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UNIFORM CONVERGENCE OF THE SCHWARZ ALTERNATING METHOD FOR OPTIMAL CONTROL PROBLEMS*

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Abstract. In this paper, we analyze the Schwarz alternating method for unconstrained elliptic optimal control problems, which is equivalent to the corresponding method for the associated saddle-point systems. A distinctive feature in this setting is that the local error propagation operators are not necessarily nonexpansive in the energy norm, which stands in marked contrast to the standard elliptic boundary-value case. We develop a rigorous uniform convergence theory in the continuous setting and then extend the analysis to finite difference discretizations. In both formulations, we prove that the Schwarz iteration converges whenever its counterpart for the underlying elliptic equation is convergent. Furthermore, we show that the contraction factor for the auxiliary elliptic equation in the maximum norm provides a uniform upper bound, which is independent of the regularization parameter α, for the contraction factor of the optimal control iteration in the same norm. The theoretical framework is also extended to cover one-level alternating Schwarz and parallel Schwarz variants. Numerical experiments are presented to validate the theoretical results.

Key words. Schwarz alternating method, saddle-point problems, optimal control problems, finite difference methods, uniform convergence

MSC codes. 65N55, 49J20, 49N20, 49N45, 65N06

1. Introduction.

In this paper, we consider the following optimal control problem

$$\min_{u\in L^2(\Omega),y\in H_0^1(\Omega)} J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to

$$\mathcal{L}y = f + u \ \text{ in } \Omega \quad \text{and} \quad y = 0 \ \text{ on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded Lipschitz domain, $u \in L^2(\Omega)$ is the control variable, $y_d \in L^2(\Omega)$ is the desired state or observation, $\alpha > 0$ is the regularization parameter, $f \in L^2(\Omega)$ and $\mathcal{L}$ is a self-adjoint and strictly elliptic operator which is defined in strong form as

$$\mathcal{L}y = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial y}{\partial x_i}\right) + c_0(x)y, \tag{1.3}$$

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with $a_{ij}(x),\ c_0(x) \in L^\infty(\Omega)$ and $c_0 \geq 0$.

The optimal control problem (1.1)-(1.2) has a unique solution (y, u) which can be characterized by its first-order optimality system (see, e.g., [33]). Using standard arguments, the first-order optimality system of (1.1)-(1.2) is

$$\begin{cases} \alpha u + p = 0 & \text{in } \Omega, \\ \mathcal{L}y = f + u & \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ \mathcal{L}p = y - y_d & \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where p is the adjoint state.

By eliminating u, it is equivalent to the following saddle-point system

$$\begin{cases} \mathcal{L}y = f - \alpha^{-1}p & \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ \mathcal{L}p = y - y_d & \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Over the past several decades, PDE-constrained optimization problems have attracted considerable attention due to their increasingly broad applications in modern science and engineering. This growing demand has in turn stimulated intensive research on efficient numerical methods. Nevertheless, the fast and reliable simulation of such problems remains highly challenging. For a comprehensive theoretical treatment, we refer to [33, 47]; for recent advances in optimization algorithms and numerical schemes, we refer to the monograph [30].

Beyond the convergence analysis of discretization schemes and the design of optimization algorithms, the fast and robust solution of the resulting discrete systems is of paramount importance. Numerous strategies have been developed to address this challenge. Preconditioned Krylov subspace methods with specially designed block preconditioners have been employed in [5, 25, 41, 55]. Multigrid methods have been investigated in [6, 7, 8, 40, 43, 48]. Domain decomposition methods (DDMs for short) constitute another powerful strategy; see, e.g., [2, 3, 4, 15, 11, 20, 29, 28, 37, 45, 44]. Parallel implementations of DDM-type algorithms are discussed in [38, 39, 54], while time-domain decomposition for time-dependent optimal control problems is addressed in [19, 31, 21, 24, 32].

In this paper, we focus on the Schwarz alternating method for optimal control problems. Originally introduced by H. A. Schwarz in [42] to prove the existence and uniqueness of solutions to Laplace's equation on general domains with Dirichlet boundary conditions, the method gained prominence as a computational tool in the 1980s. This was particularly spurred by P. L. Lions's proof of convergence via variational principles [34], which significantly simplified the convergence analysis. Lions later also established convergence properties based on the maximum principle [35]. For a historical overview of the Schwarz alternating method and its developments, we refer to [22]. Generalizations of the Schwarz alternating method in various directions have given rise to a broad class of domain decomposition methods, which have proven highly effective as fast solvers for self-adjoint positive definite PDEs. Their inherent parallelism makes them particularly attractive in large-scale applications. For a comprehensive treatment of the design and convergence analysis of DDMs for elliptic equations, we refer to the monograph [46] and the review articles [49, 52] and the references therein. For extensions to nonsymmetric and indefinite problems, we refer to [10, 9, 50].

In contrast to the case of elliptic equations, where DDMs are supported by a rich theoretical foundation and exhibit satisfactory numerical performance, the theoretical understanding of DDMs for optimal control problems remains limited and far from satisfactory. Nevertheless, numerous numerical experiments in the literature have demonstrated the efficiency and robustness of DDMs for such problems.

Roughly speaking, domain decomposition methods for optimal control problems can be classified into two categories: *PDE-level* methods, where DDMs are applied separately to the state and adjoint equations (cf. [11]), and *Optimization-level* methods, where DDM are designed to decompose the optimal control problem directly to some optimization subproblems (cf. [3, 4, 20]). In [3, 4], non-overlapping DDMs based on Robin-type transmission conditions were proposed, and convergence of the algorithms was established without explicit estimates on the contraction factor; this approach was subsequently extended within the optimized Schwarz framework in [18, 17, 53]. In [20], the authors proved convergence of several non-overlapping methods and showed that, under suitable parameter choices, the corresponding algorithms converge in at most three iterations.

To the best of our knowledge, no robust theoretical convergence results exist in the literature for *Optimization-level* overlapping DDMs applied to optimal control problems—in the sense of methods that converge uniformly with respect to the mesh size and the regularization parameter α. The extension of the existing convergence theory for DDMs from elliptic equations to optimal control problems (or, more generally, PDE-constrained optimization) is hindered by the inherent saddle-point structure of the first-order optimality system. A key ingredient in the standard DDM convergence framework is the nonexpansiveness of the local error propagation operators (cf. [51, Assumption (A2)]). However, this assumption fails for optimal control problems, and consequently, the standard approach does not yield uniform convergence. Further discussion is provided in [13].

In this paper, we establish the uniform convergence of the Schwarz alternating method for elliptic optimal control problems. To this end, we introduce suitable error merit functions (or error vectors in the discrete setting) that are naturally tied to the norms employed in [8, 25, 26, 45, 41, 55]. By invoking the maximum principle for elliptic operators, we further derive a uniform contraction factor that is strictly less than one and independent of the regularization parameter.

Our analysis rests primarily on the weak maximum principle for second-order elliptic operators, which we recall in the following theorem; for further details, we refer to [23].

THEOREM 1.1. *[23, Theorem 8.1] Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded domain and let $\phi \in H^1(\Omega)$ satisfy $\mathcal{L}\phi \le 0$ (≥ 0) in Ω. Then*

$$\sup_{\Omega} \phi \le \sup_{\partial\Omega} \phi^+ \quad (\inf_{\Omega} \phi \ge \inf_{\partial\Omega} \phi^-)$$

where $\phi^+ = \max\{\phi, 0\} \quad (\phi^- = \min\{\phi, 0\})$,

$$\sup_{\partial\Omega} \phi = \inf\{c | \phi \le c \textit{ on } \partial\Omega,\ c \in \mathbb{R}\} \textit{ and } \inf_{\partial\Omega} \phi = -\sup_{\partial\Omega}(-\phi).$$

The remainder of this paper is organized as follows. In Section 2, we formulate the Schwarz alternating method for the optimal control problems under consideration. Section 3 is devoted to the uniform convergence analysis of the method in the continuous setting, where we establish a contraction factor in the maximum norm and show that it is uniformly bounded above by the corresponding contraction factor for the elliptic equation case. Additionally, we prove convergence in the L^2 norm. The extension to multiple subdomains is presented in Section 4. In Section 5, we adapt the analysis to the finite difference discretization and demonstrate that the same convergence properties hold. Finally, numerical experiments are reported in the last section to corroborate the theoretical findings.

