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Longitudinal versus quasi-P waves in weakly anisotropic elasticity: operator comparison and long-time dephasing

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Abstract

In isotropic elasticity, P-wave displacement is longitudinal and curl-free. In anisotropic elasticity, the physical quasi-P (qP) mode is instead selected by the principal eigenvector of the Christoffel matrix and is generally not parallel to the wave normal. We compare, in homogeneous two-dimensional vertical transverse isotropy (VTI), the exact elastic qP propagator with a longitudinally constrained propagator that suppresses transverse displacement by construction. Both dynamics are placed in a periodic operator and variational framework. The qP space is an invariant spectral subspace of the elastic operator, whereas the longitudinal space yields a Galerkin restriction that is generally not invariant. The corresponding forms are coercive on mean-zero H^1 , their generators are positive self-adjoint operators of order two, and both Cauchy problems are well posed and energy conservative. A two-dimensional Rayleigh-quotient identity expresses the longitudinal squared-speed deficit as the qP–qSV spectral gap multiplied by the squared polarization mismatch. Building on classical weak-anisotropy perturbation theory, we derive an explicit VTI specialization of the qP projector and phase symbol. The projector error and qSV contamination are $O(\eta)$ in amplitude, whereas the qP phase-speed error is $O(\eta^2)$. For longitudinal initial data, the field error admits an orthogonal decomposition into modal contamination and qP dephasing. A uniform estimate bounds it by $C|\eta|\|u_0\| + Ct\eta^2\|u_0\|_{H^1}$ on bounded times. After removal of the common carrier, an abstract two-branch multiplier theorem gives field-level limits on the intermediate scale $t = \tau/\eta$ and the diffractive scale $t = \tau/\eta^2$. The first yields a finite polarization–phase crossover, whereas the second yields a generally order-one dephasing profile. Exact-in-time Fourier experiments verify the identities, asymptotic coefficients, convergence rates, and two long-time limits. The main contribution is therefore not a new local qP approximation, but a quantitative error theory describing when a longitudinal elastic model approximates the exact qP branch and when its second-order phase accuracy is exhausted by long-time dephasing.

Keywords: anisotropic elasticity; quasi-P waves; Christoffel matrix; weak anisotropy; spectral projector; mode separation; long-time asymptotics.

1 Introduction

The identification of P waves with longitudinal, curl-free displacement is exact in homogeneous isotropic elasticity. In anisotropic media, however, the propagating bulk modes are the eigenspaces of the Christoffel matrix. The fastest branch is conventionally denoted qP because its polarization is only quasi-longitudinal: it need not be parallel to the wave normal. Consequently, two natural operations that coincide in isotropy become distinct in anisotropy:

- (i) projecting displacement onto the wave-normal direction;
- (ii) projecting displacement onto the qP eigenspace.

The present work compares the two resulting homogeneous propagators at the level of their operator structure, phase symbols, and propagated fields.

Three well-developed strands of literature form the background. First, weak-anisotropy perturbation theory provides local approximations of qP phase velocity and polarization. Thomsen’s VTI parameters yield the classical first-order velocity formula [1]; general first-order eigenvalue and eigenvector perturbations were developed by Jech and Pšenčík [2], specialized to qP attributes by Pšenčík and Gajewski [5], and extended to higher-order wave attributes by Farra and Pšenčík [7]. The wave-vector-coordinate treatment of Ohanian, Snyder, and Carcione gives particularly transparent first-order expressions [8]. Thus neither first-order qP polarization nor first-order qP phase velocity is new here.

Second, anisotropic wave-mode separation by Christoffel polarizations is classical. Dellinger and Etgen introduced Fourier-domain separators for two-dimensional anisotropic fields [3]; Yan and Sava constructed spatially variable VTI separators [9]; and Cheng and Fomel developed low-rank mixed-domain implementations [12]. Cheng and Kang explicitly interpreted projection onto the wave vector as only a partial qP separation and removed the residual qS content by correcting the polarization deviation [11]. Accordingly, the observation that a longitudinal projection contains residual qSV energy is also established.

Third, a large computational literature constructs “pure-qP” equations by isolating or approximating a qP dispersion branch. Examples include the pseudoacoustic model of Alkhalifah [6], scalar or pseudo-pure-mode propagators [11, 13], Padé and separable approximations [14], pure P/S phase-velocity equations [16], and recent optimized time-space localizations [17, 19, 20]. Many of these works use the acoustic approximation, in which the axial shear-wave speed is set to zero, and focus on stable and efficient numerical realization of the resulting qP dispersion relation.

Terminology. Throughout this paper, *exact qP propagation* means the spectral restriction of the *full elastic* operator, with $v_{s0} > 0$, to its qP eigenspace. It should not be confused with a pseudoacoustic pure-qP equation obtained after setting $v_{s0} = 0$, nor with a local approximation of such an equation. The competing longitudinal model is likewise not a pseudoacoustic model: it is the Galerkin restriction of the full elastic energy to curl-free displacement fields.

The question addressed here is therefore narrower and dynamical: how accurately does this longitudinal elastic restriction approximate the exact elastic qP restriction, and for how long? The contributions are:

- (1) a common periodic operator and variational framework in which exact qP propagation is an invariant spectral restriction, whereas longitudinal propagation is a conservative Galerkin restriction to curl-free fields;
- (2) a two-dimensional exact relation between polarization mismatch, spectral gap, and longitudinal speed deficit, together with an exact orthogonal field-error decomposition into qSV contamination and qP dephasing;
- (3) an explicit VTI specialization of classical weak-anisotropy perturbation formulas, used to expose the different orders of the two error mechanisms;
- (4) a nonasymptotic bounded-time estimate and an abstract two-branch multiplier theorem yielding strong L^2 interaction-picture limits on the scales $t = O(1)$, $t = O(\eta^{-1})$, and $t = O(\eta^{-2})$;
- (5) exact-in-time spectral experiments that verify the coefficients and the complete limiting curves.

The originality claim is deliberately confined to the comparative propagated-field analysis and its long-time consequences. No novelty is claimed for Christoffel projection, local weak-anisotropy wave attributes, or the existence of scalar pure-qP pseudodifferential equations.

The paper is organized as follows. [Section 2](#) introduces the VTI Christoffel symbol and exact qP projector. [Section 3](#) develops the operator and variational framework for the exact qP and constrained longitudinal dynamics. [Section 4](#) compares their symbols and proves the exact identities. [Section 5](#) derives the VTI weak-anisotropy specialization. [Section 6](#) develops the finite-time field decomposition, while [Section 7](#) isolates the abstract multiplier mechanism and establishes the two VTI long-time limits. Numerical design and results are presented in [Sections 8](#) and [9](#). The relation to previous work and the limits of the claims are discussed in [Section 10](#).

2 Homogeneous 2D VTI elasticity and its spectral projectors

Let $\mathbf{x} = (x, z)^T$ and $\mathbf{u} = (u_x, u_z)^T$. In the symmetry plane of a homogeneous VTI medium, the stress–strain law is

$$\begin{aligned}\sigma_{xx} &= C_{11} \partial_x u_x + C_{13} \partial_z u_z, \\ \sigma_{zz} &= C_{13} \partial_x u_x + C_{33} \partial_z u_z, \\ \sigma_{xz} &= C_{44} (\partial_z u_x + \partial_x u_z).\end{aligned}\tag{1}$$

The displacement equation reads

$$\rho \partial_{tt} \mathbf{u} - \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f}.\tag{2}$$

The VTI coefficients are parameterized by the vertical velocities v_{P0} and v_{S0} and by Thomsen's parameters ϵ and δ :

$$\begin{aligned} C_{33} &= \rho v_{P0}^2, & C_{44} &= \rho v_{S0}^2, \\ C_{11} &= C_{33}(1 + 2\epsilon), & C_{13} + C_{44} &= \sqrt{(C_{33} - C_{44})^2 + 2\delta C_{33}(C_{33} - C_{44})}. \end{aligned} \quad (3)$$

The positive square-root branch is chosen, i.e., the branch continuous from the isotropic reference medium.

For a Fourier wave vector $\mathbf{k} = (k_x, k_z)^T$, the density-normalized elastic symbol is

$$\mathbf{A}(\mathbf{k}) = \frac{1}{\rho} \begin{pmatrix} C_{11}k_x^2 + C_{44}k_z^2 & (C_{13} + C_{44})k_xk_z \\ (C_{13} + C_{44})k_xk_z & C_{44}k_x^2 + C_{33}k_z^2 \end{pmatrix}. \quad (4)$$

Writing $\mathbf{k} = |\mathbf{k}|\mathbf{n}$, one has

$$\mathbf{A}(\mathbf{k}) = |\mathbf{k}|^2 \mathbf{\Gamma}(\mathbf{n}), \quad \mathbf{\Gamma}_{ik}(\mathbf{n}) = \rho^{-1} C_{ijkl} n_j n_l, \quad (5)$$

where $\mathbf{\Gamma}$ is the density-normalized Christoffel matrix. Its eigenvalues are the squared phase velocities $c_P^2(\mathbf{n})$ and $c_{SV}^2(\mathbf{n})$.

