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ANALYSIS AND APPROXIMATION OF MIXED-DIMENSIONAL PDES ON 3D-1D DOMAINS COUPLED WITH LAGRANGE MULTIPLIERS

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Abstract. Coupled partial differential equations defined on domains with different dimensionality are usually called mixed dimensional PDEs. We address mixed dimensional PDEs on three-dimensional (3D) and one-dimensional domains, giving rise to a 3D-1D coupled problem. Such problem poses several challenges from the standpoint of existence of solutions and numerical approximation. For the coupling conditions across dimensions, we consider the combination of essential and natural conditions, basically the combination of Dirichlet and Neumann conditions. To ensure a meaningful formulation of such conditions, we use the Lagrange multiplier method, suitably adapted to the mixed dimensional case. The well posedness of the resulting saddle point problem is analyzed. Then, we address the numerical approximation of the problem in the framework of the finite element method. The discretization of the Lagrange multiplier space is the main challenge. Several options are proposed, analyzed and compared, with the purpose to determine a good balance between the mathematical properties of the discrete problem and flexibility of implementation of the numerical scheme. The results are supported by evidence based on numerical experiments.

Key words. mixed dimensional PDEs, finite element approximation, essential coupling conditions, Lagrange multipliers

AMS subject classifications. n.a.

1. Introduction.

In this study we consider coupled partial differential equations on domains with mixed dimensionality, in particular we address the 3D-1D case. The mathematical structure of such problems can be represented by the following formal equations:

$$-\Delta u + u + \lambda\delta_\Lambda = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$d_s^2 u_\odot + u_\odot - \lambda = g \qquad \text{on } \Lambda, \tag{1.1b}$$

$$\mathcal{T}_\Lambda u - u_\odot = q \qquad \text{on } \Lambda. \tag{1.1c}$$

Problem (1.1) can be described as an example of *mixed dimensional PDEs*. Here, u, $u_\odot$, λ are unknowns, Ω is a bounded domain in $\mathbb{R}^3$, whereas $\Lambda \subset \Omega$ is a 1D manifold parametrized in terms of s and d_s is the derivative with respect to s. The term $\lambda\delta_\Lambda$ is a Dirac measure such that $\int_\Omega \lambda(x)\delta_\Lambda v(x)\,\mathrm{d}x = \int_\Lambda \lambda(t)v(t)\,\mathrm{d}t$ for a continuous function v and $\mathcal{T}_\Lambda : \Omega \to \Lambda$ is a suitable restriction operator from 3D to 1D. We remark that λ can be viewed as a Lagrange multiplier associated with the coupling constraint (1.1c), see Appendix §A for a precise definition.

Using models based on mixed dimensional PDEs is motivated by the fact that many problems in geo- and biophysics are characterized by slender cylindrical structures coupled to a larger 3D body, where the characteristic transverse length scale of the slender structure is many orders of magnitude smaller than the longitudinal length. For example, in geophysical applications the radii of wells are often of the order of 10 cm while the length may be several kilometers [28, 29]. Similarly, in applications involving the blood flow and oxygen transport of the micro-circulation the capillary

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radius is a few microns, while simulations are often performed on mm to cm scale, with thousands of vessels [4, 15, 18, 33]. Finally, in neuro-science applications a neuron has width of a few microns, while its length is much longer. For example, an axon of a motor neurons may be as long as a meter. Hence, at least 4 orders of magnitude in difference in transverse and longitudinal direction is common in both geo-physics, bio-mechanics and neuro-science. Meshes dictated by resolving the transverse length scale in 3D would then possibly lead to the order of 10^{12} degrees of freedom. Even if adaptive and strongly anisotropic meshes are allowed for, the computations quickly become demanding if many slender structures and their interactions are under study.

From a mathematical standpoint, the challenge involved in problem (1.1) is that neither $\mathcal{T}_\Lambda$ nor δ_Λ are well defined. That is, without extra regularity, solutions of elliptic PDEs only have well defined traces of co-dimension one. Here, $\mathcal{T}_\Lambda$ is of co-dimension two, mapping functions defined on a domain in 3D to functions defined along a 1D curve. The challenge of coupling PDEs on domains with high dimensionality gap has recently attracted the attention of many researchers. The sequence of works by D'Angelo, [11, 12, 13] have remedied the well-posedness by weakening the solution concept. The approach naturally leads to non-symmetric formulations. An alternative approach is to decompose the solution into smooth and non-smooth components, where the non-smooth component may be represented in terms of Green's functions, and then consider the well-posedness of the smooth component [17]. The numerical approximation of such equations has been also studied in a series of works. The consistent derivation of numerical approximation schemes for PDEs in mixed dimension is addressed in [6]. Concerning approximability, elliptic equations with Dirac sources represent an effective prototype case that has been addressed in [5, 19, 21], where the optimal a-priori error estimates for the finite element approximation are derived. Furthermore, the interplay between the mathematical structure of the problem and solvers, as well as preconditioners for its discretization has been studied in details in [23] for the solution of 1D differential equations embedded in 2D, and more recently extended to the 3D-1D case in [22].

Stemming from this literature, in this work we adopt and analyze a different approach, closely related to [20, 24]. That is, we exploit the fact that Λ is not strictly a 1D curve, but rather a very thin 3D structure with a cross-sectional area far below from what can be resolved. With this additional assumption, we show that robustness with respect to the cross-sectional area can be restored. The major novelty of this work is that we address essential type coupling conditions, namely Dirichlet-Neumann conditions, see in particular problem (A.1) in the Appendix. In previous works, see for example [13, 20, 24], natural type coupling conditions of Robin-Robin type were analyzed. Dirichlet-type coupling conditions pose additional difficulties as the conditions are not a natural part of the weak formulation of the problem. As shown in Appendix A, we overcome this difficulty by resorting to a weak formulation of the Dirichlet-Neumann coupling conditions across dimensions by using Lagrange multipliers.

Although the focus of the present work is mostly on the analysis and approximation of the proposed approach, we stress that it aims to build the mathematical foundations to tackle various applications involving 3D-1D mixed dimensional PDEs, such as FSI of slender bodies [27], microcirculation and lymphatics [30, 34], subsurface flow models with wells [9] and the electrical activity of neurons.

2. Preliminaries.

Let the domain $\Omega \subset \mathbb{R}^3$ be an open, connected and convex set that can be subdivided in two parts, $\Omega_\ominus$ and $\Omega_\oplus := \Omega \setminus \overline{\Omega}_\ominus$. Let $\Omega_\ominus$ be a *generalized*

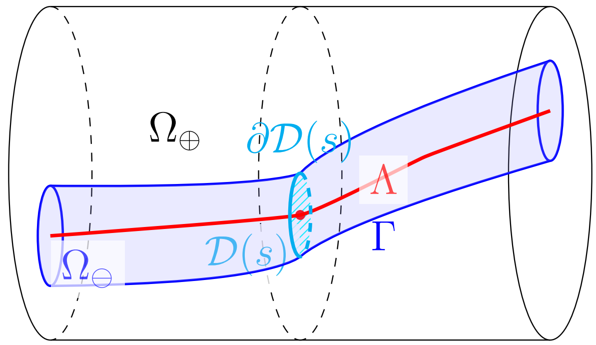

FIGURE 2.1. *Geometrical setting of the problem*

cylinder, c.f. [16], that is; the swept volume of a two dimensional set, $\partial\mathcal{D}$, moved along a curve, Λ, in the three-dimensional domain, Ω, see for Figure 2.1 for an illustration. More precisely, the curve $\Lambda = \{\boldsymbol{\lambda}(s),\ s \in (0,S)\}$, where $\boldsymbol{\lambda}(s) = [\xi(s), \tau(s), \zeta(s)],\ s \in (0,S)$ is a $\mathcal{C}^2$-regular curve in the three-dimensional domain Ω. For simplicity, let us assume that $\|\boldsymbol{\lambda}'(s)\| = 1$ such that the arc-length and the coordinate s coincide. Further, let $\mathcal{D}(s) = [x(r,t), y(r,t)] : (0, R(s)) \times (0, T(s)) \to \mathbb{R}^2$ be a parametrization of the cross section with $R(s) \geq R_0 > 0$ being R_0 the minimum cross sectional radius of the generalized cylinder and Γ be the lateral surface of $\Omega_\ominus$, i.e. $\Gamma = \{\ \partial\mathcal{D}(s) \mid s \in \Lambda\}$, while the upper and lower faces of $\Omega_\ominus$ belong to $\partial\Omega$. We assume that $\Omega_\ominus$ crosses Ω from side to side. Finally, $|\cdot|$ denotes the Lebesgue measure of a set, e.g. $|\mathcal{D}(s)|$ is the cross-sectional area of the cylinder. In general, $|\mathcal{D}(s)|$ must be strictly positive and bounded. According to the geometrical setting, we will denote with v, $v_\oplus$, $v_\ominus$, $v_\odot$, functions defined on Ω, $\Omega_\oplus$, $\Omega_\ominus$, Λ, respectively.

Let D be a generic regular bounded domain in $\mathbb{R}^3$ and X be a Hilbert space defined on D. Then $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ denote the inner product and norm of X, respectively. The duality pairing between the X and its dual X^* is denoted as $\langle\cdot,\cdot\rangle$. Let $(\cdot,\cdot)_{L^2(D)}$, $(\cdot,\cdot)_D$ or simply $(\cdot,\cdot)$ be the $L^2(D)$ inner product on D. We use the standard notation $H^q(D)$ to denote the Sobolev space of functions on D with all derivatives up to the order q in $L^2(D)$. The corresponding norm is $\|\cdot\|_{H^q(D)}$ and the seminorm is $|\cdot|_{H^q(D)}$. The space $H^q_0(D)$ represents the closure in $H^q(D)$ of smooth functions with compact support in D.

Let Σ be a Lipschitz co-dimension one subset of D. We denote with $\mathcal{T}_\Sigma : H^q(D) \to H^{q-\frac{1}{2}}(\Sigma)$ the trace operator from D to Σ. The space of functions in $H^{\frac{1}{2}}(\Sigma)$ with continuous extension by zero outside Σ is denoted $H^{\frac{1}{2}}_{00}(\Sigma)$ and we remark that $H^{\frac{1}{2}}_{00}(\Sigma) = \mathcal{T}_\Sigma H^1_0(D)$ and $H^{-\frac{1}{2}}(\Sigma) = (H^{\frac{1}{2}}_{00}(\Sigma))^*$

We will frequently use inner products and norms that are weighted. The L_2 and H^1 inner products weighted by a scalar function w, which is strictly positive and bounded almost everywhere, are defined as follows

$$(u,v)_{L^2(\Sigma),w} = \int_\Sigma w\, u\, v d\omega \quad \text{and} \quad (u,v)_{H^1(\Sigma),w} = \int_\Sigma w\, u\, v d\omega + \int_\Sigma w\, \nabla u \cdot \nabla v d\omega$$

whereas a weighted fractional space $H^s_{00}(\Sigma; w)$ is defined in terms of the interpolation of the corresponding weighted spaces (see [25, ch. 2.1] and also [2, 10]). More precisely we have $H^s_{00}(\Gamma; w) = [H^1_0(\Sigma; w), L^2(\Sigma; w)]_s$, with $s \in [0,1]$ using the notation of [2]. For the norm of such spaces, we introduce the Riesz map S such that for $u, v \in H^1_0(\Sigma)$

we have

$$\int_\Sigma w\nabla\left(Su\right)\cdot\nabla v d\omega = (u,v)_{L^2(\Sigma),w}.$$

Then $S=-\Delta^{-1}$ is a compact self-adjoint operator. Assuming that $\{\lambda_k\}_k$ is the set of eigenvalues, $\{\phi_k\}_k$ the set of eigenvectors of S orthonormal with respect to the inner product $(\cdot,\cdot)_{L^2(\Sigma),w}$ and $u\in H_0^1(\Sigma)$ can be expressed as $u=\sum_k c_k\phi_k$, then

$$\|u\|^2_{H^s_{00}(\Sigma),w} = \sum_k \lambda_k^{-s}c_k^2. \tag{2.1}$$

Owing to the positivity and boundedness of w, the weighted spaces equal the corresponding non-weighted spaces as sets, but their norms are different.

Central in our analysis are the transverse averages $\overline{w}$, $\overline{\overline{w}}$ defined as,

$$\overline{w}(s)=|\partial\mathcal{D}(s)|^{-1}\int_{\partial\mathcal{D}(s)} wd\gamma \quad\text{and}\quad \overline{\overline{w}}(s)=|\mathcal{D}(s)|^{-1}\int_{\mathcal{D}(s)} wd\sigma,$$

where $d\omega,\ d\sigma,\ d\gamma$ are the generic volume, surface and curvilinear Lebesgue measures. Clearly,

$$\int_{\Omega_\ominus} wd\omega = \int_\Lambda\int_{\mathcal{D}(s)} wd\sigma ds = \int_\Lambda |\mathcal{D}(s)|\overline{\overline{w}}(s)ds$$
$$\int_{\partial\Omega_\ominus} wd\sigma = \int_\Lambda\int_{\partial\mathcal{D}(s)} wd\gamma ds = \int_\Lambda |\partial\mathcal{D}(s)|\overline{w}(s)ds\,.$$

Analogously, for functions defined on Λ and $\Omega_\ominus$ respectively, we let d_s and ∂_s be the ordinary and partial derivative with respect to the arclength.

