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# A Hybrid High-Order method for Kirchhoff–Love plate bending problems\*

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## Abstract

We present a novel Hybrid High-Order (HHO) discretization of fourth-order elliptic problems arising from the mechanical modeling of the bending behavior of Kirchhoff–Love plates, including the biharmonic equation as a particular case. The proposed HHO method supports arbitrary approximation orders on general polygonal meshes, and reproduces the key mechanical equilibrium relations locally inside each element. When polynomials of degree  $k \geq 1$  are used as unknowns, we prove convergence in  $h^{k+1}$  (with  $h$  denoting, as usual, the meshsize) in an energy-like norm. A key ingredient in the proof are novel approximation results for the oblique biharmonic projector on local polynomial spaces. Under biharmonic regularity assumptions, a sharp estimate in  $h^{k+3}$  is also derived for the  $L^2$ -norm of the error on the deflection. The theoretical results are supported by numerical experiments, which additionally show the robustness of the method with respect to the choice of the stabilization.

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**Keywords:** Hybrid High-Order methods, Kirchhoff–Love plates, biharmonic problems, oblique biharmonic projector

## 1 Introduction

As remarked by O. C. Zienkiewicz [29], “one of the early requirements of the Finite Element (FE) approximation was the choice of shape functions which did not lead to infinite strains on element interfaces and which therefore preserved a necessary degree of continuity”. This requirement (also called of conformity) appeared easy to satisfy for simple self-adjoint problems governed by second-order equations, where  $C^0$ -continuity at interfaces is enough. The situation is different as far as it concerns the knowledge, essential in structural engineering, of the bending of plates, whose numerical treatment has always been a goal of FE computations. Since thin plate bending in the Kirchhoff–Love approximation is governed by a fourth-order equation,  $C^1$ -continuity has to be introduced (and the continuity of both the function and of its normal gradient assured at interfaces). This was difficult to achieve and computationally expensive in the classical FE framework, see e.g. Zienkiewicz [30] for a first engineering-oriented discussion and Ciarlet [15]

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for a mathematically-oriented one. In order to relax such  $C^1$ -continuity condition, many non-conforming, mixed, hybrid plates elements have been studied and tested all over the last fifty years, and the literature on this subject is very broad; a minimal and by far non-exhaustive sample includes the seminal paper by Lascaux–Lesaint [26], as well as the classical works of Bathe [4], Boffi–Brezzi–Fortin [7], Brenner [8], Brenner–Scott [9], Brezzi–Fortin [11], Ciarlet [15], Hughes [25]; see also references therein. More recent nonconforming methods which have similarities (and differences) with the one presented here include the Hybridizable Discontinuous Galerkin method [16] of Cockburn–Dong–Guzmán and the Weak Galerkin method [28] of Lin–Wang–Ye; see also [17] concerning the passage from Discontinuous Galerkin to hybrid methods. We also cite here [13], where the fourth-order operator in the Cahn–Hilliard equations is treated as a system of second-order operators.

A recent approach to the construction of FE spaces with  $C^1$ -regularity, on the other hand, has been developed in the context of the Virtual Element Method (VEM) [6], [10]. Here, global continuity requirements are enforced by renouncing an explicit expression of the basis functions at each point, and local contributions are built using computable projections thereof (a stabilization term therefore has to be added). We refer the reader to [12] [14] for an application of  $C^1$ -conforming virtual spaces to plate-bending problems similar to the ones considered here. Nonconforming versions of the VEM have also been developed for fourth order operators, see, e.g., the very recent contributions by Antonietti–Manzini–Verani [3] (with virtual functions continuous at nodes) and Zhao–Chen–Zhang [31] (with  $C^0$ -continuous virtual functions). These results suggest that a nonconforming version of HHO methods can be developed for the Kirchhoff–Love bending problem.

The Kirchhoff–Love plate bending model problem considered in this work reads

$$-\operatorname{div} \operatorname{div} \mathbf{M} = f \quad \text{in } \Omega, \quad (1a)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (1b)$$

$$\partial_{\mathbf{n}} u = 0 \quad \text{on } \partial\Omega, \quad (1c)$$

where  $\Omega \subset \mathbb{R}^2$  denotes a two-dimensional bounded and connected polygonal domain, representing the middle surface of a plate in its reference configuration, and the divergence operator is denoted by  $\operatorname{div}$  or  $\operatorname{div}$ , as to whether it acts on vector- or tensor-valued fields, respectively. In (1a),  $f$  represents a surface load orthogonal to the plane of the plate, and  $\mathbf{M}$  is the *moment tensor*, a second-order symmetric tensor field related to the scalar unknown  $u$ , the *deflection* of the plate, by the constitutive law

$$\mathbf{M} = -\mathbb{A} \nabla^2 u,$$

where  $\mathbb{A}$  is a fourth-order, symmetric and uniformly-elliptic tensor field, and  $-\nabla^2 u$  is referred to as the *curvature tensor*. For the sake of simplicity, we assume in what follows that  $\mathbb{A}$  is piecewise constant on a finite polygonal partition  $P_\Omega = \{\Omega_i : i \in I\}$  of  $\Omega$ , and that  $f \in L^2(\Omega)$ . Variational formulations are classical for problem (1). For  $X \subset \bar{\Omega}$ , we denote by  $(\cdot, \cdot)_X$  the scalar product in  $L^2(X)$ ,  $L^2(X)^2$  or  $L^2(X)^{2 \times 2}$ , depending on the context, and by  $\|\cdot\|_X$  the associated norm; we omit the subscript  $X$  whenever  $X = \Omega$ . The *primal* variational formulation of (1) reads: Find  $u \in H_0^2(\Omega)$  such that

$$(\mathbb{A} \nabla^2 u, \nabla^2 v) =: a(u, v) = (f, v) \quad \forall v \in H_0^2(\Omega). \quad (2)$$

Owing to the Lax–Milgram Lemma, problem (2) is well-posed.

In this work, we propose and analyze a novel Hybrid High-Order (HHO) method for the approximation of problem (2) which sits at the far end of the spectrum of nonconforming methods, since the underlying space does not even embed  $C^0$ -continuity. HHO methods, introduced in [20] in the context of quasi-incompressible linear elasticity, are a class of new-generation discretization methods for partial differential equations with several advantageous features. The most relevant in the context of plate bending problems are: (i) the support of arbitrary approximation orders on general polygonal meshes; (ii) the reproduction of key continuous properties (such as, e.g.,

local equilibrium relations) at the discrete level; (iii) reduced computational cost thanks to static condensation and compact stencil. We refer the reader to [23] for an introduction covering the salient aspects of HHO methods for linear and nonlinear problems.

The HHO method for problem (2) is formulated in terms of discrete unknowns defined on mesh faces and elements (whence the term *hybrid*), and such unknowns are polynomials of arbitrary degree  $k \geq 1$  (whence the expression *high-order*). The construction is conceived so that only face-based unknowns are globally coupled, whereas element-based unknowns can be eliminated by the computation of static condensation; see Remark 9 below for further details. Element-based unknowns play the role of the deflection  $u$  inside elements, whereas face unknowns play the role of the traces of  $u$  and of its gradient on faces. From these unknowns, a reconstruction of the deflection of degree  $(k+2)$  is obtained by solving a local problem inside each element. This reconstruction is conceived so that, composed with a local reduction map, it coincides with the oblique biharmonic projector and, as such, has optimal approximation properties in the space of polynomials of total degree (up to)  $(k+2)$ ; see Theorem 11 below, whose proof hinges on the recent results of [19]. The high-order deflection reconstruction is used to formulate a local contribution, which includes a carefully tailored stabilization term. The role of the latter is to ensure coercivity with respect to a  $H^2$ -like seminorm while, at the same time, preserving the approximation properties of the local deflection reconstruction.

An extensive convergence analysis of the method is carried out. Specifically, in Theorem 11 below we prove convergence in  $h^{k+1}$  (with  $h$  denoting, as usual, the meshsize) in an energy-like norm and, in Theorem 15 below, a sharp estimate in  $h^{k+3}$  for the  $L^2$ -norm under biharmonic regularity assumptions. The latter result highlights a salient feature of HHO methods, namely the fact that, by construction, element-based unknowns superconverge to the  $L^2$ -orthogonal projection of the exact solution on general meshes. As this happens by design (i.e., this behavior is not serendipitous), this phenomenon is henceforth referred to as *supercloseness* rather than *superconvergence*. We also show that the method satisfies locally inside each element a discrete version of the principle of virtual work with moments and shear forces obeying a law of action and reaction. The performance of the method is showcased on numerical examples, including a study of the robustness with respect to the choice of the stabilization.

The rest of the paper is organized as follows. In Section 2 we introduce the discrete setting: regularity for polygonal meshes, basic results thereon, and local projectors. A novel general result contained in this section is Theorem 1, where optimal approximation properties for the oblique biharmonic projector on local polynomial spaces are studied. The proof of this theorem is given in Section 6. In Section 3 we introduce the HHO method, state the main results corresponding to Theorems 11 and 15, and provide a few numerical examples. In Section 4 we prove the local equilibrium properties of the HHO method and identify discrete equilibrated counterparts of moments and shear forces at interfaces. Section 5 collects the technical proofs of the properties of the discrete bilinear form relevant to the analysis. Some indications of possible developments are given in Section 7.

## 2 Discrete setting

In this section we introduce some assumptions on the mesh, recall a few known results, and define two projectors on local polynomial spaces that will play a key role in the analysis of the method.

### 2.1 Mesh

The HHO method is built upon a polygonal mesh of the domain  $\Omega$  defined prescribing a *set of elements*  $\mathcal{T}_h$  and a *set of faces*  $\mathcal{F}_h$ .

The set of elements  $\mathcal{T}_h$  is a finite collection of open disjoint polygons  $T$  with nonzero area such that  $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}$  and  $h := \max_{T \in \mathcal{T}_h} h_T$ , with  $h_T$  the diameter of  $T$ . The set of faces  $\mathcal{F}_h$  is a finite collection of open disjoint line segments in  $\bar{\Omega}$  with nonzero length such that, for all  $F \in \mathcal{F}_h$ , (i) either there exist two distinct mesh elements  $T_1, T_2 \in \mathcal{T}_h$  such that  $F \subset \partial T_1 \cap \partial T_2$  (and  $F$  is called an *interface*) or (ii) there exists a mesh element  $T \in \mathcal{T}_h$  such that  $F \subset \partial T \cap \partial \Omega$  (and  $F$  is called a *boundary face*). We assume that  $\mathcal{F}_h$  is a partition of the mesh skeleton in the sense that  $\bigcup_{T \in \mathcal{T}_h} \partial T = \bigcup_{F \in \mathcal{F}_h} \bar{F}$ .

We denote by  $\mathcal{F}_h^i$  the set of all interfaces and by  $\mathcal{F}_h^b$  the set of all boundary faces, so that  $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$ . The length of a face  $F \in \mathcal{F}_h$  is denoted by  $h_F$ . For any  $T \in \mathcal{T}_h$ ,  $\mathcal{F}_T$  is the set of faces that lie on  $\partial T$  (the boundary of  $T$ ) and, for any  $F \in \mathcal{F}_T$ ,  $\mathbf{n}_{TF}$  is the unit normal to  $F$  pointing out of  $T$ . Symmetrically, for any  $F \in \mathcal{F}_h$ ,  $\mathcal{T}_F$  is the set containing the mesh elements sharing face  $F$  (two if  $F$  is an interface, one if  $F$  is a boundary face).

The notion of *geometric regularity* for polygonal meshes is more subtle than for standard meshes. To formulate it, we assume the existence of a *matching simplicial submesh*, meaning that there is a conforming triangulation  $\mathfrak{T}_h$  of the domain such that each mesh element  $T \in \mathcal{T}_h$  is decomposed into a finite number of triangles from  $\mathfrak{T}_h$  and each mesh face  $F \in \mathcal{F}_h$  is decomposed into a finite number of edges from the skeleton of  $\mathfrak{T}_h$ . We denote by  $\varrho$  the *regularity parameter* such that (i) for any triangle  $S \in \mathfrak{T}_h$  of diameter  $h_S$  and inradius  $r_S$ ,  $\varrho h_S \leq r_S$  and (ii) for any mesh element  $T \in \mathcal{T}_h$  and any triangle  $S \in \mathfrak{T}_h$  such that  $S \subset T$ ,  $\varrho h_T \leq h_S$ . When considering refined mesh sequences, the regularity parameter should remain bounded away from zero.