Set $a_A = A_{11}$, $d_A = A_{22}$, $b_A = A_{12}$, and

$$D(\mathbf{k}) = \sqrt{(a_A - d_A)^2 + 4b_A^2}. \quad (6)$$

Then

$$\lambda_P = \frac{a_A + d_A + D}{2}, \quad \lambda_{SV} = \frac{a_A + d_A - D}{2}, \quad (7)$$

with $\lambda_\alpha = |\mathbf{k}|^2 c_\alpha^2$. We assume a positive qP–qSV gap for $\mathbf{k} \neq 0$.

The sign-invariant projectors are

$$\boxed{\mathbf{\Pi}_P(\mathbf{k}) = \frac{\mathbf{A}(\mathbf{k}) - \lambda_{SV}(\mathbf{k})\mathbf{I}}{\lambda_P(\mathbf{k}) - \lambda_{SV}(\mathbf{k})}, \quad \mathbf{\Pi}_{SV} = \mathbf{I} - \mathbf{\Pi}_P.} \quad (8)$$

They satisfy

$$\mathbf{\Pi}_P^2 = \mathbf{\Pi}_P, \quad \mathbf{A}\mathbf{\Pi}_P = \lambda_P \mathbf{\Pi}_P, \quad \mathbf{A} = \lambda_P \mathbf{\Pi}_P + \lambda_{SV} \mathbf{\Pi}_{SV}. \quad (9)$$

Thus, in a homogeneous medium, exact qP propagation is obtained by applying $\mathbf{\Pi}_P$ and evolving with frequency $\omega_P = \sqrt{\lambda_P}$.

3 Operator and variational formulation

We place the exact qP and longitudinally constrained dynamics in a common periodic framework. This makes precise the distinction between an invariant spectral restriction and a conservative Galerkin restriction.

Let $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ and let $H_0^s(\mathbb{T}^2)$ denote the mean-zero periodic Sobolev space. The zero Fourier mode is removed only to exclude rigid translations and obtain a coercive energy norm; it can be restored as an independently evolving constant displacement. Vector-valued spaces are denoted by $H_0^s(\mathbb{T}^2; \mathbb{C}^2)$. We normalize Fourier series so that Parseval's identity has no additional constant. For a matrix multiplier $M(D)$,

$$\widehat{M(D)v}(\mathbf{k}) = M(\mathbf{k})\widehat{v}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}. \quad (10)$$

We assume uniform strong ellipticity and uniform separation of the two branches: there exist $0 < m \leq M$ and $\gamma > 0$ such that

$$m|\mathbf{k}|^2|q|^2 \leq q^* \mathbf{A}(\mathbf{k})q \leq M|\mathbf{k}|^2|q|^2, \quad (11)$$

$$\lambda_P(\mathbf{k}) - \lambda_{SV}(\mathbf{k}) \geq \gamma|\mathbf{k}|^2 \quad (12)$$

for every $q \in \mathbb{C}^2$ and $\mathbf{k} \neq 0$. Hence

$$m|\mathbf{k}|^2 \leq \lambda_{SV}(\mathbf{k}) \leq \lambda_P(\mathbf{k}) \leq M|\mathbf{k}|^2. \quad (13)$$

These conditions hold uniformly for all sufficiently small $|\eta|$ along the weak-anisotropy family introduced in [Section 5](#).

Remark 3.1 (Whole-space version). The torus is used to match the Fourier experiments and to remove the rigid zero mode. On $\mathbb{R}^2$, Fourier sums are replaced by integrals and the closed energy forms have domain $H^1(\mathbb{R}^2; \mathbb{C}^2)$, with coercivity in the homogeneous $\dot{H}^1$ seminorm. The comparison estimates and long-time limits below carry over under the same Fourier-moment assumptions; only the standard low-frequency bookkeeping for the wave energy changes. No argument uses arithmetic properties of the lattice.

3.1 Exact qP restriction

Define the global qP projection

$$\mathcal{P}_P = \mathbf{\Pi}_P(D), \quad \mathcal{H}_P^s = \mathcal{P}_P H_0^s(\mathbb{T}^2; \mathbb{C}^2). \quad (14)$$

Since $\mathbf{\Pi}_P(\mathbf{k})$ is an orthogonal projector, $\mathcal{P}_P$ is a bounded self-adjoint projection on every H_0^s . On $\mathcal{H}_P^1$ define

$$\mathfrak{a}_P(u, v) = \sum_{\mathbf{k} \neq 0} \lambda_P(\mathbf{k}) \widehat{u}(\mathbf{k}) \cdot \overline{\widehat{v}(\mathbf{k})}. \quad (15)$$

By [\(13\)](#),

$$m\|\nabla u\|_{L^2}^2 \leq \mathfrak{a}_P(u, u) \leq M\|\nabla u\|_{L^2}^2, \quad u \in \mathcal{H}_P^1. \quad (16)$$

Proposition 3.2 (Exact qP operator). *The operator associated with $\mathfrak{a}_P$ is*

$$D(\mathcal{L}_P) = \mathcal{H}_P^2, \quad \widehat{\mathcal{L}_P u}(\mathbf{k}) = \lambda_P(\mathbf{k}) \widehat{u}(\mathbf{k}). \quad (17)$$

It is positive and self-adjoint in $\mathcal{H}_P^0$. If

$$\mathcal{A} = \mathbf{A}(D), \quad D(\mathcal{A}) = H_0^2(\mathbb{T}^2; \mathbb{C}^2), \quad (18)$$

then

$$[\mathcal{P}_P, \mathcal{A}] = 0, \quad \mathcal{L}_P = \mathcal{A}|_{\mathcal{H}_P^2}. \quad (19)$$

Thus $\mathcal{H}_P^0$ is an invariant subspace of homogeneous elastodynamics.

Proof. The equivalence $\lambda_P(\mathbf{k}) \simeq |\mathbf{k}|^2$ gives the domain. Positivity and self-adjointness follow mode by mode from the real nonnegative symbol. Since $\mathbf{\Pi}_P(\mathbf{k})$ is a spectral projector of $\mathbf{A}(\mathbf{k})$, the two matrices commute, and $\mathbf{A}(\mathbf{k})\widehat{u} = \lambda_P(\mathbf{k})\widehat{u}$ on $\text{Ran } \mathbf{\Pi}_P(\mathbf{k})$. $\square$

3.2 Longitudinal Galerkin restriction

Let

$$\mathbf{\Pi}_L(\mathbf{k}) = \mathbf{n}\mathbf{n}^T, \quad \mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}, \quad \mathcal{P}_L = \mathbf{\Pi}_L(D), \quad (20)$$

and define

$$\mathcal{H}_L^s = \mathcal{P}_L H_0^s(\mathbb{T}^2; \mathbb{C}^2). \quad (21)$$

This is the space of mean-zero curl-free fields. The constrained squared-frequency symbol is

$$\lambda_L(\mathbf{k}) = \mathbf{n}^T \mathbf{A}(\mathbf{k}) \mathbf{n} = |\mathbf{k}|^2 c_L^2(\mathbf{n}). \quad (22)$$

Define

$$\mathbf{a}_L(u, v) = \sum_{\mathbf{k} \neq 0} \lambda_L(\mathbf{k}) \widehat{u}(\mathbf{k}) \cdot \overline{\widehat{v}(\mathbf{k})}, \quad u, v \in \mathcal{H}_L^1. \quad (23)$$

Since λ_L is a Rayleigh quotient of $\mathbf{A}$, the bounds in (16) hold with $\mathbf{a}_P$ replaced by $\mathbf{a}_L$. The associated operator is

$$D(\mathcal{L}_L) = \mathcal{H}_L^2, \quad \widehat{\mathcal{L}_L u}(\mathbf{k}) = \lambda_L(\mathbf{k}) \widehat{u}(\mathbf{k}), \quad (24)$$

and satisfies

$$\boxed{\mathcal{L}_L = \mathcal{P}_L \mathcal{A} \mathcal{P}_L|_{\mathcal{H}_L^2}}. \quad (25)$$

Indeed, for $u, v \in \mathcal{H}_L^1$, the form $\mathbf{a}_L(u, v)$ is exactly the full elastic energy form tested on the longitudinal subspace. Therefore $\mathcal{L}_L$ is the Galerkin restriction of $\mathcal{A}$ to curl-free displacement fields.

Theorem 3.3 (Reduced Cauchy problems). *Let $X \in \{P, L\}$. For $u_0 \in \mathcal{H}_X^1$, $u_1 \in \mathcal{H}_X^0$, and $f_X \in L^1(0, T; \mathcal{H}_X^0)$, the problem*

$$u_{tt} + \mathcal{L}_X u = f_X, \quad u(0) = u_0, \quad u_t(0) = u_1, \quad (26)$$

has a unique solution

$$u \in C([0, T]; \mathcal{H}_X^1) \cap C^1([0, T]; \mathcal{H}_X^0). \quad (27)$$

Its energy

$$\mathcal{E}_X(t) = \frac{1}{2} \|u_t(t)\|_{L^2}^2 + \frac{1}{2} \mathbf{a}_X(u(t), u(t)) \quad (28)$$

satisfies

$$\mathcal{E}_X(t) - \mathcal{E}_X(0) = \operatorname{Re} \int_0^t (f_X(s), u_t(s))_{L^2} ds. \quad (29)$$

In particular, both reduced dynamics are energy conservative when unforced.

