\binom{n}{2k+1}(-1)^k x^{2k+1}y^{n-2k-1}\right) \\ +\sum_{n=N+1}^{\infty}\left(\frac{1}{\Upsilon_1}\right)^n\left((\mathcal{A}_{\Upsilon_1}^n e_\varepsilon^N)\sum_{0\leqslant 2k\leqslant n}\binom{n}{2k}(-1)^k x^{2k}y^{n-2k}+(\mathcal{B}_{\Upsilon_1}^n e_\varepsilon^N)\sum_{0\leqslant 2k+1\leqslant n}\binom{n}{2k+1}(-1)^k x^{2k+1}y^{n-2k-1}\right).$$

We have for $K_1, K_2 \in \mathbb{N}$ such that $K_1 + K_2 = 2$ and $n \geqslant 2$,

$$\begin{cases} \partial_x^{K_1}\partial_y^{K_2}(x^{2k}y^{n-2k}) = \dfrac{(2k)!}{(2k-K_1)!}\dfrac{(n-2k)!}{(n-2k-K_2)!}x^{2k-K_1}y^{n-2k-K_2}, & \forall\, 0 \leqslant 2k \leqslant n, \\ \partial_x^{K_1}\partial_y^{K_2}(x^{2k+1}y^{n-2k-1}) = \dfrac{(2k+1)!}{(2k+1-K_1)!}\dfrac{(n-2k-1)!}{(n-2k-1-K_2)!}x^{2k+1-K_1}y^{n-2k-1-K_2}, & \forall\, 0 \leqslant 2k+1 \leqslant n, \end{cases}$$

and for $n < 2$,

$$\partial_x^{K_1}\partial_y^{K_2}(x^{2k}y^{n-2k}) = \partial_x^{K_1}\partial_y^{K_2}(x^{2k+1}y^{n-2k-1}) = 0.$$

After computation we have for $n \geqslant 2$, for $\varepsilon > 0$ sufficiently small,

$$\begin{cases} \|x^{2k}y^{n-2k}\|_{2,\omega_\varepsilon} & \leqslant C(1+n)^{\frac{3}{2}}\varepsilon^{n-1}, \\ \|x^{2k+1}y^{n-2k-1}\|_{2,\omega_\varepsilon} & \leqslant C(1+n)^{\frac{3}{2}}\varepsilon^{n-1}, \end{cases}$$

with C independent of n and ε. Noticing that

$$\sum_{0\leqslant 2k\leqslant n}\binom{n}{2k} + \sum_{0\leqslant 2k+1\leqslant n}\binom{n}{2k+1} = 2^n,$$

we have for $\varepsilon > 0$ sufficiently small,

$$|e_\varepsilon^N|_{2,\omega_\varepsilon} \leqslant C\left(\sum_{n=2}^{N}(1+n)^{\frac{3}{2}}2^n\varepsilon^{-1}\left(|\mathcal{A}_\varepsilon^n u_0| + |\mathcal{B}_\varepsilon^n u_0|\right) + \sum_{n=N+1}^{\infty}(1+n)^{\frac{3}{2}}2^n\varepsilon^{-1}\left(\frac{\varepsilon}{\Upsilon_1}\right)^n\left(|\mathcal{A}_{\Upsilon_1}^n e_\varepsilon^N| + |\mathcal{B}_{\Upsilon_1}^n e_\varepsilon^N|\right)\right).$$

Moreover, taking $\Upsilon = \Upsilon_1$ in the proof of **Lemma 3.3**, we have, for $\varepsilon > 0$ sufficiently small,

$$|\mathcal{A}_\varepsilon^n u_0| + |\mathcal{B}_\varepsilon^n u_0| \leqslant C\left(\frac{\varepsilon}{\Upsilon_1}\right)^n\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \tag{A.22}$$

with C independent of n and ε. Using the same arguments as in the proof of **Lemma 3.3**, we can also prove that, for $\varepsilon > 0$ sufficiently small,

$$|\mathcal{A}_{\Upsilon_1}^n e_\varepsilon^N| + |\mathcal{B}_{\Upsilon_1}^n e_\varepsilon^N| \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \tag{A.23}$$

