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Antonietti, P.F.; Perugia, I.; Zaliani, D.

MOX, Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox@mate.polimi.it http://mox.polimi.it

# Schwarz domain decomposition preconditioners for plane wave discontinuous Galerkin methods

Paola F. Antonietti[♮], Ilaria Perugia[♭], Davide Zaliani[†]

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[♮] MOX–*Modeling and Scientific Computing*, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
paola.antonietti@polimi.it

[♭] Fakultät für Mathematik, Univeristät Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien
ilaria.perugia@univie.ac.at

† Dipartimento di Matematica, Università di Pavia, Via Ferrata 1, 27100, Pavia, Italy
davidezal@yahoo.it

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**Abstract**

We construct Schwarz domain decomposition preconditioners for plane wave discontinuous Galerkin methods for Helmholtz boundary value problems. In particular, we consider additive and multiplicative non-overlapping Schwarz methods. Numerical tests show good performance of these preconditioners when solving the linear system of equations with GMRES.

## 1 Introduction

Over the last years, finite element methods based on non-polynomial shape functions for time harmonic wave propagation problems have become increasingly popular. The idea behind these methods is to incorporate information on the oscillatory behaviour of the solutions directly within the approximating spaces by using, instead of polynomial basis functions, Trefftz basis functions, namely, local solutions to the differential operator. For the Helmholtz equation these functions can be, for instance, plane waves or circular/shperical waves, with the same frequency as the original problem. Although these methods are not pollution-free, they can deliver more accurate results, for a given number of degrees of freedom, than standard polynomial finite element methods. The way of imposing continuity at interelement boundaries generates different Trefftz-type methods: ultra weak variational formulation/Trefftz-discontinuous Galerkin methods [8,13,14,22,23], partition of unity [6,33], least squares methods [34] or Lagrange multiplier methods [1, 37]. These methods have been extended from acoustic to electromagnetic [12, 24, 27] and elastic [28, 31] time-harmonic wave propagation problems. Here, we consider plane wave discontinuous Galerkin (PWDG) methods, of which

the ultra-weak variational formulation can be seen as a particular case, for the Helmholtz equation, and we attack the problem of preconditioning the arising algebraic linear systems by domain decomposition methods (for plane wave methods with Lagrange multipliers, domain decomposition preconditioners have been introduced in [20]).

The solution of algebraic linear systems arising from discretizations of the Helmholtz equation is a difficult problem [19] (see also the references therein for a bibliography on this topic). In case of Dirichlet or Neumann boundary conditions, classical discretization methods for the Helmholtz equation based on polynomial spaces result into real, symmetric and indefinite linear systems of equations that can be preconditioned, for example, by overlapping Schwarz, multigrid or substructuring type methods, see for example [9, 10, 18] and the references therein. A complete theory generalizing the classical Schwarz analysis for symmetric, positive definite problems to indefinite problems has been provided in [9,10] where, exploiting GMRES converge bounds [16,35], it is proven that GMRES converges uniformly (with respect to the meshsize and the number of subdomains) provided that *i)* the subdomain and coarse partitions are sufficiently fine; *ii)* the low-order term of the differential operator is a relatively compact perturbation of the second order term. For the Helmholtz problem, these methods are in general not scalable with respect to the wavenumber and become less and less effective as the wavenumber increases, unless a sufficient number of coarse points per wavelength is employed. Such a requirement may become unfeasible for practical applications. Nevertheless, they are currently employed for large scale computations, although a comprehensive and sharp theory is still missing (because it also relies on GMRES convergence bounds which are not sharp).

Using PWDG leads to a different situation: the resulting linear system of equations is complex (independently of the considered boundary conditions) and non-hermitian. Several strategies have already beed studied to cope with the severe ill-conditioning; see [7, 29, 32].

The aim of this contribution is to preliminarily explore the performance of a class of Schwarz methods to precondition the linear system of equations arising from PWDG approximation of the Helmholtz equation in a 2D cavity with impedance boundary condition. To keep as low as possible the computational effort without loosing effectiveness, we take advantage of the DG framework and consider the non-overlapping version of the classical Schwarz preconditioners. Indeed, according to [2–4, 21], non-overlapping preconditioners for DG methods converge as fast as overlapping solvers with minimal overlap for continuous discretizations. Our numerical experiments indicate that our preconditioners work well in reducing the computation effort in the solution of the resulting linear system of equations, and that the PWDG method seems to be particularly well suited for the development of solvers that are scalable with respect to the wavenumber.