The operator obtained from a combination of the average operator $\overline{(\cdot)}$ with the trace on Γ will be denoted with $\overline{\mathcal{T}}_\Lambda=\overline{(\cdot)}\circ\mathcal{T}_\Gamma$, as it maps functions on Ω to functions on Λ. Further, let the extension operator $\mathcal{E}_\Gamma: H^{\frac12}_{00}(\Lambda)\to H^{\frac12}_{00}(\Gamma)$ be defined such that $(\mathcal{E}_\Gamma v_\odot)(x)=v_\odot(s)$, for any $x\in\partial\mathcal{D}(s)$. Then, the following identity shows that the transversal uniform extension operator is the inverse of the transversal average,

$$\langle\overline{\mathcal{T}}_\Lambda u, v_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} = \int_\Lambda |\partial\mathcal{D}|\left(\frac{1}{|\partial\mathcal{D}|}\int_{\partial\mathcal{D}}\mathcal{T}_\Gamma u\, d\gamma\right) v_\odot\, ds = \langle\mathcal{T}_\Gamma u, \mathcal{E}_\Gamma v_\odot\rangle_\Gamma\,. \tag{2.2}$$

With the above notation we are now able to formulate the **Problem 3D-1D-1D**. The problem reads: given $f\in L^2(\Omega),\ g\in L^2(\Omega_\ominus),\ q\in H^{\frac12}_{00}(\Gamma)$ find $u\in H^1_0(\Omega),\ u_\odot\in H^1_0(\Lambda),\ \lambda_\odot\in H^{-\frac12}(\Lambda)$, such that

$$(u,v)_{H^1(\Omega)} + \langle\overline{\mathcal{T}}_\Lambda v,\lambda_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} = (f,v)_{L^2(\Omega)} \qquad \forall v\in H^1_0(\Omega)\,, \tag{2.3a}$$
$$(u_\odot,v_\odot)_{H^1(\Lambda),|\mathcal{D}|} - \langle v_\odot,\lambda_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} = (\overline{\overline{g}},v_\odot)_{L^2(\Lambda),|\mathcal{D}|} \qquad \forall v_\odot\in H^1_0(\Lambda)\,, \tag{2.3b}$$
$$\langle\overline{\mathcal{T}}_\Lambda u - u_\odot,\mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} = \langle\overline{q},\mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \qquad \forall\mu_\odot\in H^{-\frac12}(\Lambda)\,. \tag{2.3c}$$

In addition to the 3D-1D-1D problem we will also consider an intermediate problem where the 3D and 1D problems are coupled at an intermediate 2D surface encapsulating the 1D structure. This is referred to as the **Problem 3D-1D-2D** and it reads: given $f\in L^2(\Omega),\ g\in L^2(\Omega_\ominus),\ q\in H^{\frac12}_{00}(\Gamma)$ find $u\in H^1_0(\Omega),\ u_\odot\in H^1_0(\Lambda),\ \lambda\in H^{-\frac12}(\Gamma)$ such that

$$(u,v)_{H^1(\Omega)} + \langle\mathcal{T}_\Gamma v,\lambda\rangle_\Gamma = (f,v)_{L^2(\Omega)} \qquad \forall v\in H^1_0(\Omega)\,, \tag{2.4a}$$
$$(u_\odot,v_\odot)_{H^1(\Lambda),|\mathcal{D}|} - \langle\mathcal{E}_\Gamma v_\odot,\lambda\rangle_\Gamma = (\overline{\overline{g}},v_\odot)_{L^2(\Lambda),|\mathcal{D}|} \qquad \forall v_\odot\in H^1(\Lambda)\,, \tag{2.4b}$$
$$\langle\mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot,\mu\rangle_\Gamma = \langle q,\mu_\odot\rangle_\Gamma \qquad \forall\mu\in H^{-\frac12}(\Gamma)\,. \tag{2.4c}$$

We conclude this section with the analysis of a fundamental property for the problem formulation that we will address, namely, the characterization of the regularity of the operator $\overline{\mathcal{T}}_\Lambda$. More precisely we aim to show that $\overline{\mathcal{T}}_\Lambda : H_0^1(\Omega) \to H_{00}^{\frac{1}{2}}(\Lambda)$. This is a consequence of the following lemma.

Lemma 2.1. *Let Γ be a tensor product domain, $\Gamma = (0, X) \times (0, Y)$. For any regular $u(x, y)$ in Γ, let $\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x, y)\, dy$. Then, for any $u \in H_{00}^{\frac{1}{2}}(\Gamma)$, $\overline{u}(x) \in H_{00}^{\frac{1}{2}}((0, X))$. Moreover, if $u(x, y) \in H_{00}^{\frac{1}{2}}(\Gamma)$ is constant with respect to y, namely $u(x, y) = u(x)$, then*

$$\|u\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = Y \|u\|_{H_{00}^{\frac{1}{2}}(0,X)}.$$

The proof of Lemma 2.1 is based on the representation of fractional norms in terms of the spectrum of the Laplace operator and subsequent standard arguments in harmonic analysis. The full proof is reported in the appendix for the sake of clarity.

Under the geometric assumptions stated above for Ω, Γ, Λ, Lemma 2.1 implies the following result.

Corollary 2.2 (of Lemma 2.1). *If $u \in H_{00}^{\frac{1}{2}}(\Gamma)$ then $\overline{u} \in H_{00}^{\frac{1}{2}}(\Lambda)$ and there exists a constant C_Γ, bounded independently of $\mathcal{D}$ and $\partial\mathcal{D}$, such that*

$$\|\overline{u}\|_{H_{00}^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \leq C_\Gamma \|u\|_{H_{00}^{\frac{1}{2}}(\Gamma)}.$$

Proof. Being Γ the surface of a generalized cylinder it can be parametrized as a tensor product domain using a local coordinate system such as the Frenet frame. Then, Lemma 2.1 can be applied. The inequality above follows from inequality (B.3) in Appendix B. ∎

Furthermore, from the above Corollary, it is clear that $\overline{\mathcal{T}}_\Lambda : H_0^1(\Omega) \to H_{00}^{\frac{1}{2}}(\Lambda)$.

3. Saddle-point problem analysis.

Let $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bilinear forms. Let us consider a general saddle point problem of the form: find $u \in X$, $\lambda \in Q$ s.t.

$$\begin{aligned} a(u, v) + b(v, \lambda) &= c(v), \quad \forall v \in X, \\ b(u, \mu) &= d(\mu), \quad \forall \mu \in Q. \end{aligned} \tag{3.1}$$

The Brezzi conditions [7] ensure that the problem (3.1) is well-posed. For our purpose here, we use the following particular version of the Brezzi conditions:

Theorem 3.1. *Let $a(\cdot, \cdot) : X \times X \to \mathbb{R}$ and $b(\cdot, \cdot) : X \times Q \to \mathbb{R}$ be bounded bilinear forms satisfying the following properties:*

$$a(u, u) \geq \alpha \|u\|_X^2, \qquad u \in X, \tag{3.2}$$

$$a(u, v) \leq C_a \|u\|_X \|v\|_X, \qquad u, v \in X, \tag{3.3}$$

$$b(u, \mu) \leq C_b \|u\|_X \|\mu\|_Q, \qquad u \in X, \mu \in Q, \tag{3.4}$$

$$\sup_{v \in X} \frac{b(v, \mu)}{\|v\|_X} \geq \beta \|\mu\|_Q, \qquad \mu \in Q \tag{3.5}$$

with positive constants α, β, C_a, C_b. Then, there exists unique $u \in X$, $\lambda \in Q$, solution *of problem* (3.1) *and the following a priori estimates hold:*

$$\|u\|_X \le \frac{1}{\alpha}\|c\|_{X'} + \frac{1}{\beta}\left(1+\frac{C_a}{\alpha}\right)\|d\|_{Q'}\,, \tag{3.6}$$

$$\|\lambda\|_Q \le \frac{1}{\beta}\left(1+\frac{C_a}{\alpha}\right)\|c\|_{X'} + \frac{C_a}{\beta^2}\left(1+\frac{C_a}{\alpha}\right)\|d\|_{Q'}\,. \tag{3.7}$$

Here, the coercivity condition (3.2) applies to X, which is a particular case of Brezzi's original conditions. We also notice that the constant C_b does not play a role in the a priori estimates, but it is relevant in the a priori analysis of the numerical approximation error of the finite element method.

3.1. Problem 3D-1D-2D. We aim to find $u \in H_0^1(\Omega)$, $u_\odot \in H_0^1(\Lambda)$, $\lambda \in H^{-\frac{1}{2}}(\Gamma)$, solutions of (3.1), where

$$\begin{aligned} a([u,u_\odot],[v,v_\odot]) &= (u,v)_{H^1(\Omega)} + (u_\odot,v_\odot)_{H^1(\Lambda),|\mathcal{D}|},\\ b([v,v_\odot],\mu) &= \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma,\\ c([v,v_\odot]) &= (f,v)_{L^2(\Omega)} + (\overline{\overline{g}},v_\odot)_{L^2(\Lambda),|\mathcal{D}|},\\ d(\mu) &= \langle q,\mu\rangle_\Gamma. \end{aligned}$$

We prove that the conditions of Theorem 3.1 are fulfilled choosing $X = H_0^1(\Omega) \times H_0^1(\Lambda)$, $Q = H^{-\frac{1}{2}}(\Gamma)$, where X is equipped with the norm $|||[u,u_\odot]|||^2 = \|u\|^2_{H^1(\Omega)} + \|u_\odot\|^2_{H^1(\Lambda),|\mathcal{D}|}$. To this purpose, we recall the trace inequality relative to the operator $\mathcal{T}_\Gamma$, namely for any $v \in H^1(\Omega)$ there exists a constant C_T, depending on the diameter of Ω such that $\|\mathcal{T}_\Gamma v\|_{H_{00}^{\frac{1}{2}}(\Gamma)} \le C_T\|v\|_{H^1(\Omega)}$. We also define a lifting operator, from $H_{00}^{1/2}(\Gamma)$ to $H_0^1(\Omega)$. First, we define the harmonic extension $\mathcal{H}_{\Omega_\oplus}$ from $H_{00}^{1/2}(\Gamma)$ to $H_0^1(\Omega_\oplus)$, such that $\mathcal{H}_{\Omega_\oplus}\xi = v$ for any $\xi \in H_{00}^{1/2}(\Gamma)$ with $v \in H_0^1(\Omega_\oplus)$. Further, for this operator there exists $C_{\Omega_\oplus} \in \mathbb{R}^+$, depending only on the diameter of $\Omega_\oplus$, such that $\|v\|_{H^1(\Omega_\oplus)} \le C_{\Omega_\oplus}\|\xi\|_{H_{00}^{1/2}(\Gamma)}$. Now, to define an extension form $H_0^1(\Omega_\oplus)$ to $H_0^1(\Omega)$ we use the results of [31], in particular Theorem 2.3 for the specific case of a domain with a long hole such as $\Omega_\oplus$, where it is established that there exists a lifting operator $\mathcal{E}_\Omega$ from $H_0^1(\Omega_\oplus)$ to $H_0^1(\Omega)$ such that $\mathcal{E}_\Omega\xi = v$ for any $\xi \in H_0^1(\Omega_\oplus)$ with $v \in H_0^1(\Omega)$ and there exists $C_\Omega \in \mathbb{R}^+$ such that $\|v\|_{H^1(\Omega_\oplus)} \le C_\Omega\|\xi\|_{H^1(\Omega)}$ where C_Ω is a positive constant independent of the (minimal) radius of Γ.

LEMMA 3.2. *The bilinear forms of the problem 3D-1D-2D satisfy conditions* (3.2)*-*(3.5) *with constants* $\alpha = 1$, $\beta = (C_{\Omega_\oplus}C_\Omega)^{-1}$, $C_a = 1$, $C_b = C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}$.

Proof. We need to establish the four Brezzi conditions. The bilinear form $a(\cdot\,,\,\cdot)$ is clearly bounded and coercive with constants $\alpha = C_a = 1$ since for any $u = u_\odot$, $v = v_\odot$ we have,

$$a([u,u_\odot],[v,v_\odot]) = (u,v)_{H^1(\Omega)} + (u_\odot,v_\odot)_{H^1(\Lambda),|\mathcal{D}|} = \|u\|^2_{H^1(\Omega)} + \|u_\odot\|^2_{H^1(\Lambda),|\mathcal{D}|}.$$

Furthermore, the bilinear form $b(\cdot\ ,\ \cdot)$ is bounded because

$$
\begin{aligned}
b([v,v_\odot],\mu) &= \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma \le \|\mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot\|_{H^{\frac12}_{00}(\Gamma)} \|\mu\|_{H^{-\frac12}(\Gamma)} \\
&\le \left(\|\mathcal{T}_\Gamma v\|_{H^{\frac12}_{00}(\Gamma)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^{\frac12}_{00}(\Gamma)} \right) \|\mu\|_{H^{-\frac12}(\Gamma)} \\
&\le \left(C_T \|v\|_{H^1(\Omega)} + \|\mathcal{E}_\Gamma v_\odot\|_{H^1(\Gamma)} \right) \|\mu\|_{H^{-\frac12}(\Gamma)} \\
&\le \left(C_T \|v\|_{H^1(\Omega)} + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac12} \|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|} \right) \|\mu\|_{H^{-\frac12}(\Gamma)} \\
&\le \left(C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac12} \right) |||[v,v_\odot]||| \|\mu\|_{H^{-\frac12}(\Gamma)}.
\end{aligned}
$$

To fulfill the inf-sup condition for $b(\cdot,\cdot)$ we choose $v_\odot \in H^1_0(\Lambda)$ such that $\mathcal{E}_\Gamma v_\odot = 0$. Therefore we obtain,

$$
\sup_{\substack{v\in H^1_0(\Omega),\\ v_\odot\in H^1_0(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \mu\rangle_\Gamma}{|||[v,v_\odot]|||} \ge \sup_{v\in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}}.
$$

We notice that the trace operator is surjective from $H^1_0(\Omega)$ to $H^{\frac12}_{00}(\Gamma)$. Indeed, $\forall \xi \in H^{\frac12}_{00}(\Gamma)$, we can find $v = \mathcal{E}_\Omega \mathcal{H}_{\Omega_\oplus}\xi$. Using the stability of $\mathcal{E}_\Omega$, $\mathcal{H}_{\Omega_\oplus}$ we obtain

$$
\sup_{v\in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu\rangle_\Gamma}{\|v\|_{H^1(\Omega)}} \ge \sup_{\xi\in H^{\frac12}_{00}(\Gamma)} \frac{\langle \xi, \mu\rangle_\Gamma}{C_{\Omega_\oplus} C_\Omega \|\xi\|_{H^{\frac12}_{00}(\Gamma)}} = (C_{\Omega_\oplus} C_\Omega)^{-1} \|\mu\|_{H^{-\frac12}(\Gamma)}, \tag{3.8}
$$

where in the last inequality we exploited the fact that $H^{-\frac12}(\Gamma) = (H^{\frac12}_{00}(\Gamma))^*$. Then, (3.5) is satisfied with $\beta = (C_{\Omega_\oplus} C_\Omega)^{-1}$, a constant independent of the size of the inclusion. ∎

Corollary 3.3 (of Theorem 3.1). *The 3D-1D-2D problem admits a unique solution $u \in H^1_0(\Omega)$, $u_\odot \in H^1_0(\Lambda)$, $\lambda \in H^{-\frac12}(\Gamma)$ that satisfies the following a priori estimates, with constants independent of the minimal (transverse) diameter of Γ,*

$$
\begin{aligned}
|||[u,u_\odot]||| &\le \left(\|f\|_{L^2(\Omega)} + \|\overline{\overline{g}}\|_{L^2(\Lambda),|\mathcal{D}|}\right) + 2C_{\Omega_\oplus} C_\Omega \|q\|_{H^{\frac12}_{00}(\Gamma)}, \\
\|\lambda\|_{H^{-\frac12}(\Gamma)} &\le 2C_{\Omega_\oplus} C_\Omega \left(\|f\|_{L^2(\Omega)} + \|\overline{\overline{g}}\|_{L^2(\Lambda),|\mathcal{D}|}\right) + 2(C_{\Omega_\oplus} C_\Omega)^2 \|q\|_{H^{\frac12}_{00}(\Gamma)}.
\end{aligned}
$$

3.2. Problem 3D-1D-1D. We aim to find $u \in H^1_0(\Omega)$, $u_\odot \in H^1_0(\Lambda)$, $\lambda_\odot \in H^{-\frac12}(\Lambda)$, solution of (3.1) with

$$
\begin{aligned}
a([u,u_\odot],[v,v_\odot]) &= (u,v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|}, \\
b([v,v_\odot],\mu_\odot) &= \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}, \\
c([v,v_\odot]) &= (f,v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|}, \\
d(\mu_\odot) &= \langle \overline{q}, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}.
\end{aligned}
$$

We prove that the assumptions of Theorem 3.1 are fulfilled with the following spaces $X = H^1_0(\Omega) \times H^1_0(\Lambda)$, $Q = H^{-\frac12}(\Lambda)$. Let us consider X equipped with the norm $|||[\cdot,\cdot]|||$ and Q equipped with the norm $\|\cdot\|_{H^{-\frac12}(\Lambda),|\partial\mathcal{D}|}$.