Throughout this paper, we adopt the standard notation for differential operators, function spaces, and norms as used in [1, 23]. We shall denote by C a generic positive constant that is independent of both the regularization parameter α and the mesh size.

2. The Schwarz alternating method for elliptic optimal control problems.

In this section, we extend the Schwarz alternating method for elliptic PDEs to the optimal control problem (1.1)-(1.2).

2.1. The subproblem.

The subproblem retains the structure of an optimal control problem with an elliptic PDE constraint (see also [29]).

DEFINITION 2.1. *Let $\omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded Lipschitz domain, and let p_Γ, $y_\Gamma \in H^{1/2}(\partial\omega)$ be given functions on $\partial\omega$, and y_d, f be given functions in $L^2(\omega)$. The following optimal control problem is called a Dirichlet-Dirichlet optimal control problem*

$$\min_{u\in L^2(\omega)} \; J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\omega)} - \langle \frac{\partial y}{\partial n_{\mathcal{L}}}, p_\Gamma\rangle_{H^{-1/2}(\partial\omega),H^{1/2}(\partial\omega)} \tag{2.1}$$

subject to

$$\mathcal{L}y = f + u \quad \text{in } \omega \quad \text{and} \quad y = y_\Gamma \quad \text{on } \partial\omega, \tag{2.2}$$

where the conormal derivative $\frac{\partial y}{\partial n_{\mathcal{L}}}$ is given by

$$\frac{\partial y}{\partial n_{\mathcal{L}}} = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial y}{\partial x_i}\cos(\mathbf{n}, x_j),$$

$\cos(\mathbf{n}, x_j)$ *is the j-th direction cosine of* $\mathbf{n}$ *and* $\mathbf{n}$ *is the unit outward normal vector of* $\partial\omega$.

A standard argument gives the first order optimality system of the Dirichlet-Dirichlet optimal control problem as

$$\begin{cases} \alpha u + p = 0 & \text{in } \omega, \\ \mathcal{L}y = f + u & \text{in } \omega, \quad y = y_\Gamma \quad \text{on } \partial\omega, \\ \mathcal{L}p = y - y_d & \text{in } \omega, \quad p = p_\Gamma \quad \text{on } \partial\omega. \end{cases}$$

Eliminating u by the first equation, it gives an equivalent saddle-point system

$$\begin{cases} \mathcal{L}y = f - \alpha^{-1}p & \text{in } \omega, \quad y = y_\Gamma \quad \text{on } \partial\omega, \\ \mathcal{L}p = y - y_d & \text{in } \omega, \quad p = p_\Gamma \quad \text{on } \partial\omega. \end{cases} \tag{2.3}$$

2.2. The Schwarz alternating method.

Let $\{\Omega_i : i = 1, 2\}$ be an overlapping domain decomposition of Ω with $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$ (see Figure 1). Assume that Ω_i $(i = 1, 2)$ are two bounded Lipschitz domains.

With these ingredients, we are now in a position to define the Schwarz alternating method for the elliptic optimal control problem. Given the overlapping decomposition

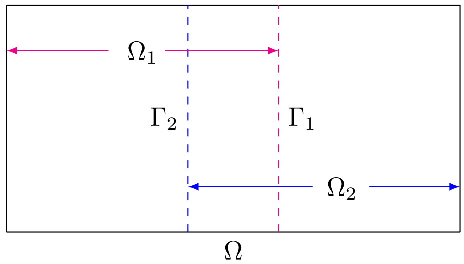

FIG. 1. *Overlapping decomposition of* Ω

Algorithm 2.1 : The Schwarz alternating method for OCPs

1. Initialization: choose $y^{(0)},\ p^{(0)} \in C(\overline{\Omega}) \cap H_0^1(\Omega)$.
2. For $k = 0, 1, \cdots$, solve the Dirichlet-Dirichlet optimal control problem on Ω_i ($i = 1, 2$) alternately

$$\min_{u_i^{(2k+i)} \in L^2(\Omega_i)} \quad J(y_i^{(2k+i)}, u_i^{(2k+i)}) = \frac{1}{2}\|y_i^{(2k+i)} - y_d\|^2_{L^2(\Omega_i)} + \frac{\alpha}{2}\|u_i^{(2k+i)}\|^2_{L^2(\Omega_i)} - \langle \frac{\partial y_i^{(2k+i)}}{\partial n_{\mathcal{L},i}}, p^{(2k+i-1)}|_{\partial\Omega_i} \rangle_{H^{-1/2}(\partial\Omega_i), H^{1/2}(\partial\Omega_i)}$$

subject to

$$\mathcal{L} y_i^{(2k+i)} = f + u_i^{(2k+i)} \text{ in } \Omega_i \quad \text{and} \quad y_i^{(2k+i)} = y^{(2k+i-1)} \text{ on } \partial\Omega_i$$

and set $u^{(2k+i)}$, $y^{(2k+i)}$, $p^{(2k+i)}$ as follows:

$$u^{(2k+i)} = \begin{cases} u_i^{(2k+i)} & \text{in } \Omega_i, \\ u^{(2k+i-1)} & \text{in } \overline{\Omega} \setminus \Omega_i; \end{cases} \qquad y^{(2k+i)} = \begin{cases} y_i^{(2k+i)} & \text{in } \Omega_i, \\ y^{(2k+i-1)} & \text{in } \overline{\Omega} \setminus \Omega_i; \end{cases}$$

$$p^{(2k+i)} = -\alpha u^{(2k+i)}.$$

$\Omega = \bigcup_{i=1}^2 \Omega_i$, the method is stated in Algorithm 2.1. We denote by $\frac{\partial y}{\partial n_{\mathcal{L},i}}$ ($i = 1, 2$) the conormal derivative as that of $\frac{\partial y}{\partial n_{\mathcal{L}}}$ where ω is replaced by Ω_i ($i = 1, 2$).

Leveraging the equivalence between the optimal control problem and its first-order optimality system, we present in Algorithm 2.2 the corresponding Schwarz alternating method applied directly to the saddle-point system (1.4). This algorithm can be interpreted as a domain decomposition approach that splits the original saddle-point problem into two coupled saddle-point subproblems.

Remark 2.2. Since each subdomain problem retains the structure of an optimal control problem, the decomposition here is an *Optimization-level* decomposition of the original problem. Notably, it simultaneously decomposes both the state equation and the objective functional, which is a distinctive feature of the algorithm.

3. Uniform convergence of the Schwarz alternating method. This section is devoted to the uniform convergence analysis of Algorithm 2.2; the convergence properties of Algorithm 2.1 then follow directly from the equivalence of the two algo-

Algorithm 2.2 : The Schwarz alternating method for saddle-point problems

1. Initialization: choose $y^{(0)},\ p^{(0)} \in C(\overline{\Omega}) \cap H_0^1(\Omega)$.
2. For $k = 0, 1, \cdots$, solve the following problem on $\Omega_i\ (i = 1, 2)$ alternately

$$\begin{cases} \mathcal{L}y^{(2k+i)} = f - \alpha^{-1}p^{(2k+i)} & \text{in } \Omega_i, \\ \mathcal{L}p^{(2k+i)} = y^{(2k+i)} - y_d & \text{in } \Omega_i, \\ y^{(2k+i)} = y^{(2k+i-1)} & \text{on } \partial\Omega_i, \\ p^{(2k+i)} = p^{(2k+i-1)} & \text{on } \partial\Omega_i, \\ y^{(2k+i)} = y^{(2k+i-1)} & \text{in } \overline{\Omega} \setminus \overline{\Omega_i}, \\ p^{(2k+i)} = p^{(2k+i-1)} & \text{in } \overline{\Omega} \setminus \overline{\Omega_i}. \end{cases}$$

rithms. Our proof of uniform convergence in the maximum norm relies on the weak maximum principle stated in Theorem 1.1.