In what follows, we also assume that the mesh is *compliant with the data*, i.e. for each mesh element  $T \in \mathcal{T}_h$  there exist a unique  $\Omega_i \in P_\Omega$  such that  $T \subset \Omega_i$ . As a result, the material tensor field  $\mathbb{A}$  is *element-wise constant*, and we set for the sake of brevity

$$\mathbb{A}_T := \mathbb{A}|_T \quad \forall T \in \mathcal{T}_h.$$

We also denote by  $\mathcal{A}_T^-$  and  $\mathcal{A}_T^+$  the smallest and largest eigenvalues of  $\mathbb{A}_T$ , regarded as an endomorphism of  $\mathbb{R}_{\text{sym}}^{2 \times 2}$ , respectively. For  $l \geq 0$  we also introduce, for later use, the *broken Sobolev space*

$$H^l(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \in H^l(T) \quad \forall T \in \mathcal{T}_h\}, \quad (3)$$

equipped, unless noted otherwise, with the broken norm  $\|\cdot\|_{H^l(\mathcal{T}_h)}$  defined by

$$\forall v \in H^l(\mathcal{T}_h), \quad \|v\|_{H^l(\mathcal{T}_h)} := \left( \sum_{T \in \mathcal{T}_h} \|v\|_{H^l(T)}^2 \right)^{1/2}. \quad (4)$$

## 2.2 Basic results

We next recall a few geometric and functional inequalities, whose proofs are straightforward adaptations of the results collected in [21, Chapter 1] (where a slightly different notion of mesh faces is considered). For any mesh element  $T \in \mathcal{T}_h$  and any face  $F \in \mathcal{F}_T$  it holds that

$$\varrho^2 h_T \leq h_F \leq h_T, \quad (5)$$

which expresses the fact that we are working on isotropic meshes. Moreover, the maximum number of faces of a mesh element is uniformly bounded: There is an integer  $N_\varrho \geq 3$  only depending on  $\varrho$  such that

$$\max_{h \in \mathcal{H}} \max_{T \in \mathcal{T}_h} \text{card}(\mathcal{F}_T) \leq N_\varrho. \quad (6)$$

Let a polynomial degree  $l \geq 0$  be fixed, let  $X$  be a mesh element or face, and denote by  $\mathbb{P}^l(X)$  the space spanned by the restrictions to  $X$  of two-variate polynomials of total degree at most  $l \geq 0$ . There exist three real numbers  $C_{\text{tr}} > 0$ ,  $C_{\text{tr},c} > 0$ , and  $C_{\text{inv}}$  depending on  $\varrho$  and possibly on  $l$ , but

independent of  $h$ , such that for any  $T \in \mathcal{T}_h$  and  $F \in \mathcal{F}_T$ , the following discrete trace, continuous trace, and inverse inequalities hold:

$$\|w\|_F \leq C_{\text{tr}} h_F^{-1/2} \|w\|_T \quad \forall w \in \mathbb{P}^l(T), \quad (7a)$$

$$h_T^{1/2} \|w\|_{\partial T} \leq C_{\text{tr},c} (\|w\|_T + h_T \|\nabla w\|_T) \quad \forall w \in H^1(T), \quad (7b)$$

$$\|\nabla w\|_T \leq C_{\text{inv}} h_T^{-1} \|w\|_T \quad \forall w \in \mathbb{P}^l(T), \quad (7c)$$

We also recall the following Poincaré inequality, valid for all  $T \in \mathcal{T}_h$  and all  $w \in H^1(T)$  such that  $(w, 1)_T = 0$ :

$$\|w\|_T \leq C_p h_T \|\nabla w\|_T, \quad (8)$$

where the real number  $C_p$  is independent of both  $h_T$  and  $T$ , but possibly depends on  $\varrho$  (for instance,  $C_p = \pi^{-1}$  for convex elements [5]).

### 2.3 Projectors on local polynomial spaces

Projectors on local polynomial spaces are an essential ingredient in the construction and analysis of our method. Let a polynomial degree  $l \geq 0$  be fixed, and let  $X$  denote a mesh element or face. The  $L^2$ -orthogonal projector  $\pi_X^l: L^2(X) \rightarrow \mathbb{P}^l(X)$  is such that, for all  $v \in L^2(X)$ ,  $\pi_X^l v$  is the unique polynomial satisfying the relation

$$(\pi_X^l v - v, w)_X = 0 \quad \forall w \in \mathbb{P}^l(X). \quad (9)$$

The corresponding vector-valued version, denoted by  $\boldsymbol{\pi}_X^l$ , acts component-wise. We recall the following approximation results that are a special case of the ones proved in [22, Lemmas 3.4 and 3.6]: There exists a real number  $C > 0$  independent of  $h$ , but possibly depending on  $\varrho$  and  $l$ , such that, for all  $T \in \mathcal{T}_h$ , all  $s \in \{0, \dots, l+1\}$ , and all  $v \in H^s(T)$ ,

$$|v - \pi_T^l v|_{H^m(T)} \leq C h_T^{s-m} |v|_{H^s(T)} \quad \forall m \in \{0, \dots, s\}, \quad (10a)$$

and, if  $s \geq 1$ ,

$$|v - \pi_T^l v|_{H^m(\partial T)} \leq C h_T^{s-m-1/2} |v|_{H^s(T)} \quad \forall m \in \{0, \dots, s-1\}. \quad (10b)$$

Here we have set, for any  $\varphi \in H^s(T)$ ,

$$|\varphi|_{H^m(T)} := \sum_{\boldsymbol{\alpha} \in \mathbb{N}^2, \|\boldsymbol{\alpha}\|_1 = m} \|\partial^{\boldsymbol{\alpha}} \varphi\|_{L^2(T)}, \quad |\varphi|_{H^m(\partial T)} := \sum_{\boldsymbol{\alpha} \in \mathbb{N}^2, \|\boldsymbol{\alpha}\|_1 = m} \|\partial^{\boldsymbol{\alpha}} \varphi\|_{L^2(\partial T)},$$

with  $m$  respectively as in (10a) and (10b),  $\|\boldsymbol{\alpha}\|_1 := \alpha_1 + \alpha_2$  and  $\partial^{\boldsymbol{\alpha}} := \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ . Notice that, in the second definition,  $\varphi$  and  $\partial^{\boldsymbol{\alpha}} \varphi$  stand for the *boundary traces* of the function and of its derivatives up to order  $m$ , respectively.

Let a mesh element  $T \in \mathcal{T}_h$  be fixed. For  $u, v \in H^2(T)$ , we let  $a|_T(u, v) := (\mathbb{A}_T \nabla^2 u, \nabla^2 v)_T$  and introduce the *oblique biharmonic projector*  $\varpi_T^l: H^2(T) \rightarrow \mathbb{P}^l(T)$  such that, for any integer  $l \geq 2$  and any function  $v \in H^2(T)$ ,

$$a|_T(\varpi_T^l v - v, w)_T = 0 \text{ for all } w \in \mathbb{P}^l(T) \quad \text{and} \quad \pi_T^1(\varpi_T^l v - v) = 0. \quad (11)$$

Optimal approximation properties for the oblique biharmonic projector are stated in the following theorem, whose proof is given in Section 6.

**Theorem 1** (Optimal approximation properties of the oblique biharmonic projector). *There is a real number  $C > 0$  independent of  $h$ , but possibly depending on  $\mathbb{A}$ ,  $\varrho$  and  $l$ , such that, for all  $T \in \mathcal{T}_h$ , all  $s \in \{2, \dots, l+1\}$ , and all  $v \in H^s(T)$ , it holds*

$$|v - \varpi_T^l v|_{H^m(T)} \leq C h_T^{s-m} |v|_{H^s(T)} \quad \forall m \in \{0, \dots, s\}, \quad (12a)$$

and

$$|v - \varpi_T^l v|_{H^m(\partial T)} \leq C h_T^{s-m-1/2} |v|_{H^s(T)} \quad \forall m \in \{0, \dots, s-1\}. \quad (12b)$$

*Remark 2* (Dependence on the material tensor). It can be checked that the constant  $C$  in the right-hand side of (12) actually depends on  $\mathbb{A}_T$  only through the ratio between  $\mathcal{A}_T^+$  and  $\mathcal{A}_T^-$ .

### 3 The Hybrid High-Order method

In this section we present the construction underlying the HHO method, state the discrete problem, and discuss the main results.

#### 3.1 Local discrete unknowns and interpolation

Let a mesh element  $T \in \mathcal{T}_h$  and a polynomial degree  $k \geq 1$  be fixed. The *local space of discrete unknowns* is defined as the set

$$\underline{\mathbb{U}}_T^k := \mathbb{P}^k(T) \times \left( \prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F)^2 \right) \times \left( \prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right). \quad (13)$$

For a general collection of discrete unknowns  $\underline{v}_T \in \underline{\mathbb{U}}_T^k$ , we use the standard HHO notation

$$\underline{v}_T = (v_T, (\mathbf{v}_{\nabla, F})_{F \in \mathcal{F}_T}, (v_F)_{F \in \mathcal{F}_T}),$$

where  $v_T$  contains the element-based discrete unknowns,  $\mathbf{v}_{\nabla, F}$  the discrete unknowns related to the trace of the gradient on the face  $F$ , and  $v_F$  the discrete unknowns related to the trace on  $F$ . For a variation of the method with boundary unknowns representing the *normal* component of the gradient, we refer the reader to Section 7.1.

The *local interpolation operator*  $\underline{\mathbb{I}}_T^k: H^2(T) \rightarrow \underline{\mathbb{U}}_T^k$  is such that, for all  $v \in H^2(T)$ ,

$$\underline{\mathbb{I}}_T^k v := (\pi_T^k v, (\boldsymbol{\pi}_F^k(\nabla v)|_F)_{F \in \mathcal{F}_T}, (\pi_F^k(v|_F))_{F \in \mathcal{F}_T}). \quad (14)$$

Since the boundary of  $T$  is piecewise smooth, the trace theorem ensures that the restrictions  $v|_F$  and  $(\nabla v)|_F$  of  $v$  appearing in (14) are both well-defined.

#### 3.2 Local deflection reconstruction

Let again a mesh element  $T \in \mathcal{T}_h$  and a polynomial degree  $k \geq 1$  be fixed. We introduce the *local deflection reconstruction operator*  $p_T^{k+2}: \underline{\mathbb{U}}_T^k \rightarrow \mathbb{P}^{k+2}(T)$  such that, for all  $\underline{v}_T \in \underline{\mathbb{U}}_T^k$ ,  $p_T^{k+2} \underline{v}_T \in \mathbb{P}^{k+2}(T)$  satisfies for all  $w \in \mathbb{P}^{k+2}(T)$

$$\begin{aligned} a|_T(p_T^{k+2} \underline{v}_T, w) = & \\ & - (v_T, \operatorname{div} \operatorname{div} \mathbf{M}_{w, T})_T - \sum_{F \in \mathcal{F}_T} (\mathbf{v}_{\nabla, F}, \mathbf{M}_{w, T} \mathbf{n}_{TF})_F + \sum_{F \in \mathcal{F}_T} (v_F, \operatorname{div} \mathbf{M}_{w, T} \cdot \mathbf{n}_{TF})_F, \end{aligned} \quad (15)$$

where  $\mathbf{M}_{w, T} := -\mathbb{A}_T \nabla^2 w$ . Here, the notation  $\mathbf{M}_{w, T}$  is used to emphasize the fact that  $\mathbf{M}_{w, T}$  is a moment tensor of *virtual* nature (with space of virtual deflections equal to  $\mathbb{P}^{k+2}(T)$ ) unlike tensor  $\mathbf{M}$  appearing in bilinear form  $a$  introduced in (2). The right-hand side of (15) is conceived so as to resemble an integration by parts formula where the roles of the function represented by  $\underline{v}_T$  and of its gradient are played by element discrete unknowns inside volumetric integrals and by face-based discrete unknowns on boundary integrals.

Since  $\ker \nabla^2 = \mathbb{P}^1(T)$ , the compatibility condition for problem (15) requires that the linear form on the right-hand side vanish on the elements of  $\mathbb{P}^1(T)$ ; since  $\mathbf{M}_{w, T} = \mathbf{0}$  for all  $w \in \mathbb{P}^1(T)$ , this

condition is satisfied. The solution of (15) is not unique: if  $p_T^{k+2}\underline{v}_T \in \mathbb{P}^{k+2}(T)$  is a solution,  $p_T^{k+2}\underline{v}_T + z_T$  for any  $z_T \in \mathbb{P}^1(T)$  also is. To ensure uniqueness, we add the *closure condition*

$$\pi_T^1 p_T^{k+2}\underline{v}_T = \pi_T^1 v_T. \quad (16)$$

Notice, in passing, that element discrete unknowns do not contribute to the right-hand side of (15) for  $k = 1$ , and they only appear in the closure condition (16).

For further use, we also observe that, since  $v_T$  is smooth, performing an integration by parts on the first term in the right-hand side of (15) and using the symmetry of  $\mathbb{A}_T$  leads to the following reformulation, which points out the non-conformity of the method:

$$a_{|T}(p_T^{k+2}\underline{v}_T, w) = a_{|T}(v_T, w) - \sum_{F \in \mathcal{F}_T} (\mathbf{v}_{\nabla, F} - \nabla v_T, \mathbf{M}_{w, T} \mathbf{n}_{TF})_F + \sum_{F \in \mathcal{F}_T} (v_F - v_T, \mathbf{div} \mathbf{M}_{w, T} \cdot \mathbf{n}_{TF})_F. \quad (17)$$

The definition of  $p_T^{k+2}$  is justified by the following proposition, which establishes a link with the oblique biharmonic projector defined by (11).

**Proposition 3** (Link with the oblique biharmonic projector). *It holds*

$$p_T^{k+2} \circ \mathbb{I}_T^k = \varpi_T^{k+2}. \quad (18)$$

*Proof.* We write (15) for  $\underline{v}_T = \mathbb{I}_T^k v$  (cf. (14) for the definition of the local interpolator). Since  $w \in \mathbb{P}^{k+2}(T)$  and  $\mathbb{A}_T$  is a constant tensor, we infer that

$$\mathbf{div} \mathbf{div} \mathbf{M}_{w, T} \in \mathbb{P}^{k-2}(T) \subset \mathbb{P}^k(T)$$

and, for all  $F \in \mathcal{F}_T$ ,

$$(\mathbf{M}_{w, T})_{|F} \mathbf{n}_{TF} \in \mathbb{P}^k(F)^2, \quad (\mathbf{div} \mathbf{M}_{w, T})_{|F} \cdot \mathbf{n}_{TF} \in \mathbb{P}^{k-1}(F) \subset \mathbb{P}^k(F).$$

Consequently, recalling the definition (9) of  $\pi_T^k$ ,  $\boldsymbol{\pi}_F^k$ , and  $\pi_F^k$ , we have

$$\begin{aligned} (\pi_T^k v, \mathbf{div} \mathbf{div} \mathbf{M}_{w, T})_T &= (v, \mathbf{div} \mathbf{div} \mathbf{M}_{w, T})_T, \\ (\boldsymbol{\pi}_F^k (\nabla v)_{|F}, \mathbf{M}_{w, T} \mathbf{n}_{TF})_F &= ((\nabla v)_{|F}, \mathbf{M}_{w, T} \mathbf{n}_{TF})_F, \\ (\pi_F^k v_{|F}, \mathbf{div} \mathbf{M}_{w, T} \cdot \mathbf{n}_{TF})_F &= (v_{|F}, \mathbf{div} \mathbf{M}_{w, T} \cdot \mathbf{n}_{TF})_F. \end{aligned}$$

Plugging the above identities into the right-hand side of (15), performing an integration by parts, and using the symmetry of  $\mathbb{A}_T$ , we arrive at the following *orthogonality condition*:

$$a_{|T}(p_T^{k+2}\mathbb{I}_T^k v - v, w) = 0. \quad (19)$$

Comparing (19) and (16) with the definition (11) of  $\varpi_T^{k+2}$  concludes the proof.  $\square$

*Remark 4* (Approximation properties for  $p_T^{k+2} \circ \mathbb{I}_T^k$ ). The above result implies that  $p_T^{k+2} \circ \mathbb{I}_T^k = \varpi_T^{k+2}$  has optimal approximation properties in  $\mathbb{P}^{k+2}(T)$ , in the sense precised by Theorem 1.

