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Trace inequalities for piecewise W^1,p functions over general polytopic meshes

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Trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions over general polytopic meshes

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Abstract

Trace inequalities are crucial tools to derive the stability of partial differential equations with inhomogeneous, natural boundary conditions. In the analysis of corresponding Galerkin methods, they are also essential to show convergence of sequences of discrete solutions to the exact one for data with minimal regularity under mesh refinements and/or degree of accuracy increase. In nonconforming discretizations, such as Crouzeix-Raviart and discontinuous Galerkin, the trial and test spaces consists of functions that are only piecewise continuous: standard trace inequalities cannot be used in this case. In this work, we prove several trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions. Compared to analogous results already available in the literature, our inequalities are established: (i) on fairly general polytopic meshes (with arbitrary number of facets and arbitrarily small facets); (ii) without the need of finite dimensional arguments (e.g., inverse estimates, approximation properties of averaging operators); (iii) for different ranges of maximal and nonmaximal Lebesgue indices.

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1 Introduction

Overture. Trace inequalities are prominent in the analysis of numerical methods for partial differential equations; they are particularly relevant, e.g., in the proof of well-posedness and convergence in presence of inhomogeneous, natural boundary conditions. While standard trace inequalities are typically enough for conforming methods, one needs more refined counterparts for nonconforming schemes, which employ piecewise regular functions. Such inequalities take the rough form (p and q are Lebesgue indices discussed in Section 2.1 below; ∇_h denotes the broken gradient with respect to a decomposition of the domain Ω as in Section 2.2 below; $|\cdot|_{J,p,q}$ is related to jump terms in the sense of (10a) below)

$$\|u\|_{\mathrm{L}^q(\partial\Omega)} \lesssim \|\nabla_h u\|_{\mathbf{L}^p(\Omega)} + |u|_{J,p,q}.$$

An example. With the notation as in (3) below, we consider the p-Laplacian problem, p in $(1,\infty)$,

$$\begin{cases} \text{find } u : \Omega \to \mathbb{R} \text{ such that} \\ -\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ |\nabla u|^{p-2}\nabla u \cdot \mathbf{n} = g_N & \text{on } \Gamma_N. \end{cases} \tag{1}$$

Given Lebesgue indices as in (5) below, for g_N in $\mathrm{L}^{(p^\sharp)'}(\Gamma_N)$, a weak formulation of (1) reads

$$\begin{cases} \text{find } u \in V := W^{1,p}_{\Gamma_D}(\Omega) \text{ such that} \\ (|\nabla u|^{p-2}\nabla u, \nabla v)_{0,\Omega} = (g_N, v)_{0,\Gamma_N} \qquad \forall v \in V. \end{cases} \tag{2}$$

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Taking $v = u$ in (2), Hölder's inequality entails

$$|u|^p_{\mathrm{W}^{1,p}(\Omega)} \le \|g_N\|_{\mathrm{L}^{(p^\sharp)'}(\Gamma_N)} \|u\|_{\mathrm{L}^{p^\sharp}(\Gamma_N)}.$$

The Sobolev embedding $\mathrm{W}^{\frac{1}{p'},p}(\Gamma_N) \hookrightarrow \mathrm{L}^{p^\sharp}(\Gamma_N)$, the trace inequality $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{W}^{\frac{1}{p'},p}(\Gamma_N)$, and a Poincaré inequality in $\mathrm{W}^{1,p}_{\Gamma_D}(\Omega)$ (the overall constant being C) give

$$|u|^p_{\mathrm{W}^{1,p}(\Omega)} \le C \|g_N\|_{\mathrm{L}^{(p^\sharp)'}(\Gamma_N)} |u|_{\mathrm{W}^{1,p}(\Omega)} \qquad \Longrightarrow \qquad |u|_{\mathrm{W}^{1,p}(\Omega)} \le C^{\frac{1}{p-1}} \|g_N\|^{\frac{1}{p-1}}_{\mathrm{L}^{(p^\sharp)'}(\Gamma_N)}.$$

If we consider a $\mathrm{W}^{1,p}$-conforming finite element discretization of (2), see, e.g., [3], then we deduce analogously a bound of the form (here u_h is the conforming finite element solution over a given simplicial/Cartesian mesh and for an arbitrary polynomial degree)

$$|u_h|_{\mathrm{W}^{1,p}(\Omega)} \le C^{\frac{1}{p-1}} \|g_N\|^{\frac{1}{p-1}}_{\mathrm{L}^{(p^\sharp)'}(\Gamma_N)}.$$

This is the starting point for proving convergence of the finite element method over sequences of meshes under minimal data regularity; cf., e.g., [8]. Convergence for nonconforming (e.g., discontinuous Galerkin, Crouzeix-Raviart) methods require more sophisticated trace inequalities since these schemes are based on finite dimensional spaces that are only piecewise $\mathrm{W}^{1,p}$; the constants appearing therein should depend neither on the size of the elements in the mesh nor on the type of discretization space that is used.

State-of-the-art. For simplicial meshes, Girault and Wheeler [12, Proposition 5] proved a trace inequality for lowest order Crouzeix–Raviart elements based on a Scott–Zhang-type regularization argument; they showed that traces of discrete functions have the same Lebesgue regularity as that appearing in the optimal conforming case. Buffa and Ortner [5, Theorem 4.4] generalized those results to the case of arbitrary order piecewise polynomial spaces, again with maximal Lebesgue regularity; the main technical tool here is the stability of a lifting operator, whose stability properties require polynomial inverse inequalities. Analogous results are given in [14, Lemma 3.6], where the main technical tool in the proof is the lowest order Crouzeix-Raviart interpolant. Trace inequalities for broken Sobolev spaces were extended to regular polytopic meshes in [10, Lemma B.24, eq. (B.58)] for lowest order and [7, Theorem 6.7] for general order broken polynomial spaces; the Lebesgue indices of the boundary and jump terms are the same as that of the broken gradient and again inverse estimates and averaging arguments are the lynchpin of the analysis therein. In the recent work [2], novel discrete trace inequalities for hybrid methods were proved, which involve a mesh-dependent $H^{\frac{1}{2}}$-seminorm on the boundary. All the references above are based on finite dimensional arguments and consider either simplicial meshes or regular polytopic meshes.

Main results and advances. The three main results of this work are Theorems 3.3, 3.4, and 3.5 below; each of them establishes trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions involving different types of Lebesgue indices; their applicability in the analysis of nonconforming methods is discussed in Remark 3 below. Their proofs are not based on finite dimensional arguments (e.g., polynomial inverse estimates, smoothing/averaging operators) but only on direct estimates; moreover, they hold true on fairly general meshes (with, e.g., arbitrarily small facets, arbitrary number of facets per element). The constants appearing in these inequalities are provided, which are: (*i*) explicit in terms of constants of other elementary inequalities (e.g., Sobolev-Poincaré inequalities for broken Sobolev spaces); (*ii*) certain "regularity" parameters of a given mesh (cf. Section 2.2); (*iii*) the involved Lebesgue indices. The first dependence is essential in the proof of convergence for the p- and hp-versions of nonconforming methods for linear and nonlinear problems, since inverse estimates typically result in bounds involving the polynomial degree of the scheme; it is also essential for nonpolynomial methods. The latter dependence is crucial while performing refinement/coarsening within an adaptive mesh refinement; meshes with elements obtained by merging smaller simplicial elements are fully covered by our theory. Another advancement of this manuscript is that we exhibit different types of trace inequalities allowing for maximal and nonmaximal Lebesgue regularity.

Structure of the paper. Standard Sobolev spaces and their broken versions on certain classes of polytopic meshes, and spaces of functions with bounded variation (BV) are detailed in Section 2; there, we also review related inequalities, including standard trace inequalities, Sobolev-Poincaré inequalities for broken $\mathrm{W}^{1,p}$ spaces, and bounds on the BV norm in terms of broken Sobolev norms. Section 3 is concerned with the proof of the novel trace inequalities for broken Sobolev spaces, based on some novel technical results.