## 2 The PWDG method for the Helmholtz problem

We consider the homogeneous Helmholtz problem in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, with impedance boundary condition along $\partial\Omega$. Given a wavenumber $k$ (the corresponding

wavelength is $\lambda = 2\pi/k$) such that $k \geq k_0 > 0$, the problem reads:

$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} + ik\, u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $i$ is the imaginary unit, $\mathbf{n}$ is the outer normal unit vector to $\partial\Omega$, and $g \in L^2(\partial\Omega)$ is given. The variational formulation of the problem reads as follows: find $u \in H^1(\Omega)$ such that, for all $v \in H^1(\Omega)$, it holds

$$\int_\Omega (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v})\, \mathrm{d}V + ik \int_{\partial\Omega} u \overline{v}\, \mathrm{d}S = \int_{\partial\Omega} g \overline{v}\, \mathrm{d}V. \tag{2}$$

By Fredholm alternative, problem (2) is well posed, and stability estimates are given by [33, Proposition 8.1.4].

In order to derive the PWDG method, we consider a shape-regular, quasi-uniform family of finite element partitions $\{\mathcal{T}_h\}$ of $\Omega$, possibly featuring hanging nodes. We assume, for simplicity, that the elements $K$ of $\mathcal{T}_h$ are convex polygons. We write $h$ for the mesh width of $\mathcal{T}_h$, i.e, $h = \max_{K\in\mathcal{T}_h} h_K$, with $h_K := \mathrm{diam}(K)$. We define the mesh skeleton $\mathcal{F}_h = \bigcup_{K\in\mathcal{T}_h} \partial K$, and set $\mathcal{F}_h^I = \mathcal{F}_h \setminus \partial\Omega$.

Given an element $K \in \mathcal{T}_h$, we denote by $PW_p(K)$ the plane wave space on $K$:

$$PW_p(K) = \{v \in L^2(K) :\ v(\mathbf{x}) = \sum_{j=1}^{p} \alpha_j \exp(ik\, \mathbf{d}_j \cdot (\mathbf{x} - \mathbf{x}_K)),\ \alpha_j \in \mathbb{C}\},$$

where $\mathbf{x}_K$ is the mass center of $K$, and $\mathbf{d}_j$, $|\mathbf{d}_j| = 1$, $1 \leq j \leq p$, are $p$ different directions. We assume these directions to be uniformly spaced. We define the plane wave discontinuous finite element spaces on $\mathcal{T}_h$ as follows:

$$PW_p(\mathcal{T}_h) = \{v_{hp} \in L^2(\Omega) :\ v_{hp}|_K \in PW_{p_K}(K)\ \forall K \in \mathcal{T}_h\}\ .$$

The functions in $PW_p(\mathcal{T}_h)$ possess the local *Trefftz property*

$$-\Delta v_{hp} - k^2 v_{hp} = 0 \qquad \forall v_{hp} \in PW_{p_K}(K)\ . \tag{3}$$

In this paper, we assume uniform local resolution, i.e., $p_K = p$ for all $K \in \mathcal{T}_h$, and we use the same directions $\mathbf{d}_j$, $1 \leq j \leq p$, in every element.

We briefly recall the derivation of the PWDG methods following [26]. We multiply the first equation of (1) by smooth test functions $v$ and integrate by parts on each $K \in \mathcal{T}_h$ obtaining

$$\int_K (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v})\, \mathrm{d}V - \int_{\partial K} \nabla u \cdot \mathbf{n}_K\, \overline{v}\, \mathrm{d}S = 0.$$

Then, we integrate by parts a second time, and replace $u$ and $v$ by discrete functions $u_{hp}, v_{hp} \in PW_p(\mathcal{T}_h)$, and the traces of $u$ and $\nabla u$ at $\partial K$ by *numerical fluxes* to be defined ($u \to \widehat{u}_{hp}$ $\nabla u \to ik\widehat{\boldsymbol{\sigma}}_{hp}$). Taking into account the Trefftz property (3) of the test functions $v_{hp}$, we obtain the elemental formulation of the PWDG method:

$$\int_{\partial K} \widehat{u}_{hp}\, \nabla \overline{v}_{hp} \cdot \mathbf{n}_K\, \mathrm{d}S - \int_{\partial K} ik\widehat{\boldsymbol{\sigma}}_{hp} \cdot \mathbf{n}_K\, \overline{v}_{hp}\, \mathrm{d}S = 0\ .$$