For this purpose, we first establish the well-posedness and regularity results of the subproblems and give a key observation of the self-adjoint and strictly elliptic operator $\mathcal{L}$.

LEMMA 3.1. *Let $\omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded Lipschitz domain, $\beta > 0$ and let $\psi_b, \phi_b \in C(\partial\omega) \cap H^{1/2}(\partial\omega)$. Consider the coupled system*

$$\begin{cases} \mathcal{L}\psi = -\beta\phi \quad \text{in} \ \ \omega, \quad \psi = \psi_b \quad \text{on} \ \ \partial\omega, \\ \mathcal{L}\phi = \psi \qquad \text{in} \ \ \omega, \quad \phi = \phi_b \quad \text{on} \ \ \partial\omega. \end{cases}$$

Under the above assumptions, the system admits a unique solution (ψ, ϕ) satisfying $\psi, \phi \in C(\overline{\omega}) \cap H^1(\omega)$. Moreover, the following inequality holds:

$$\mathcal{L}(\psi^2 + \beta\phi^2) \leq 0 \quad \text{in} \ \ \omega.$$

Proof. The existence and uniqueness of a solution $(\psi, \phi) \in H^1(\omega) \times H^1(\omega)$ follow from the trace theorem (cf. [27, Theorem 1.5.1.3]) and the Lax–Milgram theorem applied to the equivalent weak formulation of the coupled system; see, e.g., [8, 26].

To establish continuity of ψ and ϕ up to the boundary, note that $H^1(\omega) \hookrightarrow L^2(\omega)$. Since $\psi_b, \phi_b \in C(\partial\omega)$, by elliptic regularity theory for second-order operators with L^q right-hand sides and $2q > d$, we obtain $\psi, \phi \in C(\overline{\omega})$ (cf. [23, Theorem 8.30]). Consequently, $\psi, \phi \in C(\overline{\omega}) \cap H^1(\omega) \subset L^\infty(\omega) \cap H^1(\omega)$, which further implies that $\phi^2, \psi^2 \in H^1(\omega)$.

A direct calculation gives

$$\mathcal{L}(\psi^2) = 2\psi\mathcal{L}\psi - 2\sum_{i,j=1}^{d} \left(a_{ij}(x) \frac{\partial\psi}{\partial x_i} \frac{\partial\psi}{\partial x_j} \right) - c_0(x)\psi^2$$

and

$$\mathcal{L}(\phi^2) = 2\phi\mathcal{L}\phi - 2\sum_{i,j=1}^{d} \left(a_{ij}(x) \frac{\partial\phi}{\partial x_i} \frac{\partial\phi}{\partial x_j} \right) - c_0(x)\phi^2.$$

By the coupled system and using the strict ellipticity of $\mathcal{L}$, we obtain

$$\begin{aligned}&\mathcal{L}(\psi^2+\beta\phi^2)\\ &= -2\sum_{i,j=1}^{d}\left(a_{ij}(x)\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\right) - 2\beta\sum_{i,j=1}^{d}\left(a_{ij}(x)\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right) - c_0(x)(\psi^2+\beta\phi^2)\\ &\le 0.\end{aligned}$$

This completes the proof. ∎

Denote $e_y^{(j)} = y - y^{(j)}$ and $e_p^{(j)} = p - p^{(j)}$ with $y,\ p$ the solutions of (1.4) and $y^{(j)},\ p^{(j)}$ generated by Algorithm 2.2. A direct calculation shows that the errors $e_y^{(j)}$, $e_p^{(j)}$ $(j = 2k+i, i = 1, 2, k = 0, 1, 2, ...)$ satisfy the following equations

$$(3.1)\qquad \begin{cases} \mathcal{L}e_y^{(2k+i)} = -\alpha^{-1}e_p^{(2k+i)} & \text{in } \Omega_i, \quad e_y^{(2k+i)} = e_y^{(2k+i-1)} \quad \text{on } \partial\Omega_i,\\ \mathcal{L}e_p^{(2k+i)} = e_y^{(2k+i)} & \text{in } \Omega_i, \quad e_p^{(2k+i)} = e_p^{(2k+i-1)} \quad \text{on } \partial\Omega_i,\\ e_y^{(2k+i)} = e_y^{(2k+i-1)} & \text{in } \overline{\Omega}\setminus\overline{\Omega_i},\\ e_p^{(2k+i)} = e_p^{(2k+i-1)} & \text{in } \overline{\Omega}\setminus\overline{\Omega_i}. \end{cases}$$

We denote

$$(3.2)\qquad \eta^{(j)} = (e_y^{(j)})^2 + \alpha^{-1}(e_p^{(j)})^2 \ge 0,$$

with $j = 2k+i$ $(i = 1, 2, k = 0, 1, ...)$. For each $j = 2k+i$, by Lemma 3.1, $\eta^{(j)}$ satisfies

$$(3.3)\qquad \begin{cases} \mathcal{L}\eta^{(2k+i)} \le 0 & \text{in } \Omega_i, \quad \eta^{(2k+i)} = \eta^{(2k+i-1)} \quad \text{on } \partial\Omega_i,\\ \eta^{(2k+i)} = \eta^{(2k+i-1)} & \text{in } \overline{\Omega}\setminus\overline{\Omega_i}. \end{cases}$$

Similarly, we can define the Schwarz alternating method for the elliptic equation

$$\mathcal{L}\xi = 0 \ \text{ in } \ \Omega \quad \text{and} \quad \xi = 0 \ \text{ on } \ \partial\Omega$$

on Ω_i with the initial guess $\xi^{(0)} = \eta^{(0)}$. The k-th iteration is given by

$$(3.4)\qquad \begin{cases} \mathcal{L}\xi^{(2k+i)} = 0 & \text{in } \Omega_i, \quad \xi^{(2k+i)} = \xi^{(2k+i-1)} \quad \text{on } \partial\Omega_i,\\ \xi^{(2k+i)} = \xi^{(2k+i-1)} & \text{in } \overline{\Omega}\setminus\overline{\Omega_i} \end{cases}$$

with $j = 2k+i$, $i = 1, 2$, $k = 0, 1, 2, \ldots$, $\xi^{(0)} = \eta^{(0)}$ on $\overline{\Omega}$.

LEMMA 3.2. *For $i = 1, 2$ and $k = 0, 1, 2, \ldots$, suppose that $\eta^{(2k+i)}$ and $\xi^{(2k+i)}$ are defined as above. Then*

$$0 \le \eta^{(2k+i)} \le \xi^{(2k+i)} \quad \text{in } \overline{\Omega}_i.$$

Proof. We prove this by induction. For the base case, we consider $k = 0$ and $i = 1$. From (3.3), (3.4), and the fact that $\xi^{(0)} = \eta^{(0)}$ on $\overline{\Omega}$, we obtain

$$\mathcal{L}(\eta^{(1)} - \xi^{(1)}) \le 0 \quad \text{in } \Omega_1, \qquad \eta^{(1)} - \xi^{(1)} = 0 \quad \text{on } \partial\Omega_1.$$

An application of the weak maximum principle (Theorem 1.1) then yields

$$0 \le \eta^{(1)} \le \xi^{(1)} \quad \text{in } \overline{\Omega}_1,$$

which establishes the claim for $k = 0$ and $i = 1$.