2 Polytopic meshes, broken spaces, and related inequalities

We consider an open, bounded, Lipschitz domain

$$\Omega \subset \mathbb{R}^d,\ d \text{ in } \mathbb{N}\setminus\{1\}, \text{ with boundary } \Gamma := \partial\Omega, \text{ which we split into } \Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}},\ |\Gamma_{\mathrm{D}}| \neq \emptyset. \qquad (3)$$

We assume that Ω

- is either star-shaped with respect to a ball B_ρ of radius ρ; the diameter of Ω is h_Ω;
- or admits a shape-regular decomposition into simplices;
- or is the union of a finite number of star-shaped domains.

The regularity of Ω plays a role only in the proof of Lemma 2.1 below; cf. [4].

This section is structured as follows: in Section 2.1 we set the notation related to Lebesgue and Sobolev spaces, and introduce some Lebesgue indices; in Section 2.2 we introduce a class of fairly general polytopic meshes, broken Sobolev spaces, and jump operators; corresponding broken Sobolev seminorms and norms, along with their basic properties are detailed in Section 2.3; the space of functions of bounded variation, its norm, and the relation with broken Sobolev norms are discussed in Section 2.4.

2.1 Lebesgue and Sobolev spaces, and special Lebesgue indices

For a generic subset X of Ω with diameter h_X, and for all p in $[1,\infty)$, we consider the Lebesgue and Sobolev spaces $\mathrm{L}^p(X)$ and $\mathrm{W}^{1,p}(X)$, which we endow with norm, and seminorm and norm, respectively,

$$\|v\|^p_{\mathrm{L}^p(X)} := \int_X |v|^p,$$

and

$$|v|_{\mathrm{W}^{1,p}(X)} := \|\nabla v\|_{\mathbf{L}^p(X)}, \qquad \|v\|^p_{\mathrm{W}^{1,p}(X)} := h_X^{-p}\|v\|^p_{\mathrm{L}^p(X)} + |v|^p_{\mathrm{W}^{1,p}(X)}.$$

Vector version Sobolev spaces, norms, and seminorms are denoted by $\mathbf{W}^{1,p}(X)$, $\|\cdot\|_{\mathbf{W}^{1,p}(X)}$, and $|\cdot|_{\mathbf{W}^{1,p}(X)}$, respectively.

We define the subspace of $\mathrm{W}^{1,p}(X)$ of functions with zero trace over Γ_D:

$$\mathrm{W}^{1,p}_{\Gamma_D}(X) := \{v \in \mathrm{W}^{1,p}(X) \mid v_{|\Gamma_D} = 0\}.$$

We further recall the following Sobolev embeddings [11, Sect. 2.3]:

- if $p < d$, $\mathrm{W}^{1,p}(X) \hookrightarrow \mathrm{L}^q(X)$ for all q in $[p, \frac{pd}{d-p}]$;
- if $p = d$, $\mathrm{W}^{1,p}(X) \hookrightarrow \mathrm{L}^q(X)$ for all q in $[p,\infty)$.

We spell out the generic Sobolev embedding bound: for p and q as above, there exists a positive constant $C_{\mathrm{Sob}}(q,p,X)$ such that

$$\|v\|_{\mathrm{L}^q(X)} \le C_{\mathrm{Sob}}(q,p,X) h_X^{\frac{d}{q}-\frac{d}{p}+1} \|v\|_{\mathrm{W}^{1,p}(X)} \qquad \forall v \in \mathrm{W}^{1,p}(X). \qquad (4)$$

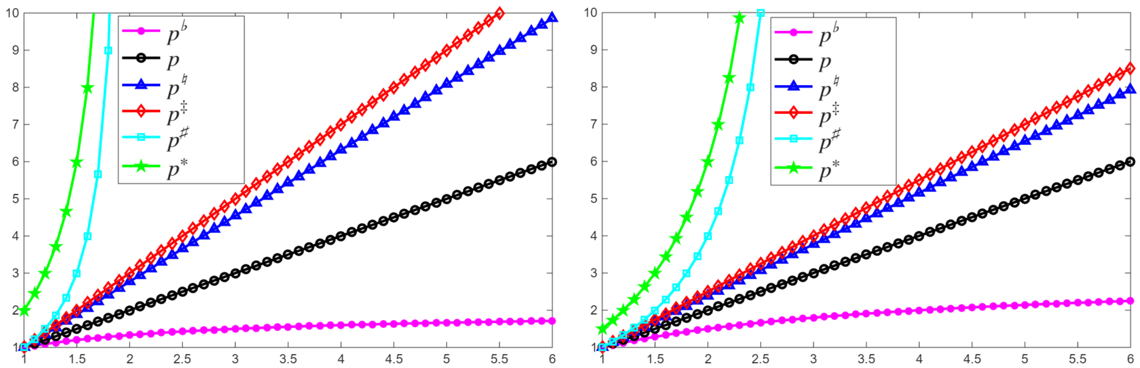
Given an index p in $[1,\infty)$, we define

$$
\begin{aligned}
&p' := \frac{p}{p-1}; \qquad && p^* := \begin{cases} \frac{pd}{d-p} & \text{if } p<d \\ \infty & \text{otherwise;} \end{cases} \qquad && p^\sharp := \begin{cases} \frac{p(d-1)}{d-p} & \text{if } p<d \\ \infty & \text{otherwise;} \end{cases} \\
&p^\ddagger := \frac{pd-1}{d-1}; \qquad && p^\natural := p + \frac{p-1}{(d-1)p'}; \qquad && p^\flat := \frac{pd}{p+d-1}.
\end{aligned} \tag{5}
$$

Observe

$$
1 \le p^\flat \le p \le p^\natural \le p^\ddagger \le p^\sharp \le p^*, \qquad (p^\flat)^\sharp = p = (p^\sharp)^\flat, \qquad \begin{cases} p^\sharp \le p1^* & \text{if } p \le \frac{2d-1}{d} \\ p^\sharp \ge p1^* & \text{if } p \ge \frac{2d-1}{d}. \end{cases} \tag{6}
$$

In Figure 1, we report the behaviour of the indices in (5) for p varying in $[1,6]$ with step $1/10$ in dimensions $d=2$ (left) and $d=3$ (right).



(a) $d=2$. (b) $d=3$.

Figure 1: Behaviour of the indices in (5) for p varying in $[1,6]$ with step $1/10$.

Remark 1 (Meaning of the Lebesgue indices). For a given p larger than 1: p' is its conjugate index; p^* is the maximal Lebesgue regularity of functions in $\mathrm{W}^{1,p}(\Omega)$; $p^\sharp$ is the maximal Lebesgue regularity of the trace on $\partial\Omega$ of functions in $\mathrm{W}^{1,p}(\Omega)$; $p^\ddagger$ is a technical index appearing in Theorem 3.4 below; $p^\natural$ is a technical index appearing in Theorem 3.3 below; $p^\flat$ is the "dual" of $p^\sharp$. ■

2.2 Meshes and broken Sobolev spaces

We consider families of meshes $\{\mathcal{T}_h\}$, where each $\mathcal{T}_h$ is a finite collection of disjoint, closed, polytopic elements such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. For each K in $\mathcal{T}_h$, ∂K and h_K denote the boundary and the diameter of K, respectively. We associate each $\mathcal{T}_h$ with a set $\mathcal{F}_h$ covering the mesh skeleton, i.e., $\bigcup_{K\in\mathcal{T}_h} \partial K = \bigcup_{F\in\mathcal{F}_h} F$. A facet F in $\mathcal{F}_h$ is a hyperplanar, closed, and connected subset of $\overline{\Omega}$ with positive measure such that

- either there exist distinct $K_{1,F}$ and $K_{2,F}$ in $\mathcal{T}_h$ such that $F \subseteq \partial K_{1,F} \cap \partial K_{2,F}$ and F is called an *internal facet*,
- or there exists K_F in $\mathcal{T}_h$ such that $F \subseteq \partial K_F \cap \partial\Omega$ and F is called a *boundary facet*.