Proof. Every nonzero Fourier coefficient solves a forced harmonic oscillator with frequency $\sqrt{\lambda_X(\mathbf{k})}$. The uniform spectral bounds give equivalence between the summed oscillator energy and the $H^1 \times L^2$ norm on the mean-zero subspace. The modewise Duhamel formula yields existence, uniqueness, and continuity. The energy identity follows first for finite Fourier sums and then by density. $\square$

Corollary 3.4 (Exact modal restriction of elasticity). *If u solves $u_{tt} + \mathcal{A}u = f$, then $u_P = \mathcal{P}_P u$ solves (26) with $X = P$ and projected data and forcing. In particular, qP initial data and qP forcing remain exactly qP for all time.*

The structural distinction is now explicit. In general,

$$[\mathcal{P}_L, \mathcal{A}] \neq 0, \quad (30)$$

so $\mathcal{H}_L^0$ is not invariant under full anisotropic elasticity. The longitudinal equation is nevertheless a well-posed, coercive, conservative constrained model.

For completeness, the qP dynamics has a scalar representation. Choose a unit qP eigenvector $\mathbf{g}_P(\mathbf{k})$ at each nonzero lattice mode, with a compatible choice at $\pm\mathbf{k}$, and write

$$\widehat{u}(\mathbf{k}) = \mathbf{g}_P(\mathbf{k})\widehat{a}(\mathbf{k}). \quad (31)$$

This gives a unitary equivalence between $\mathcal{H}_P^s$ and $H_0^s(\mathbb{T}^2)$ under which

$$a_{tt} + \Lambda_P a = f_a, \quad \widehat{\Lambda_P a}(\mathbf{k}) = \lambda_P(\mathbf{k})\widehat{a}(\mathbf{k}). \quad (32)$$

The exact homogeneous qP equation is therefore scalar and positive, but generally pseudodifferential because λ_P is homogeneous of degree two without being a quadratic polynomial in $\mathbf{k}$.

4 Longitudinal propagation and exact error identities

The longitudinal squared speed is the Rayleigh quotient

$$c_L^2(\mathbf{n}) = \mathbf{n}^T \mathbf{\Gamma}(\mathbf{n}) \mathbf{n}. \quad (33)$$

Let $\mathbf{g}_P$ and $\mathbf{g}_{SV}$ be normalized qP and qSV eigenvectors. Define the signed angle β from $\mathbf{n}$ to $\mathbf{g}_P$ by

$$\mathbf{n} = \cos \beta \mathbf{g}_P - \sin \beta \mathbf{g}_{SV}. \quad (34)$$

Proposition 4.1 (Exact speed identity). *For every direction with a positive qP–qSV gap,*

$$\boxed{c_P^2 - c_L^2 = (c_P^2 - c_{SV}^2) \sin^2 \beta.} \quad (35)$$

Consequently, $c_L \leq c_P$, with equality if and only if the qP polarization is longitudinal.

Proof. In the orthonormal eigenbasis,

$$c_L^2 = \mathbf{n}^T \mathbf{\Gamma} \mathbf{n} = c_P^2 \cos^2 \beta + c_{SV}^2 \sin^2 \beta.$$

Rearrangement gives (35). □

Remark 4.2. This is a two-dimensional Rayleigh-quotient identity, not a new eigenvalue perturbation formula. Its role is to connect the polarization geometry directly to the error of the competing longitudinal symbol.

Corollary 4.3 (qSV content of a longitudinal mode). *For a longitudinal Fourier amplitude $A_0 \mathbf{n}$,*

$$\frac{\|\mathbf{\Pi}_{SV}(A_0 \mathbf{n})\|^2}{\|A_0 \mathbf{n}\|^2} = \sin^2 \beta. \quad (36)$$

Thus a curl-free field is pure qP if and only if $\beta = 0$.

For zero forcing and zero initial velocity, compare the longitudinal solution with the exact qP solution initialized by the qP projection of the same datum:

$$\hat{\mathbf{u}}_P(t, \mathbf{k}) = \cos(\omega_P t) \mathbf{\Pi}_P \hat{\mathbf{u}}_0, \quad (37)$$

$$\hat{\mathbf{u}}_L(t, \mathbf{k}) = \cos(\omega_L t) \hat{\mathbf{u}}_0, \quad \omega_L = |\mathbf{k}| c_L. \quad (38)$$

For a longitudinal datum $\hat{\mathbf{u}}_0 = A_0 \mathbf{n}$, the difference has the exact orthogonal splitting

$$\boxed{\hat{\mathbf{u}}_L - \hat{\mathbf{u}}_P = \underbrace{\mathbf{\Pi}_P \hat{\mathbf{u}}_L - \hat{\mathbf{u}}_P}_{\text{qP phase component}} + \underbrace{\mathbf{\Pi}_{SV} \hat{\mathbf{u}}_L}_{\text{qSV contamination}}.} \quad (39)$$

The two terms are orthogonal mode by mode, and therefore also in $L^2(\mathbb{T}^2)$ after summation.

5 Weak-anisotropy expansion

Let

$$a = v_{P0}^2, \quad b = v_{S0}^2, \quad d = a - b > 0, \quad (40)$$

and introduce the analytic weak-anisotropy family

$$\epsilon = \eta \epsilon_0, \quad \delta = \eta \delta_0, \quad |\eta| \leq \eta_0. \quad (41)$$

We parameterize the wave normal by

$$\mathbf{n} = (c, s)^T = (\cos \theta, \sin \theta)^T, \quad \mathbf{s} = (-s, c)^T. \quad (42)$$

At $\eta = 0$, the Christoffel matrix has eigenpairs $(a, \mathbf{n})$ and $(b, \mathbf{s})$. Along (41),

$$\frac{C_{13} + C_{44}}{\rho} = \sqrt{d^2 + 2\eta \delta_0 a d} = d + \eta a \delta_0 + O(\eta^2). \quad (43)$$

Consequently,

$$\mathbf{\Gamma}^\eta(\mathbf{n}) = \begin{pmatrix} a(1 + 2\eta \epsilon_0)c^2 + bs^2 & (d + \eta a \delta_0 + O(\eta^2))cs \\ (d + \eta a \delta_0 + O(\eta^2))cs & bc^2 + as^2 \end{pmatrix}. \quad (44)$$

In the isotropic basis $(\mathbf{n}, \mathbf{s})$ write

$$\begin{pmatrix} \mathbf{n}^T \\ \mathbf{s}^T \end{pmatrix} \mathbf{\Gamma}^\eta(\mathbf{n}) \begin{pmatrix} \mathbf{n} & \mathbf{s} \end{pmatrix} = \begin{pmatrix} \alpha^\eta & h^\eta \\ h^\eta & \sigma^\eta \end{pmatrix}. \quad (45)$$

Then $\alpha^\eta = (c_L^\eta)^2$, $\alpha^0 = a$, $\sigma^0 = b$, and a direct multiplication of (44) gives

$$h^\eta = \eta h_1(\theta) + O(\eta^2), \quad h_1(\theta) = a c s [\delta_0 \cos(2\theta) - 2\epsilon_0 c^2]. \quad (46)$$

All remainders in this section are uniform in θ .

Theorem 5.1 (Uniform VTI weak-anisotropy expansion). *There exist $\eta_0 > 0$ and $C > 0$ such that, for $|\eta| \leq \eta_0$ and every direction $\mathbf{n}$,*

$$\beta(\theta, \eta) = \eta \beta_1(\theta) + R_\beta(\theta, \eta), \quad |R_\beta| \leq C \eta^2, \quad (47)$$

$$\mathbf{\Pi}_P^\eta = \mathbf{\Pi}_L + \eta \mathbf{\Pi}_1 + R_\Pi^\eta, \quad \|\mathbf{R}_\Pi^\eta\| \leq C \eta^2, \quad (48)$$

$$(c_P^\eta)^2 - (c_L^\eta)^2 = \eta^2 d \beta_1^2 + R_{c^2}^\eta, \quad |R_{c^2}^\eta| \leq C |\eta|^3, \quad (49)$$

$$c_P^\eta - c_L^\eta = \eta^2 \sqrt{a} r_2 + R_c^\eta, \quad |R_c^\eta| \leq C |\eta|^3, \quad (50)$$

where

$$\boxed{\beta_1(\theta) = \frac{a}{d}cs [\delta_0 \cos(2\theta) - 2\epsilon_0 c^2], \quad \Pi_1 = \beta_1(\mathbf{n}\mathbf{s}^T + \mathbf{s}\mathbf{n}^T),} \quad (51)$$

and

$$\boxed{r_2(\theta) = \frac{d}{2a}\beta_1(\theta)^2.} \quad (52)$$

Consequently, with

$$\kappa(\mathbf{k}) = |\mathbf{k}|\sqrt{a}r_2(\mathbf{n}), \quad (53)$$

the frequency difference satisfies the uniform estimate

$$|\omega_P^\eta(\mathbf{k}) - \omega_L^\eta(\mathbf{k}) - \eta^2 \kappa(\mathbf{k})| \leq C|\eta|^3 |\mathbf{k}|. \quad (54)$$

Proof. The VTI coefficients and hence $\mathbf{\Gamma}^\eta(\mathbf{n})$ are analytic in η near zero, uniformly for $\mathbf{n} \in \mathbb{S}^1$. Since the unperturbed gap is $d > 0$, it remains uniformly positive for sufficiently small $|\eta|$. In the basis of (45), the qP rotation angle satisfies

$$\tan(2\beta) = \frac{2h^\eta}{\alpha^\eta - \sigma^\eta}. \quad (55)$$

Using $h^\eta = \eta h_1 + O(\eta^2)$ and $\alpha^\eta - \sigma^\eta = d + O(\eta)$ gives $\beta = \eta h_1/d + O(\eta^2)$ uniformly, which yields (51). Expanding $\mathbf{g}_P = \cos \beta \mathbf{n} + \sin \beta \mathbf{s}$ gives the projector formula and its uniform remainder.