### 3.3 Local contribution

We introduce the local bilinear form  $a_T(\cdot, \cdot)$  on  $\underline{\mathbb{U}}_T^k \times \underline{\mathbb{U}}_T^k$  given by

$$a_T(\underline{u}_T, \underline{v}_T) := a_{|T}(p_T^{k+2}\underline{u}_T, p_T^{k+2}\underline{v}_T) + s_T(\underline{u}_T, \underline{v}_T). \quad (20)$$



Here, the first contribution is the usual Galerkin term responsible for consistency. The second contribution, in charge of stability, penalizes high-order differences between the reconstruction and the unknowns and is such that, for all  $(\underline{u}_T, \underline{v}_T) \in \underline{U}_T^k \times \underline{U}_T^k$ ,

$$\begin{aligned} s_T(\underline{u}_T, \underline{v}_T) &:= \frac{\mathcal{A}_T^+}{h_T^4} \left( \pi_T^k(p_T^{k+2}\underline{u}_T - u_T), \pi_T^k(p_T^{k+2}\underline{v}_T - v_T) \right)_T \\ &\quad + \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \left( \pi_F^k(\nabla p_T^{k+2}\underline{u}_T - \mathbf{u}_{\nabla, F}), \pi_F^k(\nabla p_T^{k+2}\underline{v}_T - \mathbf{v}_{\nabla, F}) \right)_F \\ &\quad + \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \left( \pi_F^k(p_T^{k+2}\underline{u}_T - u_F), \pi_F^k(p_T^{k+2}\underline{v}_T - v_F) \right)_F. \end{aligned} \quad (21)$$

*Remark 5* (Stabilization). Other expressions are possible for the stabilization term, and the specific choice can affect the accuracy of the results. In particular, the discussion below remains true if we replace (20) by

$$a_T(\underline{u}_T, \underline{v}_T) := a_{|T}(p_T^{k+2}\underline{u}_T, p_T^{k+2}\underline{v}_T) + \eta s_T(\underline{u}_T, \underline{v}_T), \quad (22)$$

with  $\eta > 0$  denoting a user-dependent parameter independent of  $h$ . In practice, it is important that the numerical results be only marginally affected by the specific choice of the stabilization. We refer the reader to Section 3.7 below for a study of the robustness of the method with respect to  $\eta$ .

The following proposition states a consistency result for the stabilization bilinear form (21).

**Proposition 6** (Consistency of  $s_T$ ). *There is a real number  $C > 0$  independent of  $h$ , but possibly depending on  $\mathbb{A}$ ,  $\varrho$  and  $k$ , such that, for all  $v \in H^{k+3}(T)$ ,*

$$s_T(\underline{I}_T^k v, \underline{I}_T^k v)^{1/2} \leq Ch_T^{k+1} |v|_{H^{k+3}(T)}. \quad (23)$$

*Proof.* We have

$$s_T(\underline{I}_T^k v, \underline{I}_T^k v) = \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3,$$

where, recalling Proposition 3 and using the linearity of projectors,

$$\begin{aligned} \mathfrak{T}_1 &:= \frac{\mathcal{A}_T^+}{h_T^4} \|\pi_T^k(\varpi_T^{k+2}v - \pi_T^k v)\|_T^2 = \frac{\mathcal{A}_T^+}{h_T^4} \|\pi_T^k(\varpi_T^{k+2}v - v)\|_T^2, \\ \mathfrak{T}_2 &:= \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(\nabla \varpi_T^{k+2}v - \pi_F^k(\nabla v))\|_F^2 = \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(\nabla \varpi_T^{k+2}v - \nabla v)\|_F^2, \\ \mathfrak{T}_3 &:= \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(\varpi_T^{k+2}v - \pi_F^k v)\|_F^2 = \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(\varpi_T^{k+2}v - v)\|_F^2. \end{aligned}$$

By the boundedness of  $L^2$ -projectors, along with the approximation properties (12a)–(12b) of  $\varpi_T^{k+2}$  with  $s = k + 3$  and, respectively,  $m = 0$  for  $\mathfrak{T}_1$ ,  $m = 1$  for  $\mathfrak{T}_2$ , and again  $m = 0$  for  $\mathfrak{T}_3$ , the conclusion follows.  $\square$

We equip the space  $\underline{U}_T^k$  with the following local discrete seminorm:

$$\|\underline{v}_T\|_{\mathbb{A}, T}^2 := \|\mathbb{A}_T^{1/2} \nabla^2 v_T\|_T^2 + \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\mathbf{v}_{\nabla, F} - \nabla v_T\|_F^2 + \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_F^2. \quad (24)$$

The following result shows that the bilinear form  $a_T$  induces on  $\underline{U}_T^k$  a seminorm  $\|\cdot\|_{\mathbb{A}, T}$  uniformly equivalent to  $\|\cdot\|_{\mathbb{A}, T}$ .

**Lemma 7** (Local coercivity and boundedness). *There is a real number  $C > 0$  independent of  $h$ , but possibly depending on  $\mathbb{A}$ ,  $\varrho$  and  $k$ , such that, for all  $T \in \mathcal{T}_h$ , the following inequalities hold expressing, respectively, the coercivity and boundedness of  $\mathfrak{a}_T$ :*

$$C^{-1} \|\underline{v}_T\|_{\mathbb{A},T}^2 \leq \|\underline{v}_T\|_{\mathfrak{a},T}^2 := \mathfrak{a}_T(\underline{v}_T, \underline{v}_T) \leq C \|\underline{v}_T\|_{\mathbb{A},T}^2 \quad \forall \underline{v}_T \in \underline{U}_T^k. \quad (25)$$

*Proof.* See Section 5.1. □

### 3.4 Global space, interpolation, and norm

We define the following *global space of discrete unknowns*:

$$\underline{U}_h^k := \left( \prod_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left( \prod_{F \in \mathcal{F}_h} \mathbb{P}^k(F)^2 \right) \times \left( \prod_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right). \quad (26)$$

Note that interface unknowns in  $\underline{U}_h^k$  are single-valued, i.e., their values match from one element to the adjacent one. For a collection of discrete unknowns in  $\underline{U}_h^k$ , we use the notation

$$\underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_{\nabla, F})_{F \in \mathcal{F}_h}, (v_F)_{F \in \mathcal{F}_h}),$$

and we denote by  $\underline{v}_T = (v_T, (\mathbf{v}_{\nabla, F})_{F \in \mathcal{F}_T}, (v_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$  its restriction to a mesh element  $T \in \mathcal{T}_h$ . We also denote by  $v_h$  (no underline) the broken polynomial function on  $\mathcal{T}_h$  such that

$$v_h|_T = v_T \quad \forall T \in \mathcal{T}_h.$$

We define the global interpolator  $\underline{I}_h^k: H^2(\Omega) \rightarrow \underline{U}_h^k$  such that, for all  $v \in H^2(\Omega)$ ,

$$(\underline{I}_h^k v)|_T = \underline{I}_T^k(v|_T) \quad \forall T \in \mathcal{T}_h. \quad (27)$$

The space  $\underline{U}_h^k$  is equipped with the following seminorm (cf. (24) for the definition of  $\|\cdot\|_{\mathbb{A},T}$ ):

$$\|\underline{v}_h\|_{\mathbb{A},h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{\mathbb{A},T}^2. \quad (28)$$

We notice that the couple of boundary conditions (1b)–(1c) is *equivalent* to the couple  $u = 0$  on  $\partial\Omega$  and  $\nabla u = \mathbf{0}$  on  $\partial\Omega$ . Indeed, the fact that  $u$  vanishes on  $\partial\Omega$  implies its tangential derivative to vanish on  $\partial\Omega$  as well. Accounting for this remark, we introduce the following subspace that incorporates the latter couple of boundary conditions in a strong manner:

$$\underline{U}_{h,0}^k := \{\underline{v}_h \in \underline{U}_h^k : v_F = 0, \mathbf{v}_{\nabla, F} = \mathbf{0} \text{ for any } F \in \mathcal{F}_h^b\}. \quad (29)$$

It is a simple matter to check that the image of the restriction of  $\underline{I}_h^k$  to  $H_0^2(\Omega)$  is a subset of  $\underline{U}_{h,0}^k$ .

**Proposition 8** (Norm  $\|\underline{v}_h\|_{\mathbb{A},h}$ ). *The mapping  $\underline{U}_{h,0}^k \ni \underline{v}_h \mapsto \|\underline{v}_h\|_{\mathbb{A},h} \in \mathbb{R}$  defines a norm on  $\underline{U}_{h,0}^k$ .*

*Proof.* The seminorm property is trivial. It then suffices to show that  $\|\underline{v}_h\|_{\mathbb{A},h} = 0 \implies \underline{v}_h = \underline{0} \in \underline{U}_{h,0}^k$ . Clearly,  $\|\underline{v}_h\|_{\mathbb{A},h} = 0$  implies  $\nabla^2 v_T \equiv \mathbf{0}$  for all  $T \in \mathcal{T}_h$  and  $\mathbf{v}_{\nabla, F} - \nabla v_T \equiv \mathbf{0}$  and  $v_F - v_T \equiv 0$  for all  $F \in \mathcal{F}_h$ . By definition (29), we have  $\mathbf{v}_{\nabla, F} \equiv \mathbf{0}$  and  $v_F = 0$  for all  $F \in \mathcal{F}_h^b$ ; thus, for any  $T \in \mathcal{T}_h$ , if  $\mathcal{F}_T \cap \mathcal{F}_h^b \neq \emptyset$ , then there exists  $F \in \mathcal{F}_h^b$  such that  $\nabla v_T \equiv \mathbf{0}$  and  $v_T \equiv 0$  on  $F$ . Since  $\nabla^2 v_T \equiv \mathbf{0}$  in  $T$ , these facts imply that  $v_T \equiv 0$  in  $T$ , which in turn implies that  $v_F \equiv 0$  and  $\mathbf{v}_{\nabla, F} \equiv \mathbf{0}$  for all  $F \in \mathcal{F}_T$ . Repeating this argument for inner layers of elements yields the assertion. □

### 3.5 Discrete problem

The *discrete problem* is formulated as follows: Find  $\underline{u}_h \in \underline{U}_{h,0}^k$  such that

$$\mathfrak{a}_h(\underline{u}_h, \underline{v}_h) = (f, v_h) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k \quad (30)$$

with global bilinear form  $\mathfrak{a}_h$  on  $\underline{U}_h^k \times \underline{U}_h^k$  obtained by element-by-element assembly setting

$$\mathfrak{a}_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \mathfrak{a}_T(\underline{u}_T, \underline{v}_T). \quad (31)$$

*Remark 9* (Implementation). Problem (30) can be efficiently solved by first eliminating all element-based discrete unknowns by static condensation (the computation of a local Schur complement inside each element), and then inverting a global matrix where only face unknowns appear. Thus, for a given mesh  $\mathcal{T}_h$ , the main cost corresponds to the inversion of a sparse symmetric positive-definite matrix.

The following lemma summarizes the properties of the global bilinear form  $\mathfrak{a}_h$ .

**Lemma 10** (Properties of  $\mathfrak{a}_h$ ). *The bilinear form  $\mathfrak{a}_h$  defined by (31) has the following properties:*

- (i) Coercivity and boundedness. *There is a real number  $C > 0$  independent of  $h$ , but possibly depending on  $\mathbb{A}$ ,  $\varrho$  and  $k$ , such that*

$$C^{-1} \|\underline{v}_h\|_{\mathbb{A},h}^2 \leq \|\underline{v}_h\|_{\mathfrak{a},h}^2 := \mathfrak{a}_h(\underline{v}_h, \underline{v}_h) \leq C \|\underline{v}_h\|_{\mathbb{A},h}^2 \quad \forall \underline{v}_h \in \underline{U}_h^k. \quad (32)$$

- (ii) Consistency. *There is a real number  $C > 0$  independent of  $h$ , but possibly depending on  $\mathbb{A}$ ,  $\varrho$  and  $k$ , such that, for all  $v \in H^{k+3}(\Omega) \cap H_0^2(\Omega)$ , it holds that*

$$\sup_{\underline{w}_h \in \underline{U}_{h,0}^k \setminus \{\underline{0}_h\}} \frac{(\operatorname{div} \operatorname{div} \mathbb{A} \nabla^2 v, w_h) - \mathfrak{a}_h(\mathbb{I}_h^k v, \underline{w}_h)}{\|\underline{w}_h\|_{\mathbb{A},h}} \leq C h^{k+1} |v|_{H^{k+3}(\Omega)}. \quad (33)$$

*Proof.* See Section 5.1. □

As a consequence of the first inequality in (32), the discrete problem (30) admits a unique solution.