The notation h_F is used for the diameter of the facet F in $\mathcal{F}_h$. Interfaces and boundary facets are collected in the subsets $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$, respectively. The set of facets of an element K is $\mathcal{F}^K$; we also define $\mathcal{F}^{KD} := \mathcal{F}^K \cap \mathcal{F}_h^D$ and $\mathcal{F}^{KI} := \mathcal{F}^K \cap \mathcal{F}_h^I$.

Following [6, Assumption 2.1], we demand that, for all K in any $\mathcal{T}_h$, there exists a partition $\mathfrak{T}_K$ of K into non-overlapping d-dimensional simplices. Moreover, we assume that there exists a universal, positive constant γ such that, for all K in any $\mathcal{T}_h$ and all T in $\mathfrak{T}_K$ with $\partial T \cap \partial K \neq \emptyset$, given F_K^T the $(d-1)$-dimensional simplex $\partial T \cap \partial K$,

$$
\gamma h_K \le d|T||F_K^T|^{-1}. \tag{7}
$$

Given

$$\widetilde{h}_F \qquad \text{either equal to } h_F \text{ or equal to} \qquad \begin{cases} \min(h_{K_1}, h_{K_2}) & \text{if } F \in \mathcal{F}_h^I,\ F \subset \partial K_1 \cap \partial K_2 \\ h_K & \text{if } F \in \mathcal{F}_h^D,\ F \in \mathcal{F}^K, \end{cases} \tag{8}$$

the following geometric bounds are a consequence of (7) and (8)

$$\gamma h_K |\partial K| \le d|K| \le d h_K^d, \qquad\qquad |F| \le h_F^{d-1} \le \widetilde{h}_F^{d-1} \le h_K^{d-1}. \tag{9}$$

Families of shape-regular simplicial and Cartesian meshes fall into the setting above. Broken Sobolev spaces associated with $\mathcal{T}_h$ are defined as

$$\mathrm{W}^{1,p}(\mathcal{T}_h) := \left\{ u \in \mathrm{L}^p(\Omega) \mid u_{|K} \in \mathrm{W}^{1,p}(K) \qquad \forall K \in \mathcal{T}_h \right\}.$$

We denote the piecewise gradient over $\mathcal{T}_h$ by ∇_h. For every v in $\mathrm{W}^{1,p}(\mathcal{T}_h)$ and F in $\mathcal{F}_h$, the jump operator on F is given by

$$[\![v]\!]_F := \begin{cases} v_{|K_{1,F}} \mathbf{n}_{K_{1,F}} \cdot \mathbf{n}_F + v_{|K_{2,F}} \mathbf{n}_{K_{2,F}} \cdot \mathbf{n}_F & \text{if } F \in \mathcal{F}_h^I,\ F \subset \partial K_{1,F} \cap \partial K_{2,F} \\ v_{|F} \mathbf{n}_{K_F} \cdot \mathbf{n}_F & \text{if } F \in \mathcal{F}_h^B,\ F \subset \partial K_F \cap \partial \Omega. \end{cases}$$

We omit the subscript F whenever it is clear from the context.

2.3 Broken seminorms and norms, and technical tools

We endow the space $\mathrm{W}^{1,p}(\mathcal{T}_h)$ with the seminorms

$$|v|_{\mathrm{W}^{1,p;q}(\mathcal{T}_h)} := \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + \Big(\sum_{F \in \mathcal{F}_h^I} \widetilde{h}_F^{-\frac{p}{q'} - \frac{dp}{p'} + \frac{dp}{q'}} \|[\![v]\!]\|_{\mathrm{L}^q(F)}^p \Big)^{\frac{1}{p}}, \tag{10a}$$

$$\|v\|_{\mathrm{W}^{1,p;q}_{\Gamma_D}(\mathcal{T}_h)} := \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + \Big(\sum_{F \in \mathcal{F}_h^{ID}} \widetilde{h}_F^{-\frac{p}{q'} - \frac{dp}{p'} + \frac{dp}{q'}} \|[\![v]\!]\|_{\mathrm{L}^q(F)}^p \Big)^{\frac{1}{p}}. \tag{10b}$$

Remark 2 (Comparison of broken seminorms). From definition (10b) direct Lebesgue embeddings on facets, and the second chain of inequalities in (9), we get

$$|v|_{\mathrm{W}^{1,p;s}(\mathcal{T}_h)} \le |v|_{\mathrm{W}^{1,p;t}(\mathcal{T}_h)} \qquad\qquad \forall t \in [1,\infty), \quad \forall s \in [1,t]. \tag{11}$$

Moreover, proceeding as in [9, Lemma 5.1] and using the first inequality in (9), there also exists a positive constant C_{DG} only depending on γ and Ω such that

$$|v|_{\mathrm{W}^{1,p;p}(\mathcal{T}_h)} \le C_{DG} |v|_{\mathrm{W}^{1,q;q}(\mathcal{T}_h)} \qquad\qquad \forall q \in [1,\infty), \quad \forall p \in [1,q].$$

■

The seminorm in (10b) is a norm, as a consequence of the next result, which was proven in [4].

Lemma 2.1 (Sobolev-Poincaré inequalities for piecewise $\mathrm{W}^{1,p}$ functions)**.** *Let $\{\mathcal{T}_h\}$ be a family of meshes as in Section 2.2 and p be in $[1,\infty)$. There exist positive constant C_{SP} and $C_{\mathrm{SP}}^\sharp$ may possibly depend on p, Γ_D, Ω, d, and γ such that*

$$\|v\|_{\mathrm{L}^{p1^*}(\Omega)} \le C_{\mathrm{SP}} \|v\|_{\mathrm{W}^{1,p;p}_{\Gamma_D}(\mathcal{T}_h)} \qquad\qquad \forall v \in \mathrm{W}^{1,p}(\mathcal{T}_h), \tag{12}$$

and, if p further belongs to $[1,d)$,

$$\|v\|_{\mathrm{L}^{p^*}(\Omega)} \le C_{\mathrm{SP}}^\sharp \|v\|_{\mathrm{W}^{1,p;p^\sharp}_{\Gamma_D}(\mathcal{T}_h)} \qquad\qquad \forall v \in \mathrm{W}^{1,p}(\mathcal{T}_h). \tag{13}$$

For the explicit dependence of C_{SP} and $C_{\mathrm{SP}}^\sharp$ on the parameters highlighted in the statement of Lemma 2.1, we refer to [4, Theorems 1.5-1.6].

We report the following Sobolev-trace inequalities from [4, Theorem 1.1 and eq. (26)].

Lemma 2.2 (Local Sobolev–trace inequality)**.** *Let $\{\mathcal{T}_h\}$ be a family of meshes as in Section 2.2, p be in $[1,\infty)$, and q be in $[p,\infty)$. Then, we have*

$$\|v\|^q_{\mathrm{L}^q(\partial K)} \le \frac{d}{\gamma h_K}|K|^{1-\frac{q'}{p'}}\|v\|^q_{\mathrm{L}^{p'(q-1)}(K)} + \frac{q}{\gamma}\|\nabla v\|_{\mathbf{L}^p(K)}\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(K)}. \tag{14}$$

We conclude this section with an auxiliary result.