In order to complete the definition of the method, like in [23], we mimic the general form of the fluxes defined in [11]. Using the standard DG notation [5] for averages $\{\!\{\cdot\}\!\}$ and normal jumps $[\![\cdot]\!]_N$ across interelement boundaries, and denoting by $\nabla_h$ the elementwise application of $\nabla$, we set

$$ik\widehat{\boldsymbol{\sigma}}_{hp} = \begin{cases} \{\!\{\nabla_h u_{hp}\}\!\} - \alpha\, ik\, [\![u_{hp}]\!]_N & \text{on faces in } \mathcal{F}_h^I, \\ \nabla_h u_{hp} - (1-\delta)\,(\nabla_h u_{hp} + iku_{hp}\mathbf{n} - g_R\mathbf{n}) & \text{on faces on } \partial\Omega, \end{cases}$$

$$\widehat{u}_{hp} = \begin{cases} \{\!\{u_{hp}\}\!\} - \beta\,(ik)^{-1}[\![\nabla_h u_{hp}]\!]_N & \text{on faces in } \mathcal{F}_h^I, \\ u_{hp} - \delta\left((ik)^{-1}\nabla_h u_{hp}\cdot\mathbf{n} + u_{hp} - (ik)^{-1}g_R\right) & \text{on faces on } \partial\Omega, \end{cases}$$

where the so-called flux parameters $\alpha, \beta, \delta > 0$ here are assumed to be *constant*, with $\delta \le 1/2$ (taking $\alpha = \beta = \delta = 1/2$ gives the ultra weak variational formulation [13]).

**Remark 2.1.** *More general choices of flux parameters can be useful, for instance, for improving some convergence properties in the h-version of the method [22], or when non quasi-uniform meshes are used [25, 26].*

Adding over all elements (and multiplying by $-i$ ), we obtain the following formulation of the PWDG method: find $u_{hp} \in PW_p(\mathcal{T}_h)$ such that, for all $v_{hp} \in PW_p(\mathcal{T}_h)$,

$$\mathcal{A}_h(u_{hp}, v_{hp}) := \mathcal{B}_h(u_{hp}, v_{hp}) + \mathcal{S}_h(u_{hp}, v_{hp}) = \ell_h(v_{hp})\ , \tag{4}$$

where

$$\begin{aligned} \mathcal{B}_h(u,v) =& i\left[ -\int_{\mathcal{F}_h^I} \{\!\{u\}\!\}[\![\nabla_h \overline{v}]\!]_N \,\mathrm{d}S + \int_{\mathcal{F}_h^I} \{\!\{\nabla_h u\}\!\}\cdot[\![\overline{v}]\!]_N \,\mathrm{d}S \right. \\ & \left. - \int_{\partial\Omega} (1-\delta)\, u\, \nabla_h\overline{v}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{\partial\Omega} \delta\,\nabla_h u\cdot\boldsymbol{n}\,\overline{v}\,\mathrm{d}S \right], \\ \mathcal{S}_h(u,v) =& \int_{\mathcal{F}_h^I} \beta\, k^{-1}[\![\nabla_h u]\!]_N[\![\nabla_h\overline{v}]\!]_N\,\mathrm{d}S + \int_{\mathcal{F}_h^I} \alpha\, k\,[\![u]\!]_N\cdot[\![\overline{v}]\!]_N\,\mathrm{d}S \\ & + \int_{\partial\Omega} \delta\, k^{-1}(\nabla_h u\cdot\boldsymbol{n})\,(\nabla_h\overline{v}\cdot\boldsymbol{n})\,\mathrm{d}S + \int_{\partial\Omega}(1-\delta)\,k\,u\,\overline{v}\,\mathrm{d}S, \end{aligned}$$

and

$$\ell_h(v) = \int_{\partial\Omega} \delta\, k^{-1} g\,\nabla_h\overline{v}\cdot\boldsymbol{n}\,\mathrm{d}S - i\int_{\partial\Omega}(1-\delta)\,g\,\overline{v}\,\mathrm{d}S\ .$$

The PWDG method (4) is *unconditionally* well-posed and stable; see, e.g., [8,13,23,26], where error estimates were also derived. We only recall here coercivity and continuity properties of the sesquilinear form $\mathcal{A}_h(\cdot,\cdot)$ from [23,26]. To this aim, on the mesh $\mathcal{T}_h$, we define the *Trefftz space*

$$T(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) :\ \exists s > 0 \text{ s.t. } v \in H^{\frac{3}{2}+s}(\mathcal{T}_h) \text{ and } \Delta v + k^2 v = 0 \text{ in each } K \in \mathcal{T}_h \right\},$$

where $H^r(\mathcal{T}_h)$ is a shorthand notation for elementwise $H^r$-spaces on $\mathcal{T}_h$; the solution $u$ of problem (1) actually belongs to $T(\mathcal{T}_h)$.