### 3.6 Main results

We next present the main results of the analysis, namely error estimates in an energy-like norm, in a jump-seminorm, and in the  $L^2$ -norm. Inside the proofs of this section, we often abridge as  $a \lesssim b$  the inequality  $a \leq Cb$  with  $C > 0$  independent of  $h$ , but possibly depending on  $\mathbb{A}$ ,  $\varrho$ , and  $k$ .

#### 3.6.1 Energy error estimate

We introduce the global deflection reconstruction operator  $p_h^{k+2} : \underline{U}_h^k \rightarrow L^2(\Omega)$  such that, for all  $\underline{v}_h \in \underline{U}_h^k$ ,

$$(p_h^{k+2} \underline{v}_h)|_T = p_T^{k+2} \underline{v}_T \quad \forall T \in \mathcal{T}_h.$$

We also define the stabilization seminorm  $|\cdot|_{s,h}$  on  $\underline{U}_h^k$  setting, for all  $\underline{v}_h \in \underline{U}_h^k$ ,

$$|\underline{v}_h|_{s,h}^2 := \sum_{T \in \mathcal{T}_h} s_T(\underline{v}_T, \underline{v}_T).$$

**Theorem 11** (Energy error estimate). *Let  $u \in H_0^2(\Omega)$  and  $\underline{u}_h \in \underline{U}_{h,0}^k$  denote the unique solutions to the continuous (2) and discrete (30) problems, respectively. Assume the additional regularity  $u \in H^{k+3}(\Omega)$ . Then, it holds that*

$$\|\mathbb{A}^{1/2} \nabla_h^2 (p_h^{k+2} \underline{u}_h - u)\| + |\underline{u}_h|_{s,h} \leq Ch^{k+1} |u|_{H^{k+3}(\Omega)}, \quad (34)$$

where  $\nabla_h$  denotes the usual broken gradient operator on  $\mathcal{T}_h$  and the real number  $C > 0$  is independent of  $h$  (but possibly depends on  $\mathbb{A}$ ,  $\varrho$ , and  $k$ ).

*Remark 12* (Regularity of the solution). Concerning the regularity assumptions on  $u$ , we mention as an example that, for  $k = 1$ , such regularity is satisfied by the solution of the biharmonic problem with Dirichlet boundary conditions (obtained taking  $\mathbb{A} = \mathbb{I}$  in (1)) posed on a cubic three-dimensional domain, provided the load  $f$  is square-integrable (see, e.g., Maz'ya [27, Chapter 4]). In two dimensions, under the weaker assumption that  $f \in H^{-1}(\Omega)$ , it holds that  $u \in H^3(\Omega)$  provided  $\Omega$  is convex (see, e.g., Grisvard [24, Chapter 3]). In general, a regularity assumption on the exact solution is actually the consequence of a *compatibility condition* between the datum regularity and the domain geometry. As a further reference on the regularity for the solution of fourth-order elliptic problems, we also refer the reader to Dauge [18, Chapter 4].

*Proof of Theorem 11.* Let, for the sake of brevity,  $\hat{u}_h := \mathbb{I}_h^k u$ . We start by proving that

$$\|\underline{u}_h - \hat{u}_h\|_{a,h} \lesssim h^{k+1} |u|_{H^{k+3}(\Omega)}, \quad (35)$$

with norm  $\|\cdot\|_{a,h}$  defined by (32). Using the linearity of  $a_h$  in its first argument together with the discrete problem (30), and recalling that  $\operatorname{div} \mathbf{div} \mathbb{A} \nabla^2 u = f$  a.e. in  $\Omega$ , we have, for all  $\underline{v}_h \in \underline{U}_{h,0}^k$ ,

$$a_h(\underline{u}_h - \hat{u}_h, \underline{v}_h) = (f, v_h) - a_h(\hat{u}_h, \underline{v}_h) \leq \sup_{\underline{w}_h \in \underline{U}_{h,0}^k \setminus \{\underline{0}_h\}} \frac{(\operatorname{div} \mathbf{div} \mathbb{A} \nabla^2 u, w_h) - a_h(\hat{u}_h, \underline{w}_h)}{\|\underline{w}_h\|_{\mathbb{A},h}} \|\underline{v}_h\|_{\mathbb{A},h}.$$

Thus, choosing  $\underline{v}_h = \underline{u}_h - \hat{u}_h$  and using the consistency (33) of  $a_h$  to bound the supremum in the right-hand side, the basic estimate (35) follows.

Let us now prove (34). Using the triangle inequality, we infer that

$$\begin{aligned} & \|\mathbb{A}^{1/2} \nabla_h^2 (p_h^{k+2} \underline{u}_h - u)\| + |\underline{u}_h|_{s,h} \\ & \leq \|\mathbb{A}^{1/2} \nabla_h^2 (p_h^{k+2} \underline{u}_h - \hat{u}_h)\| + |\underline{u}_h - \hat{u}_h|_{s,h} + \|\mathbb{A}^{1/2} \nabla_h^2 (p_h^{k+2} \hat{u}_h - u)\| + |\hat{u}_h|_{s,h} \\ & \leq 2\|\underline{u}_h - \hat{u}_h\|_{a,h} + \|\mathbb{A}^{1/2} \nabla_h^2 (p_h^{k+2} \hat{u}_h - u)\| + |\hat{u}_h|_{s,h}, \end{aligned}$$

where we have used the definition (32) of the  $\|\cdot\|_{a,h}$ -norm in the last line. The conclusion follows using (35) to estimate the first term in the right-hand side, the optimal approximation properties (12a) of  $p_T^{k+2} \hat{u}_T = \varpi_T^{k+2} u$  with  $s = k + 3$  and  $m = 2$  for all  $T \in \mathcal{T}_h$  to estimate the second term, and the consistency (23) of  $s_T$  for all  $T \in \mathcal{T}_h$  to estimate the third term.  $\square$

### 3.6.2 Convergence of the jumps

From the estimate of Theorem 11, one can prove that the jumps of  $p_h^{k+2} \underline{u}_h$  and of its gradient converge to zero with optimal rate. To this end, define on  $H^2(\mathcal{T}_h)$  (cf. definition (3)) the following *jump seminorm*:

$$|v|_{j,h}^2 := \sum_{F \in \mathcal{F}_h} \left( \frac{\mathcal{A}_F}{h_F} \|\pi_F^k [\nabla v]_F\|_F^2 + \frac{\mathcal{A}_F}{h_F^3} \|\pi_F^k [v]_F\|_F^2 \right),$$

where  $[\cdot]_F$  is the usual jump operator if  $F$  is an interface (the sign is irrelevant), whereas  $[\varphi]_F := \varphi|_F$  if  $F$  is a boundary face, and  $\mathcal{A}_F := \min_{T \in \mathcal{T}_F} \mathcal{A}_T^+$ .

**Corollary 13** (Convergence of the jumps). *Under the assumptions and notations of Theorem 11, there exists a real number  $C > 0$  independent of  $h$ , but possibly depending on  $\mathbb{A}$ ,  $\varrho$ , and  $k$ , such that*

$$|p_h^{k+2}\underline{u}_h|_{J,h} \leq Ch^{k+1}|u|_{H^{k+3}(\Omega)}. \quad (36)$$

*Proof.* Using the triangle inequality together with the uniform equivalence of the element and face diameters expressed by (5), it is inferred that  $|p_h^{k+2}\underline{u}_h|_{J,h}^2 \leq 2\varrho^{-6}|\underline{u}_h|_{S,h}^2$ . The desired result follows using the estimate (34) for the last factor in the right-hand side of the previous inequality.  $\square$

*Remark 14* (Nonconformity of  $p_h^{k+2}\underline{u}_h$ ). Broken polynomial functions are in  $H_0^2(\Omega)$  if, along with their gradient, they are continuous across interfaces and vanish on boundary faces. Thus, the quantity  $|p_h^{k+2}\underline{u}_h|_{J,h}$  can be interpreted as a measure of the nonconformity of  $p_h^{k+2}\underline{u}_h$ . The estimate (36) then shows that  $p_h^{k+2}\underline{u}_h$  approaches a  $H_0^2(\Omega)$ -conforming function as  $h \rightarrow 0^+$ .

### 3.6.3 $L^2$ -error estimate

A sharp  $L^2$ -norm error estimate can also be inferred assuming *biharmonic regularity*, in the following form: For all  $q \in L^2(\Omega)$ , the unique solution  $z \in H_0^2(\Omega)$  to

$$a(z, v) = (q, v) \quad \forall v \in H_0^2(\Omega) \quad (37)$$

satisfies the *a priori* estimate

$$\|z\|_{H^4(\Omega)} \leq C_{\text{bihar}}\|q\|, \quad (38)$$

with  $C_{\text{bihar}} > 0$  only depending on  $\Omega$ .

**Theorem 15** ( $L^2$ -error estimate). *Let  $u \in H_0^2(\Omega)$  and  $\underline{u}_h \in \underline{U}_{h,0}^k$  denote the unique solutions to the continuous (2) and discrete (30) problems, respectively. Assume biharmonic regularity,  $u \in H^{k+3}(\Omega)$ , and  $f \in H^{k+1}(\mathcal{T}_h)$ . Then, there exists a real number  $C > 0$  depending on  $\mathbb{A}$ ,  $\varrho$ , and  $k$ , but independent of  $h$ , such that*

$$\|p_h^{k+2}\underline{u}_h - u\| \leq Ch^{k+3} (\|u\|_{H^{k+3}(\Omega)} + \|f\|_{H^{k+1}(\mathcal{T}_h)}). \quad (39)$$

*Proof.* Let, for the sake of brevity,  $\hat{\underline{u}}_h := \mathbb{I}_h^k u$ . By the triangle inequality, we have that

$$\|p_h^{k+2}\underline{u}_h - u\| \leq \|p_h^{k+2}\hat{\underline{u}}_h - u\| + \|p_h^{k+2}(\underline{u}_h - \hat{\underline{u}}_h)\| =: \mathfrak{T}_1 + \mathfrak{T}_2.$$

By the approximation properties (12a) of  $p_T^{k+2} \circ \mathbb{I}_T^k = \varpi_T^{k+2}$  (cf. Remark 4) with  $s = k + 3$  and  $m = 0$ , we immediately have that

$$\mathfrak{T}_1 \lesssim h^{k+3}\|u\|_{H^{k+3}(\Omega)}.$$

For the second term, on the other hand, we observe that

$$\begin{aligned} \mathfrak{T}_2^2 &= \sum_{T \in \mathcal{T}_h} \|p_T^{k+2}(\hat{\underline{u}}_T - \underline{u}_T)\|_T^2 \\ &\lesssim \sum_{T \in \mathcal{T}_h} \left( h_T^4 \|\mathbb{A}_T^{1/2} \nabla^2 p_T^{k+2}(\hat{\underline{u}}_T - \underline{u}_T)\|_T^2 + \|\pi_T^1(\hat{\underline{u}}_T - \underline{u}_T)\|_T^2 \right) \\ &\lesssim h^4 \|\hat{\underline{u}}_h - \underline{u}_h\|_{\mathbb{a},h}^2 + \|\hat{\underline{u}}_h - \underline{u}_h\|^2, \end{aligned}$$

where we have used the triangle inequality and the approximation properties of  $\pi_T^1$  for  $s = 2$  and  $m = 0$ , as well as the closure condition (16) to pass to the second line, and the definition of the  $\|\cdot\|_{\mathbb{a},h}$ -norm to conclude. Using (35) and Lemma 16 below to bound respectively the first and second addend in the right-hand side, the conclusion follows.  $\square$

The following lemma, used in the proof of Theorem 15 above, shows that element-based discrete unknowns behave “almost” like the  $L^2$ -orthogonal projection of the exact solution on the space of broken polynomials of total degree at most  $k$  on  $\mathcal{T}_h$ .

**Lemma 16** (Supercloseness of element discrete unknowns). *Under the assumptions and notations of Theorem 15, it holds that*

$$\|\hat{u}_h - u_h\| \leq Ch^{k+3} (\|u\|_{H^{k+3}(\Omega)} + \|f\|_{H^{k+1}(\mathcal{T}_h)}), \quad (40)$$

where  $\hat{u}_h$  and  $u_h$  are the broken polynomial functions of total degree at most  $k$  such that  $\hat{u}_{h|T} := \hat{u}_T = \pi_T^k u$  and  $u_{h|T} := u_T$  for any mesh element  $T \in \mathcal{T}_h$ .