Lemma 2.3 (Auxiliary result)**.** *Given a mesh $\mathcal{T}_h$ in a sequence as in Section 2.2, let p be in $[1,\infty)$ and v in $\mathrm{W}^{1,p}(\mathcal{T}_h)$. Then, $|v|^p$ belongs to $\mathrm{W}^{1,1}(\mathcal{T}_h)$; $|v|^q$ also belongs to $\mathrm{W}^{1,1}(\mathcal{T}_h)$ for all q in $[p,p^\sharp]$ if p is in $[1,d)$.*

Proof. Given v in $\mathrm{W}^{1,p}(\mathcal{T}_h)$, $|v|^p$ belongs to $\mathrm{L}^1(\Omega)$ since $(|v|^p)_{|K}$ is in $\mathrm{L}^1(K)$ for all K in $\mathcal{T}_h$. Using Hölder's inequality (p and p') and $|\nabla(|v|^p)_{|K}| = p|v_{|K}|^{p-1}|\nabla v_{|K}|$ for all K in $\mathcal{T}_h$, we deduce that $|v|^p$ is in $\mathrm{W}^{1,1}(\mathcal{T}_h)$. In fact,

$$\int_K |\nabla(|v|^p)| \le p\Big(\int_K |v|^p\Big)^{\frac{p-1}{p}}\Big(\int_K |\nabla v|^p\Big)^{\frac{1}{p}} \le p\|v\|^{p-1}_{\mathrm{L}^p(K)}|v|_{\mathrm{W}^{1,p}(K)} \le p\|v\|^p_{\mathrm{W}^{1,p}(K)} \qquad \forall K \in \mathcal{T}_h.$$

In order to show that $|v|^q$ belongs to $\mathrm{W}^{1,1}(\mathcal{T}_h)$ if p is in $[1,d)$ and q is in $[p,p^\sharp]$, we first remark that $|v|^{p^\sharp}$ belongs to $\mathrm{L}^1(\Omega)$ using that $p^\sharp < p^*$ and the corresponding Sobolev embedding (4). Applying Hölder's inequality and observing that $(p^\sharp-1)p' = p^*$, we infer

$$\int_K |\nabla(|v|^{p^\sharp})| = p^\sharp \int_K |v_{|K}|^{p^\sharp-1}|\nabla v_{|K}| \le p^\sharp\Big(\int_K |v|^{p^*}\Big)^{\frac{1}{p'}}\Big(\int_K |\nabla v|^p\Big)^{\frac{1}{p}} \le p^\sharp\|v\|^{p^\sharp-1}_{\mathrm{L}^{p^*}(K)}\|v\|_{\mathrm{W}^{1,p}(K)}.$$

The Sobolev embedding in (4) gives $\|v\|_{\mathrm{L}^{p^*}(K)} \le C_{\mathrm{Sob}}(p^*,p,K)\|v\|_{\mathrm{W}^{1,p}(K)}$, whence the right-hand side above is finite and thus the assertion follows. □

2.4 Functions of bounded variation

The space $\mathrm{BV}(\Omega)$ consists of functions whose distributional derivative is a Radon measure with finite total variation, i.e.,

$$\mathrm{BV}(\Omega) := \{v \in \mathrm{L}^1(\Omega) \mid |Dv|(\Omega) < \infty\} \qquad \text{where} \qquad |Dv|(\Omega) := \sup_{\boldsymbol{\xi}\in \boldsymbol{C}^1_0(\Omega),\, \|\boldsymbol{\xi}\|_{\mathrm{L}^\infty(\Omega)}\le 1} \int_\Omega v\, \nabla\cdot\boldsymbol{\xi},$$

which we endow with the norm

$$\|v\|_{\mathrm{BV},\Omega} := \|v\|_{\mathrm{L}^1(\Omega)} + |Dv|(\Omega).$$

In what follows, we review some properties of functions of bounded variation; cf. [1] for a detailed presentation of these spaces. The following result provides the existence of a bounded, linear trace operator from $\mathrm{BV}(\Omega)$ into $\mathrm{L}^1(\partial\Omega)$; details can be found in [13, Theorem 5.5], [5, Theorem 4.2], and [1, Chapter 3].

Lemma 2.4 (Trace inequality in $\mathrm{BV}(\Omega)$)**.** *There exists a positive constant C_{BV} only depending on Ω such that*

$$\|v\|_{\mathrm{L}^1(\partial\Omega)} \le C_{\mathrm{BV}}\|v\|_{\mathrm{BV},\Omega} \qquad \forall v \in \mathrm{BV}(\Omega). \tag{15}$$

Another important result states that $\mathrm{BV}(\Omega)$ embeds continuously in broken Sobolev spaces for any mesh as in Section 2.2; the proof is rather standard but we report it for the sake of completeness.

Lemma 2.5 (Bound of the BV-norm by broken Sobolev norms)**.** *Given C_{SP} as in (12) and a polytopic mesh $\mathcal{T}_h$ as in Section 2.2, we have*

$$\|v\|_{\mathrm{BV},\Omega} \le (1+|\Omega|^{\frac{1}{d}}C_{\mathrm{SP}})\|v\|_{\mathrm{W}^{1,1}_{\Gamma_D}(\mathcal{T}_h)} \qquad \forall v \in \mathrm{W}^{1,1}(\mathcal{T}_h).$$

Proof. Let v be in $\mathrm{W}^{1,1}(\mathcal{T}_h)$. We first bound the total variation $|Dv|(\Omega)$ using an integration by parts and Hölder's inequality (1 and ∞):

$$\begin{aligned}|Dv|(\Omega) &= \sup_{\boldsymbol{\xi}\in \boldsymbol{C}_0^1(\Omega)\,,\,\|\boldsymbol{\xi}\|_{\mathrm{L}^\infty(\Omega)}\leq 1} \sum_{K\in\mathcal{T}_h}\int_K v\;\nabla\cdot\boldsymbol{\xi}\\ &= \sup_{\boldsymbol{\xi}\in \boldsymbol{C}_0^1(\Omega)\,,\,\|\boldsymbol{\xi}\|_{\mathrm{L}^\infty(\Omega)}\leq 1}\Big(-\int_\Omega \nabla_h v\cdot\boldsymbol{\xi} + \sum_{F\in\mathcal{F}_h^I}\int_F [\![v]\!]\;\boldsymbol{\xi}\cdot\boldsymbol{n}_F\Big)\\ &\leq \sup_{\boldsymbol{\xi}\in \boldsymbol{C}_0^1(\Omega)\,,\,\|\boldsymbol{\xi}\|_{\mathrm{L}^\infty(\Omega)}\leq 1}\Big(\|\nabla_h v\|_{\mathbf{L}^1(\Omega)} + \sum_{F\in\mathcal{F}_h^I}\|[\![v]\!]\|_{\mathrm{L}^1(F)}\Big)\|\boldsymbol{\xi}\|_{\mathrm{L}^\infty(\Omega)} \leq |v|_{\mathrm{W}^{1,1}(\mathcal{T}_h)}.\end{aligned}$$

Using Hölder's inequality (1^* and $(1^*)'=d$) and (12), we deduce

$$\|v\|_{\mathrm{L}^1(\Omega)}\leq |\Omega|^{\frac{1}{d}}\|v\|_{\mathrm{L}^{1^*}(\Omega)}\leq |\Omega|^{\frac{1}{d}}C_{\mathrm{SP}}\|v\|_{\mathrm{W}^{1,1}_{\Gamma_D}(\mathcal{T}_h)}.$$

The assertion follows noting

$$\|v\|_{\mathrm{BV},\Omega}\leq |v|_{\mathrm{W}^{1,1}(\mathcal{T}_h)} + |\Omega|^{\frac{1}{d}}C_{\mathrm{SP}}\|v\|_{\mathrm{W}^{1,1}_{\Gamma_D}(\mathcal{T}_h)}.$$

□

3 Trace inequalities in broken Sobolev spaces

The aim of this section is to establish several trace inequalities in broken Sobolev spaces $\mathrm{W}^{1,p}(\mathcal{T}_h)$. Some technical, preliminary results are detailed in Section 3.1; Section 3.2 is instead concerned with the proof of these trace inequalities and a discussion on their applicability in the analysis of certain nonconforming methods.

3.1 Preliminary results

We begin with an algebraic bound.