The larger eigenvalue of the rotated matrix obeys the exact identity

$$(c_P^\eta)^2 - \alpha^\eta = \frac{(h^\eta)^2}{(c_P^\eta)^2 - \sigma^\eta}. \quad (56)$$

The denominator is $d + O(\eta)$, while $h^\eta = \eta h_1 + O(\eta^2)$; hence

$$(c_P^\eta)^2 - (c_L^\eta)^2 = \eta^2 \frac{h_1^2}{d} + O(\eta^3) = \eta^2 d \beta_1^2 + O(\eta^3).$$

Since both speeds are uniformly bounded away from zero,

$$c_P^\eta - c_L^\eta = \frac{(c_P^\eta)^2 - (c_L^\eta)^2}{c_P^\eta + c_L^\eta} = \eta^2 \frac{d\beta_1^2}{2\sqrt{a}} + O(\eta^3),$$

which is (50)–(52). Multiplication by $|\mathbf{k}|$ gives (54). Uniformity follows from compactness of $\mathbb{S}^1$ and the uniform spectral gap. $\square$

Remark 5.2 (Relation to classical perturbation theory). The first-order qP polarization and velocity formulas are classical [2, 5, 8]; higher-order eigenvalue structures are treated in [7]. The role of Theorem 5.1 is to record uniform remainders in the normalization required by the propagated-field comparison.

The common first-order squared-speed term is

$$(c_P^\eta)^2 = (c_L^\eta)^2 = a + 2a\eta(\epsilon_0 c^4 + \delta_0 c^2 s^2) + O(\eta^2), \quad (57)$$

consistent with the classical weak-VTI qP formula [1]. Thus the projector is wrong at first order, whereas the phase speed is wrong only at second order.

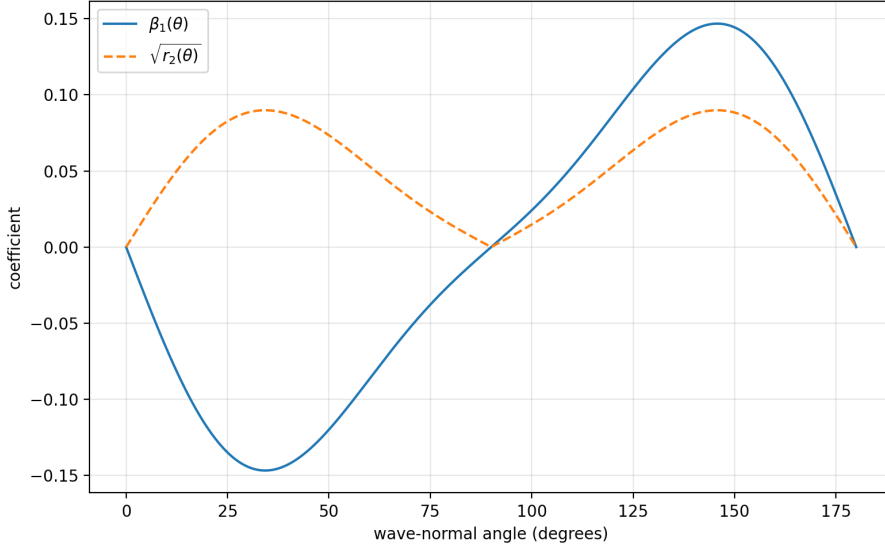


Figure 1: First-order polarization coefficient $\beta_1(\theta)$ and the square root of the second-order relative speed coefficient $r_2(\theta)$. The speed error is quadratic in the polarization mismatch.

For a radially symmetric longitudinal spectrum, the leading angular qSV fraction is

$$\frac{1}{2\pi} \int_0^{2\pi} \beta_1(\theta)^2 d\theta = \frac{a^2}{32d^2} (\delta_0^2 - 2\delta_0\epsilon_0 + 5\epsilon_0^2). \quad (58)$$

6 Finite-time field error

Let $u_0 \in \mathcal{H}_L^1$ have Fourier coefficients

$$\widehat{u}_0(\mathbf{k}) = A_0(\mathbf{k})\mathbf{n}, \quad W(\mathbf{k}) = |A_0(\mathbf{k})|^2, \quad (59)$$

and assume zero initial velocity. Let u_L^η and u_P^η be defined by (38) and (37) with the same longitudinal datum and its qP projection, respectively.

Theorem 6.1 (Uniform finite-time estimate). *There is a constant C , independent of t and sufficiently small $|\eta|$, such that*

$$\|\Pi_{SV}^\eta u_L^\eta(t)\|_{L^2} \leq C|\eta| \|u_0\|_{L^2}, \quad (60)$$

$$\|\Pi_P^\eta u_L^\eta(t) - u_P^\eta(t)\|_{L^2} \leq C|t|\eta^2 \|u_0\|_{H^1}. \quad (61)$$

Consequently,

$$\sup_{|t| \leq T} \|u_L^\eta(t) - u_P^\eta(t)\|_{L^2} \leq C(|\eta| \|u_0\|_{L^2} + T\eta^2 \|u_0\|_{H^1}). \quad (62)$$

The two terms on the left of (60)–(61) are orthogonal, so their squared norms add exactly.

Proof. The uniform projector expansion gives $\|\Pi_{SV}^\eta(\mathbf{k})\mathbf{n}\| \leq C|\eta|$. Parseval's identity yields (60). Moreover,

$$\Pi_P^\eta \widehat{u}_L^\eta - \widehat{u}_P^\eta = [\cos(\omega_L^\eta t) - \cos(\omega_P^\eta t)] \Pi_P^\eta \widehat{u}_0.$$

The Lipschitz bound for the cosine and (54) imply

$$|\cos(\omega_L^\eta t) - \cos(\omega_P^\eta t)| \leq C|t|\eta^2|\mathbf{k}|.$$

After summation this gives (61). Orthogonality follows from the ranges of $\mathbf{\Pi}_P^\eta$ and $\mathbf{\Pi}_{SV}^\eta$. $\square$

The theorem provides a genuine, nonasymptotic validity criterion for the longitudinal approximation. The leading profiles are identified next.

Corollary 6.2 (Leading bounded-time profiles). *Let $u_0 \in \mathcal{H}_L^2$ and set $\omega_0(\mathbf{k}) = |\mathbf{k}|\sqrt{a}$. For every fixed $T > 0$, the following convergences hold in $C([-T, T]; L^2)$ as $\eta \downarrow 0$:*

$$\eta^{-1} \mathbf{\Pi}_{SV}^\eta u_L^\eta(t) \longrightarrow \mathcal{F}^{-1}[-A_0 \beta_1 \mathbf{s} \cos(\omega_0 t)], \quad (63)$$

$$\eta^{-2} (\mathbf{\Pi}_P^\eta u_L^\eta(t) - u_P^\eta(t)) \longrightarrow \mathcal{F}^{-1}[A_0 t \kappa \mathbf{n} \sin(\omega_0 t)]. \quad (64)$$

Proof. The first convergence follows from the uniform projector expansion and $\omega_L^\eta \rightarrow \omega_0$. For the second, use

$$\frac{\cos(\omega_L^\eta t) - \cos(\omega_P^\eta t)}{\eta^2} = t \kappa \sin(\omega_0 t) + o(1)$$

mode by mode. On $|t| \leq T$, the scaled difference is bounded by $C_T(|\mathbf{k}| + |\mathbf{k}|^2)$ after subtraction of the limit, so the H^2 assumption and Parseval's identity give uniform dominated convergence. $\square$

Here and below $\mathcal{F}^{-1}$ denotes inverse Fourier series. For this bounded-time comparison, set

$$\begin{aligned} E_{SV}(t; \eta) &= \frac{\|\mathbf{\Pi}_{SV}^\eta u_L^\eta(t)\|_{L^2}}{\|u_0\|_{L^2}}, \\ E_{ph}(t; \eta) &= \frac{\|\mathbf{\Pi}_P^\eta u_L^\eta(t) - u_P^\eta(t)\|_{L^2}}{\|u_0\|_{L^2}}, \\ E_{tot}(t; \eta) &= \frac{\|u_L^\eta(t) - u_P^\eta(t)\|_{L^2}}{\|u_0\|_{L^2}}. \end{aligned}$$

Normalizing the limiting profiles by the same fixed norm, define

$$Q_1(t)^2 = \frac{\sum_{\mathbf{k} \neq 0} W \beta_1^2 \cos^2(\omega_0 t)}{\sum_{\mathbf{k} \neq 0} W}, \quad (65)$$

$$P_2(t)^2 = \frac{\sum_{\mathbf{k} \neq 0} W [t \kappa \sin(\omega_0 t)]^2}{\sum_{\mathbf{k} \neq 0} W}. \quad (66)$$

Then

$$E_{SV}(t; \eta) = |\eta| Q_1(t) + o(|\eta|), \quad E_{ph}(t; \eta) = \eta^2 P_2(t) + o(\eta^2), \quad (67)$$

Since the two exact components are orthogonal,

$$E_{tot}(t; \eta) = \sqrt{(|\eta| Q_1(t) + o(|\eta|))^2 + (\eta^2 P_2(t) + o(\eta^2))^2}. \quad (68)$$

This is the rigorous form of the two-term composite approximation used in the numerical comparisons.