*Proof.* Set, for the sake of brevity,  $\underline{e}_h := \hat{\underline{u}}_h - \underline{u}_h$  and  $e_h := \hat{u}_h - u_h$ . Let  $z$  solve (37) with  $q = e_h$  and set  $\hat{z}_h := \mathbb{I}_h^k z$ . Integrating by parts, using the linearity of  $\mathbf{a}_h$  in its first argument, as well as the continuity of moments and shear forces at interfaces, and letting  $\check{z}_T := \varpi_T^{k+2}(z|_T)$ , we have that  $\|e_h\|^2 = \mathfrak{T}_1 + \mathfrak{T}_2$ , with

$$\begin{aligned} \mathfrak{T}_1 &:= \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \left( (\mathbb{A}_T \nabla^2(z - \check{z}_T)) \mathbf{n}_{TF}, \mathbf{e}_{\nabla, F} - \nabla e_T \right)_F \\ &\quad - (\operatorname{div} \mathbb{A}_T \nabla^2(z - \check{z}_T) \cdot \mathbf{n}_{TF}, e_F - e_T)_F - \sum_{T \in \mathcal{T}_h} s_T(\hat{z}_T, \underline{e}_T), \\ \mathfrak{T}_2 &:= \mathbf{a}_h(\hat{\underline{u}}_h, \hat{z}_h) - (f, \pi_h^k z), \end{aligned} \quad (41)$$

where  $\pi_h^k$  is such that  $(\pi_h^k v)|_T = \pi_T^k(v|_T)$  for all  $T \in \mathcal{T}_h$  and all  $v \in H^2(\Omega)$ . The Cauchy–Schwarz inequality then yields

$$\begin{aligned} |\mathfrak{T}_1| &\lesssim \left( \sum_{T \in \mathcal{T}_h} \left( \frac{h_T}{\mathcal{A}_T^+} \sum_{F \in \mathcal{F}_T} \|\mathbb{A}_T \nabla^2(z - \check{z}_T)\|_F^2 \right. \right. \\ &\quad \left. \left. + \frac{h_T^3}{\mathcal{A}_T^+} \sum_{F \in \mathcal{F}_T} \|\operatorname{div} \mathbb{A}_T \nabla^2(z - \check{z}_T)\|_F^2 \right) + |\hat{z}_h|_{s,h}^2 \right)^{1/2} \times (\|\underline{e}_h\|_{\mathbb{A},h}^2 + |\underline{e}_h|_{s,h}^2)^{1/2}. \end{aligned}$$

The approximation properties (12) of  $\varpi_T^{k+2}$  with  $s = 4$ , and the stability of  $\mathbf{a}_h$  together with the energy error estimate (35) allow to conclude that

$$|\mathfrak{T}_1| \lesssim h^2 |z|_{H^4(\Omega)} h^{k+1} \|u\|_{H^{k+3}(\Omega)} \lesssim h^{k+3} \|u\|_{H^{k+3}(\Omega)} \|e_h\|,$$

where in the last estimate we have used the biharmonic regularity hypothesis. Turning to  $\mathfrak{T}_2$ , using the fact that  $(f, \pi_T^k z)_T = (\pi_T^k f, z)_T$  and exploiting the orthogonality property (19), we have

$$\mathfrak{T}_2 = \sum_{T \in \mathcal{T}_h} a_{|T}(\varpi_T^{k+2} u - u, \varpi_T^{k+2} z - z) + \sum_{T \in \mathcal{T}_h} s_T(\hat{\underline{u}}_T, \hat{z}_T) + (f - \pi_h^k f, z) =: \mathfrak{T}_{2,1} + \mathfrak{T}_{2,2} + \mathfrak{T}_{2,3}.$$

We have that  $|\mathfrak{T}_{2,1}| \lesssim h^{k+3} \|u\|_{H^{k+3}(\Omega)} \|e_h\|$  by the Cauchy–Schwarz inequality, the approximation properties of  $\varpi_T^{k+2}$ , and biharmonic regularity. An analogous bound can be obtained for  $|\mathfrak{T}_{2,2}|$ . Finally, we observe that  $\mathfrak{T}_{2,3} = (f - \pi_h^k f, z - \pi_h^k z)$  by the definition (9) of the  $L^2$ -orthogonal projector. Using the approximation properties (10a) of  $\pi_T^k$  with  $l = k$ ,  $m = 0$ , and  $s = k + 1$  for the first factor,  $s = 2$  for the second, we obtain

$$|\mathfrak{T}_{2,3}| \leq \|f - \pi_h^k f\| \|z - \pi_h^k z\| \lesssim h^{k+1} \|f\|_{H^{k+1}(\mathcal{T}_h)} h^2 \|z\|_{H^2(\Omega)} \lesssim h^{k+3} \|f\|_{H^{k+1}(\mathcal{T}_h)} \|e_h\|.$$

This concludes the proof.  $\square$

### 3.7 Numerical examples

We solve problem (1) for  $\mathbb{A} = \mathbb{I}$  (i.e., the biharmonic equation) in the unit square  $\Omega = (0, 1) \times (0, 1)$ , with the right-hand side  $f$  set in agreement with the exact solution

$$u(x, y) = x^2(1-x)^2y^2(1-y)^2,$$

on three different meshes: triangular, cartesian and hexagonal (cf. Fig. 1). Figures 2 and 3 show convergence results in the energy norm and in the  $L^2$ -norm, respectively, for different meshes and polynomial degrees, up to three. We consider  $\|\hat{\mathbf{u}}_h - \mathbf{u}_h\|_{a,h}$  and  $\|\pi_h^k u - u_h\|$  as measures of the error in the energy norm and in the  $L^2$ -norm, respectively. The numerical results show asymptotic convergence rates that match those predicted by the theory in all of the three cases.

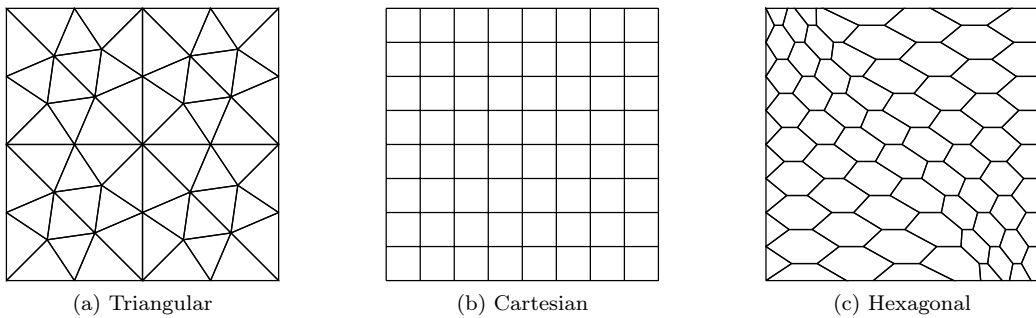


Figure 1: Meshes used for the numerical tests

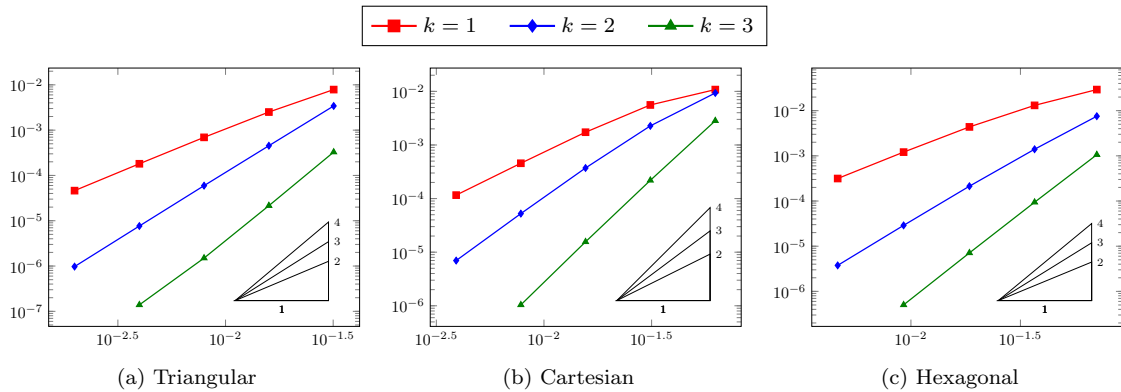


Figure 2:  $\|\hat{\mathbf{u}}_h - \mathbf{u}_h\|_{a,h}$  vs.  $h$  for three different meshes

We also test the robustness of the variant of the HHO method based on the local bilinear form (22) with respect to the user-dependent parameter  $\eta$ . In Figures 4 and 5 we plot, respectively, the energy- and  $L^2$ -norms of the error when  $\eta$  varies from  $10^{-3}$  to  $10^3$  on fixed meshes corresponding to the third refinement level of the ones in Figure 1. From these plots, the robustness of the method can be appreciated, as the energy error spans only two orders of magnitude and the  $L^2$ -error spans four orders of magnitude, while the user-dependent parameter  $\eta$  spans six orders of magnitude.

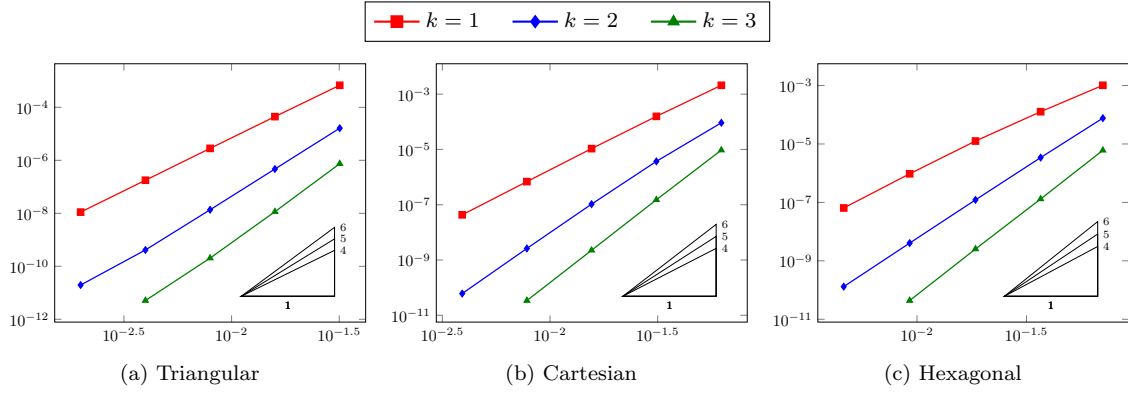


Figure 3:  $\|\pi_h^k u - u_h\|$  vs.  $h$  for three different meshes

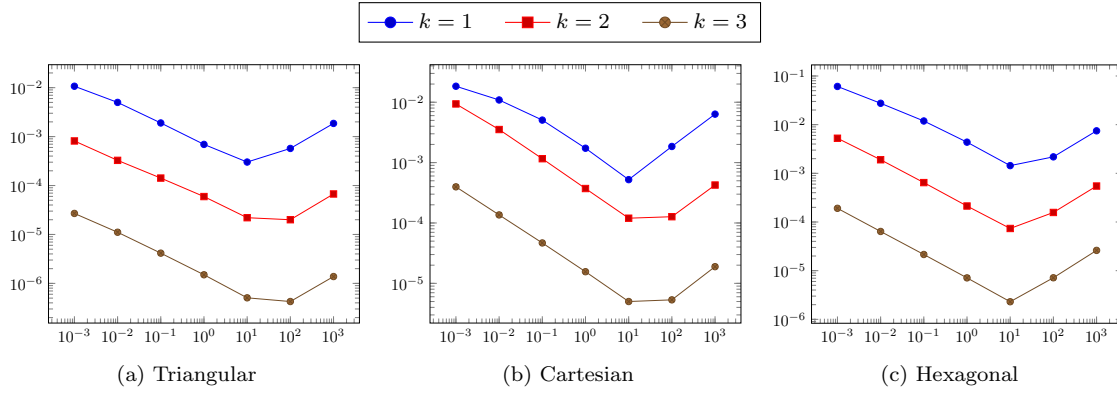


Figure 4:  $\|\hat{u}_h - \underline{u}_h\|_{a,h}$  vs.  $\eta$  for the third refinement level of the Triangular, Cartesian, and Hexagonal meshes

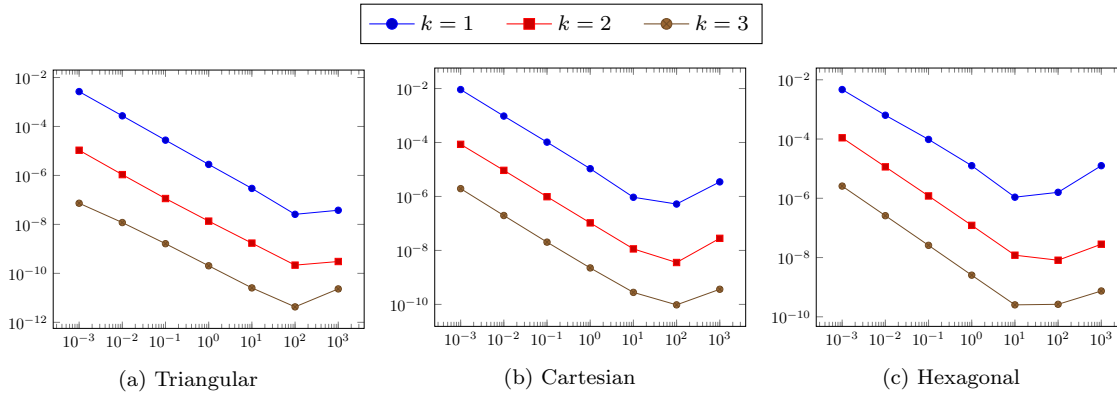


Figure 5:  $\|\pi_h^k u - u_h\|$  vs.  $\eta$  for the third refinement level of the Triangular, Cartesian, and Hexagonal meshes



## 4 Local principle of virtual work and laws of action-reaction

Let a mesh element  $T \in \mathcal{T}_h$  be fixed. At the continuous level, the deflection field  $u$  satisfies, for all  $v \in \mathbb{P}^k(T)$ ,

$$a|_T(u, v) + \sum_{F \in \mathcal{F}_T} (\mathbf{M}_T \mathbf{n}_{TF}, \nabla v)_F - \sum_{F \in \mathcal{F}_T} (\operatorname{div} \mathbf{M}_T \cdot \mathbf{n}_{TF}, v)_F = (f, v)_T, \quad (42a)$$

where  $\mathbf{M}_T := -\mathbb{A}_T \nabla^2 u$ . Equation (42a) expresses the principle of *virtual work* in the context of Kirchhoff–Love plates, written for the mesh element  $T$  and with  $\mathbb{P}^k(T)$  as the space of virtual deflections. The quantities  $\mathbf{M}_T \mathbf{n}_{TF}$  and  $\operatorname{div} \mathbf{M}_T \cdot \mathbf{n}_{TF}$  are internal actions and represent, respectively, the moment and the (scalar) shear force exerted on the face  $F \in \mathcal{F}_T$  by the adjacent element. These two facts can be viewed as two-dimensional counterparts of Cauchy’s hypothesis that the contact force density  $\mathbf{c}$  at a point of an oriented surface  $\Sigma$  in a three-dimensional continuum depend on  $\Sigma$  only through the normal  $\mathbf{n}$  to  $\Sigma$  at that point; indeed, this implies that there is a second-order tensor field, the Cauchy stress  $\mathbf{S}$ , such that, at each point of the three-dimensional body,  $\mathbf{c} = \mathbf{S}\mathbf{n}$ .