Lemma 3.1 (Algebraic bound)**.** *Given $a\geq b\geq 0$ and $r\geq 1$, we have*

$$a^r - b^r \leq r(a-b)\frac{(a^{r-1}+b^{r-1})}{2}.$$

Proof. If $b=0$, then the assertion is trivial. Otherwise, we have

$$a^r - b^r = (a-b)(a^{r-1}+a^{r-2}b+\cdots+ab^{r-2}+b^{r-1}).$$

In order to conclude, it suffices to observe that the function $f:\mathbb{R}^+\to\mathbb{R}^+$ defined such that

$$f(t)=\frac{t^{r-1}+t^{r-2}+\cdots+t+1}{t^{r-1}+1}$$

attains its maximum in 1 and point out $f(1)=r/2$. □

We prove next an ancillary result.

Proposition 3.2 (Preliminary Sobolev-trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions)**.** *Let p be in $[1,\infty)$, q in $[p,p^\sharp]\cap[p,\infty)$, and s in $[p,q]$. Then, for all v in $\mathrm{W}^{1,p}_{\Gamma_D}(\mathcal{T}_h)$, we have*

$$\|v\|^q_{\mathrm{L}^q(\partial\Omega)}\leq C_1(q,\Omega)\left[\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(\Omega)} + \Big(\sum_{K\in\mathcal{T}_h} h_K^{d-\frac{(d-1)p'}{s'}}\|v\|^{p'(q-1)}_{\mathrm{L}^{s'(q-1)}(\partial K)}\Big)^{\frac{1}{p'}}\right]\|v\|_{\mathrm{W}^{1,p;s}_{\Gamma_D}(\mathcal{T}_h)},$$

where, given C_{SP} in (12) and C_{BV} in (15), the constant C_1 is defined as

$$C_1(q,\Omega):=q\,C_{\mathrm{BV}}(1+|\Omega|^{\frac{1}{d}}C_{\mathrm{SP}}). \tag{16}$$

Proof. Since q belongs to $[p,p^\sharp]\cap[p,\infty)$, Lemmas 2.3 and 2.5 imply that $|v|^q$ is in $\mathrm{W}^{1,1}(\mathcal{T}_h)\subset \mathrm{BV}(\Omega)$ for any v in $\mathrm{W}^{1,p}(\mathcal{T}_h)$. Thus, using Proposition 2.4 and Lemma 2.5, we infer

$$\begin{aligned}\|v\|^q_{\mathrm{L}^q(\partial\Omega)} &= \||v|^q\|_{\mathrm{L}^1(\partial\Omega)} \overset{(15)}{\le} C_{\mathrm{BV}}\||v|^q\|_{\mathrm{BV},\Omega} \overset{(12)}{\le} C_{\mathrm{BV}}(1+|\Omega|^{\frac{1}{d}}C_{\mathrm{SP}})\|\,|v|^q\|_{\mathrm{W}^{1,1}_{\Gamma_D}(\mathcal{T}_h)}\\ &\overset{(10a)}{=} C_{\mathrm{BV}}(1+|\Omega|^{\frac{1}{d}}C_{\mathrm{SP}})\Big(\sum_{K\in\mathcal{T}_h}\int_K|\nabla(|v|^q)|+\sum_{F\in\mathcal{F}_h^{ID}}\int_F|\,[\![|v|^q]\!]\,|\Big).\end{aligned}\tag{17}$$

For all K in $\mathcal{T}_h$, $|\nabla(|v_{|K}|^q)| = q\ |v_{|K}|^{q-1}\ |\nabla v_{|K}|$. Lemma 3.1 and the triangle inequality entail

$$[\![|v|^q]\!] \le q\{|v|^{q-1}\}_F\,[\![|v|]\!] \le q\{|v|^{q-1}\}_F|\,[\![v]\!]\,| \qquad \forall F\in\mathcal{F}_h^I.$$

Given C_1 as in (16), this allows us to write

$$\|v\|^q_{\mathrm{L}^q(\partial\Omega)} \le C_1\Big(\sum_{K\in\mathcal{T}_h}\int_K|v|^{q-1}|\nabla v|+\sum_{F\in\mathcal{F}_h^{ID}}\int_F\{|v|^{q-1}\}_F|\,[\![v]\!]\,|\Big) = C_1(\mathcal{I}_1+\mathcal{I}_2).\tag{18}$$

Hölder's inequality (p and p') and Hölder's inequality for sequences (p and p') give

$$\mathcal{I}_1 \le \sum_{K\in\mathcal{T}_h}\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(K)}\|\nabla v\|_{\mathbf{L}^p(K)} \quad \le \|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(\Omega)}\|\nabla_h v\|_{\mathbf{L}^p(\Omega)}.$$

As for the term $\mathcal{I}_2$, we let s be in $[p,q]$ and define

$$\alpha := \frac{1}{s'}+\frac{d}{p'}-\frac{d}{s'}>0.\tag{19}$$

We use Hölder's inequality (s and s'), Hölder's inequality for sequences (p and p'), and the fact that $\widetilde{h}_F$ in (8) is smaller than h_K for all F in $\mathcal{F}^K$, and get

$$\begin{aligned}\mathcal{I}_2 &\le \sum_{K\in\mathcal{T}_h}\Big(\frac{1}{2}\sum_{F\in\mathcal{F}^{KI}}\|v_{|K}\|^{q-1}_{\mathrm{L}^{s'(q-1)}(F)}\|[\![v]\!]\|_{\mathrm{L}^s(F)}+\sum_{F\in\mathcal{F}^{KD}}\|v_{|K}\|^{q-1}_{\mathrm{L}^{s'(q-1)}(F)}\|[\![v]\!]\|_{\mathrm{L}^s(F)}\Big)\\
&\le\Big(\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h^K}\widetilde{h}_F^{\alpha p'}\|v_{|K}\|^{p'(q-1)}_{\mathrm{L}^{s'(q-1)}(F)}\Big)^{\frac{1}{p'}}\Big(\sum_{F\in\mathcal{F}_h^I}(2\widetilde{h}_F)^{-\alpha p}\|[\![v]\!]\|^p_{\mathrm{L}^s(F)}+\sum_{F\in\mathcal{F}_h^D}\widetilde{h}_F^{-\alpha p}\|[\![v]\!]\|^p_{\mathrm{L}^s(F)}\Big)^{\frac{1}{p}}\\
&\le\Big(\sum_{K\in\mathcal{T}_h}h_K^{\alpha p'}\|v\|^{p'(q-1)}_{\mathrm{L}^{s'(q-1)}(\partial K)}\Big)^{\frac{1}{p'}}\Big(\sum_{F\in\mathcal{F}_h^{ID}}\widetilde{h}_F^{-\alpha p}\|[\![v]\!]\|^p_{\mathrm{L}^s(F)}\Big)^{\frac{1}{p}}\\
&\le\Big(\sum_{K\in\mathcal{T}_h}h_K^{\alpha p'}\|v\|^{p'(q-1)}_{\mathrm{L}^{s'(q-1)}(\partial K)}\Big)^{\frac{1}{p'}}\|v\|_{\mathrm{W}^{1,p;s}_{\Gamma_D}(\mathcal{T}_h)}.\end{aligned}$$

The conclusion follows by plugging the bounds on $\mathcal{I}_1$ and $\mathcal{I}_2$ in (18). □

3.2 Trace theorems

We are now in a position to prove the main results of this work. We consider three cases and postpone their applicability in the analysis of nonconforming methods to Remark 3 below:

- **case 1**: p in $[1,\infty)$ and q in $[1,p^\natural]$;
- **case 2**: p in $[1,d)$ and q in $[p,p^\sharp]$;
- **case 3**: p in $[d,\infty)$ and q in $[p,\infty)$.