Now assume that the result holds for $2k + i - 1$; we prove it for $2k + i$. Observe that $\eta^{(2k+i)} - \xi^{(2k+i)}$ satisfies

$$\mathcal{L}(\eta^{(2k+i)} - \xi^{(2k+i)}) \leq 0 \quad \text{in } \Omega_i \text{ and } \eta^{(2k+i)} - \xi^{(2k+i)} = \eta^{(2k+i-1)} - \xi^{(2k+i-1)} \quad \text{on } \partial\Omega_i.$$

By the induction hypothesis,

$$\eta^{(2k+i-1)} - \xi^{(2k+i-1)} \leq 0 \quad \text{on } \overline{\Omega}.$$

Applying Theorem 1.1, we conclude that

$$\eta^{(2k+i)} - \xi^{(2k+i)} \leq 0 \quad \text{in } \overline{\Omega}_i.$$

Together with (3.2), this yields $0 \leq \eta^{(2k+i)} \leq \xi^{(2k+i)}$ in $\overline{\Omega}_i$. The induction is thus complete. ∎

Lemma 3.2 establishes the relationship between the errors $\eta^{(j)}$ of Algorithm 2.2 and the auxiliary solutions $\xi^{(j)}$, namely,

$$0 \leq \eta^{(j)} \leq \xi^{(j)} \quad \text{on } \overline{\Omega}, \qquad \forall\ j = 2k + i,\ k = 0, 1, 2, \ldots,\ i = 1, 2,$$

provided that $\xi^{(0)} = \eta^{(0)}$ on $\overline{\Omega}$. This inequality immediately implies the uniform convergence of Algorithm 2.2 whenever $\xi^{(j)} \to 0$ as $j \to +\infty$. Moreover, it enables us to derive a uniform upper bound on the contraction factor of Algorithm 2.2. These convergence results are summarized in the following theorem.

THEOREM 3.3. *Let an overlapping domain decomposition of Ω into two Lipschitz subdomains be given. If the Schwarz alternating method for the homogeneous elliptic equation*

$$\mathcal{L}\xi = 0 \quad \textit{in } \Omega, \qquad \xi = 0 \quad \textit{on } \partial\Omega \tag{3.5}$$

is convergent, then the Schwarz alternating method for the optimality system (1.4), i.e., Algorithm 2.2, is uniformly convergent as well. Moreover, if the contraction factor of the Schwarz method for the elliptic equation (3.5) in the maximum norm is denoted by $\rho_e \in (0, 1)$, satisfying

$$\sup_{x\in\overline{\Omega}} \xi^{(2k)} \leq \rho_e \sup_{x\in\overline{\Omega}} \xi^{(2(k-1))}, \qquad k = 1, 2, \ldots, \tag{3.6}$$

then for all $k \geq 1$, the following contraction estimate holds:

$$\sup_{x\in\overline{\Omega}} \eta^{(2k)} \leq \rho_e \sup_{x\in\overline{\Omega}} \eta^{(2(k-1))}. \tag{3.7}$$

Proof. It suffices to establish the contraction estimate (3.7).

For any $k \geq 1$, we take the $2(k-1)$-th iterate as the initial step by setting

$$\xi^{(2(k-1))} = \eta^{(2(k-1))} \quad \text{on } \overline{\Omega}$$

in the auxiliary systems (3.3) and (3.4). Lemma 3.2 then yields

$$\eta^{(2k)} \leq \xi^{(2k)} \quad \text{on } \overline{\Omega}.$$

Combining this with the contraction estimate (3.6) for the elliptic equation, we obtain

$$\sup_{x\in\overline{\Omega}} \eta^{(2k)} \leq \sup_{x\in\overline{\Omega}} \xi^{(2k)} \leq \rho_e \sup_{x\in\overline{\Omega}} \xi^{(2(k-1))} = \rho_e \sup_{x\in\overline{\Omega}} \eta^{(2(k-1))}.$$

This completes the proof. ∎

By applying the above theorem, we can also obtain the convergence of the algorithm under the L^2 norm.

COROLLARY 3.4. *Suppose the assumptions in Theorem 3.3 hold, then it holds*

$$\|e_y^{(2k)}\|^2_{L^2(\Omega)} + \alpha^{-1}\|e_p^{(2k)}\|^2_{L^2(\Omega)} \le C\rho_e^k \sup_{x\in\overline{\Omega}} \eta^{(0)},$$

where $C > 0$ is a constant independent of ρ_e and α.

Proof. According to Theorem 3.3, we have

$$\|e_y^{(2k)}\|^2_{L^2(\Omega)} + \alpha^{-1}\|e_p^{(2k)}\|^2_{L^2(\Omega)} = \int_\Omega \eta^{(2k)} dx \le |\Omega| \sup_{x\in\overline{\Omega}} \eta^{(2k)} \le C\rho_e^k \sup_{x\in\overline{\Omega}} \eta^{(0)}.$$

This gives the results. ∎

Remark 3.5. Following are some remarks.

1. One can refer to [42, 35, 12] for the convergence analysis and the contraction factor of the Schwarz alternating method for elliptic equations under the maximum norm.
2. Estimate (3.7) gives a uniform upper bound ρ_e for the contraction factor of the method for optimal control problems. Our numerical results in Section 6 indicate this upper bound is not optimal and the optimal one might be ρ_e^2. More details can be found in [12, 13].

4. Extension to multiple subdomains.

Let $\{\Omega_i : \ i = 1, 2, \ldots, N\}$ be an overlapping decomposition of Ω into Lipschitz subdomains and $\{\chi_i : \ i = 1, 2, \ldots, N\}$ be a partition of unity based on this decomposition, which satisfies

$$0 \le \chi_i \le 1 \quad \text{and} \quad \sum_{i=1}^N \chi_i = 1. \tag{4.1}$$

The one-level alternating Schwarz method and the parallel Schwarz method are given in Algorithm 4.1 and Algorithm 4.2, respectively.

In the previous section we have shown that the convergence of the Schwarz alternating method for the optimal control problem follows from the convergence of its counterpart for the state equation. This is also true for the one-level alternating Schwarz method and the parallel Schwarz method.

Actually, for the one-level alternating Schwarz method we can conduct the convergence analysis following the same process as that of the Schwarz alternating method in Theorem 3.3 by Lemma 3.2.

For the parallel Schwarz method, by the convexity of the quadratic functional and (4.1), we have

$$\begin{aligned}
&(y - y^{(k+1)})^2 + \alpha^{-1}(p - p^{(k+1)})^2 \\
&= \left[\sum_{i=1}^N \chi_i (y - y^{(k+\frac{i}{N})})\right]^2 + \alpha^{-1}\left[\sum_{i=1}^N \chi_i (p - p^{(k+\frac{i}{N})})\right]^2 \\
&\le \sum_{i=1}^N \chi_i [(y - y^{(k+\frac{i}{N})})^2 + \alpha^{-1}(p - p^{(k+\frac{i}{N})})^2] \\
&= \sum_{i=1}^N \chi_i [(e_y^{(k+\frac{i}{N})})^2 + \alpha^{-1}(e_p^{(k+\frac{i}{N})})^2],
\end{aligned}$$

Algorithm 4.1 : The alternating Schwarz method

1. Initialization: choose $y^{(0)},\ p^{(0)} \in C(\overline{\Omega}) \cap H_0^1(\Omega)$.
2. For $k = 0, 1, \cdots$, solve the Dirichlet-Dirichlet optimal control problem on Ω_i ($i = 1, 2, \ldots, N$) sequentially

$$\min_{u_i^{(k+\frac{i}{N})} \in L^2(\Omega_i)} \quad J(y_i^{(k+\frac{i}{N})}, u_i^{(k+\frac{i}{N})}) = \frac{1}{2}\|y_i^{(k+\frac{i}{N})} - y_d\|^2_{L^2(\Omega_i)} + \frac{\alpha}{2}\|u_i^{(k+\frac{i}{N})}\|^2_{L^2(\Omega_i)} - \langle \frac{\partial y_i^{(k+\frac{i}{N})}}{\partial n_{\mathcal{L},i}}, p^{(k+\frac{i-1}{N})}|_{\partial\Omega_i} \rangle_{H^{-1/2}(\partial\Omega_i), H^{1/2}(\partial\Omega_i)}$$

subject to

$$\mathcal{L} y_i^{(k+\frac{i}{N})} = f + u_i^{(k+\frac{i}{N})} \ \text{ in } \Omega_i \quad \text{and} \quad y_i^{(k+\frac{i}{N})} = y^{(k+\frac{i-1}{N})} \ \text{ on } \partial\Omega_i,$$