with C independent of n, N and ε. We then conclude that for $0 < \varepsilon < \frac{\rho}{2} < \Upsilon_1$,

$$|e_\varepsilon^N|_{2,\omega_\varepsilon} \leqslant C\varepsilon^{-1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right)\left(\sum_{n=2}^{\infty}\left(\frac{\rho}{\Upsilon_1}\right)^n(1+n)^{\frac{3}{2}}\right) \leqslant C\varepsilon^{-1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right).$$

with C independent of N and ε. Using the estimates we have on the H^1-norm of e_ε^N given by **Theorem 3.5**, we can conclude that, for $\varepsilon > 0$ sufficiently small,

$$\|e_\varepsilon^N\|_{2,\omega_\varepsilon} \leqslant C\varepsilon^{-1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of N and ε. We will now look at the regularity of e_ε^N in $\omega_{\Upsilon_1}\backslash\overline{\omega_\varepsilon}$ where e_ε^N is given by (A.18). If we set $\alpha_{\varepsilon,0} = C_{\varepsilon,0}$, $\beta_{\varepsilon,0} = D_{\varepsilon,0}$, for $1 \leqslant n \leqslant N$, $\alpha_{\varepsilon,n} = \varepsilon^n C_{\varepsilon,n}$, $\xi_{\varepsilon,n} = \varepsilon^{-n}E_{\varepsilon,n}$, $\beta_{\varepsilon,n} = \varepsilon^n D_{\varepsilon,n}$, $\zeta_{\varepsilon,n} = \varepsilon^{-n}F_{\varepsilon,n}$ and for $n \geqslant N+1$, $\alpha_{\varepsilon,n} = \Upsilon_1^n C_{\varepsilon,n}$, $\beta_{\varepsilon,n} = \Upsilon_1^n D_{\varepsilon,n}$, we have for $\varepsilon < r < \Upsilon_1$ and $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,

$$e_\varepsilon^N(r\mathbf{x}) = \alpha_{\varepsilon,0} + \beta_{\varepsilon,0}\log(r) + \sum_{n=1}^{N}\left(\alpha_{\varepsilon,n}\left(\frac{r}{\varepsilon}\right)^n + \xi_{\varepsilon,n}\left(\frac{\varepsilon}{r}\right)^n\right)\cos(n\theta) + \left(\beta_{\varepsilon,n}\left(\frac{r}{\varepsilon}\right)^n + \zeta_{\varepsilon,n}\left(\frac{\varepsilon}{r}\right)^n\right)\sin(n\theta),$$
$$+ \sum_{n=N+1}^{\infty}\left(\frac{r}{\Upsilon_1}\right)^n\left(\alpha_{\varepsilon,n}\cos(n\theta) + \beta_{\varepsilon,n}\sin(n\theta)\right).$$

We then deduce, using inequality (A.22) and (A.23) themselves derived from **Lemma 3.3**, and looking at the expression (A.19), (A.20) and (A.21), that for $\varepsilon > 0$ sufficiently small,

$$\begin{cases} |\alpha_{\varepsilon,0}| & \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \\ |\beta_{\varepsilon,0}| & \leqslant C(-\log(\varepsilon))^{-1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right), \end{cases} \tag{A.24}$$

for $1 \leqslant n \leqslant N$,

$$|\alpha_{\varepsilon,n}| + |\xi_{\varepsilon,n}| + |\beta_{\varepsilon,n}| + |\zeta_{\varepsilon,n}| \leqslant C \left(\frac{\varepsilon}{\Upsilon_1}\right)^n \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

and for $n \geqslant N+1$,

$$|\alpha_{\varepsilon,n}| + |\xi_{\varepsilon,n}| + |\beta_{\varepsilon,n}| + |\zeta_{\varepsilon,n}| \leqslant C \left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of n and ε. We see that the dependence of the H^2-norm of e_ε^N with respect to ε can be directly deduced from the limiting term f_0 defined for $\varepsilon < r < \Upsilon_1$ by