Theorem 3.3 (Sobolev-trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions: **case 1**). *Given p in $[1,\infty)$ and q in $[1,p^\natural]$, we have*

$$\|v\|_{\mathrm{L}^q(\partial\Omega)} \le C_{\mathrm{S}}^\natural(p,q,\Omega)\|v\|_{\mathrm{W}^{1,p;p}_{\Gamma_D}(\mathcal{T}_h)} \qquad \forall v\in\mathrm{W}^{1,p}_{\Gamma_D}(\mathcal{T}_h),\tag{20}$$

where, given C_{SP} as in (12) and C_1 as in (16),

$$C_{\mathrm{S}}^\natural(p,q,\Omega) := |\partial\Omega|^{\frac{p^\natural-q}{p^\natural q}}C_1^{\frac{1}{p^\natural}}(p^\natural,\Omega)C_{\mathrm{SP}}^{\frac{1}{(p^\natural)'}}\Big[|\Omega|^{\frac{(p1^*-p^\natural)}{p1^*(p^\natural)'}}+\Big(\frac{d}{\gamma}C_{\mathrm{SP}}|\Omega|^{\frac{p1^*}{p1^*-p^\ddagger}}+\frac{p^\ddagger}{\gamma}\max_{K\in\mathcal{T}_h}(h_K^p)\Big)^{\frac{1}{p'}}\Big]^{\frac{1}{p^\natural}}.$$

Proof. We select $q = p^\natural$ and $s = p$ in Proposition 3.2 and observe that $p'(p^\natural - 1) = p^\ddagger$ to obtain

$$\|v\|^{p^\natural}_{\mathrm{L}^{p^\natural}(\partial\Omega)} \le C_1(p^\natural, \Omega)\left[\|v\|^{p^\natural-1}_{\mathrm{L}^{p^\ddagger}(\Omega)} + \Big(\sum_{K\in\mathcal{T}_h} h_K \|v\|^{p^\ddagger}_{\mathrm{L}^{p^\ddagger}(\partial K)}\Big)^{\frac{1}{p'}}\right]\|v\|_{\mathrm{W}^{1,p;p}_{\Gamma_D}(\mathcal{T}_h)} \tag{21}$$

Applying (14) with $q = p^\ddagger$, and observing that $p'(p^\ddagger - 1) = p1^*$ and $1 - (p^\ddagger)'/p' = 1/(dp)$, we infer

$$h_K\|v\|^{p^\ddagger}_{\mathrm{L}^{p^\ddagger}(\partial K)} \le \frac{d}{\gamma}|K|^{\frac{1}{dp}}\|v\|^{p^\ddagger}_{\mathrm{L}^{p1^*}(K)} + \frac{p^\ddagger}{\gamma} h_K\|\nabla v\|_{\mathbf{L}^p(K)}\|v\|^{p^\ddagger-1}_{\mathrm{L}^{p1^*}(K)}.$$

Using Hölder's inequality for sequences $(p1^*/(p1^* - p^\ddagger)$ and $p1^*/p^\ddagger)$, we remark that

$$\frac{d}{\gamma}\sum_{K\in\mathcal{T}_h}|K|^{\frac{1}{dp}}\|v\|^{p^\ddagger}_{\mathrm{L}^{p1^*}(K)} \le \frac{d}{\gamma}\|v\|^{p^\ddagger}_{\mathrm{L}^{p1^*}(\Omega)}\big(\sum_{K\in\mathcal{T}_h}|K|\big)^{\frac{p1^*}{p1^*-p^\ddagger}} = \frac{d}{\gamma}|\Omega|^{\frac{p1^*}{p1^*-p^\ddagger}}\|v\|^{p^\ddagger}_{\mathrm{L}^{p1^*}(\Omega)}.$$

Thus, using the previous bounds and again Hölder's inequality for sequences (p and p'), we have

$$\begin{aligned}
\Big(\sum_{K\in\mathcal{T}_h} h_K\|v\|^{p^\ddagger}_{\mathrm{L}^{p^\ddagger}(\partial K)}\Big)^{\frac{1}{p'}} &\le \Big(\frac{d}{\gamma}\sum_{K\in\mathcal{T}_h}|K|^{\frac{1}{dp}}\|v\|^{p^\ddagger}_{\mathrm{L}^{p1^*}(K)} + \frac{p^\ddagger}{\gamma}\sum_{K\in\mathcal{T}_h} h_K\|\nabla v\|_{\mathbf{L}^p(K)}\|v\|^{p^\ddagger-1}_{\mathrm{L}^{p1^*}(K)}\Big)^{\frac{1}{p'}}\\
&\le \Big[\frac{d}{\gamma}|\Omega|^{\frac{p1^*}{p1^*-p^\ddagger}}\|v\|^{p^\ddagger}_{\mathrm{L}^{p1^*}(\Omega)} + \frac{p^\ddagger}{\gamma}\Big(\sum_{K\in\mathcal{T}_h} h_K^p\|\nabla v\|^p_{\mathbf{L}^p(K)}\Big)^{\frac{1}{p}}\Big(\sum_{K\in\mathcal{T}_h}\|v\|^{p1^*}_{\mathrm{L}^{p1^*}(K)}\Big)^{\frac{1}{p'}}\Big]^{\frac{1}{p'}}\\
&\le \Big[\frac{d}{\gamma}|\Omega|^{\frac{p1^*}{p1^*-p^\ddagger}}\|v\|^{p^\ddagger}_{\mathrm{L}^{p1^*}(\Omega)} + \frac{p^\ddagger}{\gamma}\max_{K\in\mathcal{T}_h}(h_K^p)\|\nabla v\|_{\mathbf{L}^p(\Omega)}\|v\|^{p^\ddagger-1}_{\mathrm{L}^{p1^*}(\Omega)}\Big]^{\frac{1}{p'}}.
\end{aligned}$$

Finally, we apply (12) to deduce

$$\Big(\sum_{K\in\mathcal{T}_h} h_K\|v\|^{p^\ddagger}_{\mathrm{L}^{p^\ddagger}(\partial K)}\Big)^{\frac{1}{p'}} \le \Big(\frac{d}{\gamma}C_{\mathrm{SP}}^{p^\ddagger}|\Omega|^{\frac{p1^*}{p1^*-p^\ddagger}} + \frac{p^\ddagger}{\gamma}\max_{K\in\mathcal{T}_h}(h_K^p)C_{\mathrm{SP}}^{p^\ddagger-1}\Big)^{\frac{1}{p'}}\|v\|^{\frac{p^\ddagger}{p'}}_{\mathrm{W}^{1,p;p}(\mathcal{T}_h)}$$

and

$$\|v\|^{p^\natural-1}_{\mathrm{L}^{p^\natural}(\Omega)} \le |\Omega|^{\frac{(p1^*-p^\natural)}{p1^*(p^\natural)'}}\|v\|^{p^\natural-1}_{\mathrm{L}^{p1^*}(\Omega)} \le |\Omega|^{\frac{(p1^*-p^\natural)}{p1^*(p^\natural)'}}C_{\mathrm{SP}}^{p^\natural-1}\|v\|^{p^\natural-1}_{\mathrm{W}^{1,p;p}(\Omega)}.$$

We insert these inequalities in (21), recall again that $p^\ddagger/p' = p^\natural - 1$, and obtain

$$\|v\|^{p^\natural}_{\mathrm{L}^{p^\natural}(\partial\Omega)} \le C_1\Big[|\Omega|^{\frac{(p1^*-p^\natural)}{p1^*(p^\natural)'}}C_{\mathrm{SP}}^{p^\natural-1} + \Big(\frac{d}{\gamma}C_{\mathrm{SP}}^{p^\ddagger}|\Omega|^{\frac{p1^*}{p1^*-p^\ddagger}} + \frac{p^\ddagger}{\gamma}\max_{K\in\mathcal{T}_h}(h_K^p)C_{\mathrm{SP}}^{p^\ddagger-1}\Big)^{\frac{1}{p'}}\Big]\|v\|^{p^\natural}_{\mathrm{W}^{1,p;p}(\Omega)}.$$