For an interface  $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$ , with  $T_1, T_2$  distinct elements of  $\mathcal{T}_h$ , since  $\mathbf{n}_{T_2F} = -\mathbf{n}_{T_1F}$ , both moments and shear forces obey the following *laws of action-reaction*:

$$\mathbf{M}_{T_1} \mathbf{n}_{T_1F} + \mathbf{M}_{T_2} \mathbf{n}_{T_2F} = \mathbf{0}, \quad \operatorname{div} \mathbf{M}_{T_1} \cdot \mathbf{n}_{T_1F} + \operatorname{div} \mathbf{M}_{T_2} \cdot \mathbf{n}_{T_2F} = 0. \quad (42b)$$

The denomination for equations (42b) emphasizes the fact that the moment (resp., shear force) exerted on element  $T_1$  by element  $T_2$  through the common interface  $F$  is the opposite of the moment (resp., shear force) exerted on  $T_2$  by  $T_1$  through  $F$ .

We next show that the solution to discrete problem (30) satisfies discrete counterparts of (42a) and (42b). This requires a reformulation of the stabilization contribution in terms of the differences between face-based and element-based discrete unknowns. Define the space

$$\underline{\mathbb{D}}_{\partial T}^k := \left( \prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F)^2 \right) \times \left( \prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

and the boundary difference operator  $\underline{\delta}_{\partial T}^k: \underline{\mathbb{U}}_T^k \rightarrow \underline{\mathbb{D}}_{\partial T}^k$  such that, for all  $\underline{v}_T \in \underline{\mathbb{U}}_T^k$ ,

$$\underline{\delta}_{\partial T}^k \underline{v}_T \equiv ((\delta_{\nabla, F}^k \underline{v}_T)_{F \in \mathcal{F}_T}, (\delta_F^k \underline{v}_T)_{F \in \mathcal{F}_T}) := ((\mathbf{v}_{\nabla, F} - \nabla v_T)_{F \in \mathcal{F}_T}, (v_F - v_T)_{F \in \mathcal{F}_T}).$$

**Proposition 17** (Boundary difference reformulation of  $s_T$ ). *The local stabilization bilinear form  $s_T$  defined by (21) can be rewritten as*

$$s_T(\underline{u}_T, \underline{v}_T) = s_T((0, \underline{\delta}_{\partial T}^k \underline{u}_T), (0, \underline{\delta}_{\partial T}^k \underline{v}_T)). \quad (43)$$

*Proof.* As a consequence of (18), for all  $v_T \in \mathbb{P}^k(T) \subset \mathbb{P}^{k+2}(T)$  it holds

$$p_T^{k+2} \mathbb{I}_T^k v_T = \varpi_T^{k+2} v_T = v_T, \quad (44)$$

where we have used the fact that, as a projector,  $\varpi_T^{k+2}$  preserves polynomials up to degree  $(k+2)$ . Now, using (44) and the linearity of  $p_T^{k+2}$ , we have

$$p_T^{k+2} \underline{v}_T - v_T = p_T^{k+2} (\underline{v}_T - \mathbb{I}_T^k v_T) = p_T^{k+2} (0, \underline{\delta}_{\partial T}^k \underline{v}_T). \quad (45)$$

Also, for all  $F \in \mathcal{F}_T$ , it holds

$$\nabla p_T^{k+2} \underline{v}_T - \mathbf{v}_{\nabla, F} = \nabla (p_T^{k+2} \underline{v}_T - p_T^{k+2} \mathbb{I}_T^k v_T) - (\mathbf{v}_{\nabla, F} - \nabla v_T) = \nabla p_T^{k+2} (0, \underline{\delta}_{\partial T}^k \underline{v}_T) - \delta_{\nabla, F}^k \underline{v}_T \quad (46)$$

and, analogously,

$$p_T^{k+2} \underline{v}_T - v_F = (p_T^{k+2} \underline{v}_T - p_T^{k+2} \mathbb{I}_T^k v_T) - (v_F - v_T) = p_T^{k+2} (0, \underline{\delta}_{\partial T}^k \underline{v}_T) - \delta_F^k \underline{v}_T. \quad (47)$$

Using (45), (46), and (47) respectively in the first, second, and third term in the right-hand side of (21), the conclusion follows.  $\square$

Define now the residual operator

$$\underline{\mathbf{R}}_{\partial T}^k \equiv ((\mathbf{R}_{\nabla, F}^k)_{F \in \mathcal{F}_T}, (\mathbf{R}_F^k)_{F \in \mathcal{F}_T}) : \underline{\mathbf{U}}_T^k \rightarrow \underline{\mathbf{D}}_{\partial T}^k$$

such that, for all  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$  and all  $\underline{\alpha}_{\partial T} \equiv ((\boldsymbol{\alpha}_{\nabla, F})_{F \in \mathcal{F}_T}, (\alpha_F)_{F \in \mathcal{F}_T}) \in \underline{\mathbf{D}}_{\partial T}^k$ ,

$$(\underline{\mathbf{R}}_{\partial T}^k \underline{\mathbf{v}}_T, \underline{\alpha}_{\partial T})_{0, \partial T} := \sum_{F \in \mathcal{F}_T} \left( (\mathbf{R}_{\nabla, F}^k \underline{\mathbf{v}}_T, \boldsymbol{\alpha}_{\nabla, F})_F + (\mathbf{R}_F^k \underline{\mathbf{v}}_T, \alpha_F)_F \right) = s_T((0, \underline{\delta}_{\partial T}^k \underline{\mathbf{v}}_T), (0, \underline{\alpha}_{\partial T})) \quad (48)$$

Problem (48) is well-posed as a consequence of the Riesz representation theorem for the  $L^2$ -like product in the left-hand side.

**Lemma 18** (Local principle of virtual work and laws of action-reaction). *Denote by  $\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_{h,0}^k$  the unique solution to (30) and, for all  $T \in \mathcal{T}_h$  and all  $F \in \mathcal{F}_T$ , define the discrete moment and shear force*

$$\begin{aligned} \mathcal{M}_{TF}^k(\underline{\mathbf{u}}_T) &:= -((\mathbb{A} \nabla^2 p_T^{k+2} \underline{\mathbf{u}}_T) \mathbf{n}_{TF} + \mathbf{R}_{\nabla, F}^k \underline{\mathbf{u}}_T), \\ \mathcal{S}_{TF}^k(\underline{\mathbf{u}}_T) &:= -\mathbf{div} \mathbb{A} \nabla^2 p_T^{k+2} \underline{\mathbf{u}}_T \cdot \mathbf{n}_{TF} + \mathbf{R}_F^k \underline{\mathbf{u}}_T. \end{aligned} \quad (49)$$

Then, the following discrete counterparts of (42a) and (42b) hold, respectively: For any mesh element  $T \in \mathcal{T}_h$ ,

$$a_{|T}(p_T^{k+2} \underline{\mathbf{u}}_T, v_T) + \sum_{F \in \mathcal{F}_T} (\mathcal{M}_{TF}^k(\underline{\mathbf{u}}_T), \nabla v_T)_F - \sum_{F \in \mathcal{F}_T} (\mathcal{S}_{TF}^k(\underline{\mathbf{u}}_T), v_T)_F = (f, v_T)_T, \quad \forall v_T \in \mathbb{P}^k(T), \quad (50a)$$

and, for any interface  $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$ , with  $T_1, T_2$  distinct elements of  $\mathcal{T}_h$ ,

$$\mathcal{M}_{T_1 F}^k(\underline{\mathbf{u}}_{T_1}) + \mathcal{M}_{T_2 F}^k(\underline{\mathbf{u}}_{T_2}) = \mathbf{0}, \quad \mathcal{S}_{T_1 F}^k(\underline{\mathbf{u}}_{T_1}) + \mathcal{S}_{T_2 F}^k(\underline{\mathbf{u}}_{T_2}) = 0. \quad (50b)$$

*Proof.* Recalling the definition (20) of  $a_T$ , and using the reformulation (43) of  $s_T$  together with the definition (48) of the residual operator, it is inferred from the discrete problem (30) that, for all  $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k$ , it holds

$$\sum_{T \in \mathcal{T}_h} \left( a_{|T}(p_T^{k+2} \underline{\mathbf{u}}_T, p_T^{k+2} \underline{\mathbf{v}}_T) + (\underline{\mathbf{R}}_{\partial T}^k \underline{\mathbf{u}}_T, \underline{\delta}_{\partial T}^k \underline{\mathbf{v}}_T)_{0, \partial T} \right) = (f, v_h). \quad (51)$$

Using the definition (17) of  $p_T^{k+2} \underline{\mathbf{v}}_T$  with  $w = p_T^{k+2} \underline{\mathbf{u}}_T$  for the first term, and recalling (48) and (49), we can rewrite (51) as

$$\sum_{T \in \mathcal{T}_h} \left( a_{|T}(p_T^{k+2} \underline{\mathbf{u}}_T, v_T) - \sum_{F \in \mathcal{F}_T} (\mathcal{M}_{TF}^k(\underline{\mathbf{u}}_T), \mathbf{v}_{\nabla, F} - \nabla v_T)_F + \sum_{F \in \mathcal{F}_T} (\mathcal{S}_{TF}^k(\underline{\mathbf{u}}_T), v_F - v_T)_F \right) = (f, v_h). \quad (52)$$

Thus, for a given mesh element  $T \in \mathcal{T}_h$ , choosing in (52)  $\underline{\mathbf{v}}_h$  such that  $v_T$  spans  $\mathbb{P}^k(T)$ ,  $v_{T'} \equiv 0$  for all  $T' \in \mathcal{T}_h \setminus \{T\}$ ,  $\mathbf{v}_{\nabla, F} \equiv \mathbf{0}$  and  $v_F \equiv 0$  for all  $F \in \mathcal{F}_h$  immediately yields (50a). Next, for a given interface  $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$ , choosing in (52)  $\underline{\mathbf{v}}_h$  such that  $v_T \equiv 0$  for all  $T \in \mathcal{T}_h$ ,  $\mathbf{v}_{\nabla, F'} \equiv \mathbf{0}$  for all  $F' \in \mathcal{F}_h \setminus \{F\}$ ,  $v_F \equiv 0$  for all  $F \in \mathcal{F}_h$ , and letting  $\mathbf{v}_{\nabla, F}$  span  $\mathbb{P}^k(F)^2$  yields the first equation in (50b). Similarly, choosing in (52)  $\underline{\mathbf{v}}_h$  such that  $v_T \equiv 0$  for all  $T \in \mathcal{T}_h$ ,  $\mathbf{v}_{\nabla, F} \equiv \mathbf{0}$  for all  $F \in \mathcal{F}_h \setminus \{F\}$ ,  $v_{F'} \equiv 0$  for all  $F' \in \mathcal{F}_h \setminus \{F\}$ , and letting  $v_F$  span  $\mathbb{P}^k(F)$  yields the second equation in (50b).  $\square$

## 5 Properties of the discrete bilinear form

This section contains the proofs of the technical Lemmas 7 and 10.

## 5.1 Local coercivity and boundedness

*Proof of Lemma 7.* Let a mesh element  $T \in \mathcal{T}_h$  be fixed, and let  $\underline{v}_T \in \underline{\mathbb{U}}_T^k$ .

(i) *Coercivity.* Taking  $w = v_T \in \mathbb{P}^k(T) \subset \mathbb{P}^{k+2}(T)$  in (17) gives

$$a|_T(v_T, v_T) = a|_T(p_T^{k+2}\underline{v}_T, v_T) + \sum_{F \in \mathcal{F}_T} (\mathbf{v}_{\nabla, F} - \nabla v_T, \mathbf{M}_{v_T} \mathbf{n}_{TF})_F - \sum_{F \in \mathcal{F}_T} (v_F - v_T, \mathbf{div} \mathbf{M}_{v_T} \cdot \mathbf{n}_{TF})_F.$$

Using the Cauchy–Schwarz inequality to bound the first term in the right-hand side, the Cauchy–Schwarz and discrete trace (7a) inequalities to bound the second, and the Cauchy–Schwarz, discrete trace (7a) and inverse (7c) inequalities to bound the third, and simplifying we obtain:

$$\|\mathbb{A}_T^{1/2} \nabla^2 v_T\|_T \lesssim \left( \|\mathbb{A}_T^{1/2} \nabla^2 p_T^{k+2}\underline{v}_T\|_T^2 + \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\mathbf{v}_{\nabla, F} - \nabla v_T\|_F^2 + \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_F^2 \right)^{1/2}. \quad (53)$$

It remains to estimate the boundary terms inside the parentheses in terms of the  $\|\cdot\|_{a, T}$ -seminorm.