and set $u^{(k+\frac{i}{N})}$, $y^{(k+\frac{i}{N})}$, $p^{(k+\frac{i}{N})}$ as follows:

$$u^{(k+\frac{i}{N})} = \begin{cases} u_i^{(k+\frac{i}{N})} & \text{in } \Omega_i, \\ u^{(k+\frac{i-1}{N})} & \text{in } \overline{\Omega} \setminus \Omega_i; \end{cases} \quad y^{(k+\frac{i}{N})} = \begin{cases} y_i^{(k+\frac{i}{N})} & \text{in } \Omega_i, \\ y^{(k+\frac{i-1}{N})} & \text{in } \overline{\Omega} \setminus \Omega_i; \end{cases}$$

$$p^{(k+\frac{i}{N})} = -\alpha u^{(k+\frac{i}{N})}.$$

---

where $e_y^{(k+\frac{i}{N})} = y - y^{(k+\frac{i}{N})}$ and $e_p^{(k+\frac{i}{N})} = p - p^{(k+\frac{i}{N})}$ for $i = 1, \ldots, N$. Note that $(e_y^{(k+\frac{i}{N})})^2 + \alpha^{-1}(e_p^{(k+\frac{i}{N})})^2$ can be bounded by using Lemma 3.2. This will give the desired results.

We omit the details of the proof and give the results in the following theorem.

Theorem 4.1. *For a given domain decomposition with multiple Lipschitz subdomains, the alternating Schwarz method and the parallel Schwarz method for optimal control problems converge under the maximum norm if their counterparts for the equation*

$$\mathcal{L}\xi = 0 \ \textit{ in } \ \Omega \quad \textit{and} \quad \xi = 0 \ \textit{ on } \ \partial\Omega$$

are convergent under the maximum norm, respectively.

Moreover, in the alternating Schwarz case, the contraction factor for the equation gives a uniform upper bound of that for optimal control problems. In the parallel Schwarz case, if the contraction factor for the equation case is $\rho_a \in (0,1)$, it gives

$$\sup_{x\in\overline{\Omega}}[(y - y^{(Nk)})^2 + \alpha^{-1}(p - p^{(Nk)})^2] \le \rho_a \sup_{x\in\overline{\Omega}}[(y - y^{((N-1)k)})^2 + \alpha^{-1}(p - p^{((N-1)k)})^2].$$

Remark 4.2. In [14], the authors discussed the scalability of the parallel Schwarz method for the elliptic equation. Combining with the results in Theorem 4.1, we can also expect the scalability of the method for optimal control problems.

5. Extension to the discrete case. In this section, we will consider the extension of the arguments in the previous section to the discrete case.

For illustration purposes, we will consider the problem in two dimensional case and use finite difference methods to discretize the optimal control problem. More

Algorithm 4.2 : The parallel Schwarz method

1. Initialization: choose $y^{(0)},\ p^{(0)} \in C(\overline{\Omega}) \cap H_0^1(\Omega)$.
2. For $k = 0, 1, \cdots$, solve the Dirichlet-Dirichlet optimal control problem on Ω_i ($i = 1, \ldots, N$) parallelly

$$\min_{u_i^{(k+\frac{i}{N})} \in L^2(\Omega_i)} J(y_i^{(k+\frac{i}{N})}, u_i^{(k+\frac{i}{N})}) = \frac{1}{2}\|y_i^{(k+\frac{i}{N})} - y_d\|^2_{L^2(\Omega_i)} + \frac{\alpha}{2}\|u_i^{(k+\frac{i}{N})}\|^2_{L^2(\Omega_i)} - \langle \frac{\partial y_i^{(k+\frac{i}{N})}}{\partial n_{\mathcal{L},i}}, p^{(k)}|_{\partial\Omega_i} \rangle_{H^{-1/2}(\partial\Omega_i), H^{1/2}(\partial\Omega_i)}$$

subject to

$$\mathcal{L} y_i^{(k+\frac{i}{N})} = f + u_i^{(k+\frac{i}{N})} \ \text{ in } \Omega_i \quad \text{and} \quad y_i^{(k+\frac{i}{N})} = y^{(k)} \ \text{ on } \partial\Omega_i$$

and set $u^{(k+1)}$, $y^{(k+1)}$, $p^{(k+1)}$ as follows:

$$u^{(k+1)} = \sum_{i=1}^N \chi_i u^{(k+\frac{i}{N})}, \quad y^{(k+1)} = \sum_{i=1}^N \chi_i y^{(k+\frac{i}{N})} \ \text{ and } \ p^{(k+1)} = -\alpha u^{(k+1)}.$$

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precisely, we will take

$$\Omega = (0,1)^2 \subset \mathbb{R}^2 \quad \text{and} \quad \mathcal{L} = -\Delta.$$

5.1. The discrete problem. We adopt the five-point finite difference scheme to discretize the state equation and approximate the objective functional by the trapezoidal rule. We refer to [6, 36] for the numerical analysis of the finite difference scheme.

For a given step size $h = 1/N$, we define the uniform grid (x_i, y_j) on $\Omega = (0,1)^2$, with $0 \le i, j \le N$ and $x_i = ih,\ y_j = jh$. For a given function z, we take $z_{i,j} = z(x_i, y_j)$ or some approximation of it and set $\{z_{i,j} :\ 0 \le i, j \le N\}$, which can be mapped to a vector $Z \in \mathbb{R}^{(N+1)^2}$. We denote by Z_I the part corresponding to the interior grid points in Ω and Z_∂ the part corresponding to the boundary grid points on $\partial\Omega$.

The five-point finite difference scheme for the state equation is given by

$$-\frac{y_{i-1,j} + y_{i+1,j} - 4y_{i,j} + y_{i,j-1} + y_{i,j+1}}{h^2} = f_{i,j} + u_{i,j} \quad 1 \le i, j \le N-1,$$

and the discrete problem of (1.1)-(1.2) reads

$$\text{(5.1)} \qquad \min_{U \in \mathbb{R}^n} J(Y, U) = \frac{h^2}{2}\|Y_I - Y_{d,I}\|^2_{\mathbb{R}^n} + \frac{\alpha h^2}{2}\|U_I\|^2_{\mathbb{R}^n}$$

subject to

$$\text{(5.2)} \qquad \mathcal{L}_h Y = F_I + U_I \ \text{ and } \ Y_\partial = 0,$$

where $\mathcal{L}_h$ is the corresponding coefficient matrix of the five-point finite difference scheme and $n = (N-1)^2$ and $\|\cdot\|_{\mathbb{R}^n}$ is the standard norm on $\mathbb{R}^n$.

Following a standard argument, the first order optimality system of the discrete problem reads

$$\begin{cases} \mathcal{L}_h Y = F_I + U_I, \quad Y_\partial = 0; \\ \mathcal{L}_h P = h^2(Y_I - Y_{d,I}), \quad P_\partial = 0; \\ \alpha h^2 U_I + P_I = 0. \end{cases}$$

Similarly, we can eliminate U_I and after doing some reformulation, we have

$$\begin{cases} \mathcal{L}_h Y = F_I - \frac{1}{\alpha}(\frac{1}{h^2}P_I), \quad Y_\partial = 0; \\ \mathcal{L}_h(\frac{1}{h^2}P) = Y_I - Y_{d,I}, \quad P_\partial = 0. \end{cases} \tag{5.3}$$

5.2. The Schwarz alternating method for the discrete problem. As in the continuous case, we can define the Schwarz alternating algorithms analogous to Algorithm 2.1 and Algorithm 2.2 in this discrete setting. We give the counterpart of Algorithm 2.2 and prove the uniform convergence of the algorithm. The convergence properties of the equivalent algorithm in the discrete case, which corresponds to Algorithm 2.1, follow directly.