The mesh-dependent quantity

$$|||v|||^2_{\mathcal{A}_h} := \mathrm{Re}[\mathcal{A}_h(v,v)] = \mathcal{S}_h(v,v) \tag{5}$$

defines a norm in $T(\mathcal{T}_h)$ (coercivity). Moreover, setting

$$|||v|||^2_{\mathcal{A}_h^+} = |||v|||^2_{\mathcal{A}_h} + k \left\| \beta^{-\frac{1}{2}} \{\!\{ v \}\!\} \right\|^2_{0,\mathcal{F}_h^I} + k^{-1} \left\| \alpha^{-\frac{1}{2}} \{\!\{ \nabla_h v \}\!\} \right\|^2_{0,\mathcal{F}_h^I} + k \left\| \delta^{-\frac{1}{2}} v \right\|^2_{0,\partial\Omega},$$

for all $v, w \in T(\mathcal{T}_h)$, we have (continuity)

$$|\mathcal{A}_h(v,w)| \leq 2 \|v\|_{\mathcal{A}_h^+} \|w\|_{\mathcal{A}_h}.$$

In order to give an idea of the error behavior, we report in Figure 1 the diagram of the $L^2$-error for increasing local number $p$ of plane waves, for two different values of the wavenumber $k$, for a test case with smooth analytical solution. After a preasymptotic region of amplitude proportional to $k$, the convergence is exponential, until onset of numerical instability, which due to the fact that, for high $p$, the local basis functions are close to be linearly dependent (this region is delayed for high $k$).

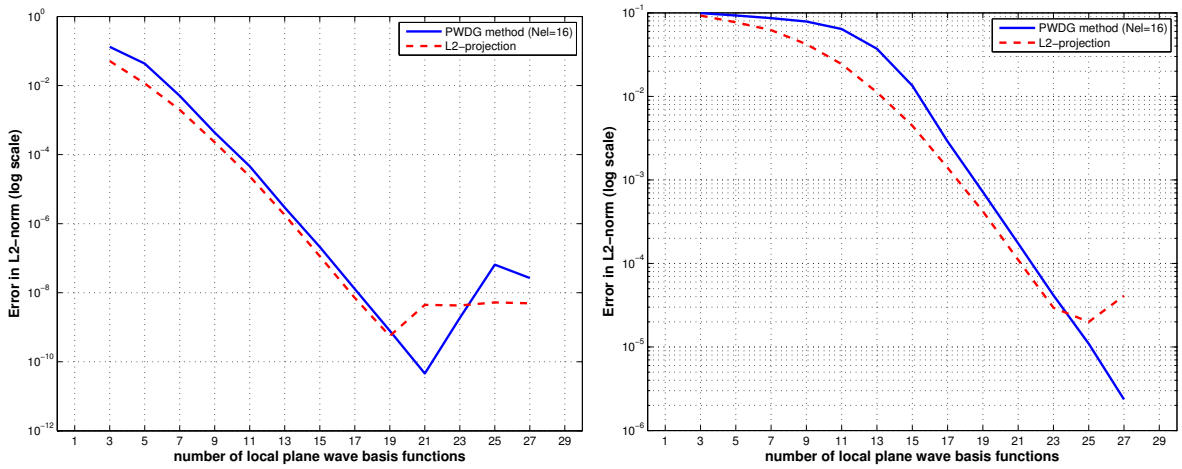


Figure 1: $L^2$-error of PWDG method versus the local number $p$ of plane waves (loglog scale). Test problem: $\Omega = (0,1)^2$, $g$ such that the analytical solution is given, in polar coordinates $\mathbf{x} = (r\cos\theta, r\sin\theta)$, by $u(\mathbf{x}) = J_1(kr)\cos\theta$, with wavenumber $k = 10$ (left) and $k = 40$ (right), on a uniform mesh of 16 squares.

Setting $N_h := \dim(PW_p(\mathcal{T}_h))$, the algebraic linear system associated with the PWDG method (4) on the mesh $\mathcal{T}_h$ is

$$A\mathbf{u} = \mathbf{b}, \tag{6}$$

where $A \in \mathbb{C}^{N_h \times N_h}$ is the matrix associated with the sesquilinear form $\mathcal{A}_h(\cdot,\cdot)$, and $\mathbf{b} \in \mathbb{C}^{N_h}$ is the vector associated with the functional $\ell_h(\cdot)$. The GMRES iteration counts to a given tolerance, for a fixed mesh and local number of plane wave directions $p$, increases with $k$, provided that the mesh is fine enough.

We consider the same test case as in the caption of Figure 1, fixing the mesh and varying $k$. In each test, $p$ is selected as the smallest value for which the $L^2$-error of the PWDG method (numerical solution computed by a direct solver) is $< 10^{-3}$. We have accelerated the GMRES choosing as preconditioner the incomplete LU factorization of $A$ with no fill-in and no pivoting (PGMRES). We report in Table 1 and in Table 2 the number of GMRES and PGMRES iterations needed to achieve convergence up to a (relative) tolerance of $10^{-8}$. The

| | $k = 10$ ($p = 9$) | $k = 20$ ($p = 13$) | $k = 30$ ($p = 17$) | $k = 40$ ($p = 19$) | $k = 50$ ($p = 23$) |
|---|---|---|---|---|---|
| syst size | 144 | 208 | 272 | 304 | 368 |
| nnz(A) | 5119 | 10747 | 18379 | 23014 | 33677 |
| GMRES | 114 | 172 | 233 | 186 | 233 |
| PGMRES | 17 | 16 | 15 | 15 | 15 |