The Sobolev trace inequality (20) follows by taking the $p^\natural$-th root on both sides and from Hölder's inequality $(p^\natural/q$ and $(p^\natural/q)')$:

$$\|v\|_{\mathrm{L}^q(\partial\Omega)} \le |\partial\Omega|^{\frac{p^\natural-q}{p^\natural q}}\|v\|_{\mathrm{L}^{p^\natural}(\partial\Omega)}.$$

□

Theorem 3.4 (Sobolev-trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions: **case 2**)**.** *Given p in $[1, d)$, we have*

$$\|v\|_{\mathrm{L}^q(\partial\Omega)} \le C_{\mathrm{S}}^\ddagger(p, q, \Omega)\|v\|_{\mathrm{W}^{1,p;p^\ddagger}_{\Gamma_D}(\mathcal{T}_h)} \qquad \forall v \in \mathrm{W}^{1,p}_{\Gamma_D}(\mathcal{T}_h), \qquad \textit{for } q \in [p, p^\ddagger], \tag{22}$$

$$\|v\|_{\mathrm{L}^q(\partial\Omega)} \le C_{\mathrm{S}}^\sharp(p, q, \Omega)\|v\|_{\mathrm{W}^{1,p;p^\sharp}_{\Gamma_D}(\mathcal{T}_h)} \qquad \forall v \in \mathrm{W}^{1,p}_{\Gamma_D}(\mathcal{T}_h), \qquad \textit{for } q \in [p^\ddagger, p^\sharp], \tag{23}$$

where, given C_{SP}, $C_{\mathrm{SP}}^\sharp$, and C_1 as in (12), (13), and (16), and C_4 as in (25) below,

$$C_{\mathrm{S}}^\ddagger(p, q, \Omega) := |\partial\Omega|^{\frac{p^\ddagger-q}{p^\ddagger q}}\frac{C_1^{\frac{1}{p^\ddagger}}(p^\ddagger, \Omega)}{\gamma^{\frac{d(p-1)}{p^\ddagger(dp-1)}}}\Big(\Big(\gamma^{\frac{d(p-1)}{dp-1}} + C_4(p, p^\ddagger)\Big)(C_{\mathrm{SP}})^{p^\ddagger-1} + \max_{K\in\mathcal{T}_h}\Big(h_K^{\frac{d(p-1)^2}{p(d-1)}}\Big)\Big)^{\frac{1}{p^\ddagger}},$$

$$C_{\mathrm{S}}^\sharp(p, q, \Omega) := |\partial\Omega|^{\frac{p^\sharp-q}{p^\sharp q}}\frac{C_1^{\frac{1}{p^\sharp}}(p^\sharp, \Omega)}{\gamma^{\frac{d(p-1)}{p^\sharp p(d-1)}}}\Big(\Big(\gamma^{\frac{d(p-1)}{p(d-1)}} + C_4(p, p^\sharp)\Big)(C_{\mathrm{SP}}^\sharp)^{p^\sharp-1} + 1\Big)^{\frac{1}{p^\sharp}}.$$

Proof. Taking $s = q$ in Proposition 3.2 yields

$$\|v\|^q_{\mathrm{L}^q(\partial\Omega)} \le C_1(q,\Omega)\left[\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(\Omega)} + \Big(\sum_{K\in\mathcal{T}_h} h_K^{d-\frac{(d-1)p'}{q'}}\|v\|^{p'(q-1)}_{\mathrm{L}^q(\partial K)}\Big)^{\frac{1}{p'}}\right]\|v\|_{\mathrm{W}^{1,p;q}_{\Gamma_D}(\mathcal{T}_h)}. \tag{24}$$

Given the unit ball $\mathcal{B}_d$ in $\mathbb{R}^d$, we recall that $|K| \le h_K^d|\mathcal{B}_d| = h_K^d\pi^{\frac{d}{2}}\Gamma\big(\frac{d}{2}+1\big)^{-1}$. Using this bound followed by Young's inequality, and introducing the constant $C_4 = C_4(p,q)$ defined as

$$C_4(p,q) := d^{\frac{1}{q'}}\left(\frac{\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}+1\big)}\right)^{\frac{1}{q'}-\frac{1}{p'}} + (q-1)^{\frac{1}{q'}}, \tag{25}$$

we deduce

$$\begin{aligned}
&h_K^{d-\frac{(d-1)p'}{q'}}\|v\|^{p'(q-1)}_{\mathrm{L}^q(\partial K)} = \Big(h_K^{\frac{dq'}{p'}-d+1}\|v\|^q_{\mathrm{L}^q(\partial K)}\Big)^{\frac{p'}{q'}}\\
&\overset{(14)}{\le} \Big(\frac{d}{\gamma}h_K^{d\left(\frac{q'}{p'}-1\right)}|K|^{1-\frac{q'}{p'}}\|v\|^q_{\mathrm{L}^{p'(q-1)}(K)} + \frac{q}{\gamma}h_K^{\frac{dq'}{p'}-d+1}\|\nabla v\|_{\mathbf{L}^p(K)}\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(K)}\Big)^{\frac{p'}{q'}}\\
&\le \Big(\frac{d}{\gamma}\Big(\frac{\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}+1\big)}\Big)^{1-\frac{q'}{p'}}\|v\|^q_{\mathrm{L}^{p'(q-1)}(K)} + \frac{q}{\gamma}h_K^{\frac{dq'}{p'}-d+1}\|\nabla v\|_{\mathbf{L}^p(K)}\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(K)}\Big)^{\frac{p'}{q'}}\\
&\le \Big[\frac{1}{\gamma}\Big(d\Big(\frac{\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}+1\big)}\Big)^{1-\frac{q'}{p'}} + q - 1\Big)\|v\|^q_{\mathrm{L}^{p'(q-1)}(K)} + \frac{h_K^{q-dq+dq\frac{q'}{p'}}}{\gamma}\|\nabla v\|^q_{\mathbf{L}^p(K)}\Big]^{\frac{p'}{q'}}\\
&\le \gamma^{-\frac{p'}{q'}}\Big(C_4^{p'}\|v\|^{p'(q-1)}_{\mathrm{L}^{p'(q-1)}(K)} + h_K^{(q-1)(p'+dq'-dp')}\|\nabla v\|^{p'(q-1)}_{\mathbf{L}^p(K)}\Big).
\end{aligned}$$

A consequence of the previous bound is

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} h_K^{d-\frac{(d-1)p'}{q'}}\|v\|^{p'(q-1)}_{\mathrm{L}^q(\partial K)} &\le \gamma^{-\frac{p'}{q'}}\sum_{K\in\mathcal{T}_h}\Big(C_4^{p'}\|v\|^{p'(q-1)}_{\mathrm{L}^{p'(q-1)}(K)} + h_K^{(q-1)(p'+dq'-dp')}\|\nabla v\|^{p'(q-1)}_{\mathbf{L}^p(K)}\Big)\\
&\le \gamma^{-\frac{p'}{q'}}\Big(C_4^{p'}\|v\|^{p'(q-1)}_{\mathrm{L}^{p'(q-1)}(\Omega)} + \max_{K\in\mathcal{T}_h}(h_K^{(q-1)(p'+dq'-dp')})\|\nabla_h v\|^{p'(q-1)}_{\mathbf{L}^p(\Omega)}\Big).
\end{aligned}$$