LEMMA 3.4. *The bilinear forms of the problem 3D-1D-1D satisfy conditions* (3.2)-(3.5) *with constants* $\alpha = 1$, $\beta = (C_{\Omega_\oplus} C_\Omega)^{-1}$, $C_a = 1$, $C_b = C_\Gamma C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}$, *where* C_Γ *is the constant of Lemma* 2.2.

Proof. The proof for the bilinear form $a(\cdot,\cdot)$ does not change with respect to the previous case.

The bound on $b(\cdot\ ,\ \cdot)$ is established as

$$
\begin{aligned}
b([v, v_\odot], \mu_\odot) &= \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|} \le \|\overline{\mathcal{T}}_\Lambda v - v_\odot\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|} \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \\
&\le \left(\|\overline{\mathcal{T}}_\Lambda v\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|} + \|v_\odot\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|} \right) \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \\
&\le \left(C_\Gamma \|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}_{00}(\Gamma)} + \|v_\odot\|_{H^1(\Lambda),|\partial\mathcal{D}|} \right) \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \\
&\le \left(C_\Gamma C_T \|v\|_{H^1(\Omega)} + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}} \|v_\odot\|_{H^1(\Lambda),|\mathcal{D}|} \right) \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \\
&\le \left(C_\Gamma C_T + \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}} \right) |||[v, v_\odot]||| \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.
\end{aligned}
$$

To show that the inf-sup condition holds we choose $v_\odot = 0$ and obtain

$$
\sup_{\substack{v \in H^1_0(\Omega),\\ v_\odot \in H^1_0(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v, v_\odot]|||} \ge \sup_{v \in H^1_0(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}}.
$$

For any $q \in H^{\frac{1}{2}}_{00}(\Lambda)$, we consider the uniform extension to Γ named as $\mathcal{E}_\Gamma q$ and then we consider the extension operator from $H^{\frac{1}{2}}_{00}(\Gamma)$ to $H^1_0(\Omega)$ defined before, namely $\mathcal{E}_\Omega \mathcal{H}_{\Omega_\oplus}$ such that $v = \mathcal{E}_\Omega \mathcal{H}_{\Omega_\oplus} \mathcal{E}_\Gamma q \in H^1_0(\Omega)$. It follows that for any $q \in H^{\frac{1}{2}}_{00}(\Lambda)$ there exists $v \in H^1_0(\Omega)$ such that $\overline{\mathcal{T}}_\Lambda v = q$. Therefore we have,

$$
\sup_{v \in H^1_0(\Omega)} \langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|} \ge \sup_{q \in H^{\frac{1}{2}}_{00}(\Lambda)} \langle q, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|}.
$$

Moreover, using Lemma 2.1 we obtain

$$
\|v\|_{H^1_0(\Omega)} \le C_{\Omega_\oplus} C_\Omega \|\mathcal{E}_\Gamma q\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = C_{\Omega_\oplus} C_\Omega \|q\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}.
$$

We conclude the proof with the following inequalities,

$$
\begin{aligned}
\sup_{v \in H^1_0(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} &\ge \sup_{q \in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle q, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v\|_{H^1(\Omega)}} \\
&\ge \frac{1}{C_{\Omega_\oplus} C_\Omega} \sup_{q \in H^{\frac{1}{2}}_{00}(\Lambda)} \frac{\langle q, \mu_\odot \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}} = \frac{1}{C_{\Omega_\oplus} C_\Omega} \|\mu_\odot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}. \qquad \square
\end{aligned}
$$

Corollary 3.5 (of Theorem 3.1). *The 3D-1D-1D problem admits a unique solution $u \in H_0^1(\Omega)$, $u_\odot \in H_0^1(\Lambda)$, $\lambda \in H^{-\frac{1}{2}}(\Lambda)$ that satisfies the following a priori estimates, with constants independent of the minimal (transverse) diameter of Γ,*

$$|||[u, u_\odot]||| \leq \left(\|f\|_{L^2(\Omega)} + \|\overline{\overline{g}}\|_{L^2(\Lambda),|\mathcal{D}|}\right) + 2C_{\Omega_\oplus} C_\Omega \|\overline{q}\|_{H_{00}^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|},$$
$$\|\lambda\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \leq 2C_{\Omega_\oplus} C_\Omega \left(\|f\|_{L^2(\Omega)} + \|\overline{\overline{g}}\|_{L^2(\Lambda),|\mathcal{D}|}\right) + 2(C_{\Omega_\oplus} C_\Omega)^2 \|\overline{q}\|_{H_{00}^{\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}.$$

Remark 3.1. *Corollaries 3.3 and 3.5 show that the stability of the continuous problem is not affected by the size of the inclusion, because all the stability constants are uniformly independent of $|\mathcal{D}|$, $|\partial\mathcal{D}|$. Referring for example to the 3D-1D-1D problem, formally taking the limit for $|\mathcal{D}|$, $|\partial\mathcal{D}| \to 0$, we observe that the weak formulation of the problem would tend to the trivial case $(u, v)_{H^1(\Omega)} = (f, v)_{L^2(\Omega)}$ and in a similar way the a priori estimates would consistently reduce to $\|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}$. In other words, the weak formulation of the problem and the a priori estimates are robust for arbitrarily small size of the inclusion.*

4. Finite element approximation.

In this section we consider the discretization of the Problems 3D-1D-2D and 3D-1D-1D by means of the finite element method. We address two main objectives; first we aim to identify a suitable approximation space for the Lagrange multiplier and to analyze the stability of the discrete saddle point problem; second we aim to derive a stable discretization method that uses independent computational meshes for Ω and Λ, not necessarily conforming to Γ. The latter objective is particularly relevant for the application of this approach in the case of very small inclusions, because it possibly allows us to use a computational mesh on Ω with a characteristic size h that is larger than the (cross sectional) diameter of the inclusion.

Let us introduce a shape-regular triangulation $\mathcal{T}_h^\Omega$ of Ω and an admissible partition $\mathcal{T}_\mathfrak{h}^\Lambda$ of Λ. We analyze two different cases: the conforming case, where compatibility constraints are satisfied by $\mathcal{T}_h^\Omega$ and $\mathcal{T}_h^\Lambda$ with respect to Γ and consequently $h = \mathfrak{h}$; and the non conforming case, where it is possible to choose $\mathcal{T}_h^\Omega$ and $\mathcal{T}_\mathfrak{h}^\Lambda$ arbitrarily.

Remark 4.1. *The mesh conformity assumptions between $\mathcal{T}_h^\Omega$, $\mathcal{T}_\mathfrak{h}^\Lambda$ and Γ (see below for a precise definition) necessarily imply that $h = \mathfrak{h} \leq R_0$, being R_0 the minimum cross sectional radius of the inclusion $\Omega_\ominus$ that is shaped as a generalized cylinder, as shown in Figure 2.1.*

4.1. Analysis of the case where $\mathcal{T}_h^\Omega$ conforms to $\mathcal{T}_\mathfrak{h}^\Lambda$ and to Γ.

As conformity conditions between $\mathcal{T}_h^\Omega$, $\mathcal{T}_\mathfrak{h}^\Lambda$ and Γ, we require that the intersection of $\mathcal{T}_h^\Omega$ and Γ is made of entire faces of elements $K \in \mathcal{T}_h^\Omega$. Furthermore, we also set a restriction between $\mathcal{T}_h^\Omega$ and $\mathcal{T}_\mathfrak{h}^\Lambda$. We assume that Λ is a piecewise linear manifold. We want that for any internal node of $\mathcal{T}_\mathfrak{h}^\Lambda$ a cross sectional plane intersecting Γ is defined. We require that all the nodes of $\mathcal{T}_h^\Omega$ laying on Γ fall on the intersection of Γ with such cross sectional planes. As a result of the latter condition we have $h \simeq \mathfrak{h}$. For this reason, from now on throughout this section we denote as $\mathcal{T}_h^\Lambda$ the mesh on Λ.

In this case, the discrete equivalent of (3.1) reads as finding $u_h \in X_h \subset X$, $\lambda_h \in Q_h \subset Q$ s.t.

$$a(u_h, v_h) + b(v_h, \lambda_h) = c(v_h) \quad \forall v_h \in X_h,$$
$$b(u_h, \mu_h) = d(\mu_h) \quad \forall \mu_h \in Q_h, \tag{4.1}$$

where with little abuse of notation we use h as the sub-index for all the discretization spaces. This discrete problem is well-posed if the conditions (3.2)-(3.5) apply to X_h and Q_h. Since $X_h \subset X$ and $Q_h \subset Q$, (3.2)-(3.4) follow immediately and only the inf-sup condition needs consideration, see for example [14, Theorem 2.42]. Furthermore, Ceá type approximation estimates can be easily derived, as shown in [14, Theorem 2.44]. We summarize these results in the Theorem below.

THEOREM 4.1. *Let $X_h \subset X$, $Q_h \subset Q$, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy the conditions* (3.2)-(3.4) *then the problem* (4.1) *is well-posed if the discrete counterpart of* (3.5) *is satisfied, i.e. there exists a constant $\beta_h > 0$, independent of the mesh discretization size h, such that*

$$\sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X} \geq \beta_h \|\mu_h\|_Q \,, \quad \forall \mu_h \in Q_h. \tag{4.2}$$

Furthermore the following a priori error estimates hold:

$$\begin{aligned}
\|u - u_h\|_X &\leq \left(1 + \frac{C_a}{\alpha}\right)\left(1 + \frac{C_b}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \frac{C_b}{\alpha} \inf_{\mu_h \in Q_h} \|\lambda - \mu_h\|_Q \,, \\
\|\lambda - \lambda_h\|_Q &\leq \frac{C_a}{\beta_h}\left(1 + \frac{C_a}{\alpha}\right)\left(1 + \frac{C_b}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X \\
&\quad + \left(1 + \frac{C_b}{\beta_h} + \frac{C_b}{\alpha}\frac{C_a}{\beta_h}\right) \inf_{\mu_h \in Q_h} \|\lambda - \mu_h\|_Q \,.
\end{aligned}$$

Before proceeding we state an auxiliary result used in the forthcoming analysis. From now on, C denotes a generic constant independent of the mesh size.

LEMMA 4.2. *Let $\mathcal{P}_h : H^{\frac{1}{2}}_{00}(\Sigma; w) \to Q_h$ be the orthogonal projection operator defined for any $v \in H^{\frac{1}{2}}_{00}(\Sigma; w)$ by $(\mathcal{P}_h v, \psi_h)_{\Sigma,w} = (v, \psi_h)_{\Sigma,w}$ for any $\psi_h \in Q_h$, where w is a bounded and positive weight function. Then, $\mathcal{P}_h$ is continuous on $H^{\frac{1}{2}}_{00}(\Sigma; w)$, namely $\|\mathcal{P}_h v\|_{H^{\frac{1}{2}}_{00}(\Sigma),w} \leq C\|v\|_{H^{\frac{1}{2}}_{00}(\Sigma),w}$.*

Proof. We show that $\mathcal{P}_h$ is continuous on $L^2(\Sigma; w)$ and on $H^1_0(\Sigma; w)$ following [14, Section 1.6.3]. Then, the desired result can be proved by interpolation between spaces, since $H^{\frac{1}{2}}_{00}(\Sigma; w) = [H^1_0(\Sigma; w), L^2(\Sigma; w)]_{\frac{1}{2}}$, namely the interpolation space between $L^2(\Sigma; w)$ and $H^1_0(\Sigma; w)$. For the L^2-continuity, we exploit the fact that, from the definition of $\mathcal{P}_h$, $(v - \mathcal{P}_h v, \mathcal{P}_h v)_{\Sigma,w} = 0$. Therefore, by Pythagoras identity,

$$\|v\|^2_{L^2(\Sigma),w} = \|v - \mathcal{P}_h v\|^2_{L^2(\Sigma),w} + \|\mathcal{P}_h v\|^2_{L^2(\Sigma),w} \geq \|\mathcal{P}_h v\|^2_{L^2(\Sigma),w}.$$

Let us now consider $v \in H^1_0(\Sigma; w)$. The Scott-Zhang interpolation operator $\mathcal{SZ}_h$ from $H^1_0(\Sigma; w)$ to Q_h satisfies the following inequalities (see [32] and also [14] Lemma 1.130, inequalities (i) and (ii) for (4.3) and (4.4) respectively),

$$\|\mathcal{SZ}_h v\|_{H^1(\Sigma),w} \leq C_1 \|v\|_{H^1(\Sigma),w}, \tag{4.3}$$

$$\|v - \mathcal{SZ}_h v\|_{L^2(\Sigma),w} \leq C_2 h \|v\|_{H^1(\Sigma),w}. \tag{4.4}$$

Therefore, using (4.3), (4.4), the L^2 stability of $\mathcal{P}_h$ and the discrete inverse inequality, we obtain,

$$\begin{aligned}\|\nabla \mathcal{P}_h v\|_{L^2(\Sigma),w} &\leq \|\nabla(\mathcal{P}_h v - \mathcal{S}\mathcal{Z}_h v)\|_{L^2(\Sigma),w} + \|\nabla \mathcal{S}\mathcal{Z}_h v\|_{L^2(\Sigma),w} \\ &\leq \|\nabla(\mathcal{P}_h v - \mathcal{S}\mathcal{Z}_h v)\|_{L^2(\Sigma),w} + C_1 \|v\|_{H^1(\Sigma),w} \\ &\leq \frac{C_3}{h} \|\mathcal{P}_h(v - \mathcal{S}\mathcal{Z}_h v)\|_{L^2(\Sigma),w} + C_1 \|v\|_{H^1(\Sigma),w} \\ &\leq \frac{C_3}{h} \|v - \mathcal{S}\mathcal{Z}_h v\|_{L^2(\Sigma),w} + C_1 \|v\|_{H^1(\Sigma),w} \\ &\leq (C_2 C_3 + C_1) \|v\|_{H^1(\Sigma),w} \,.\end{aligned}$$

As a result of the previous inequalities we obtain that

$$\|\mathcal{P}_h v\|^2_{L^2(\Sigma),w} \leq C \|v\|^2_{L^2(\Sigma),w}\,, \quad \|\mathcal{P}_h v\|_{H^1(\Sigma),w} \leq C \|v\|^2_{H^1(\Sigma),w}\,.$$

It remains to show that $\|\mathcal{P}_h v\|_{H^{\frac{1}{2}}_{00}(\Sigma),w} \leq C\|v\|_{H^{\frac{1}{2}}_{00}(\Sigma),w}$. To this end we use the interpolation theory for operators in Banach spaces. Given two separable Hilbert spaces, let us denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators from X to Y. Then, by L^2 and H^1 continuity of $\mathcal{P}_h$ we have that $\mathcal{P}_h \in \mathcal{L}(L^2(\Sigma; w), L^2(\Sigma; w)) \cap \mathcal{L}(H^1_0(\Sigma; w), H^1_0(\Sigma; w))$. Recalling that we define $H^{1/2}_{00}(\Sigma; w) = [H^1_0(\Sigma; w), L^2(\Sigma; w)]_{\frac{1}{2}}$ and Applying [2, Theorem 2.2] it follows that $P_h \in \mathcal{L}\left(H^{1/2}_{00}(\Sigma; w), H^{1/2}_{00}(\Sigma; w)\right)$, which implies the desired inequality. We remark that [2, Theorem 2.2] applies directly to our setting as the interpolation spaces therein are considered with the spectral norm rather than the K-interpolation norm. ∎

4.1.1. Problem 3D-1D-2D. We denote by $X^k_{h,0}(\Omega) \subset H^1_0(\Omega)$, with $k > 0$, the conforming finite element space of continuous piecewise polynomials of degree k defined on Ω satisfying homogeneous Dirichlet conditions on the boundary and by $X^k_{h,0}(\Lambda) \subset H^1_0(\Lambda)$ the space of continuous piecewise polynomials of degree k defined on Λ, satisfying homogeneous Dirichlet conditions on $\Lambda \cap \partial\Omega$. The space Q_h must be suitably chosen such that (4.2) holds. Let Q_h be the trace space of $X^k_{h,0}(\Omega)$, namely the space of continuous piecewise polynomials of degree k defined on Γ which satisfy homogeneous Dirichlet conditions on $\partial\Omega$. As a result, $Q_h = X^k_{h,0}(\Gamma) \subset H^{\frac{1}{2}}_{00}(\Gamma)$. The discrete version of the 3D-1D-2D problem is: find $u_h \in X^k_{h,0}(\Omega)$, $u_{\odot h} \in X^k_{h,0}(\Lambda)$, $\lambda_h \in Q_h \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\text{(4.5a)} \qquad \begin{aligned}&(u_h, v_h)_{H^1(\Omega)} + (u_{\odot h}, v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Lambda v_{\odot h}, \lambda_h \rangle_\Gamma \\ &\quad = (f, v_h)_{L^2(\Omega)} + (\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X^k_{h,0}(\Omega),\ v_{\odot h} \in X^k_{h,0}(\Lambda)\,,\end{aligned}$$

$$\text{(4.5b)} \qquad \langle \mathcal{T}_\Gamma u_h - \mathcal{E}_\Lambda u_{\odot h}, \mu_h \rangle_\Gamma = \langle q, \mu_h \rangle_\Gamma \quad \forall \mu_h \in Q_h\,.$$

In what follows, we analyze the well-posedness of the discrete problem.