Table 1: GMRES and PGMRES iteration counts: uniform mesh of 16 squares.

| | $k = 10$ ($p = 7$) | $k = 20$ ($p = 9$) | $k = 30$ ($p = 11$) | $k = 40$ ($p = 13$) | $k = 50$ ($p = 15$) |
|---|---|---|---|---|---|
| syst size | 448 | 576 | 704 | 832 | 960 |
| nnz(A) | 13861 | 23052 | 34461 | 48347 | 64342 |
| GMRES | 287 | 384 | 444 | 554 | 606 |
| PGMRES | 28 | 28 | 28 | 27 | 28 |

| | $k = 60$ ($p = 17$) | $k = 70$ ($p = 17$) | $k = 80$ ($p = 19$) | $k = 90$ ($p = 21$) | $k = 100$ ($p = 23$) |
|---|---|---|---|---|---|
| syst size | 1088 | 1088 | 1216 | 1344 | 1472 |
| nnz(A) | 82643 | 82802 | 103561 | 126460 | 151561 |
| GMRES | 716 | 359 | 371 | 456 | 477 |
| PGMRES | 27 | 27 | 27 | 27 | 27 |

Table 2: GMRES and PGMRES iteration counts: uniform mesh of 64 squares.

number of GMRES iterations remains of the same order of magnitude and that of PGMRES is constant, and much lower than that of GMRES.

While we leave the preconditioning of the $p$-version of PWDG to future investigation, we develop in the following sections Schwarz domain decomposition preconditioners for the $h$-version, addressing the issue of their scalability. Specific features of PWDG spaces (or of more general Trefftz spaces) could also be considered in order to improve the condition number of the linear systems. We refer to [29, 32] for results in this direction, but we do not elaborate on that here.

# 3 Domain decomposition preconditioners

To solve efficiently (6), we consider two-level Schwarz domain decomposition preconditioners. Let $\mathcal{T}_S$ be a partition of $\Omega$ into $N_S$ non-overlapping subdomains: $\overline{\Omega} = \cup_{j=1}^{N_S} \overline{\Omega}_j$, and let $\{\mathcal{T}_H\}_{H>0}$ and $\{\mathcal{T}_h\}_{h>0}$ be two families of coarse and fine partitions, respectively. We assume all the partitions to be shape-regular and quasi-uniform, and such that $\mathcal{T}_S \subseteq \mathcal{T}_H \subseteq \mathcal{T}_h$, i.e., each $\Omega_j \in \mathcal{T}_S$ is union of elements $D \in \mathcal{T}_H$, and in turn each element $D \in \mathcal{T}_H$ is union of elements $K \in \mathcal{T}_h$. From here on, we omit the index $p$ and set, for brevity, $PW_h = PW_p(\mathcal{T}_h)$; we recall that $N_h = \dim(PW_h)$.

## 3.1 Local and coarse spaces, prolongation and restriction operators

We define, for each subdomain $\Omega_j \in \mathcal{T}_S$, the local PWDG space $PW_h^j$ defined as

$$PW_h^j = \{v \in L^2(\Omega_j) : \ v|_K \in PW(K) \ \forall K \in \mathcal{T}_h\}, \qquad N_j := \dim(PW_h^j).$$

We denote by $\mathcal{R}_j^T : PW_h^j \hookrightarrow PW_h$ the inclusion operator (*prolongation operator*) and by $R_j^T \in \mathbb{R}^{N_h \times N_j}$ its matrix representation. The *restriction operator* $\mathcal{R}_j : PW_h \to PW_h^j$ is defined as the operator whose matrix representation is $\overline{R}_j \in \mathbb{R}^{N_j \times N_h}$, the conjugate transpose of $R_j^T$. Clearly, $R_j^T(r,s) = 1$ whenever the $r$-th basis function of $PW_h$ coincides with the $s$-th basis function of $PW_h^j$, and $R_j^T(r,s) = 0$ otherwise (and consequently $\overline{R}_j = R_j$).

We define the local sesquilinear forms $\mathcal{A}_h^j(\cdot,\cdot) : PW_h^j \times PW_h^j \to \mathbb{C}$ by

$$\mathcal{A}_h^j(u_j, v_j) = \mathcal{A}_h(\mathcal{R}_j^T u_j, \mathcal{R}_j^T v_j) \quad \forall\, u_j, v_j \in PW_h^j; \tag{7}$$

their associated matrices are

$$A_j = R_j A R_j^T \in \mathbb{C}^{N_j \times N_j},$$

We observe that the restriction of the formulation to each subdomain coincides with the PWDG formulation of the Helmholtz problem with impedance boundary condition on the subdomain boundary. Therefore, in the present situation, *exact* local solvers [4, 21] and *inexact* local solvers [2, 21] coincide.