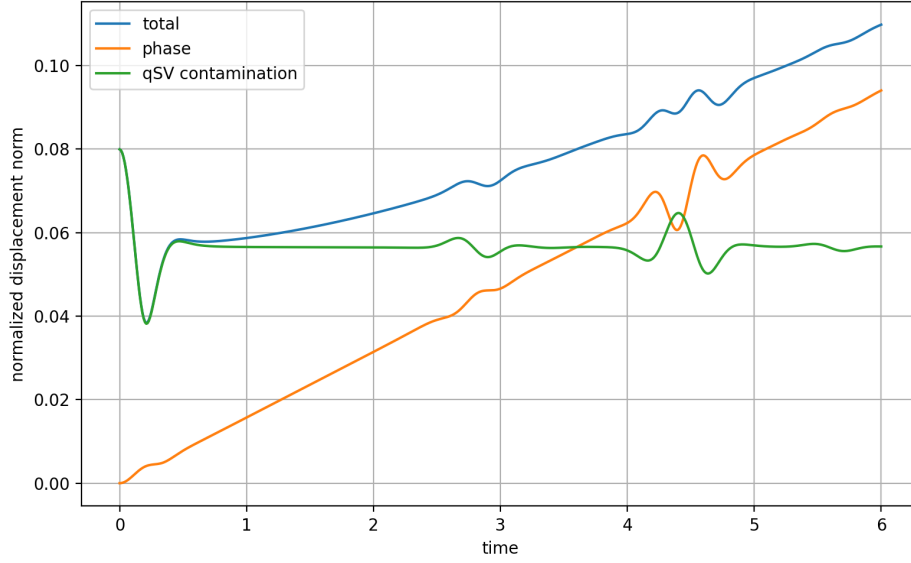


Figure 2: Finite-time error decomposition for the baseline VTI medium. The qSV component is present immediately, whereas the qP phase component accumulates in time. The squared norms add because the components are orthogonal mode by mode.

7 Long-time asymptotic scales

The real cosine propagators contain a rapidly oscillating carrier on long time scales. To expose the slowly accumulating difference, we use positive-frequency propagators and remove the common longitudinal phase. Define

$$\widehat{u}_P^+(t, \mathbf{k}) = e^{-i\omega_P^\eta(\mathbf{k})t} \mathbf{\Pi}_P^\eta(\mathbf{k}) \widehat{u}_0(\mathbf{k}), \quad (69)$$

$$\widehat{u}_L^+(t, \mathbf{k}) = e^{-i\omega_L^\eta(\mathbf{k})t} \widehat{u}_0(\mathbf{k}), \quad (70)$$

and the interaction-picture error

$$\widehat{\mathcal{R}_\eta(t)u_0(\mathbf{k})} = e^{i\omega_L^\eta(\mathbf{k})t} (\widehat{u}_L^+(t, \mathbf{k}) - \widehat{u}_P^+(t, \mathbf{k})). \quad (71)$$

It has the exact orthogonal decomposition

$$\widehat{\mathcal{R}_\eta(t)u_0} = \mathbf{\Pi}_{SV}^\eta \widehat{u}_0 + (1 - e^{-i\Delta\omega^\eta t}) \mathbf{\Pi}_P^\eta \widehat{u}_0, \quad \Delta\omega^\eta = \omega_P^\eta - \omega_L^\eta. \quad (72)$$

The interaction transformation is unitary, so it does not change any L^2 error norm.

7.1 An abstract two-branch multiplier theorem

The long-time mechanism does not depend on the explicit VTI formulas. We isolate the exact assumptions used below. Let $P_0(\mathbf{k})$ and $P_\eta(\mathbf{k})$ be rank-one orthogonal projectors for $\mathbf{k} \neq 0$, and let $\nu_\eta(\mathbf{k})$ and $\omega_\eta(\mathbf{k})$ be real frequencies. Assume that, uniformly in $\mathbf{k}/|\mathbf{k}|$,

$$P_\eta(\mathbf{k}) = P_0(\mathbf{k}) + \eta P_1(\mathbf{k}) + R_{P,\eta}(\mathbf{k}), \quad \|R_{P,\eta}(\mathbf{k})\| \leq C\eta^2, \quad (73)$$

$$\omega_\eta(\mathbf{k}) - \nu_\eta(\mathbf{k}) = \eta^2 \kappa(\mathbf{k}) + r_\eta(\mathbf{k}), \quad |r_\eta(\mathbf{k})| \leq C|\eta|^3 |\mathbf{k}|, \quad (74)$$

where $|\kappa(\mathbf{k})| \leq C|\mathbf{k}|$. For $v_0 = P_0(D)v_0$, define the interaction error

$$\widehat{\mathcal{R}_\eta(t)v_0}(\mathbf{k}) = (\mathbf{I} - P_\eta(\mathbf{k}))\widehat{v}_0(\mathbf{k}) + (1 - e^{-i[\omega_\eta(\mathbf{k}) - \nu_\eta(\mathbf{k})]t}) P_\eta(\mathbf{k})\widehat{v}_0(\mathbf{k}). \quad (75)$$

Differentiating $P_\eta^2 = P_\eta$ at $\eta = 0$ gives

$$P_0 P_1 P_0 = 0, \quad (\mathbf{I} - P_0) P_1 (\mathbf{I} - P_0) = 0. \quad (76)$$

Thus $P_1(D)v_0$ lies in the orthogonal complement of v_0 mode by mode.

Theorem 7.1 (Abstract finite- and long-time comparison). *Under (73)–(74), the following statements hold.*

(i) For $v_0 \in H_0^1$,

$$\|\mathcal{R}_\eta(t)v_0\|_{L^2} \leq C(|\eta|\|v_0\|_{L^2} + |t|\eta^2\|v_0\|_{H^1}). \quad (77)$$

(ii) If $v_0 \in H_0^1$, then for every $T > 0$,

$$\sup_{|\tau| \leq T} \|\eta^{-1}\mathcal{R}_\eta(\tau/\eta)v_0 + P_1(D)v_0 - i\tau\kappa(D)v_0\|_{L^2} \longrightarrow 0. \quad (78)$$

If $v_0 \in H_0^2$, the left-hand side is bounded by $C_T|\eta|\|v_0\|_{H^2}$.

(iii) If $v_0 \in H_0^0$, then for every fixed $\tau \in \mathbb{R}$,

$$\mathcal{R}_\eta(\tau/\eta^2)v_0 \longrightarrow (1 - e^{-i\tau\kappa(D)})v_0 \quad \text{in } L^2. \quad (79)$$

If $v_0 \in H_0^1$, the convergence is uniform for $|\tau| \leq T$ and

$$\sup_{|\tau| \leq T} \|\mathcal{R}_\eta(\tau/\eta^2)v_0 - (1 - e^{-i\tau\kappa(D)})v_0\|_{L^2} \leq C_T|\eta|\|v_0\|_{H^1}. \quad (80)$$

In the intermediate limit, the two fields $-P_1(D)v_0$ and $i\tau\kappa(D)v_0$ are orthogonal in L^2 .

Proof. The modal term in (75) is bounded by $C|\eta||\widehat{v}_0|$. For the phase term, the inequality $|1 - e^{-ix}| \leq |x|$ and (74) give

$$|1 - e^{-i(\omega_\eta - \nu_\eta)t}| \leq C|t|\eta^2|\mathbf{k}|,$$

which proves (77) by Parseval.

At $t = \tau/\eta$,

$$\eta^{-1}(\mathbf{I} - P_\eta)\widehat{v}_0 = -P_1\widehat{v}_0 + O(|\eta|)|\widehat{v}_0|,$$

and

$$\frac{1 - e^{-i(\omega_\eta - \nu_\eta)\tau/\eta}}{\eta} \longrightarrow i\tau\kappa$$

for each mode. On $|\tau| \leq T$ the quotient is bounded by $C_T|\mathbf{k}|$. Uniform convergence follows by first restricting to finitely many modes and then controlling the tail by the H^1 norm. For the rate, set $x = (\omega_\eta - \nu_\eta)\tau/\eta$. The bounds $|1 - e^{-ix} - ix| \leq x^2/2$ and $|x/\eta - \tau\kappa| \leq C_T|\eta||\mathbf{k}|$ yield

$$\left| \frac{1 - e^{-ix}}{\eta} - i\tau\kappa \right| \leq C_T|\eta|(|\mathbf{k}| + |\mathbf{k}|^2),$$

which is summable against $|\widehat{v}_0|^2$ for $v_0 \in H^2$. The projector factor contributes only $C_T |\eta| |\mathbf{k}| |\widehat{v}_0|$.