LEMMA 4.3. *There exists a constant $\gamma_{h,1} > 0$ such that for any $\mu_h \in Q_h$*

$$\sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} \geq \gamma_{h,1} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

As for the inf-sup constants, we notice that $\gamma_{h,1}$ depends on the discrete functional spaces, but is is uniformly independent of the mesh characteristic size h.

Proof. From the continuous case, in particular from (3.8), we have

$$(C_{\Omega_\oplus} C_\Omega)^{-1} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|v\|_{H^1(\Omega)}} \quad \forall \mu_h \in Q_h,$$

and by the trace inequality $\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_T \|v\|_{H^1(\Omega)}$ (see [1, 7.56]), we obtain

$$\sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|v\|_{H^1(\Omega)}} \leq C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}_{00}(\Gamma)}}.$$

Using Lemma 4.2 with $\Sigma = \Gamma$ and $w = 1$ we obtain,

$$\begin{aligned} C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{T}_\Gamma v, \mu_h \rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} &= C_T \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{P}_h(\mathcal{T}_\Gamma v), \mu_h \rangle}{\|\mathcal{T}_\Gamma v\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} \\ &\leq C \sup_{v \in H^1_0(\Omega)} \frac{\langle \mathcal{P}_h(\mathcal{T}_\Gamma v), \mu_h \rangle}{\|\mathcal{P}_h(\mathcal{T}_\Gamma v)\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} = C \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle}{\|q_h\|_{H^{\frac{1}{2}}_{00}(\Gamma)}}. \quad \square \end{aligned}$$

THEOREM 4.4 (Discrete inf-sup). *The inequality (4.2) holds true, namely there exists a positive constant $\beta_{h,1}$ such that,*

$$\sup_{\substack{v_h \in X^k_{h,0}(\Omega), \\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Gamma v_{\odot h}, \mu_h \rangle_\Gamma}{|||[v_h, v_{\odot h}]||| \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}} \geq \beta_{h,1}, \quad \forall\ \mu_h \in Q_h. \tag{4.6}$$

Proof. As in the continuous case, we choose $v_{\odot h} = 0$ and we have

$$\sup_{\substack{v_h \in X^k_{h,0}(\Omega), \\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \mathcal{T}_\Gamma v_h - \mathcal{E}_\Gamma v_{\odot h}, \mu_h \rangle_\Gamma}{|||[v_h, v_{\odot h}]|||} \geq \sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h, \mu_h \rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}}.$$

Following the approach of the continuous case, we need to construct an extension operator from Q_h to $X^k_{h,0}(\Omega)$. Thanks to the conformity of $\mathcal{T}^\Omega_h$ to the interface Γ, the existence and stability of such extension operator, named $\mathcal{E}^h_\Omega$ (as it is the discrete analogue of $\mathcal{E}_\Omega \mathcal{H}_{\Omega_\oplus}$ used before), is proved using the results of [35]. In particular, as Γ splits Ω into $\Omega_\oplus$ and $\Omega_\ominus$ as well as the corresponding meshes comply with this partition, we introduce $\mathcal{E}^h_{\Omega_\oplus}$ and $\mathcal{E}^h_{\Omega_\ominus}$ as the extension operators from Q_h to $X^k_{h,0}(\Omega_\oplus)$ and $X^k_h(\Omega_\ominus)$, respectively. Then, we set (with little abuse of notation) $\mathcal{E}^h_\Omega q_h := (\mathcal{E}^h_{\Omega_\oplus} q_h + \mathcal{E}^h_{\Omega_\ominus} q_h + \mathcal{T}_\Gamma q_h) \in X^k_{h,0}(\Omega)$. By definition, we obtain that $\mathcal{T}_\Gamma \mathcal{E}^h_\Omega$ is the identity operator on Q_h and, owing to the results of [35], there exists a constant $C_\mathcal{D}$ uniformly independent of h but possibly dependent on the size of the inclusion, namely diam$(\mathcal{D})$, such that $\|\mathcal{E}^h_\Omega q_h\|_{H^1(\Omega)} \leq C_\mathcal{D} \|q_h\|_{H^{\frac{1}{2}}_{00}(\Gamma)}$.

Using Lemma 4.3 and the boundedness of the extension operator $\mathcal{E}^h_\Omega$ we have

$$\gamma_{h,1} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} \leq C_\mathcal{D} \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle_\Gamma}{\|\mathcal{E}^h_\Omega q_h\|_{H^1(\Omega)}}.$$

Then, for any $q_h \in Q_h$ we have $q_h = \mathcal{T}_\Gamma \mathcal{E}^h_\Omega q_h$ and owing to this property we obtain the following inequality, which proves the condition, with $\beta_{h,1} = \gamma_{h,1} C_{\mathcal{D}}^{-1}$,

$$\gamma_{h,1} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \le \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle_\Gamma}{\|q_h\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} \le C_{\mathcal{D}} \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_h \rangle_\Gamma}{\|\mathcal{E}^h_\Omega q_h\|_{H^1(\Gamma)}}$$
$$= C_{\mathcal{D}} \sup_{q_h \in Q_h} \frac{\langle \mathcal{T}_\Gamma \mathcal{E}^h_\Omega q_h, \mu_h \rangle_\Gamma}{\|\mathcal{E}^h_\Omega q_h\|_{H^1(\Omega)}} \le C_{\mathcal{D}} \sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \mathcal{T}_\Gamma v_h, \mu_h \rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} . \quad \square$$

COROLLARY 4.5 (of Theorem 4.1). *Problem (4.5) admits a unique solution $u_h \in X^k_{h,0}(\Omega)$, $u_{\odot h} \in X^k_{h,0}(\Lambda)$, $\lambda_h \in X^k_{h,0}(\Gamma)$ and the following a priori error estimates are satisfied:*

$$|||[u - u_h, u_\odot - u_{\odot h}]||| \le C_{1,\mathcal{D}}\, \mathcal{ERR}(u, u_\odot, \lambda) ,$$
$$\|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma)} \le C_{2,\mathcal{D}}\, \mathcal{ERR}(u, u_\odot, \lambda) ,$$

where $C_{1,\mathcal{D}}, C_{2,\mathcal{D}} \simeq \left(\frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{\frac{1}{2}}$ and $\mathcal{ERR}(u, u_\odot, \lambda)$ is the approximation error

$$\mathcal{ERR}(u, u_\odot, \lambda) = \inf_{\substack{v_h \in X^k_{h,0}(\Omega) \\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} |||[u - v_h, u_\odot - v_{\odot h}]||| + \inf_{\mu_h \in X^k_{h,0}(\Gamma)} \|\lambda - \mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} .$$

4.1.2. Problem 3D-1D-1D. In this case, we use the same spaces $X^k_{h,0}(\Omega)$, $X^k_{h,0}(\Lambda)$ defined previously. For the multiplier space we choose $Q_h = X^k_{h,0}(\Lambda)$, therefore we impose homogeneous Dirichlet boundary condition on $\Lambda \cap \partial\Omega$ also for the Lagrange multiplier. We aim to find $u_h \in X^k_{h,0}(\Omega)$, $u_{\odot h} \in X^k_{h,0}(\Lambda)$, $\lambda_{\odot h} \in Q_h \subset H^{-\frac{1}{2}}(\Lambda)$, such that

$$\text{(4.7a)} \qquad (u_h, v_h)_{H^1(\Omega)} + (u_{\odot h}, v_{\odot h})_{H^1(\Lambda),|\mathcal{D}|} + \langle \overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, \lambda_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}$$
$$= (f, v_h)_{L^2(\Omega)} + (\overline{\overline{g}}, v_{\odot h})_{L^2(\Lambda),|\mathcal{D}|} \quad \forall v_h \in X_h(\Omega),\ v_{\odot h} \in X_h(\Lambda) ,$$

$$\text{(4.7b)} \qquad \langle \overline{\mathcal{T}}_\Lambda u_h - u_{\odot h}, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|} = \langle \overline{q}, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|} \quad \forall \mu_{\odot h} \in Q_h .$$

Below we address the well-posedness of the 3D-1D-1D discrete problem with this alternative choice of multiplier space.

LEMMA 4.6. *There exist a constant $\gamma_{h,2} > 0$ such that for any $\mu_h \in Q_h$,*

$$\sup_{q_h \in Q_h} \frac{\langle q_h, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}} \ge \gamma_{h,2} \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} .$$

The proof of this Lemma follows the one of Lemma 4.2, used with $\Sigma = \Lambda$ and $w = |\partial\mathcal{D}|$, and Lemma 4.3 with the only difference that the arguments are applied to Λ instead of Γ.

THEOREM 4.7 (Discrete inf-sup). *The inequality (4.2) holds, namely there exists a positive constant $\beta_{h,2}$ such that,*

$$\text{(4.8)} \qquad \sup_{\substack{v_h \in X^k_{h,0}(\Omega), \\ v_{\odot h} \in X^k_{h,0}(\Lambda)}} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h - v_{\odot h}, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}}{|||[v_h, v_{\odot h}]||| \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}} \ge \beta_{h,2}, \quad \forall\ \mu_{\odot h} \in Q_h.$$

Proof. Again, we choose $v_{\odot h} = 0$, so that the proof reduces to showing that there exists $\beta_{h,2}$ such that

$$\sup_{v_h \in X^k_{h,0}(\Omega)} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \geq \beta_{h,2} \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \quad \forall \mu_{\odot h} \in Q_h.$$

For any $w \in H^{\frac{1}{2}}(\Lambda)$, Lemma 2.1 ensures that $\|\mathcal{E}_\Gamma w\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = \|w\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}$. We use the extension operator $\mathcal{E}^h_\Omega$ from $X^k_{h,0}(\Gamma)$ to $X^k_{h,0}(\Omega)$ and we combine it with $\mathcal{E}^h_\Gamma$, namely the discrete uniform extension operator from Λ to Γ that for each node of $\mathcal{T}^\Lambda_{\mathfrak{h}}$ spans the nodal value of $q_h \in Q_h$ to the nodes of $\mathcal{T}^\Omega_h$ laying on the cross section of Γ that intersects the chosen node on $\mathcal{T}^\Lambda_{\mathfrak{h}}$ (see Figure 5.2 for a visualization). We call $\mathcal{E}^h_\Omega \mathcal{E}^h_\Gamma : \; Q_h := X^k_{h,0}(\Lambda) \to X^k_{h,0}(\Omega)$ the combination of these two extensions. Through this construction, it is straightforward to see that $\overline{\mathcal{T}}_\Lambda \mathcal{E}^h_\Omega \mathcal{E}^h_\Gamma$ coincides with the identity operator on Q_h.

As a result, from Lemma 4.6, we obtain the following inequality

$$\begin{aligned}
\gamma_{h,2} \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} &\leq \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|q_h\|_{H^{\frac{1}{2}}_{00}(\Lambda),|\partial\mathcal{D}|}} = \sup_{q_h \in Q_h} \frac{\langle q_h, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{E}^h_\Gamma q_h\|_{H^{\frac{1}{2}}_{00}(\Gamma)}} \\
&= C_{\mathcal{D}} \sup_{q_h \in Q_h} \frac{\langle \overline{\mathcal{T}}_\Lambda \mathcal{E}^h_\Omega \mathcal{E}^h_\Gamma q_h, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|\mathcal{E}^h_\Omega \mathcal{E}^h_\Gamma q_h\|_{H^1(\Omega)}} \\
&\leq C_{\mathcal{D}} \sup_{v_h \in X^k_{h,0}} \frac{\langle \overline{\mathcal{T}}_\Lambda v_h, \mu_{\odot h} \rangle_{\Lambda,|\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}},
\end{aligned}$$

that concludes the proof with $\beta_{h,2} = \gamma_{h,2} C^{-1}_{\mathcal{D}}$. ∎

It is straightforward to see that problem (4.7a) satisfies properties equivalent to Corollary 4.5, with the only difference that the Lagrange multiplier space is $X^k_{h,0}(\Lambda)$ and that the approximation error of the Lagrange multiplier is measured in the norm of $\|\cdot\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|}$. When $\mathcal{T}^\Omega_h$ conforms to $\mathcal{T}^\Lambda_{\mathfrak{h}}$ and to Γ, the discrete 3D-1D-2D and 3D-1D-1D problems may converge with optimal rates to the corresponding continuous problems, provided that the approximation error $\mathcal{ERR}(u, u_\odot, \lambda)$ features optimal properties. Such properties depend on the regularity of the solution $u, u_\odot, \lambda$. Assuming that such functions are poorly regular on the points of Γ solely, it is reasonable to expect that optimal convergence rates can be observed when the edges of the computational meshes resolve the surface Γ, for example as in the *conforming* case. The numerical experiments shown in Table 5.1 provide good evidence of such behavior. However, we remark that this result is not interesting in practice, because the conformity assumptions require that $h \leq R_0$, being R_0 the minimal cross sectional radius of the inclusion. As a result, in this case the computational cost of the proposed scheme would be almost equivalent to the one of resolving the full 3D-3D problem. To overcome this limitation, we develop in the next section an approximation method where $\mathcal{T}^\Omega_h$ and $\mathcal{T}^\Lambda_{\mathfrak{h}}$ do not conform to Γ.

4.2. Analysis of the case where $\mathcal{T}^\Omega_h$ and $\mathcal{T}^\Lambda_{\mathfrak{h}}$ do not conform to Γ. We analyze now the case in which the elements of the 3D mesh $\mathcal{T}^\Omega_h$ do not conform with the surface Γ nor with Λ. As the 3D-1D-1D formulation is more suitable for this purpose, we solely focus on the analysis of the discrete version of Problem 3D-1D-1D.