Now the coarse mesh $\mathcal{T}_H$ comes into play. We define the coarse PWDG space as

$$PW_h^0 = PW(\mathcal{T}_H) = \{v \in L^2(\Omega) : \ v|_D \in PW(D) \ \forall D \in \mathcal{T}_D\}, \qquad N_0 := \dim(PW_h^0).$$

We also introduce the coarse space prolongation and restriction operators: $\mathcal{R}_0^T : PW_h^0 \hookrightarrow PW_h$, with associated matrix $R_0^T \in \mathbb{C}^{N_h \times N_0}$, and $\mathcal{R}_0 : PW_h \to PW_0$, which is the operator

whose associated matrix is $\overline{R}_0 \in \mathbb{C}^{N_0 \times N_h}$. We also define

$$A_0 = \overline{R}_0 A R_0^T \in \mathbb{C}^{N_0 \times N_0},$$

associated with the sesquilinear form

$$\mathcal{A}_h^0(u_0, v_0) = \mathcal{A}_h(\mathcal{R}_0^T u_0, \mathcal{R}_0^T v_0) \quad \forall\, u_0, v_0 \in PW_h^0. \tag{8}$$

We show how the operator $R_0^T$ is constructed. Let $D \in \mathcal{T}_H$ be an element of the coarse mesh, and let $\{K_r\}_{r=1}^{N_D}$ be the elements of the fine mesh $\mathcal{T}_h$ contained in $D$. Consider one of the $p$ basis function of $PW_h^0$ supported within $D$: $\Phi_\ell(\mathbf{x}) = exp(ik\,\mathbf{d}_\ell \cdot (\mathbf{x} - \mathbf{x}_D))$. Now, we express $\Phi_\ell(\mathbf{x})$ as linear combination of the basis functions of $PW_h$ supported within the elements $\{K_r\}_{r=1}^{N_D}$, i.e., we compute the coefficients $\alpha_j^r \in \mathbb{C}$ such that

$$\Phi_\ell(\mathbf{x}) = \sum_{r=1}^{N_D} \sum_{j=1}^{p} \alpha_j^r \phi_h^{j,r}(\mathbf{x}), \tag{9}$$

where $\phi_h^{j,r}(\mathbf{x}) = exp(ik\,\mathbf{d}_j \cdot (\mathbf{x} - \mathbf{x}_{K_r}))$ for $\mathbf{x} \in K_r$, and $\phi_h^{j,r}(\mathbf{x}) = 0$ outside $K_r$. Clearly, $\alpha_j^r = 0$ for every $j \neq \ell$, i.e., the basis functions of $PW_h$ with direction different from $\mathbf{d}_\ell$ do not enter the expression (9); therefore

$$\Phi_\ell(\mathbf{x}) = \sum_{r=1}^{N_D} \alpha_\ell^r \phi_h^{\ell,r}(\mathbf{x}),$$

with

$$\alpha_\ell^r = \exp(ik\,\mathbf{d}_\ell \cdot (\mathbf{x}_{K_r} - \mathbf{x}_D)).$$

Requiring that $\overline{R}_{0,D} R_{0,D}^T = I$, where $I$ is the $N_0 \times N_0$ identity matrix, assuming that the unknowns are ordered by an external loop over all elements, and an internal loop over the $p$ directions, the elemental contribution $R_{0,D}^T \in \mathbb{C}^{(r*p)\times p}$ to the matrix $R_0^T$ is given by

$$R_{0,D}^T((r-1)*p + \ell, \ell) = \frac{1}{\sqrt{N_D}} \alpha_\ell^r, \qquad 1 \leq \ell \leq p.$$

### 3.2 Schwarz operators

For $j = 0, \ldots, N_S$, we define $\widetilde{\mathcal{P}}_j : PW_h \to PW_h^j$ as the (unique) solution of the following problem

$$\mathcal{A}_h^j(\widetilde{\mathcal{P}}_j u, v_j) = \mathcal{A}_h(u, \mathcal{R}_j^T v_j) \quad \forall v_j \in PW_h^j.$$

Well-posedness of the local sesquilinear forms $\mathcal{A}_h^j(\cdot,\cdot)$, $0 \leq j \leq N_S$, defined in (7) and (8) follows from the fact that $\mathrm{Re}[\mathcal{A}_h^j(v,v)]$ are norms in the spaces $PW_h^j$. Therefore the projection operators $\widetilde{\mathcal{P}}_j$ are well defined. We define the operators $\mathcal{P}_j = \mathcal{R}_j^T \widetilde{\mathcal{P}}_j : PW_h \to PW_h$, $j = 0, \ldots, N_S$, denote by $P_j$ their matrix representations, and observe that

$$P_j = R_j^T A_j^{-1} \overline{R}_j A \quad 0 \leq j \leq N_S.$$

Since $A_j = \overline{R}_j A R_j^T$, $0 \leq j \leq N_S$, $P_j^2 = P_j$, i.e., the $P_j$'s are projectors.