For $i = 1, 2$, we denote by $Z_I,\ Z_\partial,\ Z_i,\ Z_{i,I},\ Z_{i,\partial}$ the vector components of Z corresponding to the grid points related to $\Omega,\ \partial\Omega,\ \overline{\Omega_i},\ \Omega_i,\ \partial\Omega_i$, respectively, and $\mathcal{L}_h^{(i)}$ the restriction of $\mathcal{L}_h$ to the grid points related to $\overline{\Omega_i}$.

Algorithm 5.1 : The Schwarz alternating method for the discrete problem

1. Initialization: choose $Y^{(0)},\ P^{(0)} \in \mathbb{R}^{(N+1)^2}$ with $Y_\partial^{(0)} = 0$ and $P_\partial^{(0)} = 0$.
2. For $k = 0, 1, \cdots$, solve the following problem alternately

$$\begin{cases} \mathcal{L}_h^{(i)} Y_i^{(2k+i)} = F_{i,I} - \alpha^{-1}(\frac{1}{h^2}P_{i,I}^{(2k+i)}), \quad Y_{i,\partial}^{(2k+i)} = Y_{i,\partial}^{(2k+i-1)}, \\ \mathcal{L}_h^{(i)}(\frac{1}{h^2}P_i^{(2k+i)}) = Y_{i,I}^{(2k+i)} - Y_{d,i,I}, \quad P_{i,\partial}^{(2k+i)} = P_{i,\partial}^{(2k+i-1)}, \\ Y^{(2k+i)} = Y^{(2k+i-1)} \text{ on } \overline{\Omega} \setminus \overline{\Omega_i}, \\ P^{(2k+i)} = P^{(2k+i-1)} \text{ on } \overline{\Omega} \setminus \overline{\Omega_i}. \end{cases}$$

5.3. Convergence analysis. In the following, we prove the convergence results of Algorithm 5.1. We first write down the error system of the algorithm and then prove a discrete counterpart of Lemma 3.1 for $\mathcal{L}_h$ for a proper error merit vector. The remaining analysis can be carried out identically in parallel with the continuous case by recalling the maximum principle of $\mathcal{L}_h$.

5.3.1. The error equations. We denote $E_y^{(j)} = Y - Y^{(j)}$, $E_p^{(j)} = P - P^{(j)}$ $(j = 2k+i,\ i = 1, 2,\ k = 0, 1, 2, \dots)$ with $Y,\ P$ the solutions of (5.3) and $Y^{(j)},\ P^{(j)}$

generated by Algorithm 5.1. Then the errors $E_y^{(j)}$ and $E_p^{(j)}$ satisfy

$$\begin{cases} \mathcal{L}_h^{(i)} E_{y,i}^{(2k+i)} = -\alpha^{-1}(\frac{1}{h^2} E_{p,i,I}^{(2k+i)}), & E_{y,i,\partial}^{(2k+i)} = E_{y,i,\partial}^{(2k+i-1)}, \\ \mathcal{L}_h^{(i)}(\frac{1}{h^2} E_{p,i}^{(2k+i)}) = E_{y,i,I}^{(2k+i)}, & E_{p,i,\partial}^{(2k+i)} = E_{p,i,\partial}^{(2k+i-1)}, \\ E_y^{(2k+i)} = E_y^{(2k+i-1)} \text{ on } \overline{\Omega} \setminus \overline{\Omega_i}, & \\ E_p^{(2k+i)} = E_p^{(2k+i-1)} \text{ on } \overline{\Omega} \setminus \overline{\Omega_i}. & \end{cases} \tag{5.4}$$

5.3.2. A property of $\mathcal{L}_h$. For $Z = (Z_k)_{m\times 1} \in \mathbb{R}^m$, we define $Z^2 \in \mathbb{R}^m$ in a componentwise sense, i.e., $(Z^2)_k = (Z_k)^2$.

LEMMA 5.1. *Let $\beta > 0$ and $\Phi, \Psi \in \mathbb{R}^{(N+1)^2}$ satisfy*

$$\mathcal{L}_h \Psi = -\beta \Phi_I \quad \textit{and} \quad \mathcal{L}_h \Phi = \Psi_I,$$

then $\mathcal{L}_h(\Psi^2 + \beta\Phi^2) \le 0$, which is understood componentwisely.

Proof. Let $\{\phi_{i,j} :\ 0 \le i, j \le N\}$ be the set corresponding to Φ. Since

$$\begin{aligned} &-\frac{\phi_{i-1,j}^2 + \phi_{i+1,j}^2 - 4\phi_{i,j}^2 + \phi_{i,j-1}^2 + \phi_{i,j+1}^2}{h^2} \\ &= -2\phi_{i,j}\frac{\phi_{i-1,j} + \phi_{i+1,j} - 4\phi_{i,j} + \phi_{i,j-1} + \phi_{i,j+1}}{h^2} \\ &\quad -\frac{(\phi_{i-1,j} - \phi_{i,j})^2 + (\phi_{i+1,j} - \phi_{i,j})^2}{h^2} - \frac{(\phi_{i,j-1} - \phi_{i,j})^2 + (\phi_{i,j+1} - \phi_{i,j})^2}{h^2}, \end{aligned}$$

it gives

$$\mathcal{L}_h(\Phi^2) \le -2\Phi_I^T \mathcal{L}_h \Phi.$$

Analogously, we have

$$\mathcal{L}_h(\Psi^2) \le -2\Psi_I^T \mathcal{L}_h \Psi.$$

Combing with the assumption, it implies that

$$\mathcal{L}_h(\Psi^2 + \beta\Phi^2) \le -2\Psi_I^T \mathcal{L}_h \Psi - 2\beta\Phi_I^T \mathcal{L}_h \Phi = 0,$$

which completes the proof. ∎

Remark 5.2. The above lemma also applies to $\mathcal{L}_h^{(i)}$ $(i = 1, 2)$ for vectors related to $\overline{\Omega_i}$.

5.3.3. The maximum principle of $\mathcal{L}_h$. The maximum principle is a key ingredient in the proof of uniform convergence of the Schwarz alternating method in the continuous case. An analogous discrete maximum principle is valid for the finite difference operator $\mathcal{L}_h$, which we state in the theorem below; we refer to [16, Theorem 3] for a detailed treatment.

THEOREM 5.3. *The finite difference operator $\mathcal{L}_h$ satisfies the discrete maximum principle, i.e., for any $Z \in \mathbb{R}^{(N+1)^2}$ with $\mathcal{L}_h Z \le 0$ (≥ 0), we have*

$$\max\{Z\} \le \max\{0, Z_\partial\} \quad (\min\{Z\} \ge \min\{0, Z_\partial\}),$$

where max *and* min *are understood in a componentwise sense.*

Remark 5.4. Note that this theorem also applies to $\mathcal{L}_h^{(i)}$ $(i = 1, 2)$ for vectors related to $\overline{\Omega_i}$.

5.3.4. The uniform convergence results. We define the error merit vectors as

$$E^{(2k+i)} = (E_y^{(2k+i)})^2 + \frac{\alpha^{-1}}{h^4}(E_p^{(2k+i)})^2, \quad i = 1, 2, \ k = 0, 1, 2, \ldots.$$

Applying Lemma 5.1 to $\mathcal{L}_h^{(i)}$ and $E_i^{(2k+i)}$, we have

$$\mathcal{L}_h^{(i)} E_i^{(2k+i)} \leq 0, \quad i = 1, 2, \ k = 0, 1, 2, \ldots. \tag{5.5}$$

Using (5.4), (5.5), and Theorem 5.3, the uniform convergence of Algorithm 5.1 can be established by following the same line of argument as in the proof of Theorem 3.3. We therefore state the convergence results below and omit the details of the proof.

THEOREM 5.5. *For a given overlapping domain decomposition with two subdomains, if the Schwarz alternating method for*

$$\mathcal{L}_h W = 0 \quad \textit{and} \quad W_\partial = 0 \tag{5.6}$$

is convergent, then the Schwarz alternating method (see Algorithm 5.1) for the system (5.3) is uniformly convergent. Moreover, if the contraction factor of the Schwarz alternating method for (5.6) is $\rho_{e,d} \in (0, 1)$ *under the maximum norm, then for* $k = 1, 2, ...,$ *we have*

$$\max\{E^{(2k)}\} \leq \rho_{e,d} \max\{E^{(2(k-1))}\}.$$

Remark 5.6. As in the continuous case, the uniform upper bound $\rho_{e,d}$ here is not optimal, which has been observed in the numerical tests.