4.2.1. Problem 3D-1D-1D. Let $u_h \in X^1_{h,0}(\Omega)$ be the approximation of the 3D problem and let $u_{\odot\mathfrak{h}} \in X^1_{\mathfrak{h},0}(\Lambda)$ the one of the 1D problem. In contrast to the conforming case, here we limit the analysis to the case of piecewise-linear finite elements. With little abuse of notation, we use the sub-index h for the product space $X_h = X^1_{h,0}(\Omega) \times X^1_{\mathfrak{h},0}(\Lambda)$. Concerning the multiplier space, let $\mathcal{G}_h = \{K \in \mathcal{T}^\Omega_h : K \cap \Lambda \neq \emptyset\}$, be the set of the 3D elements that intersect Λ. Then we define $Q_h = \{\lambda_{\odot h} : \lambda_{\odot h} \in P^0(K)\, \forall K \in \mathcal{G}_h\}$. We notice that the multiplier functions are defined on the 3D elements. Again with a little abuse of notation, we denote with Q_h also the restriction to Λ of the space of piecewise constant functions defined in 3D. As a result, we have $Q_h \subset L^2(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$. However, with this choice of multipliers the problem is not inf-sup stable, therefore the idea is to add a stabilization term $s(\lambda_{\odot h}, \mu_{\odot h}) :\ Q_h \times Q_h \to \mathbb{R}$ to (4.7a) following the approach introduced in [8].

The objective of this section is to analyze the stabilized version of the 3D-1D-1D problem: find $[u_h, u_{\odot\mathfrak{h}}] \in X_h$ and $\lambda_{\odot h} \in Q_h$ such that

$$
\begin{aligned}
(4.9)\quad & a([u_h, u_{\odot\mathfrak{h}}], [v_h, v_{\odot\mathfrak{h}}]) + b([v_h, v_{\odot\mathfrak{h}}], \lambda_{\odot h}) + b([u_h, u_{\odot\mathfrak{h}}], \mu_{\odot h}) \\
& \qquad - s_h(\lambda_{\odot h}, \mu_{\odot h}) = c(v_h) + d(\mu_{\odot h}) \quad \forall [v_h, v_{\odot\mathfrak{h}}] \in X_h,\ \forall \mu_{\odot h} \in Q_h\,.
\end{aligned}
$$

The idea of the stabilization strategy proposed in [8] is to identify a new multiplier space Q_H, which is never implemented in practice, such that inf-sup stability with X_h holds true. Then, the stabilization operator is designed to control the distance between Q_h and Q_H through the following inequality

$$
\|\mu_{\odot h} - \pi_H \mu_{\odot h}\|_{Q_H} \leq C s_h(\mu_{\odot h}, \mu_{\odot h})\,,
$$

being π_H a suitable projection operator $Q_h \to Q_H$. Applying the results obtained in [8], the well posedness of problem (4.9) is governed by the following lemma.

Lemma 4.8 (Lemma 2.3 of [8]).

1. *If the $b : X_h \times Q_H \to \mathbb{R}$ is inf-sup stable.*
2. *If the stabilization operator $s_h : Q_h \times Q_h \to \mathbb{R}$ is such that*

$$
\beta_h \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \leq \sup_{v_h \in X_h} \frac{b(v_h, \mu_{\odot h})}{|||v_h|||} + s_h(\mu_{\odot h}, \mu_{\odot h}), \quad \forall \mu_{\odot h} \in Q_h
$$

   where β_h is a positive constant independent of the mesh characteristic size.
3. *If for any $[v_h, v_{\odot\mathfrak{h}}] \in X_h$ there exists a function $\xi_h \in Q_h$ depending on $[v_h, v_{\odot\mathfrak{h}}]$, namely $\xi_h = \xi_h([v_h, v_{\odot\mathfrak{h}}])$, s.t.*

$$
(4.10)\qquad a([v_h, v_{\odot\mathfrak{h}}], [v_h, v_{\odot\mathfrak{h}}]) + b([v_h, v_{\odot\mathfrak{h}}], \xi_h) \geq \alpha_\xi |||[v_h, v_{\odot\mathfrak{h}}]|||_{X_h}\,,
$$

$$
(4.11)\qquad (s_h(\xi_h, \xi_h))^{\frac{1}{2}} \leq c_s |||[v_h, v_{\odot\mathfrak{h}}]|||_{X_h}\,,
$$

   being $|||[\cdot\,,\,\cdot]|||_{X_h}$ a suitable discrete norm.

Then, problem (4.9) admits a unique solution.

For the proof of this result we refer the reader to Lemma 2.3 of [8]. In the remainder of this section, we show how to find a multiplier space Q_H and a stabilization operator s_h such that all the assumptions of Lemma 4.8 are satisfied.

The first step consists of showing that there exists a discrete space Q_H that satisfies the first assumption of Lemma 4.8. We recall that in the case of Problem 3D-1D-1D,

$$
b([u_h, v_{\odot\mathfrak{h}}], \mu_{\odot h}) = \left(\overline{\mathcal{T}}_\Lambda v_h - v_{\odot\mathfrak{h}}, \mu_{\odot h}\right)_{\Lambda,|\partial\mathcal{D}|}\,.
$$

The construction of the inf-sup stable space Q_H is based on macro elements of diameter H, where H is sufficiently large. In particular, we assume that there exists positive constants c_h and c_H such that $c_h h \le H \le c_H^{-1} h$. The space is constructed assembling the 3D elements of $\mathcal{G}_h$ into macro patches ω_j such that $H \le |\omega_j \cap \Lambda| \le cH$ with $H = \min_j |\omega_j \cap \Lambda|$ and $c \ge 1$. Let M_j be the number of elements of the patch ω_j, namely, $\omega_j = \cup_{i=0}^{M_j} K_i$, where $K_i \in \mathcal{G}_h$. We assume that M_j is uniformly bounded in j by some $M \in \mathbb{N}$ and that the interiors of the patches ω_j are disjoint. We define Q_H as the space of piecewise-constant functions on the patches, namely $Q_H = \left\{\mu_{\odot H} : \mu_{\odot H} \in P^0(\omega_j)\, \forall j\right\}$. As previously pointed out for Q_h, we denote with Q_H also the restriction of the multiplier space to Λ, namely say $Q_H \subset L^2(\Lambda) \subset H^{-\frac{1}{2}}(\Lambda)$. Moreover, we associate to each patch ω_j a shape-regular extended patch (using the classical definition of shape-regularity, see for example [14]), still denoted by ω_j for notational simplicity, which is built by adding to ω_j a sufficient number of elements of $\mathcal{T}_h^\Omega$ and we assume that the interiors of the new extended patches ω_j are still disjoint (see Figure 4.1). The extended patches ω_j are built such that they fulfill the conditions $\mathrm{meas}(\omega_j) = \mathcal{O}(H^3)$ and $\mathrm{diam}(\Gamma_{\omega_j \cap \Lambda} \cap \omega_j) = \mathcal{O}(H)$ ($\mathcal{O}(X)$ means $cX \le \mathcal{O}(X) \le CX$), where $\Gamma_{\omega_j \cap \Lambda}$ is the portion of Γ with centerline $\omega_j \cap \Lambda$. The latter assumption is required to ensure that the intersection of $\Gamma_{\omega_j \cap \Lambda}$ and ω_j is not too small and it will be needed later on to prove the inf-sup stability of the space Q_H in Lemma 4.9. A representation of this construction in the simple case in which ω_j is composed just by one tetrahedron is shown in Figure 4.1. Thanks to the shape regularity of these extended patches, the following discrete trace inequality holds true for any function $v \in H^1(\omega_j)$,

$$\|\mathcal{T}_\Gamma v\|_{L^2(\Gamma \cap \omega_j)} \le C_I H^{-\frac{1}{2}} \|v\|_{L^2(\omega_j)} \tag{4.12}$$

Moreover, $\forall u_h \in X_{h,0}^1(\Omega)$ we have the following average inequality, which is a consequence of the definition of $\overline{\mathcal{T}}_\Lambda$, Jensen inequality, and the fact that the patches are disjoint

$$\begin{aligned} \sum_j \|\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j \cap \Lambda), |\partial\mathcal{D}|} &= \int_\Lambda |\partial\mathcal{D}| \left(\frac{1}{|\partial\mathcal{D}|} \int_{\partial\mathcal{D}} \mathcal{T}_\Gamma u_h \right)^2 \\ &\le \int_\Lambda \int_{\partial\mathcal{D}} (\mathcal{T}_\Gamma u_h)^2 = \int_\Gamma (\mathcal{T}_\Gamma u_h)^2 = \sum_j \int_{\omega_j \cap \Gamma} (\mathcal{T}_\Gamma u_h)^2 = \sum_j \|\mathcal{T}_\Gamma u_h\|^2_{L^2(\omega_j \cap \Gamma)}. \end{aligned} \tag{4.13}$$

We are now ready to prove that the space Q_H is inf-sup stable.

LEMMA 4.9. *The space Q_H is inf-sup stable, namely there exists $\beta_H > 0$ independent of the characteristic size of macro-patches such that*

$$\sup_{\substack{v_h \in X_{h,0}^1(\Omega),\\ v_{\odot\mathfrak{h}} \in X_{\mathfrak{h},0}^1(\Lambda)}} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h - v_{\odot\mathfrak{h}}, \mu_{\odot H}\right)_{\Lambda, |\partial\mathcal{D}|}}{|||[v_h, v_{\odot\mathfrak{h}}]|||} \ge \beta_H \|\mu_{\odot H}\|_{H^{-\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|} \quad \forall \mu_{\odot H} \in Q_H.$$

Proof. We choose $v_{\odot\mathfrak{h}} = 0$ and we prove that

$$\sup_{v_h \in X_{h,0}^1(\Omega)} \frac{\left(\overline{\mathcal{T}}_\Lambda v_h, \mu_{\odot H}\right)_{\Lambda, |\partial\mathcal{D}|}}{\|v_h\|_{H^1(\Omega)}} \ge \beta_H \|\mu_{\odot H}\|_{H^{-\frac{1}{2}}(\Lambda), |\partial\mathcal{D}|}.$$

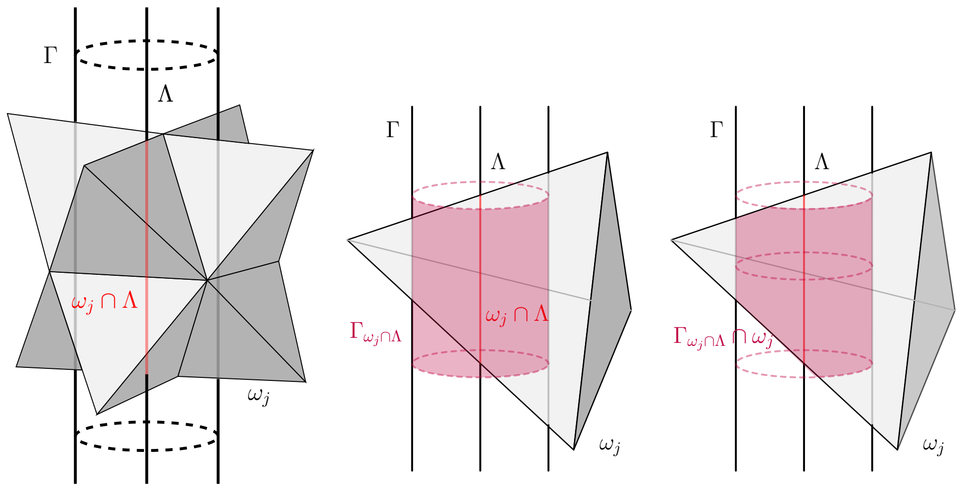

FIGURE 4.1. *(Left) Extended patches* ω_j. *(Middle)* $\Gamma_{\omega_j \cap \Lambda}$*, the portion of* Γ *generated by* $\omega_j \cap \Lambda$*. (Right) the intersection between* $\Gamma_{\omega_j \cap \Lambda}$ *and* ω_j*. Here for simplicity* ω_j *is represented as a single tetrahedron but actually it is a collection of tetrahedra as shown in left panel.*

Proving the last inequality is equivalent to finding the Fortin operator $\pi_F : H^1_0(\Omega) \to X^1_{h,0}(\Omega)$, such that

$$\left(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, \mu_{\odot H}\right)_{\Lambda, |\partial\mathcal{D}|} = 0, \quad \forall v \in H^1_0(\Omega),\ \mu_{\odot H} \in Q_H\,, \tag{4.14}$$

$$\|\pi_F v\|_{H^1(\Omega)} \leq C \|v\|_{H^1(\Omega)}\,. \tag{4.15}$$

We define

$$\pi_F v = I_h v + \sum_j \alpha_j \varphi_j \qquad \text{with } \alpha_j = \frac{\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)}{\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j}$$

where $I_h : H^1(\Omega) \to X^1_{h,0}(\Omega)$ denotes an $H^1(\Omega)$-stable interpolant and $\varphi_j \in X^1_{h,0}(\Omega)$ is such that $\mathrm{supp}(\varphi_j) \subset \omega_j$, $\mathrm{supp}(\mathcal{T}_\Gamma \varphi_j) \subset \Gamma_{\omega_j \cap \Lambda} \cap \omega_j$, $\varphi_j = 0$ on $\partial\omega_j$ and

$$\int_{\omega_j \cap \Lambda} |\partial\mathcal{D}| \overline{\mathcal{T}}_\Lambda \varphi_j = \mathcal{O}(H) \text{ and } \|\nabla \varphi_j\|_{L^2(\omega_j)} = \mathcal{O}(1). \tag{4.16}$$

We notice that $\mathrm{supp}(\mathcal{T}_\Gamma \varphi_j) \subset \Gamma_{\omega_j \cap \Lambda} \cap \omega_j$ ensures that $\overline{\mathcal{T}}_\Lambda \varphi_j \subset \omega_j \cap \Lambda$. Therefore, since the interiors of $\omega_j \cap \Lambda$ are disjoint and $\varphi_j = 0$ on $\partial\omega_j$, the functions $\overline{\mathcal{T}}_\Lambda \varphi_j\ \forall j$ have all disjoint supports. Provided H is sufficiently larger that h, the functions φ_j and their traces $\mathcal{T}_\Gamma \varphi_j$ have a sufficiently large support thanks to the fact that $\mathrm{meas}(\omega_j) = \mathcal{O}(H^3)$ and $\mathrm{diam}(\Gamma_{\omega_j \cap \Lambda} \cap \omega_j) = \mathcal{O}(H)$. Owing to these properties it is possible to satisfy (4.16). Then, by construction,

$$
\begin{aligned}
\left(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda \pi_F v, \mu_{\odot H}\right)_{\Lambda,|\partial\mathcal{D}|} &= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \sum_i \alpha_i \overline{\mathcal{T}}_\Lambda \varphi_i\right] \mu_{\odot H} \\
&= \sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| \left[\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v - \alpha_j \overline{\mathcal{T}}_\Lambda \varphi_j\right] \mu_{\odot H} \\
= \sum_j \Bigg[\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)\mu_{\odot H} &- [\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)\mu_{\odot H}\Bigg] = 0.
\end{aligned}
$$