We define the *additive* and *multiplicative* Schwarz operators as

$$\mathcal{Q}_{ad} := \sum_{j=0}^{N_S} \mathcal{P}_j, \quad \mathcal{Q}_{mu} := I - \mathcal{E}_{mu},$$

where the error propagation operator $\mathcal{E}_{mu}$ is defined as

$$\mathcal{E}_{mu} = (I - \mathcal{P}_{N_S}) \cdots (I - \mathcal{P}_1)(I - \mathcal{P}_0).$$

From the algebraic point of view, the Schwarz operators can be seen as preconditioned operators for the original operator $A$, and can be written as the product of a suitable preconditioner and $A$. For example, for the additive operator $Q_{ad}$ we have

$$Q_{ad} = P_{ad}^{-1} A, \qquad P_{ad}^{-1} = \sum_{j=0}^{N_S} R_j^T A_j^{-1} \overline{R}_j;$$

analogously we can write $Q_{mu}$ as $Q_{mu} = P_{mu}^{-1} A$.

Then, the preconditioned linear system of equations we are interested in solving is

$$Q\mathbf{u} = \mathbf{g}, \tag{10}$$

where $Q = P^{-1}A$ and $\mathbf{g} = P^{-1}\mathbf{b}$, with either $P = P_{ad}$ or $P = P_{mu}$.

# 4 Numerical results

We investigate the performance of our preconditioners when varying the fine and coarse grids, the number of subdomains $N_S$, as well as the wavenumber $k$ and the number of directions $p$. We use a uniform subdomain partition of $\Omega = (0,1)^2$ consisting of $N_S = 4, 16, 64$ square subdomains. We have solved our problem on a sequence of Cartesian grids with $1/h = 4, 8, 16, 32$; for the coarse grids we have also considered Cartesian partitions with $1/H = 2, 4, 8, 16, 32$.

We choose the impedance datum $g$ so that the analytical solution of problem (1) is given, in polar coordinates $\mathbf{x} = (r\cos\theta, r\sin\theta)$, by $u(\mathbf{x}) = J_1(kr)\cos(\theta)$, where $J_1(\cdot)$ is the Bessel function of the first type. The flux parameters have been chosen as $\alpha = \beta = \delta = \frac{1}{2}$, cf. Section 2.

Throughout this section, the linear systems of equations have been solved by GMRES with a (relative) tolerance set equal to $10^{-6}$.

We first investigate whether the additive and multiplicative Schwarz preconditioners are scalable, i.e., the iteration counts needed to reduce the residual up to a (user defined) tolerance are independent of the number of subdomains. In Table 3 and Table 4 we report the iteration counts for $k = 30$, $p = 15$ and $N_S = 4, 16, 64$ computed with the additive and multiplicative preconditioners, respectively. The proposed preconditioners seem to be asymptotically scalable, indeed for $N_S \geq 16$ the number of iterations seems to be quite independent of the number of subdomains. Moreover, in all the cases the preconditioners are effective, as

confirmed by a comparison with the iterations counts needed to solve the unpreconditioned system, cf. last line of Table 3, and, as expected, the multiplicative preconditioner performs much better than the additive one. In Tables 3 to 6, the symbol "-" refers to the case where the hypothesis $\mathcal{T}_S \subseteq \mathcal{T}_H \subseteq \mathcal{T}_h$ is not satisfied, and therefore the construction of the preconditioner is meaningless. The lower diagonals reported in Table 3 (and also in the tables below) correspond to the limit case $h = H$. In this case, the coarse component of the preconditioner

| $H^{-1}$ \ $h^{-1}$ | $N_S = 4$ | | | | $N_S = 16$ | | | | $N_S = 64$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 |
| 2 | 22 | 35 | 48 | 64 | - | - | - | - | - | - | - | - |
| 4 | 13 | 29 | 41 | 56 | 15 | 34 | 46 | 63 | - | - | - | - |
| 8 | - | 16 | 33 | 46 | - | 17 | 37 | 53 | - | 19 | 41 | 56 |
| 16 | - | - | 16 | 39 | - | - | 17 | 43 | - | - | 17 | 46 |
| iter($A$) | 134 | 670 | 1510 | 3552 | 134 | 670 | 1510 | 3552 | 134 | 670 | 1510 | 3552 |