At $t = \tau/\eta^2$,

$$(\omega_\eta - \nu_\eta)t = \tau\kappa + O_T(|\eta| |\mathbf{k}|).$$

Pointwise convergence and the uniform bound by a constant multiple of $|\widehat{v}_0|$ give (79) by dominated convergence. The Lipschitz continuity of the exponential, together with the projector estimate, gives (80). Finally, (76) shows that $P_1(D)v_0$ belongs to $\text{Ran}(\mathbf{I} - P_0(D))$, while $\kappa(D)v_0$ belongs to $\text{Ran } P_0(D)$, proving orthogonality. $\square$

Remark 7.2 (Second-order Cauchy data). On the mean-zero torus, every energy solution of $u_{tt} + \mathcal{L}_X u = 0$ admits the standard decomposition

$$u(t) = e^{-it\mathcal{L}_X^{1/2}} a_X^+ + e^{it\mathcal{L}_X^{1/2}} a_X^-, \quad a_X^\pm = \frac{1}{2} \left(u_0 \pm i\mathcal{L}_X^{-1/2} u_1 \right).$$

Thus the one-way formulation isolates one temporal branch rather than imposing a special wave equation. The real finite-time theorem above treats common displacement data with zero initial velocity; general displacement-velocity data can be handled by applying the one-way estimate to the two branches, with additional but routine bookkeeping caused by the different square roots of the two reduced operators.

For the positive-frequency long-time analysis, we use the qP-projected initial norm. Set

$$N_\eta = \|\Pi_P^\eta u_0\|_{L^2}, \quad N_0 = \|u_0\|_{L^2} > 0. \quad (81)$$

By (76), $\langle u_0, \Pi_1 u_0 \rangle = 0$; hence $N_\eta^2 = N_0^2 + O(\eta^2)N_0^2$ and $N_\eta/N_0 = 1 + O(\eta^2)$. Define

$$E_{\text{ph}}^2(t; \eta) = \frac{\sum_{\mathbf{k} \neq 0} 4 \sin^2\left(\frac{\Delta\omega_\eta t}{2}\right) |\Pi_P^\eta \widehat{u}_0|^2}{N_\eta^2}, \quad (82)$$

$$E_S^2(\eta) = \frac{\sum_{\mathbf{k} \neq 0} |\Pi_{SV}^\eta \widehat{u}_0|^2}{N_\eta^2}, \quad (83)$$

$$E_{\text{tot}}^2(t; \eta) = E_{\text{ph}}^2(t; \eta) + E_S^2(\eta). \quad (84)$$

Finally,

$$S_1^2 = \frac{\sum W \beta_1^2}{\sum W}, \quad M_2^2 = \frac{\sum W \kappa^2}{\sum W}. \quad (85)$$

Corollary 7.3 (Intermediate-scale VTI field limit). *Let $u_0 \in \mathcal{H}_L^1$. As $\eta \downarrow 0$, for every $T > 0$,*

$$\sup_{|\tau| \leq T} \left\| \eta^{-1} \mathcal{R}_\eta(\tau/\eta) u_0 - U_{\text{int}}(\tau) \right\|_{L^2} \longrightarrow 0, \quad (86)$$

where

$$\widehat{U_{\text{int}}(\tau)}(\mathbf{k}) = A_0(\mathbf{k}) [-\beta_1(\mathbf{n}) \mathbf{s} + i\tau\kappa(\mathbf{k}) \mathbf{n}]. \quad (87)$$

If $u_0 \in \mathcal{H}_L^2$, the left-hand side of (86) is bounded by $C_T |\eta| \|u_0\|_{H^2}$. Consequently, uniformly for bounded $\tau \geq 0$,

$$\frac{E_S(\eta)}{|\eta|} \longrightarrow S_1, \quad (88)$$

$$\frac{E_{\text{ph}}(\tau/\eta; \eta)}{|\eta|} \longrightarrow \tau M_2, \quad (89)$$

$$\frac{E_{\text{tot}}(\tau/\eta; \eta)}{|\eta|} \longrightarrow \sqrt{S_1^2 + \tau^2 M_2^2}. \quad (90)$$

If $M_2 > 0$, the two limiting mechanisms have equal size at

$$\tau_c = \frac{S_1}{M_2}, \quad t_c \sim \frac{\tau_c}{|\eta|}. \quad (91)$$

Proof. Apply [Theorem 7.1](#) with $P_0 = \mathbf{\Pi}_L$, $P_\eta = \mathbf{\Pi}_P^\eta$, $P_1 = \mathbf{\Pi}_1$, $\nu_\eta = \omega_L^\eta$, and $\omega_\eta = \omega_P^\eta$. The hypotheses are precisely (48) and (54). Since $\Pi_1(A_0 \mathbf{n}) = A_0 \beta_1 \mathbf{s}$, the abstract profile is (87). Orthogonality and $N_\eta/N_0 \rightarrow 1$ give the norm limits. $\square$

Corollary 7.4 (Diffractive-scale VTI field limit). *Let $u_0 \in \mathcal{H}_L^0$. As $\eta \downarrow 0$, for every fixed $\tau \in \mathbb{R}$,*

$$\mathcal{R}_\eta(\tau/\eta^2)u_0 \longrightarrow U_{\text{diff}}(\tau) \quad \text{in } L^2(\mathbb{T}^2), \quad (92)$$

where

$$\widehat{U_{\text{diff}}(\tau)}(\mathbf{k}) = A_0(\mathbf{k}) (1 - e^{-i\tau\kappa(\mathbf{k})}) \mathbf{n}. \quad (93)$$

If $u_0 \in \mathcal{H}_L^1$, convergence is uniform for $|\tau| \leq T$ and

$$\sup_{|\tau| \leq T} \|\mathcal{R}_\eta(\tau/\eta^2)u_0 - U_{\text{diff}}(\tau)\|_{L^2} \leq C_T |\eta| \|u_0\|_{H^1}. \quad (94)$$

Consequently,

$$E_{\text{ph}}(\tau/\eta^2; \eta) \longrightarrow F(\tau), \quad E_{\text{tot}}(\tau/\eta^2; \eta) \longrightarrow F(\tau), \quad (95)$$

where

$$F(\tau)^2 = \frac{\sum_{\mathbf{k} \neq 0} 4 \sin^2(\kappa(\mathbf{k})\tau/2) W(\mathbf{k})}{\sum_{\mathbf{k} \neq 0} W(\mathbf{k})}. \quad (96)$$

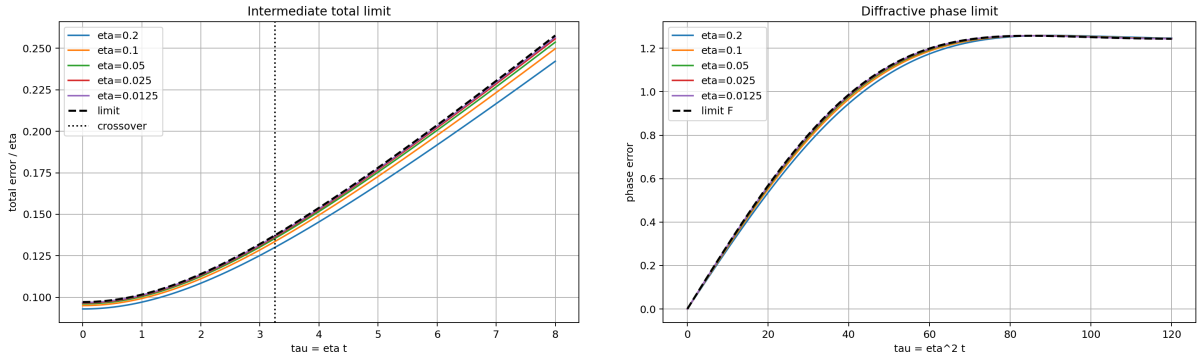
If, in addition, $u_0 \in \mathcal{H}_L^2$ and $M_2 > 0$, then for $\tau \downarrow 0$,

$$F(\tau) = \tau M_2 + O(\tau^3). \quad (97)$$

Proof. The field convergence and rate follow from [Theorem 7.1](#) with the same identifications as in [Theorem 7.3](#). Taking norms and using $N_\eta/N_0 \rightarrow 1$ gives (95)–(96). Finally, $4 \sin^2(\kappa\tau/2) = \kappa^2\tau^2 + O(\kappa^4\tau^4)$; the H^2 assumption makes the fourth moment summable, and division by $F(\tau) + \tau M_2$ yields (97). $\square$

The three regimes are

$$\begin{aligned} t = O(1) & : E_{\text{tot}} = O(|\eta|), \quad \text{polarization dominates,} \\ t = O(|\eta|^{-1}) & : E_{\text{tot}} = O(|\eta|), \quad \text{polarization and phase are comparable,} \\ t = O(\eta^{-2}) & : E_{\text{tot}} = O(1), \quad \text{finite qP dephasing survives.} \end{aligned} \quad (98)$$



(a) Intermediate scale $t = \tau/\eta$.

(b) Diffractive scale $t = \tau/\eta^2$.

Figure 3: Convergence to the two long-time limiting curves. On the intermediate scale the error remains $O(\eta)$; on the diffractive scale it converges to a finite order-one dephasing profile.