Concerning the continuity of π_F, we exploit the assumptions that the interiors of ω_j are disjoint, $\mathrm{supp}(\varphi_j) \subset \omega_j$ and the H^1-stability of I_h to show that

$$
\|\nabla \pi_F v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)} + \left(\sum_j \alpha_j^2 \|\nabla\varphi_j\|^2_{L^2(\omega_j)}\right)^{\frac{1}{2}} .
$$

For the second term, using that $\|\nabla\varphi_j\|_{L^2(\omega_j)} = \mathcal{O}(1)$, $\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda\varphi_j = \mathcal{O}(H)$ and that $|\omega_j \cap \Lambda| \le cH$, exploiting Jensen's average inequality (4.13) and trace inequality (4.12), and finally applying the approximation properties of I_h, the following upper bound holds true (where all the constants have been condensed into C),

$$
\begin{aligned}
\sum_j \alpha_j^2 \|\nabla\varphi_j\|^2_{L^2(\omega_j)} &\le C \sum_j \frac{\left(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)\right)^2}{\left(\int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|\overline{\mathcal{T}}_\Lambda\varphi_j\right)^2} \\
&\le \frac{C}{H^2} \sum_j |\omega_j\cap\Lambda| \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|^2 (\overline{\mathcal{T}}_\Lambda v - \overline{\mathcal{T}}_\Lambda I_h v)^2 \\
\le \frac{C}{H} \sum_j \|\overline{\mathcal{T}}_\Lambda (v - I_h v)\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &\le \frac{C}{H} \sum_j \|\mathcal{T}_\Gamma (v - I_h v)\|^2_{L^2(\omega_j\cap\Gamma)} \\
&\le \frac{C}{H^2} \sum_j \|v - I_h v\|^2_{L^2(\omega_j)} \le C\frac{1}{H^2}\|v - I_h v\|^2_{L^2(\Omega)} \le C\|\nabla v\|^2_{L^2(\Omega)}
\end{aligned}
$$

that is the H^1-stability of π_F. We notice that the constant in the inequality (4.15) is independent of how Λ cuts the elements of the mesh $\mathcal{T}_h^\Omega$. ◻

For the second assumption of Lemma 4.8, we recall that $b(v_h, \mu_{\odot h})$ is continuous with respect to the norms $|||v_h|||$, $\|\mu_{\odot h}\|_{L^2(\Lambda)}$. Using Lemma 4.9, and in particular the existence of a Fortin projector, there exists a constant β_h such that (the proof is analogous to the one of Lemma 2.1 in [8])

(4.17)

$$
\beta_h \|\mu_{\odot h}\|_{H^{-\frac{1}{2}}(\Lambda),|\partial\mathcal{D}|} \le \sup_{v_h\in X_h} \frac{b(v_h,\mu_{\odot h})}{|||v_h|||} + \|\mu_{\odot h} - \pi_H \mu_{\odot h}\|_{L^2(\Lambda)}, \quad \forall \mu_{\odot h} \in Q_h .
$$

We define $\pi_H = \sum_j \pi_H^j : L^2(\Lambda) \to Q_H$, where π_H^j is the operator

(4.18)
$$
\pi_H^j w_{|\omega_j\cap\Lambda} = \frac{1}{|\Gamma_{\omega_j\cap\Lambda}|} \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}| w \quad \forall j.
$$

Since $\cup_j \omega_j \cap \Lambda = \Lambda$ and $\omega_j \cap \Lambda$ are not overlapping, we obtain that π_H is an orthogonal projection, namely $(w - \pi_H w, \pi_H w) = 0$. Moreover, for any $w \in L^2(\Lambda)$ the following Poincarè inequality holds true, see for example [14, Corollary B.65],

$$\|w - \pi_H w\|_{L^2(\omega_j \cap \Lambda),|\partial\mathcal{D}|} \leq C_P H \|\partial_s w\|_{L^2(\omega_j \cap \Lambda),|\partial\mathcal{D}|} \,. \tag{4.19}$$

We consider the following stabilization operator

$$s_h(\lambda_{\odot h}, \mu_{\odot h}) = \sum_{K \in \mathcal{G}_h} \int_{\partial K \setminus \partial \mathcal{G}_h} h [\![\lambda_{\odot h}]\!] [\![\mu_{\odot h}]\!], \tag{4.20}$$

being $[\![\lambda_{\odot h}]\!]$ the jump of $\lambda_{\odot h}$ across the internal faces of $\mathcal{G}_h$. Then, we use the result of [8], Section III to show that

$$\|\mu_{\odot h} - \pi_H \mu_{\odot h}\|_{L^2(\Lambda)} \leq C s_h(\mu_{\odot h}, \mu_{\odot h}) \,,$$

which combined with (4.17) shows that the second assumption of Lemma 4.8 holds true.

The third step of the analysis consists of showing that (4.10) and (4.11) are satisfied. We introduce the following discrete norms

$$\| \, \lambda \, \|_{\pm\frac{1}{2},h,\Lambda} = \|h^{\mp\frac{1}{2}} \lambda\|_{L^2(\Lambda)},$$

recalling that h is the mesh size of $\mathcal{T}_h^{\Omega}$. We equip the space X_h with the discrete norm

$$|\!|\!|[u_h, u_{\odot\mathfrak{h}}]|\!|\!|^2_{X_h} = \|u_h\|^2_{H^1(\Omega)} + \|u_{\odot\mathfrak{h}}\|^2_{H^1(\Lambda),|\mathcal{D}|} + \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|},$$

and the space Q_H with the L^2 norm $\|\mu_{\odot H}\|_{L^2(\Lambda)}$.

Also, the function $\xi_h([v_h, v_{\odot\mathfrak{h}}]) \in Q_H \subset Q_h \subset L^2(\Lambda)$ is defined as follows

$$\xi_{h|\omega_j \cap \Lambda} = \frac{\delta}{H} \pi_H (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}})_{|\omega_j \cap \Lambda},$$

where δ is an arbitrarily small parameter. Then the following result holds true.

Lemma 4.10. *Given π_H, $s_h(\cdot,\cdot)$, ξ_h defined above, choosing δ small enough, the inequalities* (4.10) *and* (4.11) *are satisfied.*

Proof. Concerning the coercivity property (4.10), we show that $\forall [u_h, u_{\odot\mathfrak{h}}]$, there exists $\xi_h \in Q_h$ such that,

$$(u_h, u_h)_{H^1(\Omega)} + (u_{\odot\mathfrak{h}}, u_{\odot\mathfrak{h}})_{H^1(\Lambda),|\mathcal{D}|} + (\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}, \xi_h)_{\Lambda,|\partial\mathcal{D}|} \geq \alpha_\xi |\!|\!|[u_h, u_{\odot\mathfrak{h}}]|\!|\!|^2_{X_h}.$$

Using the definitions of π_H and $\xi_h([u_h, u_{\odot\mathfrak{h}}])$ previously presented and recalling that

$\xi_h \in Q_H \subset Q_h$, we obtain

$$
\begin{aligned}
&\left(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}, \xi_h\right)_{\Lambda,|\partial\mathcal{D}|} = \frac{\delta}{H}\sum_j \pi_H^j(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}) \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}) \\
&= \frac{\delta}{H}\sum_j \int_{\omega_j\cap\Lambda} |\partial\mathcal{D}|(\pi_H(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}))^2 = \frac{\delta}{H}\sum_j \|\pi_H(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}})\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&= \frac{\delta}{H}\sum_j \left(\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} - \|(\pi_H - \mathcal{I})(\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}})\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|}\right) \\
&\quad \geq \frac{\delta}{H}\sum_j \Big(\|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} - \|(\pi_H - \mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\qquad\qquad - \|(\pi_H - \mathcal{I})u_{\odot\mathfrak{h}}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|}\Big).
\end{aligned}
$$

Now, we seek an upper bound of the second and third (negative) terms of the last inequality. For the second term, we apply the additional assumption that the operators $\overline{\mathcal{T}}_\Lambda$ and ∂_s commute. This is true if the cross section $\mathcal{D}$ does not depend on the arclength s. Then, we use the Poincaré inequality (4.19), the average inequality (4.13) and the trace inequality (4.12) to show that,

$$
\begin{aligned}
\sum_j \|(\pi_H - \mathcal{I})\overline{\mathcal{T}}_\Lambda u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &\leq C_P^2 H^2 \sum_j \|\overline{\mathcal{T}}_\Lambda \partial_s u_h\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\leq C_P^2 H^2 \sum_j \|\mathcal{T}_\Gamma \partial_s u_h\|^2_{L^2(\omega_j\cap\Gamma)} \leq C_P^2 C_I^2 H \sum_j \|\nabla u_h\|^2_{L^2(\omega_j)}.
\end{aligned}
$$

For the third term, the following upper bound holds true,

$$
\begin{aligned}
\sum_j \|(\pi_H - \mathcal{I})u_{\odot\mathfrak{h}}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} &\leq C_P^2 H^2 \sum_j \|\partial_s u_{\odot\mathfrak{h}}\|^2_{L^2(\omega_j\cap\Lambda),|\partial\mathcal{D}|} \\
&\leq C_P^2 H^2 \frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|} \sum_j \|\partial_s u_{\odot\mathfrak{h}}\|^2_{L^2(\omega_j\cap\Lambda),|\mathcal{D}|}.
\end{aligned}
$$

Combining the last three inequalities, reminding that $c_h h \leq H \leq c_H^{-1} h$, we obtain

$$
\begin{aligned}
a([u_h, u_{\odot\mathfrak{h}}], [u_h, u_{\odot\mathfrak{h}}]) + b([u_h, u_{\odot\mathfrak{h}}], \xi_h([u_h, u_{\odot\mathfrak{h}}])) &\geq (1 - \delta C_P^2 C_I^2)\|\nabla u_h\|^2_{L^2(\Omega)} \\
+ \left(1 - \delta C_P^2 H \frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right) \|\partial_s u_{\odot\mathfrak{h}}\|^2_{L^2(\Lambda),|\mathcal{D}|} &+ \delta c_H \|\overline{\mathcal{T}}_\Lambda u_h - u_{\odot\mathfrak{h}}\|^2_{\frac{1}{2},h,\Lambda,|\partial\mathcal{D}|}
\end{aligned}
$$

and choosing $\delta = \frac{1}{2}\min\left[(C_P^2 C_I^2)^{-1}, \left(C_P^2 H \frac{\max|\partial\mathcal{D}|}{\min|\mathcal{D}|}\right)^{-1}\right]$ we obtain the desired inequality. Concerning inequality (4.11), the proof is analogous to the one in [8]. ∎

5. A benchmark problem with analytical solution. Let $\Omega = [0,1]^3$, $\Lambda = \{x = \frac{1}{2}\} \times \{y = \frac{1}{2}\} \times [0,1]$ and $\Omega_\ominus = [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}] \times [0,1]$. As a benchmark for the two formulations (2.4) and (2.3) we consider the case in which the source terms are defined as

$$
f = 8\pi^2 \sin(2\pi x)\sin(2\pi y), \quad \overline{\overline{g}} = \pi^2 \sin(\pi z)
$$

and q_1 for (2.4) and $\overline{q}_2$ for (2.3) are given by

$$q_1 = \sin(2\pi x)\sin(2\pi y) - \sin(\pi z), \quad \overline{q}_2 = -\sin(\pi z).$$

At the boundary $\partial\Omega$, non-homogeneous Dirichlet conditions are imposed

$$u = u_b \text{ on } \partial\Omega \quad \text{ with } u_b = \sin(2\pi x)\sin(2\pi y).$$

Under these conditions, the solution of (2.4) and (2.3) is given by

$$u = \sin(2\pi x)\sin(2\pi y), \quad u_\odot = \sin(\pi z), \quad \lambda = \lambda_\odot = 0. \tag{5.1}$$

We show that (5.1) is solution of (2.3). We notice that, regardless of the coupling constraints, u and $u_\odot$ are solutions of the following problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{5.2a}$$
$$-d^2_{zz}u_\odot = \overline{\overline{g}} \quad \text{on } \Lambda, \tag{5.2b}$$
$$u = u_b \quad \text{on } \partial\Omega. \tag{5.2c}$$

Using the integration by part formula and homogeneous boundary conditions on Ω and Λ, from (2.3) we have

$$\begin{aligned} &-(\Delta u, v)_{L^2(\Omega)} - |\mathcal{D}|(d^2_{ss}u_\odot, v_\odot)_{L^2(\Lambda)} + |\mathcal{D}|\langle \overline{v} - v_\odot, \lambda_\odot\rangle_\Lambda \\ &\qquad = (f, v)_{L^2(\Omega)} + |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)} \quad \forall v \in H^1_0(\Omega), v_\odot \in H^1_0(\Lambda). \end{aligned}$$

Since $\lambda_\odot = 0$ and the first of (5.1) satisfies (5.2a) and the second satisfies (5.2b), we have that

$$\begin{aligned} -(\Delta u, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)}, \\ -|\mathcal{D}|(d^2_{ss}u_\odot, v_\odot)_{L^2(\Lambda)} &= |\mathcal{D}|(\overline{\overline{g}}, v_\odot)_{L^2(\Lambda)}. \end{aligned}$$

Thus (5.1) satisfy equations (2.3a), (2.3b). The fact that the solution satisfy (2.3c) follows from (5.1) and the definition of $\overline{q}_2$.

We can prove in a similar way that (5.1) satisfies (2.4). Note in particular that q_1 is such that $\mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot = q_1$ on Γ.

5.1. Numerical experiments. $\mathcal{T}^\Omega_h$ conforming to Γ. Using the benchmark solution (5.1) we now investigate convergence properties of the two formulations. To this end we consider a *uniform* mesh of $\mathcal{T}^\Omega_h$ of Ω consisting of tetrahedra with diameter h. Further, the discretization shall be geometrically *conforming* to both Λ and Γ such that the meshes $\mathcal{T}^\Gamma_h$, $\mathcal{T}^\Lambda_h$ are made up of facets and edges of $\mathcal{T}^\Omega_h$ respectively, cf. Figure 5.1 for illustration.