(i.a) *Bound on  $\frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\mathbf{v}_{\nabla, F} - \nabla v_T\|_F^2$ .* For all  $F \in \mathcal{F}_T$ , inserting  $\pm \pi_F^k \nabla (p_T^{k+2}\underline{v}_T - \pi_T^k p_T^{k+2}\underline{v}_T)$  into the norm and using the linearity of  $\pi_F^k$  and the fact that it preserves polynomials in  $\mathbb{P}^k(F)^2$  as a projector, we obtain

$$\begin{aligned} & \|\mathbf{v}_{\nabla, F} - \nabla v_T\|_F \\ &= \|\pi_F^k (\mathbf{v}_{\nabla, F} - \nabla p_T^{k+2}\underline{v}_T) + \pi_F^k \nabla (p_T^{k+2}\underline{v}_T - \pi_T^k p_T^{k+2}\underline{v}_T) + \nabla (\pi_T^k p_T^{k+2}\underline{v}_T - v_T)\|_F \\ &\lesssim \|\pi_F^k (\mathbf{v}_{\nabla, F} - \nabla p_T^{k+2}\underline{v}_T)\|_F + \|\pi_F^k \nabla (p_T^{k+2}\underline{v}_T - \pi_T^k p_T^{k+2}\underline{v}_T)\|_F + \|\nabla \pi_T^k (p_T^{k+2}\underline{v}_T - v_T)\|_F \\ &=: \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3, \end{aligned} \quad (54)$$

where we have used the triangle inequality to pass to the second line. By the definition (21) of  $s_T$ , we readily infer that

$$h_T^{-1/2} \sqrt{\mathcal{A}_T^+} |\mathfrak{I}_1| \lesssim \|\underline{v}_T\|_{a, T}.$$

Using the  $L^2(F)^2$ -boundedness of  $\pi_F^k$  followed by the discrete trace inequality (7a), we can write  $|\mathfrak{I}_2| \lesssim h_T^{-1/2} \|\nabla (p_T^{k+2}\underline{v}_T - \pi_T^k p_T^{k+2}\underline{v}_T)\|_T$ . Then, by the approximation properties (10a) of  $\pi_T^k$  with  $l = k$ ,  $m = 1$ , and  $s = 2$ , we infer that

$$|\mathfrak{I}_2| \lesssim h_T^{1/2} |p_T^{k+2}\underline{v}_T|_{H^2(T)} \lesssim h_T^{1/2} \|\nabla^2 p_T^{k+2}\underline{v}_T\|_T,$$

so that

$$h_T^{-1/2} \sqrt{\mathcal{A}_T^+} |\mathfrak{I}_2| \lesssim \|\underline{v}_T\|_{a, T}.$$

Finally, the third term in the right-hand side of (54) can be estimated using the discrete trace (7a) and inverse (7c) inequalities together with the definition (21) of  $s_T$  as follows:

$$h_T^{-1/2} \sqrt{\mathcal{A}_T^+} |\mathfrak{I}_3| \lesssim h_T^{-2} \sqrt{\mathcal{A}_T^+} \|\pi_T^k (p_T^{k+2}\underline{v}_T - v_T)\|_T \leq \|\underline{v}_T\|_{a, T}.$$

Hence, multiplying (54) by  $h_T^{-1/2} \sqrt{\mathcal{A}_T^+}$ , squaring, summing over  $F \in \mathcal{F}_T$ , using the above estimates for  $\mathfrak{I}_1$ ,  $\mathfrak{I}_2$ ,  $\mathfrak{I}_3$ , and recalling the uniform bound (6) on  $\text{card}(\mathcal{F}_T)$ , we have

$$\frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\mathbf{v}_{\nabla, F} - \nabla v_T\|_F^2 \lesssim \|\underline{v}_T\|_{a, T}^2. \quad (55)$$

(i.b) *Bound on  $\frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_F^2$ .* For all  $F \in \mathcal{F}_T$ , inserting  $\pm \pi_F^k (p_T^{k+2} \underline{v}_T - \pi_T^k p_T^{k+2} \underline{v}_T)$  into the norm, and using the linearity of  $\pi_F^k$  and  $\pi_T^k$  together with the fact that they preserve polynomials up to degree  $k$  as projectors, we have that

$$\begin{aligned} \|v_F - v_T\|_F &= \|\pi_F^k (v_F - p_T^{k+2} \underline{v}_T) + \pi_F^k (p_T^{k+2} \underline{v}_T - \pi_T^k p_T^{k+2} \underline{v}_T) + \pi_T^k (p_T^{k+2} \underline{v}_T - v_T)\|_F \\ &\leq \|\pi_F^k (v_F - p_T^{k+2} \underline{v}_T)\|_F + \|\pi_F^k (p_T^{k+2} \underline{v}_T - \pi_T^k p_T^{k+2} \underline{v}_T)\|_F + \|\pi_T^k (p_T^{k+2} \underline{v}_T - v_T)\|_F \\ &=: \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3. \end{aligned} \quad (56)$$

By the definition (21) of  $s_T$ , it is readily inferred that

$$h_T^{-3/2} \sqrt{\mathcal{A}_T^+} |\mathfrak{T}_1| \lesssim \|\underline{v}_T\|_{a,T}.$$

The second term can be estimated as follows:

$$|\mathfrak{T}_2| \lesssim h_T^{-1/2} \|p_T^{k+2} \underline{v}_T - \pi_T^k p_T^{k+2} \underline{v}_T\|_T \lesssim h_T^{-1/2} h_T^2 |p_T^{k+2} \underline{v}_T|_{H^2(T)} \lesssim h_T^{3/2} \|\nabla^2 p_T^{k+2} \underline{v}_T\|_T,$$

where we have used the  $L^2(F)$ -boundedness of  $\pi_F^k$ , the discrete trace inequality (7a), the uniform equivalence of face and element diameters (5) to replace  $h_F$  with  $h_T$ , and the approximation property (10a) with  $l = k$ ,  $s = 2$ , and  $m = 0$ . Hence,

$$h_T^{-3/2} \sqrt{\mathcal{A}_T^+} |\mathfrak{T}_2| \lesssim \|\underline{v}_T\|_{a,T}.$$

Finally, using the discrete trace inequality (7a) followed by the definition (21) of  $s_T$ , we have

$$h_T^{-3/2} \sqrt{\mathcal{A}_T^+} |\mathfrak{T}_3| \lesssim \|\underline{v}_T\|_{a,T}.$$

Multiplying (56) by  $h_T^{-3/2} \sqrt{\mathcal{A}_T^+}$ , squaring, summing over  $F \in \mathcal{F}_T$ , using the above estimates for  $\mathfrak{T}_1$ ,  $\mathfrak{T}_2$ ,  $\mathfrak{T}_3$ , and recalling the uniform bound (6) on  $\text{card}(\mathcal{F}_T)$ , we arrive at

$$\frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_F^2 \lesssim \|\underline{v}_T\|_{a,T}^2. \quad (57)$$

(i.c) *Conclusion.* Combining (53), (57), and (55), the first inequality in (25) follows.

(ii) *Boundedness.* Taking  $w = p_T^{k+2} \underline{v}_T$  in (17), using the Cauchy–Schwarz, discrete trace (7a) and inverse inequalities (7c), and simplifying, we get

$$\|\mathbb{A}_T^{1/2} \nabla^2 p_T^{k+2} \underline{v}_T\|_T \lesssim \|\underline{v}_T\|_{\mathbb{A},T}, \quad (58)$$

which bounds the portion of  $\|\underline{v}_T\|_{a,T}$  stemming from the consistency term in (20).

It remains to bound on the local stabilization terms in  $s_T(\underline{v}_T, \underline{v}_T)$ .

(ii.a) *Bound on  $\frac{\mathcal{A}_T^+}{h_T^4} \|\pi_T^k (p_T^{k+2} \underline{v}_T - v_T)\|_T^2$ .* Inserting  $\pm p_T^{k+2} \underline{v}_T$  into the norm and using the triangle inequality, we have that

$$\|\pi_T^k (p_T^{k+2} \underline{v}_T - v_T)\|_T \leq \|\pi_T^k p_T^{k+2} \underline{v}_T - p_T^{k+2} \underline{v}_T\|_T + \|p_T^{k+2} \underline{v}_T - v_T\|_T =: \mathfrak{T}_1 + \mathfrak{T}_2. \quad (59)$$

For the first term, using the approximation property (10a) with  $l = k$ ,  $m = 0$ , and  $s = 2$ , and (58), we get

$$h_T^{-2} \sqrt{\mathcal{A}_T^+} |\mathfrak{T}_1| \lesssim \|\underline{v}_T\|_{\mathbb{A},T}.$$

For the second term, inserting  $0 = -\pi_T^1 p_T^{k+2} \underline{v}_T + \pi_T^1 v_T$  into the norm (see (16)) and using the triangle inequality, we obtain

$$|\mathfrak{T}_2| = \|p_T^{k+2} \underline{v}_T - \pi_T^1 p_T^{k+2} \underline{v}_T + \pi_T^1 v_T - v_T\|_T \leq \|p_T^{k+2} \underline{v}_T - \pi_T^1 p_T^{k+2} \underline{v}_T\|_T + \|\pi_T^1 v_T - v_T\|_T.$$

The approximation property (10a) with  $l = 1$ ,  $m = 0$ , and  $s = 2$  gives  $\|p_T^{k+2}\underline{v}_T - \pi_T^1 p_T^{k+2}\underline{v}_T\|_T \lesssim h_T^2 \|\nabla^2 p_T^{k+2}\underline{v}_T\|_T$  and  $\|v_T - \pi_T^1 v_T\|_T \lesssim h_T^2 \|\nabla^2 v_T\|_T$  so that, accounting for (58),

$$h_T^{-2} \sqrt{\mathcal{A}_T^+} |\mathfrak{I}_2| \lesssim \|\underline{v}_T\|_{\mathbb{A},T}.$$

Squaring (59), multiplying the resulting inequality by  $\mathcal{A}_T^+/h_T^4$ , and using the above estimates for  $\mathfrak{I}_1$  and  $\mathfrak{I}_2$  together with the uniform bound (6) on  $\text{card}(\mathcal{F}_T)$ , we conclude that

$$\frac{\mathcal{A}_T^+}{h_T^4} \|\pi_T^k(p_T^{k+2}\underline{v}_T - v_T)\|_T^2 \lesssim \|\underline{v}_T\|_{\mathbb{A},T}^2.$$

(ii.b) *Bound on  $\frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(\nabla p_T^{k+2}\underline{v}_T - \mathbf{v}_{\nabla,F})\|_F^2$ .* For any  $F \in \mathcal{F}_T$ , inserting  $\pm \nabla v_T$  into the norm, invoking the linearity of  $\pi_F^k$  together with the fact that it preserves polynomials in  $\mathbb{P}^k(F)^2$  as a projector, and using the triangle inequality, we have that

$$\begin{aligned} \|\pi_F^k(\nabla p_T^{k+2}\underline{v}_T - \mathbf{v}_{\nabla,F})\|_F &\leq \|\pi_F^k \nabla(p_T^{k+2}\underline{v}_T - v_T)\|_F + \|\nabla v_T - \mathbf{v}_{\nabla,F}\|_F \\ &\lesssim h_T^{-3/2} \|p_T^{k+2}\underline{v}_T - v_T\|_T + \|\nabla v_T - \mathbf{v}_{\nabla,F}\|_F \\ &\lesssim \frac{h_T^{1/2}}{\sqrt{\mathcal{A}_T^+}} \|\underline{v}_T\|_{\mathbb{A},T} + \|\nabla v_T - \mathbf{v}_{\nabla,F}\|_F \end{aligned} \quad (60)$$

where to pass to the second line we have used the  $L^2(F)^2$ -boundedness of  $\pi_F^k$ , the discrete trace inequality (7a), and the inverse inequality (7c), while to pass to the third line we have estimated the first addend as the term  $\mathfrak{I}_2$  in (59). Thus, squaring the above inequality, summing over  $F \in \mathcal{F}_T$ , multiplying it by  $\mathcal{A}_T^+/h_T$ , and using the uniform bound (6) on  $\text{card}(\mathcal{F}_T)$ , we finally infer

$$\frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(\nabla p_T^{k+2}\underline{v}_T - \mathbf{v}_{\nabla,F})\|_F^2 \lesssim \|\underline{v}_T\|_{\mathbb{A},T}^2. \quad (61)$$

(ii.c) *Bound on  $\frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(p_T^{k+2}\underline{v}_T - v_F)\|_F^2$ .* For any  $F \in \mathcal{F}_T$ , inserting  $\pm v_T$  into the norm, invoking the linearity of  $\pi_F^k$  together with the fact that it preserves polynomials in  $\mathbb{P}^k(F)$  as a projector, and using the triangle inequality, we infer that

$$\begin{aligned} \|\pi_F^k(p_T^{k+2}\underline{v}_T - v_F)\|_F &\leq \|\pi_F^k(p_T^{k+2}\underline{v}_T - v_T)\|_F + \|v_F - v_T\|_F \\ &\lesssim h_T^{-1/2} \|p_T^{k+2}\underline{v}_T - v_T\|_T + \|v_F - v_T\|_F \\ &\lesssim \frac{h_T^{3/2}}{\sqrt{\mathcal{A}_T^+}} \|\underline{v}_T\|_{\mathbb{A},T} + \|v_F - v_T\|_F, \end{aligned} \quad (62)$$

where to pass to the second line we have used the  $L^2(F)$ -boundedness of  $\pi_F^k$  followed by the discrete trace inequality (7a) and the uniform equivalence of the element and face diameters expressed by (5), while to pass to the third line we have estimated the first addend as the term  $\mathfrak{I}_2$  in (59). Hence, multiplying (62) by  $h_T^{-3/2} \sqrt{\mathcal{A}_T^+}$ , squaring, summing over  $F \in \mathcal{F}_T$ , recalling (59), and using the uniform bound (6) on  $\text{card}(\mathcal{F}_T)$ , we conclude that

$$\frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \|\pi_F^k(p_T^{k+2}\underline{v}_T - v_F)\|_F^2 \lesssim \|\underline{v}_T\|_{\mathbb{A},T}^2. \quad (63)$$

(ii.d) *Conclusion.* The second inequality in (25) then follows combining (58), (63), and (61) and recalling the definition (24) of  $\|\cdot\|_{\mathbb{A},T}$ .  $\square$

## 5.2 Global coercivity, boundedness, and consistency

*Proof of Lemma 10.* (i) *Coercivity and boundedness.* The norm equivalence (32) is an immediate consequence of Lemma 7 together with the definition (28) of the  $\|\cdot\|_{\mathbb{A},h}$ -norm.