$$f_0(r) = \alpha_{\varepsilon,0} + \beta_{\varepsilon,0}\log(r).$$

For $K \in \{1,2\}$ and $\varepsilon < r < \Upsilon_1$,

$$f_0^{(K)}(r) = \beta_{\varepsilon,0}(-1)^{K-1}\frac{(K-1)!}{r^K},$$

where $f_0^{(K)}$ is the K-th derivative of f_0. The L^2-norm of $f_0^{(K)}$ in $\omega_{\Upsilon_1}\backslash\overline{\omega_\varepsilon}$ writes

$$\|f_0^{(K)}\|^2_{0,\omega_{\Upsilon_1}\backslash\overline{\omega_\varepsilon}} = 2\pi(\beta_{\varepsilon,0})^2((K-1)!)^2\int_\varepsilon^{\Upsilon_1} r^{-2K+1}\mathrm{d}r = \pi(\beta_{\varepsilon,0})^2((K-1)!)((K-2)!)\left(\varepsilon^{-2(K-1)} - \Upsilon_1^{-2(K-1)}\right).$$

It follows from equations (A.24) that for $\varepsilon > 0$ sufficiently small,

$$\|f_0^{(K)}\|_{0,\omega_\Upsilon\backslash\overline{\omega_\varepsilon}} \leqslant C\varepsilon^{-(K-1)}\beta_{\varepsilon,0}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right) \leqslant C\varepsilon^{-1}(-\log(\varepsilon))^{-1}\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of ε. □

Let us now look at the domain Ω_{Υ_1} on which we have the following result.

Lemma AA2. *There exist constants $C > 0$ and $\rho > 0$ such that for all $0 < \varepsilon < \rho$,*

$$\|e_\varepsilon^N\|_{2,\Omega_{\Upsilon_1}} \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of N and ε.

Proof. Let us first note that by elliptic regularity, we have

$$\|e_\varepsilon^N\|_{2,\Omega_{\Upsilon_1}} \leqslant C\|e_\varepsilon^N\|_{\frac{3}{2},\partial\omega_{\Upsilon_1}}, \tag{A.25}$$

with C independent of N and ε. Proceeding as in **Lemma AA1**, as e_ε^N is harmonic in $\omega_{\Upsilon_2}\backslash\overline{\omega_{\Upsilon_1}}$, we have for $\theta \in [0, 2\pi[$, if $\mathbf{x} = (\cos(\theta), \sin(\theta)) \in \partial\omega$,

$$e_\varepsilon^N(\Upsilon_1\mathbf{x}) = \alpha_{0,\varepsilon} + \sum_{n=1}^{\infty}\left(\frac{\Upsilon_1}{\Upsilon_2}\right)^n(\alpha_{\varepsilon,n}\cos(n\theta) + \beta_{\varepsilon,n}\sin(n\theta)),$$

with for $n \geqslant 1$,

$$\begin{cases}\alpha_{\varepsilon,n} &= \mathcal{A}^n_{\Upsilon_2}e_\varepsilon^N,\\ \beta_{\varepsilon,n} &= \mathcal{B}^n_{\Upsilon_2}e_\varepsilon^N,\end{cases}$$

and for $\varepsilon > 0$ sufficiently small,

$$|\mathcal{A}^n_{\Upsilon_2}e_\varepsilon^N| + |\mathcal{B}^n_{\Upsilon_2}e_\varepsilon^N| \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of n, N and ε. We then deduce

$$\|e_\varepsilon^N\|_{\frac{3}{2},\partial\omega_{\Upsilon_1}} \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right)\left(\sum_{n=0}^{\infty}\left(\frac{\Upsilon_1}{\Upsilon_2}\right)^{2n}(1+n)^3\right)^{\frac{1}{2}},$$

$$\|e_\varepsilon^N\|_{\frac{3}{2},\partial\omega_{\Upsilon_1}} \leqslant C\left(\|\phi\|_{\frac{1}{2},\partial\Omega} + \|f\|_{0,\Omega}\right),$$

with C independent of N and ε. We can conclude on the proof of the lemma using estimate (A.25). □

Gathering results of **Lemma AA1** and **Lemma AA2**, we can conclude on the proof of the theorem.

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