Considering inf-sup stable discretization in terms of continuous linear Lagrange (P_1) elements (for all the spaces), Table 5.1 lists the errors of formulations (2.4) and (2.3) on the benchmark problem. It can be seen that the error in u and $u_\odot$ in H^1 norm converges linearly (as can be expected due to P_1 element discretization). Moreover, the error of the Lagrange multiplier approximation in $H^{-1/2}$ norm decreases quadratically. In the light of P_1 discretization this rate appears superconvergent. We speculate that the result is due to the fact that the exact solution is particularly simple, $\lambda = \lambda_\odot = 0$. We remark that for u and $u_\odot$ the error is interpolated into

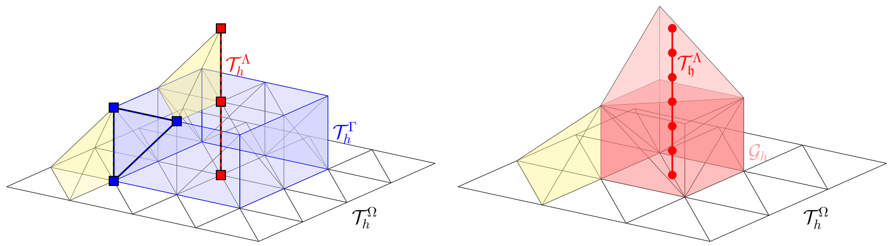

FIGURE 5.1. *(Left) The conforming discretization of Λ, Γ and Ω used for (2.4) and (2.3) is highlighted. Each cell of $\mathcal{T}_h^\Gamma$ (in blue, filled marker vertices) and $\mathcal{T}_h^\Lambda$ (in red, filled marker vertices) is a facet, respectively edge, of $\mathcal{T}_h^\Omega$ (in black, empty square marker vertices). (Right) Sample discretization of the benchmark geometry in the non-conforming case for (2.3).*

$\mathcal{T}_h^\Omega$ conforming to Γ, Λ				
h^{-1}	$\lVert u - u_h \rVert_{H^1(\Omega)}$	$\lVert u_\odot - u_{\odot h} \rVert_{H^1(\Lambda)}$	$\lVert \lambda - \lambda_h \rVert_{H^{-1/2}(\Gamma)}$	$\lVert \lambda - \lambda_h \rVert_{L^2(\Gamma)}$
4	3.4E0(−)	5.3E-1(−)	2.9E0(−)	8.7E0(−)
8	1.7E0(0.99)	2.6E-1(1.06)	6.1E-1(2.25)	1.9E0(2.21)
16	8.7E-1(0.99)	1.3E-1(1.02)	1.4E-1(2.13)	4.7E-1(1.99)
32	4.4E-1(1.00)	6.3E-2(1.00)	3.4E-2(2.03)	1.3E-1(1.80)
64	2.2E-1(1.00)	3.1E-2(1.00)	8.6E-3(2.00)	4.2E-2(1.68)
h^{-1}	$\lVert u - u_h \rVert_{H^1(\Omega)}$	$\lVert u_\odot - u_\odot \rVert_{H^1(\Lambda)}$	$\lVert \lambda_\odot - \lambda_{\odot h} \rVert_{H^{-1/2}(\Lambda)}$	$\lVert \lambda_\odot - \lambda_{\odot h} \rVert_{L^2(\Lambda)}$
4	3.1E0(−)	5.4E-1(−)	4.4E-2(−)	7.8E-2(−)
8	1.7E0(0.87)	2.6E-1(1.06)	1.1E-2(2.01)	1.9E-2(2.01)
16	8.6E-1(0.96)	1.3E-1(1.02)	2.7E-3(2.01)	4.8E-3(2.02)
32	4.4E-1(0.99)	6.3E-2(1.00)	6.7E-4(2.01)	1.2E-3(2.01)
64	2.2E-1(1.00)	3.1E-2(1.00)	1.7E-4(2.01)	3.0E-4(2.01)
128	1.1E-1(1.00)	1.6E-2(1.00)	4.1E-5(2.01)	7.4E-5(2.00)
$\mathcal{T}_h^\Omega$ non conforming to Γ, Λ				
h^{-1}	$\lVert u - u_h \rVert_{H^1(\Omega)}$	$\lVert u_\odot - u_{\odot\mathfrak{h}} \rVert_{H^1(\Lambda)}$	$\lVert \lambda_\odot - \lambda_{\odot h} \rVert_{L^2(\mathcal{G}_h)}$	
5	2.6E0(−)	2.3E-1(−)	1.7E-1(−)	
9	1.5E0(0.84)	9.4E-2(1.42)	7.1E-2(1.36)	
17	8.1E-1(0.94)	4.3E-2(1.18)	2.9E-2(1.37)	
33	4.2E-1(0.98)	2.1E-2(1.06)	7.9E-3(1.91)	
65	2.1E-1(0.99)	1.1E-2(1.02)	2.6E-3(1.64)	
129	1.1E-1(1.00)	5.2E-3(1.01)	8.5E-4(1.61)	

TABLE 5.1

Error convergence on a benchmark problem (5.2). (Top) problem (2.4), (middle) (2.3) with conforming discretization and (bottom) (2.3) in case $\mathcal{T}_h^\Omega$ does not conform to Λ using stabilized formulation (4.9). Continuous linear Lagrange elements are used for u_h, $u_{\odot h}$ and $u_{\odot\mathfrak{h}}$ and $\lambda_{\odot h}$ in conforming case, while in nonconforming case $\lambda_{\odot h}$ is piecewise constant on elements of $\mathcal{G}_h$.

the finite element space of piecewise quadratic *discontinous* functions. For (2.3) we evaluate the fractional norm and interpolate the error using piecewise continuous cubic functions. For the sake of comparison with non-conforming formulation of (2.3) from §4.2 Table 5.1 also lists the error of the Lagrange multiplier in the L^2 norm. Here, quadratic convergence is observed for (2.3). For (2.4) the rate is between 1.5 and 2.

We plot the numerical solution of problem (2.4) and (2.3) in Figure 5.2.

5.2. Numerical experiments. $\mathcal{T}_h^\Omega$ non-conforming to Γ. Using the proposed benchmark problem we consider (2.3) in the setting of §4.2. To this end we let $\mathcal{T}_h^\Omega$ be a uniform mesh of Ω such that no cell $\mathcal{T}_h^\Omega$ has any edge lying on Λ. Further we let $\mathfrak{h} = h/3$ in $\mathcal{T}_\mathfrak{h}^\Lambda$, cf. Figure 5.1.

Using discretization in terms of P_1-P_1-P_0 element Table 5.1 lists the error of the stabilized formulation of (2.3). A linear convergence in the H^1 norm can be observed in the error of u and $u_\odot$. We remark that the norms were computed as in §5.1. For simplicity the convergence of the multiplier is measured in the L^2 norm rather then the $H^{-1/2}(\Gamma)$ norm used in the analysis. Then, convergence exceeding order 1.5 can

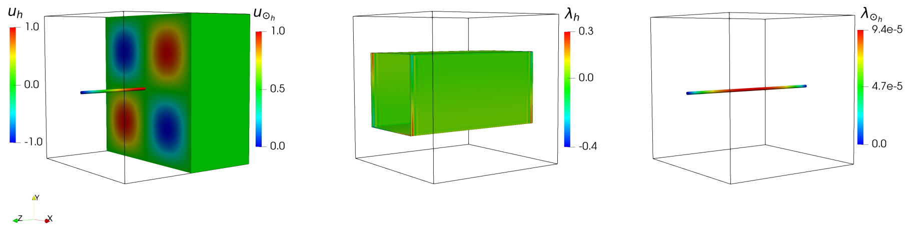

FIGURE 5.2. *Numerical solution of problem (2.4) and (2.3). (Left) functions u_h, $u_{\odot h}$ (practically identical in both problems). (Middle) Lagrange multiplier for (2.4) and (right) for (2.3).*

be observed, however, the rates are rather unstable.

5.3. Comparison. In Tables 5.1 one can observe that all the formulations yield practically identically accurate approximations of u. Further, compared to the conforming case, the stabilized formulation (2.3) results in a greater accuracy of $u_{\odot h}$ as the underlying mesh $\mathcal{T}_h^\Lambda$ is here finer. Due to the different definitions in the three formulations, comparision of the Lagrange multiplier convergence is not straightforward. We therefore limit ourselves to a comment that in the L^2 norm all the formulations yield faster than linear convergence. In order to discuss solution cost of the formulations we consider the resulting preconditioned linear systems. In particular, we shall compare spectral condition numbers and the time to convergence of the preconditioned minimal residual (MinRes) solver with the with stopping criterion requiring the relative preconditioned residual norm to be less than 10^{-8}. We remark that we shall ignore the setup cost of the preconditioner. Following operator preconditioning technique [26] we propose as preconditioners for (2.4) and (2.3) in the conforming case the (approximate) Riesz mapping with respect to the inner products of the spaces in which the two formulations were proved to be well posed. In particular, the preconditioner for the Lagrange multiplier relies on (the inverse of) the fractional Laplacian $-\Delta^{-1/2}$ on Γ for (2.4) and Λ for (2.3). A detailed analysis of the preconditioners will be presented in a separate work. We remark that in both cases the fractional Laplacian was here realized by spectral decomposition [23]. For the unfitted stabilized formulation (2.3) the Lagrange multiplier preconditioner uses a Riesz map with respect to the inner product due to $L^2(\mathcal{G}_h)$ and the stabilization (4.20), i.e.

$$(\lambda_{\odot h}, \mu_{\odot h}) \mapsto \sum_{K\in\mathcal{G}_h}\int_K \lambda_{\odot h}\mu_{\odot h} + \sum_{K\in\mathcal{G}_h}\int_{\partial K\setminus\partial\mathcal{G}_h} h[\![\lambda_{\odot h}]\!][\![\mu_{\odot h}]\!].$$

This simple choice does not yield bounded iterations. However, establishing a robust preconditioner in this case is beyond the scope of the paper and shall be pursued in the future works. In Table 5.2 we compare solution time, number of iterations and condition numbers of the (linear systems due to the) three formulations. Let us first note that the proposed preconditioners for (2.4) and (2.3) in the conforming case seem robust with respect to discretization parameter as the iteration counts and condition numbers are bounded in h. We then see that the solution time for (2.4) is about 2 times longer compared to (2.3) which is about 4 times more expensive than the solution of the Poisson problem (5.2) (which does not include any coupling, i.e. solved only for u and $u_\odot$). This is in addition to the higher setup costs of the preconditioner, which in our implementation involve solving an eigenvalue problem for the fractional Laplacian. Therefore it is advantageous to keep the multiplier space as small as possible. We remark that the missing results for (2.4) in Table 5.2 are due

l	(2.4)			(2.3)			Stabilized (2.3)			(5.2)	
	#	$T\,[s]$	κ	#	$T\,[s]$	κ	#	$T\,[s]$	κ	#	$T\,[s]$
1	20	0.03	15.56	9	0.02	3.79	21	0.01	9.70	3	< 0.01
2	35	0.06	16.28	17	0.03	6.04	31	0.03	15.87	4	< 0.01
3	38	0.14	16.64	22	0.06	8.28	53	0.15	32.93	5	0.01
4	39	1.70	16.75	24	0.89	9.42	110	4.54	61.48	5	0.12
5	38	12.04	16.78	20	5.21	6.52	232	59.43	94.25	5	0.90
6	–	–	–	17	28.77	–	507	832.90	–	6	7.75

TABLE 5.2

Cost comparison of the formulations across refinement levels l. Number of Krylov iterations (preconditioned conjugate gradient for (5.2), MinRes otherwise) and the condition number of the preconditioned problem is denoted by # and κ respectively. Time till convergence of the iterative solver (excluding the setup) is shown as T.

to the memory limitations encountered when solving the eigenvalue problem for the Laplacian, which for finest mesh involves cca 32 thousand eigenvalues, cf. Appendix C. Due to the missing proper preconditioner for the Lagrange multiplier block the number of iterations in the third, unfitted formulation can be seen to approximately double on refinement.

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Appendix A. Derivation of the model. This section provides a rigorous derivation of 3D-1D-1D problem (2.3) and 3D-1D-2D problem (2.4). The steps are similar to the derivation presented in [24], however, here the coupling conditions are different, giving rise to coupled problems featuring Lagrange multipliers. Precisely, the starting point is the problem arising from *Dirichlet-Neumann* conditions. Find $u_\oplus, u_\ominus$ s.t.:

$$-\Delta u_\oplus + u_\oplus = f \qquad \text{in } \Omega_\oplus, \tag{A.1a}$$
$$-\Delta u_\ominus + u_\ominus = g \qquad \text{in } \Omega_\ominus, \tag{A.1b}$$
$$u_\oplus - u_\ominus = q \qquad \text{on } \Gamma, \tag{A.1c}$$
$$\nabla (u_\oplus - u_\ominus) \cdot \boldsymbol{n}_\oplus = 0 \qquad \text{on } \Gamma, \tag{A.1d}$$
$$u_\oplus = 0 \qquad \text{on } \partial\Omega\,. \tag{A.1e}$$

The coupling constraints defined on Γ involve essential or strong conditions. Such conditions will be enforced weakly by using the method of Lagrange multipliers [3]. Then, the variational formulation of problem (A.1) is to find $u_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ u_\ominus \in H^1_{\partial\Omega_\ominus\setminus\Gamma}(\Omega_\ominus),\ \lambda \in H^{-\frac{1}{2}}(\Gamma)$ s.t.

$$\begin{aligned}&(u_\oplus, v_\oplus)_{H^1(\Omega_\oplus)} + (u_\ominus, v_\ominus)_{H^1(\Omega_\ominus)} + \langle v_\oplus - v_\ominus, \lambda\rangle_\Gamma \\ &\quad = (f, v_\oplus)_{L^2(\Omega_\oplus)} + (g, v_\ominus)_{L^2(\Omega_\ominus)} \quad \forall v_\oplus \in H^1_{\partial\Omega}(\Omega_\oplus),\ v_\ominus \in H^1_{\partial\Omega_\ominus\setminus\Gamma}(\Omega_\ominus),\end{aligned} \tag{A.2a}$$
$$\langle u_\oplus - u_\ominus, \mu\rangle_\Gamma = 0 \quad \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,. \tag{A.2b}$$

where λ is the Lagrange multiplier and it is equivalent to $\nabla u_\ominus \cdot \boldsymbol{n}_\ominus$.

Model reduction of the problem on $\Omega_\ominus$. We apply the averaging technique to equation (A.1b). In particular, we consider an arbitrary portion $\mathcal{P}$ of the cylinder $\Omega_\ominus$, with lateral surface $\Gamma_\mathcal{P}$ and bounded by two perpendicular sections to Λ, namely $\mathcal{D}(s_1),\ \mathcal{D}(s_2)$ with $s_1 < s_2$. We have,

$$\begin{aligned}\int_\mathcal{P} -\Delta u_\ominus + u_\ominus d\omega = -\int_{\partial\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus \, d\sigma + \int_\mathcal{P} u_\ominus d\omega = \\ \int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma - \int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma + \int_\mathcal{P} u_\ominus d\omega\end{aligned}$$

By the fundamental theorem of integral calculus

$$\int_{\mathcal{D}(s_1)} \partial_s u_\ominus d\sigma - \int_{\mathcal{D}(s_2)} \partial_s u_\ominus d\sigma = -\int_{s_1}^{s_2} d_s \int_{\mathcal{D}(s)} \partial_s u_\ominus d\sigma ds = -\int_{s_1}^{s_2} d_s \left(|\mathcal{D}(s)|\overline{\overline{\partial_s u_\ominus}}\right)$$

Moreover, we have

$$\int_{\Gamma_\mathcal{P}} \nabla u_\ominus \cdot \boldsymbol{n}_\ominus d\sigma = \int_{\Gamma_\mathcal{P}} \lambda \, d\sigma = \int_{s_1}^{s_2} \int_{\partial\mathcal{D}(s)} \lambda d\gamma \, ds = \int_{s_1}^{s_2} |\partial\mathcal{D}(s)|\overline{\lambda} \, ds\,.$$

From the combination of all the above terms with the right hand side, we obtain that the solution $u_\ominus$ of (A.1b) satisfies,

$$\int_{s_1}^{s_2} \left[-d_s(|\mathcal{D}(s)|\overline{\overline{\partial_s u_\ominus}}) + |\mathcal{D}(s)|\overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)|\overline{\lambda} - |\mathcal{D}(s)|\overline{\overline{g}}\right] ds = 0.$$

Since the choice of the points s_1, s_2 is arbitrary, we conclude that the following equation holds true,

$$-d_s(|\mathcal{D}(s)|\overline{\overline{\partial_s u_\ominus}}) + |\mathcal{D}(s)|\overline{\overline{u}}_\ominus - |\partial\mathcal{D}(s)|\overline{\lambda} = |\mathcal{D}(s)|\overline{\overline{g}} \quad \text{on } \Lambda\,, \tag{A.3}$$

which is complemented by the following conditions at the boundary of Λ,

$$|\mathcal{D}(s)|\overline{\overline{\partial_s u_\ominus}} = 0, \quad \text{on} \quad s = 0, S. \tag{A.4}$$

Then, we consider variational formulation of the averaged equation (A.3). After multiplication by a test function $v_\odot \in H^1(\Lambda)$, integration on Λ and suitable application of integration by parts, we obtain,

$$\begin{aligned}\int_\Lambda |\mathcal{D}(s)|\overline{\overline{\partial_s u_\ominus}} d_s v_\odot\, ds - (|\mathcal{D}(s)|\overline{\overline{\partial_s u_\ominus}})v_\odot|_{s=0}^{s=S} - \int_\Lambda |\partial\mathcal{D}(s)|\overline{\lambda} v_\odot\, ds + \int_\Lambda |\mathcal{D}(s)|\overline{\overline{u}}_\ominus v_\odot \\ = \int_\Lambda |\mathcal{D}(s)|\overline{\overline{g}} V\, ds\,.\end{aligned}$$

Using boundary conditions, we obtain,

$$(\overline{\overline{\partial_s u_\ominus}}, d_s v_\odot)_{\Lambda,|\mathcal{D}|} + (\overline{\overline{u}}_\ominus, v_\odot)_{\Lambda,|\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda,|\partial\mathcal{D}|} = (\overline{\overline{g}}, V)_{\Lambda,|\mathcal{D}|}\,. \tag{A.5}$$

Let us now formulate the modelling assumption that allows us to reduce equation (A.5) to a solvable one-dimensional (1D) model.