Parallel to Corollary 3.4, we have the following corollary.

COROLLARY 5.7. *Suppose that the assumptions in Theorem 5.5 hold, then it holds*

$$h^2 \|E^{(2k)}\|^2_{\mathbb{R}^{(N+1)^2}} \leq C \rho_{e,d}^k \max\{E^{(0)}\},$$

where $C > 0$ *is a constant independent of* h, $\rho_{e,d}$ *and* α.

Remark 5.8. Similar to the continuous case, we can also give the one-level alternating Schwarz method and the parallel Schwarz method in the discrete case and we will obtain similar convergence results as those in Theorem 4.1.

6. Numerical experiments. In this section, we present numerical experiments to assess the performance of the proposed methods: the Schwarz alternating method, the alternating Schwarz method, and the parallel Schwarz method. We consider the optimal control problem

$$\min_{u \in L^2(\Omega)} \ J(y, u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

subject to

$$-\Delta y = f + u \ \text{ in } \Omega \quad \text{and} \quad y = 0 \ \text{ on } \partial\Omega,$$

where $\Omega = (0,1)^2$, $y_d = \sin(\pi x_1)\sin(\pi x_2)$, $f = 2\pi^2\sin(\pi x_1)\sin(\pi x_2)$. The exact solution is $y = \sin(\pi x_1)\sin(\pi x_2)$, $p = 0$ and $u = 0$ for any $\alpha > 0$.

The problem is discretized on a uniform grid of size h using the five-point finite difference scheme, leading to the discrete problem

$$\min_{U\in\mathbb{R}^n} J(Y,U) = \frac{h^2}{2}\|Y_I - Y_{d,I}\|^2_{\mathbb{R}^n} + \frac{\alpha h^2}{2}\|U_I\|^2_{\mathbb{R}^n}$$

subject to

$$\mathcal{L}_h Y = F_I + U_I \quad \text{and} \quad Y_\partial = 0,$$

with Y_d and F denoting the grid values of y_d and f respectively, and the notation follows that introduced in the previous section. Although we present results only for this particular discrete problem, we have also tested random choices of Y_d, F, which yielded similar performance.

To construct the overlapping domain decomposition, we first partition the domain into non-overlapping subdomains and then add several layers of grid points to each subdomain. For instance, in Figure 2, the domain is first partitioned into non-overlapping subdomains (delineated by the black lines). Overlapping subdomains are then obtained by adding two layers of grid points to each subdomain, as indicated by the blue and red lines. The left panel illustrates a two-subdomain decomposition, while the right panel shows a multi-subdomain decomposition.

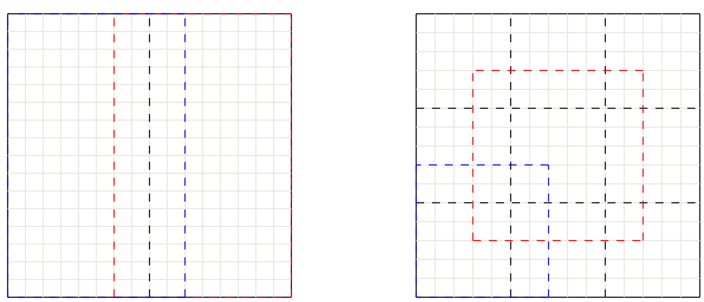


FIG. 2. *Grid and decompositions of* Ω.

In our numerical experiments, we set the mesh size to $h = 1/64$ and consider the values $\alpha = 10^{-2},\ 10^{-4},\ 10^{-6}$ for the regularization parameter. The overlapping size is controlled by the parameter δ, which denotes the number of mesh layers added to each subdomain; we choose $\delta = 1,\ 2,\ 3,\ 4$, corresponding to overlapping sizes of $h,\ 2h,\ 3h$ and $4h$, respectively. We investigate the influence of both the overlapping size δ and the regularization parameter α on the convergence behavior of the algorithm, and we compare its performance between the elliptic equation case and the optimal control problem case. The exact solution of the discrete problem is obtained by solving the first-order optimality system. We define $\|E\|_\infty$ as the maximum norm of the error between the iterative solution of the Schwarz alternating method and the exact discrete solution. For the optimal control problem, $\|E\|_\infty := \|E_y^2 + \alpha^{-1}E_p^2\|_\infty$, where E_y and E_p denote the error vectors for the state and adjoint variables, respectively. The contraction factors for the elliptic equation and the optimal control problem are denoted by $\rho_{e,d}$ and $\rho_{c,d}$, respectively. The numerical results are reported in Table 1.

Table 1 reports the maximum-norm errors of the first five iterations of the Schwarz alternating method for both the elliptic equation and the optimal control problem,

TABLE 1
The errors and contraction factors of the Schwarz alternating method versus α, δ and the iteration number k.

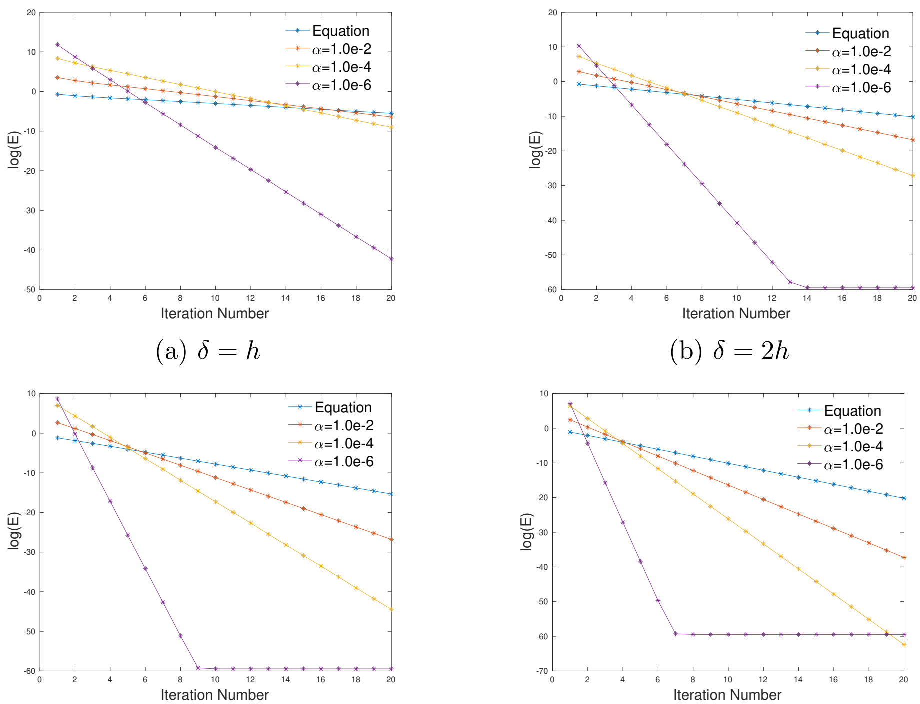
		Elliptic PDE		$\alpha = 10^{-2}$		$\alpha = 10^{-4}$		$\alpha = 10^{-6}$	
δ	k	$\|E\|_\infty$	$\rho_{e,d}$	$\|E\|_\infty$	$\rho_{c,d}$	$\|E\|_\infty$	$\rho_{c,d}$	$\|E\|_\infty$	$\rho_{c,d}$
1	1	9.2738e-1	-	8.6604e1	-	7.2285e3	-	2.5279e5	-
	2	7.9016e-1	8.5204e-1	6.1720e1	7.1267e-1	2.7918e3	3.8623e-1	1.6005e4	6.3314e-2
	3	6.6541e-1	8.4212e-1	4.2779e1	6.9311e-1	1.1760e3	4.2122e-1	9.8908e2	6.1797e-2
	4	5.5466e-1	8.3357e-1	2.9006e1	6.7803e-1	5.1234e2	4.3566e-1	5.9773e1	6.0434e-2
	5	4.5850e-1	8.2663e-1	1.9388e1	6.6842e-1	2.1107e2	4.1198e-1	3.4274e0	5.7340e-2
2	1	8.5724e-1	-	7.3447e1	-	4.5124e3	-	6.3505e4	-
	2	6.0803e-1	7.0929e-1	3.5317e1	4.8085e-1	7.6078e2	1.6860e-1	2.3274e2	3.6649e-3
	3	4.1559e-1	6.8351e-1	1.5790e1	4.4708e-1	1.2923e2	1.6986e-1	8.1402e-1	3.4975e-3
	4	2.7760e-1	6.6795e-1	6.7994e0	4.3063e-1	2.2636e1	1.7516e-1	3.0695e-3	3.7709e-3
	5	1.8311e-1	6.5963e-1	2.8540e0	4.1974e-1	3.5541e0	1.5701e-1	1.0338e-5	3.3680e-3
3	1	7.8980e-1	-	6.1795e1	-	2.7926e3	-	1.6005e4	-
	2	4.5793e-1	5.7981e-1	1.9350e1	3.1314e-1	2.1125e2	7.5645e-2	3.4274e0	2.1414e-4
	3	2.5008e-1	5.4609e-1	5.4692e0	2.8264e-1	1.5174e1	7.1830e-2	7.2018e-4	2.1012e-4
	4	1.3321e-1	5.3268e-1	1.4701e0	2.6879e-1	9.0707e-1	5.9778e-2	1.5694e-7	2.1792e-4
	5	7.0335e-2	5.2800e-1	3.8813e-1	2.6402e-1	5.7030e-2	6.2872e-2	3.1882e-11	2.0315e-4
4	1	7.2530e-1	-	5.1592e1	-	1.8031e3	-	3.8839e3	-
	2	3.3954e-1	4.6814e-1	1.0356e1	2.0074e-1	5.3161e1	2.9483e-2	5.0780e-2	1.3074e-5
	3	1.4774e-1	4.3511e-1	1.8249e0	1.7621e-1	1.4308e0	2.6915e-2	6.2383e-7	1.2285e-5
	4	6.3005e-2	4.2647e-1	3.0868e-1	1.6915e-1	3.5916e-2	2.5101e-2	7.9164e-12	1.2690e-5
	5	2.6743e-2	4.2446e-1	5.1728e-2	1.6758e-1	8.7228e-4	2.4287e-2	1.0084e-16	1.2739e-5