8 Numerical design

All experiments are performed in the periodic square $[-10, 10]^2$ and evaluated directly in Fourier space. There is no time-stepping error. With $L = 20$, the Fourier wavenumbers are

$$k_j = \frac{2\pi}{L}j, \quad j = 0, \dots, N/2 - 1, -N/2, \dots, -1.$$

The baseline snapshot and energy checks use $N = 256$ modes in each direction. The parameter sweeps and long-time tests use $N = 64$. The initial potential has width $\sigma = 0.65$. In the sweep and asymptotic calculations it is prescribed spectrally by

$$\widehat{\phi}_0(\mathbf{k}) = e^{-\sigma^2|\mathbf{k}|^2/2}, \quad \widehat{u}_0(\mathbf{k}) = i\mathbf{k} \widehat{\phi}_0(\mathbf{k}), \quad \widehat{u}_0(0) = 0, \quad (99)$$

so it is exactly longitudinal. The baseline physical-space snapshot uses the corresponding sampled centered Gaussian and its spectral gradient.

Angular extrema are evaluated on 1441 uniformly spaced directions in $[0, \pi]$. The near-isotropy and two-parameter temporal sweeps use 121 times in $[0, 6]$; the parameter map contains 23×19 points in (ϵ, δ) . The long-time tests use $\eta \in \{0.2, 0.1, 0.05, 0.025, 0.0125\}$, 321 values of $\tau \in [0, 8]$ on the intermediate scale, and 481 values of $\tau \in [0, 120]$ on the diffractive scale. The scripts used to generate all reported data and figures accompany the manuscript. They were rerun without numerical modification in MATLAB R2026a Update 3 on 64-bit Windows 11; every value quoted below was reproduced to the displayed precision, while the monitored projector, energy, and orthogonal-decomposition residuals remained at or near machine precision.

The baseline material parameters are

$$\rho = 1, \quad v_{P0} = 3, \quad v_{S0} = 1.5, \quad \epsilon = 0.2, \quad \delta = 0.1, \quad (100)$$

corresponding to

$$C_{11} = 12.6, \quad C_{33} = 9, \quad C_{44} = 2.25, \quad C_{13} = 5.3468744. \quad (101)$$

The datum in (99), or its baseline sampled counterpart, is longitudinal to machine precision. Three propagations are compared:

- (i) full elastic propagation using both qP and qSV spectral branches;
- (ii) pure qP propagation using $\mathbf{\Pi}_P$ and ω_P ;
- (iii) longitudinal propagation using $\mathbf{\Pi}_L$ and ω_L .

Projector, eigenvalue, energy, and orthogonal-decomposition identities are monitored. A weak-anisotropy sweep uses

$$(\epsilon, \delta) = \eta(0.2, 0.1), \quad (102)$$

while a two-parameter map covers

$$-0.15 \leq \epsilon \leq 0.40, \quad -0.15 \leq \delta \leq 0.30. \quad (103)$$

Long-time tests use the positive-frequency formulation in [Section 7](#).

9 Numerical results

9.1 Baseline case

The minimum squared-speed gap is 6.75. The maximum longitudinal qP speed underestimate is 0.543856%, attained near 38.375° , while the maximum polarization mismatch is 6.854863° . The exact identity (35) is satisfied with residual 3.9×10^{-15} . The discrete projector defects are

$$\|\Pi_P^2 - \Pi_P\|_\infty = 2.2 \times 10^{-16}, \quad \|\mathbf{A}\Pi_P - \lambda_P \Pi_P\|_\infty = 7.3 \times 10^{-12}. \quad (104)$$

Although the initial field is longitudinal to relative error 1.4×10^{-16} , its qSV squared fraction is 6.35×10^{-3} , or approximately 0.635%.

At $t = 2$, the longitudinal-versus-qP displacement error normalized by the instantaneous qP norm is 9.14%. Over $0 \leq t \leq 6$, the maximum error normalized by $\|u_P(0)\|$ is 10.98%. Its maximum qP phase component is 9.40%, while the maximum qSV contamination amplitude is 7.99%. Energy drifts remain below 1.2×10^{-15} .

9.2 Near-isotropy scaling

Power-law fits on $10^{-3} \leq \eta \leq 10^{-1}$ give

Quantity	fitted exponent	predicted exponent
maximum polarization mismatch	0.99596	1
maximum relative speed error	1.99224	2
initial qSV squared fraction	1.99230	2
maximum total temporal error	0.99615	1
maximum qP phase component	1.99431	2
maximum qSV contamination	0.99615	1

The fitted exponents confirm that the dominant near-isotropic field error is the first-order polarization mismatch, not the second-order phase-speed mismatch.

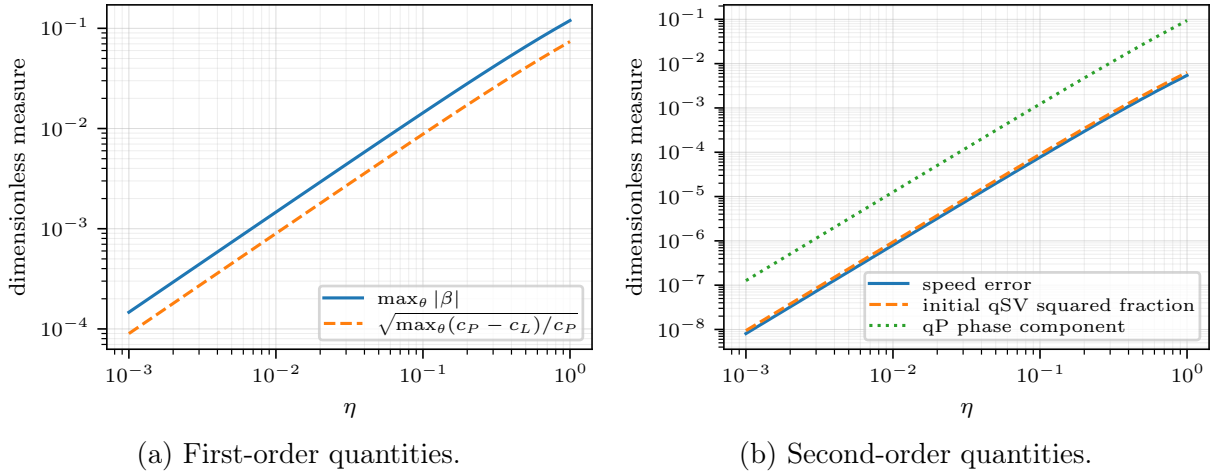
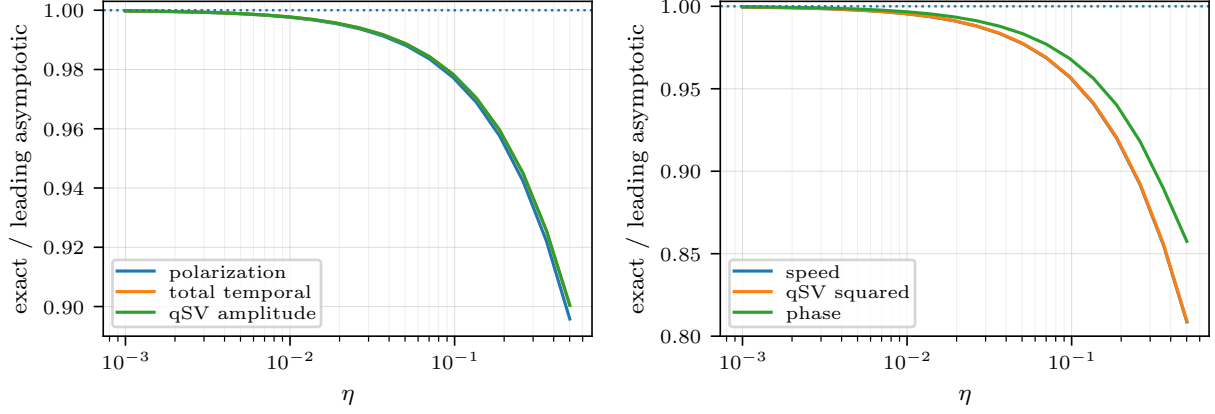


Figure 4: Weak-anisotropy scaling along $(\epsilon, \delta) = \eta(0.2, 0.1)$. The slopes are numerically indistinguishable from the predicted orders one and two.

The leading coefficients are also predicted without fitting. For the baseline ray,

$$\max_{\theta} |\beta_1| = 0.1467889544, \quad \max_{\theta} r_2 = 0.00808012393, \quad (105)$$

while the discrete initial qSV squared coefficient is 0.00943858536. At $\eta = 10^{-3}$, all exact-to-asymptotic ratios lie between 0.99954 and 0.99977.



(a) First-order exact/asymptotic ratios.

(b) Second-order exact/asymptotic ratios.

Figure 5: Coefficient-level validation of the weak-anisotropy expansions. All ratios converge to one.

9.3 Two-parameter VTI map

Across the sampled rectangle, the qP–qSV gap remains positive, with minimum 4.05. The maximum polarization mismatch is 18.18° , the maximum longitudinal speed underestimate is 3.32%, and the maximum initial qSV squared fraction is 3.39%. The maximum temporal displacement error reaches 50.12%, with a maximum phase component of 48.34% and qSV contamination amplitude of 18.73%. The strongest errors occur for positive ϵ and negative δ , where the two contributions to the first-order polarization rotation reinforce one another.