We assume that the function $u_\ominus$ has a *uniform profile* on each cross section $\mathcal{D}(s)$, namely $u_\ominus(r, s, t) = u_\odot(s)$. Therefore, observing that $u_\odot = \overline{u}_\ominus = \overline{\overline{u}}_\ominus$, and that $\overline{\overline{\partial_s u_\ominus}} = \overline{\overline{\partial_s u_\odot}} = d_s u_\odot$, problem (A.5) turns out to: find $u_\odot \in H^1(\Lambda)$ such that

$$(d_s u_\odot, d_s v_\odot)_{\Lambda,|\mathcal{D}|} + (u_\odot, v_\odot)_{\Lambda,|\mathcal{D}|} - (\overline{\lambda}, v_\odot)_{\Lambda,|\partial\mathcal{D}|} = (\overline{\overline{g}}, v_\odot)_{\Lambda,|\mathcal{D}|} \quad \forall v_\odot \in H^1(\Lambda)\,. \tag{A.6}$$

Topological model reduction of the problem on $\Omega_\oplus$. We focus here on the subproblem of (A.1a) related to $\Omega_\oplus$. We multiply both sides of (A.1a) by a test function $v \in H_0^1(\Omega)$ and integrate on $\Omega_\oplus$. Integrating by parts and using boundary and interface conditions, we obtain

$$\begin{aligned}\int_{\Omega_\oplus} fv\, d\omega &= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v\, d\omega - \int_{\partial\Omega_\oplus} \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v\, d\sigma + \int_{\Omega_\oplus} u_\oplus v\, d\omega \\ &= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v\, d\omega - \int_\Gamma \nabla u_\oplus \cdot \boldsymbol{n}_\oplus v\, d\sigma + \int_{\Omega_\oplus} u_\oplus v\, d\omega \\ &= \int_{\Omega_\oplus} \nabla u_\oplus \cdot \nabla v\, d\omega + \int_\Gamma \lambda v\, d\sigma + \int_{\Omega_\oplus} u_\oplus v\, d\omega.\end{aligned}$$

Then, we make the following modelling assumption: we identify the domain $\Omega_\oplus$ with the entire Ω, and we correspondingly omit the subscript $\oplus$ to the functions defined on $\Omega_\oplus$, namely

$$\int_{\Omega_\oplus} u_\oplus\, d\omega \simeq \int_\Omega u\, d\omega\,.$$

Therefore, we obtain

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega + (\lambda, v)_\Gamma = (f, v)_\Omega$$

550 and combining with (A.6) we obtain the first formulation of the reduced problem.

Hence, we have obtained the Problem 3D-1D-2D, equation (2.4): Find $u \in H_0^1(\Omega),\ \lambda \in H^{-\frac{1}{2}}(\Gamma),\ u_\odot \in H_0^1(\Lambda)$, such that

$$
\begin{aligned}
&(u,v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} + \langle \mathcal{T}_\Gamma v - \mathcal{E}_\Gamma v_\odot, \lambda\rangle_\Gamma \\
&\qquad = (f,v)_{L^2(\Omega)} + (\overline{\overline{g}}, v_\odot)_{L^2(\Lambda),|\mathcal{D}|}\,, && \forall v \in H_0^1(\Omega),\ v_\odot \in H^1(\Lambda)\,,\\
&\langle \mathcal{T}_\Gamma u - \mathcal{E}_\Gamma u_\odot, \mu\rangle_\Gamma = \langle q, \mu\rangle_\Gamma\,, && \forall \mu \in H^{-\frac{1}{2}}(\Gamma)\,.
\end{aligned}
$$

This coupled problem is classified as 3D-1D-2D because the unknowns u, $u_\odot$, λ belong to $\Omega \subset \mathbb{R}^3$, $\Lambda \subset \mathbb{R}$ and $\Gamma \subset \mathbb{R}^2$ respectively. Then, we apply a topological model reduction of the interface conditions, namely we go from a 3D-1D-2D formulation involving sub-problems on Ω and Λ and coupling operators defined on Γ to a 3D-1D-1D formulation where the coupling terms are set on Λ. To this purpose, let us write the Lagrange multiplier and the test functions on every cross section $\partial\mathcal{D}(s)$ as their average plus some fluctuation,

$$
\lambda = \overline{\lambda} + \tilde{\lambda}, \qquad v = \overline{v} + \tilde{v}, \quad \text{on } \partial\mathcal{D}(s)\,,
$$

where $\overline{\tilde{\lambda}} = \overline{\tilde{v}} = 0$. Therefore, the coupling term on Γ can be decomposed as,

$$
\int_\Gamma \lambda v \, d\sigma = \int_\Lambda \int_{\partial\mathcal{D}(s)} (\overline{\lambda} + \tilde{\lambda})(\overline{v} + \tilde{v}) d\gamma ds = \int_\Lambda |\partial\mathcal{D}(s)| \overline{\lambda}\overline{v}\, ds + \int_\Lambda \int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma ds\,.
$$

Thanks to the additional assumption that the product of fluctuations is small,

$$
\int_{\partial\mathcal{D}(s)} \tilde{\lambda}\tilde{v} d\gamma \simeq 0
$$

the term $(\mathcal{T}_\Gamma v, \lambda)_\Gamma$ becomes $\left(\overline{\mathcal{T}}_\Lambda v, \overline{\lambda}\right)_{\Lambda,|\partial\mathcal{D}|}$, where $\overline{\mathcal{T}}_\Lambda$ denotes the composition of operators $\overline{(\cdot)} \circ \mathcal{T}_\Gamma$. Combined with (A.6), this leads to the 3D-1D-1D formulation of the reduced problem, namely equation (2.3): find $u \in H_0^1(\Omega)$, $u_\odot \in H_0^1(\Lambda)$, $\lambda_\odot \in H^{-\frac{1}{2}}(\Lambda)$, such that

$$
\begin{aligned}
&(u,v)_{H^1(\Omega)} + (u_\odot, v_\odot)_{H^1(\Lambda),|\mathcal{D}|} + \langle \overline{\mathcal{T}}_\Lambda v - v_\odot, \lambda_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} \\
&\qquad = (f,v)_{L^2(\Omega)} + (\overline{\overline{g}}, V)_{L^2(\Lambda),|\mathcal{D}|}\,, && \forall v \in H_0^1(\Omega),\ v_\odot \in H_0^1(\Lambda)\,,\\
&\langle \overline{\mathcal{T}}_\Lambda u - u_\odot, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|} = \langle q, \mu_\odot\rangle_\Gamma = \langle \overline{q}, \mu_\odot\rangle_{\Lambda,|\partial\mathcal{D}|}\,, && \forall \mu_\odot \in H^{-\frac{1}{2}}(\Lambda)\,.
\end{aligned}
$$

551 **Appendix B. Proof of Lemma 2.1.**

Proof. Let us consider the eigenvalue problem for the Laplace operator on Γ with homogeneous Dirichlet conditions at $x = 0$, X and periodic boundary conditions at $y = 0$, Y. Let us also consider the Laplace eigenproblem on $(0, X)$ with homogeneous Dirichlet conditions. Let us denote as $\phi_{ij}(x,y)$ and ρ_{ij}, for $i = 1, 2, \ldots$, $j = 0, 1, \ldots$, the eigenfunctions and the eigenvalues of the Laplacian on Γ, and with $\phi_i(x)$ and ρ_i the eigenfunctions and the eigenvalues of the Laplacian on $(0, X)$. In particular,

$$
\begin{aligned}
&\phi_{ij}(x,y) = \sin\left(\frac{i\pi x}{X}\right)\left(\cos\left(\frac{j2\pi y}{Y}\right) + \sin\left(\frac{j2\pi y}{Y}\right)\right), && \rho_{ij} = \left(\frac{i\pi}{X}\right)^2 + \left(\frac{j2\pi}{Y}\right)^2,\\
&\phi_i(x) = \sin\left(\frac{i\pi x}{X}\right), && \rho_i = \left(\frac{i\pi}{X}\right)^2.
\end{aligned}
$$

We use here the following representation of the fractional norms,

$$
\text{(B.1)}\qquad
\begin{aligned}
\|u\|_{H^{\frac{1}{2}}_{00}(\Lambda)} &= \left(\sum_{i=1}^{\infty} (1+\rho_i)^{\frac{1}{2}}\, |a_i|^2\right)^{\frac{1}{2}},\\
\|u\|_{H^{\frac{1}{2}}_{00}(\Gamma)} &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}} |a_{i,j}|^2
\end{aligned}
$$

with $a_i = (u,\phi_i)_\Lambda$ and $a_{ij} = (u,\phi_{ij})_\Gamma$. It is easy to verify that

$$
\text{(B.2)}\qquad \int_0^Y \phi_{ij}(x,y) = 0 \quad \forall j>0, \forall i\,, \qquad \int_0^Y \phi_{ij}(x,y) = Y\,\sin\left(\frac{i\pi x}{X}\right) \quad \text{if } j=0, \forall i.
$$

Moreover we recall that $\phi_{i,j}(x,y)$ and $\phi_i(x)$ form an orthogonal basis of $L^2(\Gamma)$ and $L^2(0,X)$ respectively. Therefore,

$$
\overline{u}(x) = \frac{1}{Y}\int_0^Y u(x,y)\,dy = \frac{1}{Y}\sum_{i,j} a_{i,j}\int_0^Y \phi_{i,j}(x,y)\,dy = \sum_i a_{i,0}\phi_i(x).
$$

Let the constant C be equal to $C = C(X) = \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}$. Then, from (B.1) we have

$$
\begin{aligned}
\text{(B.3)}\qquad \|\overline{u}\|^2_{H^{\frac{1}{2}}_{00}(0,X)} &= \sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}\, a_i^2\\
&= C\left(\int_0^X \overline{u}(x)\sin\left(\frac{i\pi x}{X}\right)\,dx\right)^2 = C\left(\sum_{j=1}^{\infty} a_{j,0}\int_0^X \sin\left(\frac{j\pi x}{X}\right)\sin\left(\frac{i\pi x}{X}\right)\,dx\right)\\
&= \sum_{i=1}^{\infty}\frac{X^2}{4}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}} a_{i,0}^2 \le \frac{X^2}{4}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}|a_{i,j}|^2\\
&= \frac{X^2}{4}\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)},
\end{aligned}
$$

where we have used the orthogonality property

$$
\int_0^X \sin\left(\frac{i\pi x}{X}\right)\sin\left(\frac{j\pi x}{X}\right)\,dx = \begin{cases} 0 & i\neq j\\ \frac{X}{2} & i=j\end{cases}
$$

and we have applied (B.1) in the last equality. As a result of the previous inequality, we have proved the first statement of the Corollary, namely $u\in H^{\frac{1}{2}}_{00}(\Gamma)\to\overline{u}\in H^{\frac{1}{2}}_{00}(\Lambda)$.

The second statement of the Corollary addresses the case of the function u constant with respect to y. Precisely, we have

$$
\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)} &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(1+\rho_{ij})^{\frac{1}{2}}\,|a_{ij}|^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X\int_0^Y u(x,y)\phi_{ij}(x,y)\right)^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2+\left(\frac{j2\pi}{Y}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X u(x)\int_0^Y \phi_{ij}(x,y)\right)^2,
\end{aligned}
$$

and using (B.2) we obtain

$\square$

$$
\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)} &= \sum_{i=1}^{\infty}\left(1+\left(\frac{i\pi}{X}\right)^2\right)^{\frac{1}{2}}\left(\int_0^X Y u(x)\,\sin\left(\frac{i\pi x}{X}\right)\right)^2 \\
&= Y^2\sum_{i=1}^{\infty}(1+\rho_i)^{\frac{1}{2}}\,|a_i|^2 = Y^2\|u\|^2_{H^{\frac{1}{2}}_{00}(0,X)}.
\end{aligned}
$$

Appendix C. System sizes in benchmark formulations. In Table C.1 we list dimensions of the finite element spaces used to discretize formulations (2.4), (2.3) and stabilized (2.3) on different levels of refinement. The number of degrees of freedom in subspace $W_{i,h}$ is denote as $|W_{i,h}|$. We recall that the discrete spaces are $X^1_{h,0}(\Omega)\times X^1_{h,0}(\Lambda)\times Q_h(\Gamma)$ for the 3D-1D-2D problem (2.4), $X^1_{h,0}(\Omega)\times X^1_{h,0}(\Lambda)\times Q_h(\Lambda)$ for the 3D-1D-1D problem (2.3), and $X^1_{h,0}(\Omega)\times X^1_{\mathfrak{h},0}(\Lambda)\times Q_h(\mathcal{G}_h)$ for the stabilized 3D-1D-1D problem.

l	(2.4)			(2.3)			Stabilized (2.3)		
	$\lvert W_{1,h}\rvert$	$\lvert W_{2,h}\rvert$	$\lvert W_{3,h}\rvert$	$\lvert W_{1,h}\rvert$	$\lvert W_{2,h}\rvert$	$\lvert W_{3,h}\rvert$	$\lvert W_{1,h}\rvert$	$\lvert W_{2,\mathfrak{h}}\rvert$	$\lvert W_{3,h}\rvert$
1	125	5	40	125	5	5	180	13	24
2	729	9	144	729	9	9	900	25	48
3	4913	17	544	4913	17	17	5508	49	96
4	35937	33	2112	35937	33	33	38148	97	192
5	275K	65	8320	275K	65	65	283K	193	384
6	–	–	-	2.15M	129	129	2.18M	385	768

Table C.1
Number of degrees of freedom of the discrete spaces used in the numerical experiments.

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