for various overlapping sizes and values of α. The corresponding contraction factors are also provided for each test setting. The results show that, for a fixed domain decomposition, the contraction factor for the elliptic equation uniformly dominates that for the optimal control problem with respect to α. In both cases, increasing the overlap reduces the contraction factor, while decreasing α also leads to faster convergence for the optimal control problem. For further theoretical discussion, we refer to [13]. To facilitate comparison, the convergence curves for the different test configurations are plotted in Figure 3. These results are fully consistent with our theoretical analysis.

The numerical results for the multiple-subdomain case are reported in Table 2, which are in full agreement with our theoretical predictions. In this experiment, we use a decomposition consisting of 9 subdomains (cf. Figure 2) and set $\alpha = 10^{-4}$. The convergence of both the alternating and parallel Schwarz methods is clearly observed as the iteration count increases. As in the two-subdomain case, we present the results for the first five iterations only.

7. Conclusion. In this paper, we have established the uniform convergence of the Schwarz alternating method for unconstrained elliptic optimal control problems using the maximum principle for elliptic operators. We proved convergence in the maximum norm with a contraction factor that is uniformly bounded by a constant strictly less than one, independent of the regularization parameter α. These results were extended to multiple subdomains for both one-level alternating and parallel Schwarz methods.

As noted in Remark 3.5, the uniform upper bound obtained here is not sharp. Improving this estimate to the optimal contraction factor is an interesting problem. The proposed framework may also be extended to local optimal control problems, where the control acts only on a subset of the domain, as well as to problems with additional constraints. Furthermore, proving the geometric convergence of the method in the finite element discretization setting, which has been observed numerically (cf. [44]), remains an important direction for future research.



(a) $\delta = h$

(b) $\delta = 2h$

(c) $\delta = 3h$

(d) $\delta = 4h$

FIG. 3. *The convergence results*

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TABLE 2

The errors and contraction factors of the alternating Schwarz method (ASM) and the parallel Schwarz method (PSM) versus δ and the iteration number k.

		Elliptic PDE (ASM)		OCP (ASM)		Elliptic PDE (PSM)		OCP(PSM)	
δ	k	$\|E\|_\infty$	$\rho^m_{e,d}$	$\|E\|_\infty$	$\rho^m_{c,d}$	$\|E\|_\infty$	$\rho^a_{e,d}$	$\|E\|_\infty$	$\rho^a_{c,d}$
1	1	9.9943e-1	-	9.2749e3	-	1.0000e0	-	9.8069e3	-
	2	9.8044e-1	9.8099e-1	7.2583e3	7.8258e-1	9.9849e-1	9.9849e-1	9.5131e3	9.7005e-1
	3	9.3498e-1	9.5364e-1	5.0861e3	7.0073e-1	9.9581e-1	9.9732e-1	9.1386e3	9.6062e-1
	4	8.7425e-1	9.3505e-1	3.3998e3	6.6846e-1	9.9200e-1	9.9617e-1	8.7030e3	9.5234e-1
	5	8.0710e-1	9.2319e-1	2.2299e3	6.5588e-1	9.8708e-1	9.9504e-1	8.2249e3	9.4506e-1
2	1	9.9531e-1	-	7.6729e3	-	1.0000e0	-	9.2912e3	-
	2	9.2072e-1	9.2506e-1	3.7640e3	4.9056e-1	9.9499e-1	9.9499e-1	8.3728e3	9.0115e-1
	3	7.8463e-1	8.5219e-1	1.6230e3	4.3118e-1	9.8597e-1	9.9094e-1	7.3608e3	8.7913e-1
	4	6.4686e-1	8.2441e-1	6.9475e2	4.2808e-1	9.7343e-1	9.8728e-1	6.3430e3	8.6173e-1
	5	5.2804e-1	8.1632e-1	2.9489e2	4.2445e-1	9.5777e-1	9.8392e-1	5.3778e3	8.4782e-1
3	1	9.8418e-1	-	5.9204e3	-	1.0000e0	-	8.5890e3	-
	2	8.3020e-1	8.4355e-1	1.7166e3	2.8995e-1	9.9067e-1	9.9067e-1	7.0400e3	8.1965e-1
	3	6.1622e-1	7.4225e-1	4.9077e2	2.8589e-1	9.7390e-1	9.8307e-1	5.5747e3	7.9186e-1
	4	4.4620e-1	7.2410e-1	1.3790e2	2.8098e-1	9.5113e-1	9.7663e-1	4.2947e3	7.7039e-1
	5	3.2346e-1	7.2493e-1	3.6813e1	2.6696e-1	9.2365e-1	9.7111e-1	3.2369e3	7.5369e-1
4	1	9.6347e-1	-	4.3421e3	-	1.0000e0	-	7.8515e3	-
	2	7.2201e-1	7.4938e-1	7.4869e2	1.7243e-1	9.8598e-1	9.8598e-1	5.8252e3	7.4192e-1
	3	4.6418e-1	6.4291e-1	1.4201e2	1.8968e-1	9.6107e-1	9.7474e-1	4.1391e3	7.1054e-1
	4	2.9641e-1	6.3856e-1	2.4538e1	1.7279e-1	9.2806e-1	9.6565e-1	2.8498e3	6.8851e-1
	5	1.9104e-1	6.4451e-1	3.7718e0	1.5371e-1	8.8935e-1	9.5829e-1	1.9182e3	6.7309e-1

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