Plugging the previous result into (24) implies

$$\begin{aligned}
&C_1^{-1}\|v\|^q_{\mathrm{L}^q(\partial\Omega)}\\
&\le \Big[\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(\Omega)} + \gamma^{-\frac{1}{q'}}\Big(C_4\|v\|^{(q-1)}_{\mathrm{L}^{p'(q-1)}(\Omega)} + \max_{K\in\mathcal{T}_h}\Big(h_K^{(q-1)(1-d+\frac{dq'}{p'})}\Big)\|\nabla_h v\|^{(q-1)}_{\mathbf{L}^p(\Omega)}\Big)\Big]\|v\|_{\mathrm{W}^{1,p;q}_{\Gamma_D}(\mathcal{T}_h)}\\
&\le \frac{\gamma^{\frac{1}{q'}}+C_4}{\gamma^{\frac{1}{q'}}}\|v\|^{q-1}_{\mathrm{L}^{p'(q-1)}(\Omega)}\|v\|_{\mathrm{W}^{1,p;q}_{\Gamma_D}(\mathcal{T}_h)} + \frac{1}{\gamma^{\frac{1}{q'}}}\max_{K\in\mathcal{T}_h}\Big(h_K^{(q-1)(1-d+\frac{dq'}{p'})}\Big)\|v\|^q_{\mathrm{W}^{1,p;q}_{\Gamma_D}(\mathcal{T}_h)}.
\end{aligned}$$

Taking $q = p^\ddagger$, and using the broken Sobolev-Poincaré inequality (12) together with bound (11) and the identity

$$(p^\ddagger-1)\Big(1-d+d\frac{(p^\ddagger)'}{p'}\Big) = \frac{d(p-1)^2}{p(d-1)},$$

we get

$$\|v\|^{p^\ddagger}_{\mathrm{L}^{p^\ddagger}(\partial\Omega)} \le \frac{C_1}{\gamma^{\frac{d(p-1)}{dp-1}}}\Big(\Big(\gamma^{\frac{d(p-1)}{dp-1}} + C_4\Big)(C_{\mathrm{SP}})^{p^\ddagger-1} + \max_{K\in\mathcal{T}_h}\Big(h_K^{\frac{d(p-1)^2}{p(d-1)}}\Big)\Big)\|v\|^{p^\ddagger}_{\mathrm{W}^{1,p;p^\ddagger}_{\Gamma_D}(\mathcal{T}_h)}.$$

If instead we take $q = p^\sharp$, use the broken Sobolev-Poincaré inequality (13), and observe

$$(p^\sharp-1)\Big(1-d+d\frac{(p^\sharp)'}{p'}\Big) = 0,$$

then we obtain

$$\|v\|^{p^\sharp}_{\mathrm{L}^{p^\sharp}(\partial\Omega)} \leq \frac{C_1}{\gamma^{\frac{d(p-1)}{p(d-1)}}} \Big(\Big(\gamma^{\frac{d(p-1)}{p(d-1)}} + C_4\Big) (C^\sharp_{\mathrm{SP}})^{p^\sharp-1} + 1\Big) \|v\|^{p^\sharp}_{\mathrm{W}^{1,p;p^\sharp}_{\Gamma_D}(\mathcal{T}_h)}.$$

Bounds (22) and (23) follow using Hölder's inequality (r/q and $(r/q)'$) with $r = p^\ddagger$ and $r = p^\sharp$. $\square$

Theorem 3.5 (Sobolev-trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions: **case 3**). *Given p in $[d,\infty)$ and q in $[p,\infty)$, we have*

$$\|v\|_{\mathrm{L}^q(\partial\Omega)} \leq C^\flat_{\mathrm{S}}(p,q,\Omega)\|v\|_{\mathrm{W}^{1,p;q}_{\Gamma_D}(\mathcal{T}_h)} \qquad \forall v \in \mathrm{W}^{1,p}_{\Gamma_D}(\mathcal{T}_h), \tag{26}$$

where, given $C^\sharp_{\mathrm{S}}$ as in (23),

$$C^\flat_{\mathrm{S}}(p,q,\Omega) := C^\sharp_{\mathrm{S}}(q^\flat,q,\Omega) \max_{K\in\mathcal{T}_h} \big(|K|^{-1}h_K^d \operatorname{card}(\mathcal{F}^K)\big)^{\frac{p-q^\flat}{pq^\flat}} |\Omega|^{\frac{p-q^\flat}{pq^\flat}}.$$

Proof. Recall that $q^\flat$ in $(1,d)$ is such that, cf. (5)–(6),

$$q^\flat := \frac{qd}{q+d-1} < d \leq p. \tag{27}$$

Using (23), we write

$$\|v\|_{\mathrm{L}^q(\partial\Omega)} \leq C^\sharp_{\mathrm{S}}(q^\flat,q,\Omega)\|v\|_{\mathrm{W}^{1,q^\flat;q}_{\Gamma_D}(\mathcal{T}_h)}.$$

To conclude, we need to estimate from above the two terms in the DG norm appearing on the right-hand side; cf. (10b). We begin with the gradient term:

$$\|\nabla_h v\|_{\mathbf{L}^{q^\flat}(\Omega)} \overset{(27)}{\leq} |\Omega|^{\frac{p-q^\flat}{pq^\flat}} \|\nabla_h v\|_{\mathbf{L}^p(\Omega)}.$$

We now focus on the jump terms: applying Hölder's inequality for sequences ($p/(p-q^\flat)$ and $p/q^\flat$), we obtain

$$\begin{aligned}
\Big(\sum_{F\in\mathcal{F}_h^{ID}} \|[\![v]\!]\|^{q^\flat}_{\mathrm{L}^q(F)}\Big)^{\frac{1}{q^\flat}} &= \Big(\sum_{F\in\mathcal{F}_h^{ID}} \widetilde{h}_F^{\frac{d(p-q^\flat)}{p}} \widetilde{h}_F^{-\frac{d(p-q^\flat)}{p}} \|[\![v]\!]\|^{q^\flat}_{\mathrm{L}^{q^\flat}(F)}\Big)^{\frac{1}{r}} \\
&\leq \Big(\sum_{F\in\mathcal{F}_h^{ID}} \widetilde{h}_F^d\Big)^{\frac{p-q^\flat}{pq^\flat}} \Big(\sum_{F\in\mathcal{F}_h^{ID}} \widetilde{h}_F^{-\frac{d(p-q^\flat)}{q^\flat}} \|[\![v]\!]\|^p_{\mathrm{L}^q(F)}\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}^K} \widetilde{h}_F^d\Big)^{\frac{p-q^\flat}{pq^\flat}} \Big(\sum_{F\in\mathcal{F}_h^{ID}} \widetilde{h}_F^{-\frac{d(p-q^\flat)}{q^\flat}} \|[\![v]\!]\|^p_{\mathrm{L}^q(F)}\Big)^{\frac{1}{p}} \\
&\leq \max_{K\in\mathcal{T}_h} \big(|K|^{-1}h_K^d \operatorname{card}(\mathcal{F}^K)\big)^{\frac{p-q^\flat}{pq^\flat}} |\Omega|^{\frac{p-q^\flat}{pq^\flat}} \Big(\sum_{F\in\mathcal{F}_h^{ID}} \widetilde{h}_F^{-\frac{d(p-q^\flat)}{q^\flat}} \|[\![v]\!]\|^p_{\mathrm{L}^q(F)}\Big)^{\frac{1}{p}}.
\end{aligned}$$

The assertion follows noting that

$$-\frac{d(p-q^\flat)}{q^\flat} = -\frac{p}{q'} - \frac{dp}{p'} + \frac{dp}{q'}.$$

$\square$

Remark 3 (Applicability of trace inequalities for piecewise $\mathrm{W}^{1,p}$ functions in the analysis of nonconforming Galerkin methods). Inequality (20) is suited for the analysis of interior penalty discontinuous Galerkin methods for nonlinear problems; such methods in fact involve the $\mathbf{L}^p$ norm of the broken gradient and the L^p norm of the jumps. For $p < d$, inequality (23) exhibits the maximal Lebesgue regularity indices on the left-hand side and on the jump terms, while inequality (22) is its counterpart with weaker norms; a variant of both inequalities (namely that with a projection onto constants over facets of the jumps) can be employed in the analysis of Crouzeix-Raviart schemes, as it allows for improved Lebesgue norms on the left-hand side compared to (20). Inequality (26) is instead concerned with the case $p \geq d$ and involves the same Lebesgue norms on $\partial\Omega$ and on the facet jumps. ■

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