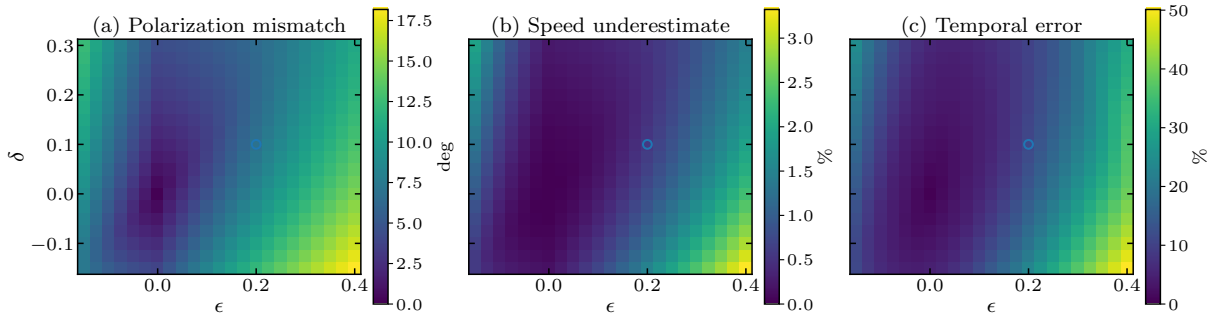


Figure 6: Parameter maps in the (ϵ, δ) plane: (a) maximum qP polarization mismatch, (b) maximum longitudinal speed underestimate, and (c) maximum temporal displacement error. The geometries of the polarization and temporal-error maps show that modal mismatch is the leading near-isotropic mechanism, whereas accumulated phase error dominates in the strongly anisotropic corner.

9.4 Long-time limits

For the Gaussian datum,

$$S_1 = 0.09715238215, \quad M_2 = 0.02984007282, \quad \tau_c = 3.2557689365. \quad (106)$$

At $\eta = 0.0125$, the supremum errors in the intermediate limiting curves are approximately 10^{-3} , while those in the diffractive limiting curves are approximately 2.7×10^{-3} . Fits using $\eta \leq 0.1$ give convergence exponents between 0.988 and 0.994, confirming the expected $O(\eta)$ remainders after scaling.

The diffractive limit takes the values

$$F(10) = 0.29471, \quad F(40) = 0.98655, \quad F(80) = 1.25608, \quad (107)$$

and reaches 1.25737 near $\tau = 84.75$ over $0 \leq \tau \leq 120$. The bounded, oscillatory behavior is dephasing rather than instability; the relative difference between normalized complex fields can exceed one and is bounded above by two.

10 Discussion and relation to previous work

The literature audit separates the local spectral ingredients from the propagated-field result.

Local wave attributes. First-order qP velocity and polarization formulas are classical consequences of perturbation theory [2, 5, 8]. Farra and Pšenčík further describe higher-order eigenvalue and eigenvector approximations, including the familiar quadratic contribution of off-diagonal Christoffel couplings to the qP eigenvalue [7]. The VTI coefficients β_1 and r_2 derived in Section 5 should therefore be read as an explicit specialization and normalization for the present comparison, not as a new perturbation formalism.

Mode separation and projection deviation. Projection onto Christoffel polarizations, and the failure of divergence or wave-normal projection to produce a pure anisotropic qP field, are established in [3, 9]. Cheng and Kang go further by propagating a field projected onto an isotropic reference direction and then correcting the deviation from the true qP polarization in order to remove residual qS energy [11]. Thus neither $\Pi_P \neq \Pi_L$ nor residual qSV content under longitudinal projection is claimed as new. The exact orthogonal splitting (39) is used here because it allows the two errors to be propagated and estimated separately.

Pure-qP equations. Scalar qP pseudodifferential equations and energy-conscious numerical realizations predate the present work. Xu and Zhou explicitly design a self-adjoint differential component to improve stability and energy behavior [13]. Subsequent studies use Padé, rational, optimized, separable, or scalar-operator decompositions to approximate a qP dispersion branch [14, 16, 17, 19, 20]. These works are primarily computational and, in many cases, start from the acoustic TI approximation. By contrast, the qP operator in Section 3 is the exact spectral restriction of the full elastic operator. Positive self-adjointness and energy conservation are natural consequences of this restriction and are not, by themselves, advanced as new results.

The 2026 weak-anisotropy decomposition of Yang et al. is particularly instructive [20]. It uses a weak-anisotropy differential symbol whose phase-velocity error is of second order and supplements it with a scalar correction to recover an accurate pseudoacoustic qP wavefield. This is fully consistent with the local order found here. The present question is complementary: rather than correcting the second-order phase discrepancy, we quantify how the uncorrected discrepancy accumulates in time, while simultaneously retaining the first-order elastic polarization error absent from a scalar pseudoacoustic model.

Scope of the contribution. Within the papers examined above and a targeted search of the qP propagation literature, we did not find a prior analysis that combines all of the following for the longitudinal and exact elastic qP propagators:

- (i) a common invariant-subspace/Galerkin operator comparison;
- (ii) an orthogonal decomposition of the propagated-field error into modal contamination and qP dephasing;
- (iii) the quantitative separation $O(\eta)$ versus $O(t\eta^2)$;
- (iv) the crossover limit on $t = O(\eta^{-1})$ and the nontrivial dephasing limit on $t = O(\eta^{-2})$.

This is a bounded literature statement, not a claim that no related result can exist. The strongest and safest contribution of the paper is the multiscale dynamic error theory, rather than any individual local spectral formula. [Theorem 7.1](#) also shows that this mechanism is not tied to VTI: it follows for homogeneous two-branch multipliers whenever the projector mismatch is first order and the frequency mismatch is second order.

The results also suggest a constructive corrected model. The longitudinal projector requires an $O(\eta)$ correction,

$$\Pi_P \approx \Pi_L + \eta\beta_1(\mathbf{n}\mathbf{s}^T + \mathbf{s}\mathbf{n}^T), \quad (108)$$

whereas the phase symbol requires an $O(\eta^2)$ correction,

$$c_P^2 \approx c_L^2 + \eta^2 d\beta_1^2. \quad (109)$$

These corrections address different mechanisms. Correcting only the speed leaves the leading qSV contamination intact; correcting only the projector removes the dominant bounded-time modal error but does not prevent long-time dephasing. Existing rational, scalar-operator, and low-rank techniques provide possible numerical realizations, but developing another pure-qP solver is outside the scope of this paper.

The principal limitation is homogeneity. In a variable medium, the local qP projector depends on both position and wave vector and does not commute with the elastic operator. Mode conversion and lower-order commutator terms then arise. Extending the present analysis to slowly varying coefficients would require a microlocal or semiclassical diagonalization. Boundaries, attenuation, and three-dimensional qS degeneracies are likewise excluded from the current setting.

11 Conclusions

A longitudinal, curl-free wave is not generally a pure qP wave in anisotropic elasticity. In homogeneous 2D VTI media, the difference can be characterized completely. On a periodic

domain, the qP projector defines a closed invariant subspace of the elastic operator and an exact positive self-adjoint reduced dynamics. The longitudinal model is instead the positive self-adjoint Galerkin restriction of the same elastic energy to the curl-free subspace. This establishes well-posedness and energy conservation for both models while isolating their essential structural difference: invariance versus constraint.

The longitudinal squared-speed deficit equals the qP–qSV gap multiplied by the squared polarization mismatch. Near isotropy, the qP projector differs from the longitudinal projector at first order, whereas the phase-speed difference is second order. A longitudinal initial field therefore contains an $O(\eta)$ qSV amplitude even though its qP speed is accurate through first order.

At the field level, modal contamination and qP dephasing are orthogonal. On bounded times a uniform estimate separates an $O(|\eta|)$ modal term from an $O(t\eta^2)$ phase term. An abstract two-branch multiplier theorem isolates the underlying mechanism. In an interaction representation, the rescaled fields converge strongly in L^2 on $t = O(|\eta|^{-1})$, where a finite crossover occurs, and on $t = O(\eta^{-2})$, where the qSV component vanishes but the qP phase error has a nontrivial order-one limit. Thus, second-order phase-speed accuracy does not imply uniform-in-time accuracy of the propagated field.

Exact Fourier experiments confirm every identity, asymptotic coefficient, and limiting curve. The results give both a rigorous interpretation of longitudinal P-wave models and a precise blueprint for correcting their projector and phase symbols separately.

CRediT authorship contribution statement

Stefano Micheletti: Conceptualization, Methodology, Formal analysis, Software, Validation, Investigation, Visualization, Writing – original draft, Writing – review and editing.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the author used ChatGPT, an AI-assisted tool by OpenAI, to support literature organization, manuscript editing, LaTeX formatting, consistency checks, and preparation of submission files. The author reviewed, verified, and edited all AI-assisted output and takes full responsibility for the content of the article.

Code and data availability

The self-contained MATLAB scripts used for all numerical experiments accompany the manuscript. They use exact Fourier-mode propagation and require no time integrator or additional toolbox. A validation archive contains the MATLAB environment record, console logs, CSV tables, MAT result structures, and generated figures from the clean rerun described in [Section 8](#).

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