(ii) *Consistency.* Let us prove (33). An element-wise integration by parts yields

$$\begin{aligned} (\operatorname{div} \operatorname{div} \mathbb{A} \nabla^2 v, w_h) &= \sum_{T \in \mathcal{T}_h} \left( (\mathbb{A}_T \nabla^2 v, \nabla^2 w_T)_T - \sum_{F \in \mathcal{F}_T} (\operatorname{div} \mathbb{A}_T \nabla^2 v \cdot \mathbf{n}_{TF}, w_F - w_T)_F \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_T} ((\mathbb{A}_T \nabla^2 v) \mathbf{n}_{TF}, \mathbf{w}_{\nabla, F} - \nabla w_T)_F \right), \end{aligned} \quad (64)$$

where we have used the fact that moments and Kirchhoff shear forces are continuous at interfaces owing to the regularity of  $v$  (see (42b) for the expression of these continuity properties for the exact solution  $u$ ) and that homogeneous boundary conditions are embedded in  $\underline{U}_{h,0}^k$ . Now, let

$$\hat{\underline{v}}_h := \mathbb{I}_h^k v, \quad \hat{\underline{v}}_T := \mathbb{I}_T^k(v|_T) \quad \text{and} \quad \check{v}_T := p_T^{k+2} \hat{\underline{v}}_T = p_T^{k+2} \mathbb{I}_T^k(v|_T); \quad (65)$$

we have

$$\begin{aligned} a_h(\hat{\underline{v}}_h, \underline{w}_h) &= \sum_{T \in \mathcal{T}_h} \left( (\mathbb{A}_T \nabla^2 \check{v}_T, \nabla^2 w_T)_T - \sum_{F \in \mathcal{F}_T} (\operatorname{div} \mathbb{A}_T \nabla^2 \check{v}_T \cdot \mathbf{n}_{TF}, w_F - w_T)_F \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_T} ((\mathbb{A}_T \nabla^2 \check{v}_T) \mathbf{n}_{TF}, \mathbf{w}_{\nabla, F} - \nabla w_T)_F + s_T(\hat{\underline{v}}_T, \underline{w}_T) \right). \end{aligned} \quad (66)$$

Thus, letting  $\mathcal{E}_h(\underline{w}_h) := (\operatorname{div} \operatorname{div} \mathbb{A} \nabla^2 v, w_h) - a_h(\hat{\underline{v}}_h, \underline{w}_h)$ , (64) and (66) yield

$$\begin{aligned} \mathcal{E}_h(\underline{w}_h) &= \sum_{T \in \mathcal{T}_h} \left( (\mathbb{A}_T \nabla^2 (\check{v}_T - v), \nabla^2 w_T)_T + ((\mathbb{A}_T \nabla^2 (\check{v}_T - v)) \mathbf{n}_{TF}, \mathbf{w}_{\nabla, F} - \nabla w_T)_F \right. \\ &\quad \left. - \sum_{F \in \mathcal{F}_T} (\operatorname{div} \mathbb{A}_T \nabla^2 (\check{v}_T - v) \cdot \mathbf{n}_{TF}, w_F - w_T)_F \right) + \sum_{T \in \mathcal{T}_h} s_T(\hat{\underline{v}}_T, \underline{w}_T) \\ &=: \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3 + \mathfrak{T}_4. \end{aligned}$$

By the definition (11) of the oblique biharmonic projector, we have that

$$\mathfrak{T}_1 = 0. \quad (67)$$

Also, using the approximation properties (12) with  $l = k + 2$ ,  $s = k + 3$ , and  $m = 2, 3$ , we infer that

$$|\mathfrak{T}_2 + \mathfrak{T}_3| \lesssim h^{k+1} |v|_{H^{k+3}(\Omega)} \|\underline{w}_h\|_{\mathbb{A},h}. \quad (68)$$

Moreover, we have  $s_T(\hat{\underline{v}}_T, \underline{w}_T) \leq s_T(\hat{\underline{v}}_T, \hat{\underline{v}}_T)^{1/2} s_T(\underline{w}_T, \underline{w}_T)^{1/2}$ ; as for the first factor, by (23) we have  $s_T(\hat{\underline{v}}_T, \hat{\underline{v}}_T)^{1/2} \lesssim \sqrt{\mathcal{A}_T^+} h_T^{k+1} |v|_{H^{k+3}(T)}$ , whereas the second inequality in (25) gives  $s_T(\underline{w}_T, \underline{w}_T)^{1/2} \lesssim \|\underline{w}_T\|_{\mathbb{A},T}$ , so that

$$|\mathfrak{T}_4| \lesssim h^{k+1} |v|_{H^{k+3}(\Omega)} \|\underline{w}_h\|_{\mathbb{A},h}. \quad (69)$$

Using (67), (68), and (69) to estimate  $\mathcal{E}_h(\underline{w}_h)$ , and using the resulting bound in the supremum in (33) concludes the proof.  $\square$

## 6 Proof of Theorem 1

(i) *Proof of (12a)*. We apply [19, Lemma 3]. Therefore, proving (12a) amounts to proving the following estimates:

$$\|\nabla^2 \varpi_T^l v\|_T \lesssim \|\nabla^2 v\|_T, \quad (70a)$$

$$\|\nabla \varpi_T^l v\|_T \lesssim (\|\nabla v\|_T + h_T \|\nabla^2 v\|_T), \quad (70b)$$

$$\|\varpi_T^l v\|_T \lesssim (\|v\|_T + h_T \|\nabla v\|_T + h_T^2 \|\nabla^2 v\|_T). \quad (70c)$$

where  $a \lesssim b$  means  $a \leq Cb$  with  $C > 0$  as in (12).

We start by proving (70a). The definition (11) of  $\varpi_T^l$  implies that

$$\begin{aligned} \sqrt{\mathcal{A}_T^-} \|\nabla^2(v - \varpi_T^l v)\|_T &\leq \|\mathbb{A}_T^{1/2} \nabla^2(v - \varpi_T^l v)\|_T \\ &\leq \|\mathbb{A}_T^{1/2} \nabla^2(v - \pi_T^l v)\|_T \lesssim \sqrt{\mathcal{A}_T^+} \|\nabla^2 v\|_T, \end{aligned} \quad (71)$$

where we have used the definition of  $\mathcal{A}_T^-$  (see Section 2.1) in the first line, the characterization of  $\varpi_T^l$  as  $\arg \min_{z \in \mathbb{P}^l(T)} \|\mathbb{A}_T^{1/2} \nabla^2(v - z)\|_T$  in the second line, along with the definition of  $\mathcal{A}_T^+$  and the  $H^2$ -stability of the  $L^2$ -orthogonal projector (resulting from (10a) with  $s = m = 2$ ) to conclude. Thus, using again the triangle inequality, we have that

$$\|\nabla^2 \varpi_T^l v\|_T \leq \|\nabla^2(\varpi_T^l v - v)\|_T + \|\nabla^2 v\|_T \lesssim \|\nabla^2 v\|_T,$$

and (70a) is proved.

To prove (70b), we introduce the quantities  $0 = -\nabla \pi_T^1 \varpi_T^l v + \nabla \pi_T^1 v$  (recall the second condition in (11)) and  $\pm \nabla v$  inside the  $L^2(T)$ -norm of  $\nabla \varpi_T^l v$  to infer that

$$\begin{aligned} \|\nabla \varpi_T^l v\|_T &\leq \|\nabla(\varpi_T^l v - \pi_T^1 \varpi_T^l v)\|_T + \|\nabla(v - \pi_T^1 v)\|_T + \|\nabla v\|_T \\ &\lesssim h_T \|\nabla^2 \varpi_T^l v\|_T + h_T \|\nabla^2 v\|_T + \|\nabla v\|_T \lesssim (\|\nabla v\|_T + h_T \|\nabla^2 v\|_T), \end{aligned}$$

where we have used the approximation estimate (10a) for  $\pi_T^1$  with  $m = 1$  and  $s = 2$  together with the fact that, for any  $w \in H^2(T)$ ,  $|w|_{H^2(T)} \lesssim \|\nabla^2 w\|_T$  to estimate the first two terms, and (70a) to conclude.

The proof of (70c) is completely analogous. We obtain

$$\begin{aligned} \|\varpi_T^l v\|_T &\leq \|\varpi_T^l v - \pi_T^1 \varpi_T^l v\|_T + \|v - \pi_T^1 v\|_T + \|v\|_T \\ &\lesssim h_T^2 \|\nabla^2 \varpi_T^l v\|_T + h_T \|\nabla v\|_T + \|v\|_T \\ &\lesssim (h_T^2 \|\nabla^2 v\|_T + h_T \|\nabla v\|_T + \|v\|_T), \end{aligned}$$

where we have used (10a) to estimate the first two addends in the first line, with  $m = 0$  and  $s = 2$  for the first one and with  $m = 0$  and  $s = 1$  for the second one. This concludes the proof of (12a).

(ii) *Proof of (12b)*. For  $m \leq s - 1$ , by applying the continuous trace inequality (7b) to  $w = \partial^\alpha(v - \varpi_T^l v) \in H^1(T)$  for all  $\alpha \in \mathbb{N}^2$  such that  $\alpha_1 + \alpha_2 = m$ , we have

$$h_T^{1/2} |v - \varpi_T^l v|_{H^m(\partial T)} \lesssim |v - \varpi_T^l v|_{H^m(T)} + h_T |v - \varpi_T^l v|_{H^{m+1}(T)}.$$

The conclusion follows using (12a) for  $m$  and  $m + 1$  to bound the terms in the right-hand side.

## 7 Concluding remarks

Some concluding remarks are in order.

### 7.1 Other choices for the boundary gradient unknowns

It is possible to devise a local deflection reconstruction by using scalar face-based unknowns playing the role of the normal derivative of the deflection, in place of vector face-based unknowns, playing the role of its gradient. For this, one has to replace the local space of unknowns defined by (13) by

$$\begin{aligned}\tilde{\mathbf{U}}_T^k &:= \mathbb{P}^k(T) \times \left( \prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right) \times \left( \prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right), \\ \tilde{\mathbf{U}}_T^k \ni \tilde{\mathbf{v}}_T &= (v_T, (v_{n,F})_{F \in \mathcal{F}_T}, (v_F)_{F \in \mathcal{F}_T}),\end{aligned}\tag{72}$$

and reformulate (15) as follows: Find  $\tilde{p}_T^{k+2}: \tilde{\mathbf{U}}_T^k \rightarrow \mathbb{P}^{k+2}(T)$  such that, for all  $w \in \mathbb{P}^{k+2}(T)$ ,

$$\begin{aligned}a_{|T}(\tilde{p}_T^{k+2} \tilde{\mathbf{v}}_T, w) &= -(v_T, \operatorname{div} \operatorname{div} \mathbf{M}_w)_T - \sum_{F \in \mathcal{F}_T} (v_{n,F}, \mathbf{M}_w \mathbf{n}_{TF} \cdot \mathbf{n}_{TF})_F + \\ &+ \sum_{F \in \mathcal{F}_T} (v_F, \operatorname{div} \mathbf{M}_w \cdot \mathbf{n}_{TF} + \partial_\tau(\mathbf{M}_w \mathbf{n}_{TF} \cdot \boldsymbol{\tau}))_F,\end{aligned}\tag{73}$$

where  $\boldsymbol{\tau}$  denotes the unit tangent vector to the boundary of the element. Correspondingly, the expression (21) of stabilization bilinear form has to be modified as follows (compare, in particular, the terms in the second line): For all  $(\tilde{\mathbf{u}}_T, \tilde{\mathbf{v}}_T) \in \tilde{\mathbf{U}}_T^k \times \tilde{\mathbf{U}}_T^k$  (cf. (72)),

$$\begin{aligned}\tilde{s}_T(\tilde{\mathbf{u}}_T, \tilde{\mathbf{v}}_T) &:= \frac{\mathcal{A}_T^+}{h_T^4} \left( \pi_T^k(\tilde{p}_T^{k+2} \tilde{\mathbf{u}}_T - u_T), \pi_T^k(\tilde{p}_T^{k+2} \tilde{\mathbf{v}}_T - v_T) \right)_T \\ &+ \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \left( \pi_F^k(\partial_{\mathbf{n}_{TF}} \tilde{p}_T^{k+2} \tilde{\mathbf{u}}_T - u_{n,F}), \pi_F^k(\partial_{\mathbf{n}_{TF}} \tilde{p}_T^{k+2} \tilde{\mathbf{v}}_T - v_{n,F}) \right)_F \\ &+ \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \left( \pi_F^k(\tilde{p}_T^{k+2} \tilde{\mathbf{u}}_T - u_F), \pi_F^k(\tilde{p}_T^{k+2} \tilde{\mathbf{v}}_T - v_F) \right)_F.\end{aligned}\tag{74}$$

### 7.2 Mixed formulations

The results of this paper concern the primal formulation (1) of the Kirchhoff–Love plate bending model problem. As it is well known, this problem admits dual and mixed formulations that have been the basis for the development of mixed and hybrid nonconforming finite elements (see, e.g., [11]). A HHO discretization based on a mixed formulation will make the object of a future work, as well as the study of its relation with the method presented here and its variations. We notice, in passing, that a similar study for a second-order elliptic problem has been carried out in [1].

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