Table 3: Additive preconditioner: GMRES iteration counts. Wavenumber $k = 30$, number of directions $p = 15$.

| $H^{-1}$ \ $h^{-1}$ | $N_S = 4$ | | | | $N_S = 16$ | | | | $N_S = 64$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 |
| 2 | 12 | 19 | 25 | 34 | - | - | - | - | - | - | - | - |
| 4 | 2 | 14 | 21 | 30 | 2 | 17 | 25 | 34 | - | - | - | - |
| 8 | - | 2 | 16 | 24 | - | 2 | 17 | 26 | - | 2 | 21 | 29 |
| 16 | - | - | 2 | 18 | - | - | 2 | 20 | - | - | 3 | 21 |
| iter($A$) | 134 | 670 | 1510 | 3552 | 134 | 670 | 1510 | 3552 | 134 | 670 | 1510 | 3552 |

Table 4: Multiplicative preconditioner: GMRES iteration counts. Wavenumber $k = 30$, number of directions $p = 15$.

is an exact solver for the original linear system (6), indeed, if $A_0 = A$, then $R_0 = I$ and the operator $P_0$ becomes $P_0 = R_0^T A^{-1} \overline{R}_0 A = I$. Thus, in principle, we should obtain convergence in one iteration, and the fact that this does not happen indicates that the local solutions "spoil" the result (see also [2]). The results for $h = H$ are reported in order to give an idea of the performance of the preconditioners also in such limit case.

We have repeated the same set of experiments taking the wavenumber $k = 50$, cf. Table 5 and Table 6 for the additive and multiplicative preconditioners, respectively. We observe that the iteration counts of the preconditioned system do not seem to vary significantly as the wavenumber increases, whereas the iteration counts of the unpreconditioned one increase, at least when $1/h$ becomes sufficiently large. Moreover, also in this case the multiplicative preconditioner outperforms the additive one.

We next address the performance of our preconditioners when varying the number of directions $p$, fixing, for the sake of simplicity, $N_S = 16$ and $1/h = 16$. In Table 7 and Table

| $H^{-1}$ \ $h^{-1}$ | $N_S = 4$ | | | | $N_S = 16$ | | | | $N_S = 64$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 |
| 2 | 16 | 29 | 42 | 55 | - | - | - | - | - | - | - | - |
| 4 | 10 | 27 | 42 | 57 | 15 | 37 | 52 | 70 | - | - | - | - |
| 8 | - | 14 | 31 | 43 | - | 16 | 32 | 45 | - | 19 | 37 | 50 |
| 16 | - | - | 17 | 37 | - | - | 18 | 37 | - | - | 19 | 40 |
| iter($A$) | 33 | 508 | 1977 | 4461 | 33 | 508 | 1977 | 4461 | 33 | 508 | 1977 | 4461 |

Table 5: Additive preconditioner: GMRES iteration counts. Wavenumber $k = 50$, number of directions $p = 15$.

| $H^{-1}$ \ $h^{-1}$ | $N_S = 4$ | | | | $N_S = 16$ | | | | $N_S = 64$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 | 4 | 8 | 16 | 32 |
| 2 | 8 | 15 | 23 | 30 | - | - | - | - | - | - | - | - |
| 4 | 2 | 14 | 22 | 30 | 2 | 18 | 27 | 36 | - | - | - | - |
| 8 | - | 2 | 15 | 22 | - | 2 | 16 | 24 | - | 2 | 20 | 37 |
| 16 | - | - | 1 | 16 | - | - | 2 | 17 | - | - | 2 | 19 |
| iter($A$) | 33 | 508 | 1977 | 4461 | 33 | 508 | 1977 | 4461 | 33 | 508 | 1977 | 4461 |

Table 6: Multiplicative preconditioner: GMRES iteration counts. Wavenumber $k = 50$, number of directions $p = 15$.

8 we report the GMRES iteration counts for $p = 13, 15, 17, 19$ and $k = 30, 40, 50$ computed with the additive and multiplicative preconditioners, respectively. The last row of Table 7 and Table 8 shows the corresponding iteration counts needed to solve the unpreconditioned systems. From the numerical results, we can observe that the iteration counts needed to solve

| | $k = 30$ | | | | $k = 40$ | | | | $k = 50$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H^{-1}$ \ $p$ | 13 | 15 | 17 | 19 | 13 | 15 | 17 | 19 | 13 | 15 | 17 | 19 |
| 4 | 47 | 46 | 47 | 49 | 50 | 48 | 48 | 47 | 49 | 52 | 52 | 50 |
| 8 | 35 | 37 | 39 | 41 | 33 | 35 | 36 | 37 | 32 | 32 | 34 | 34 |
| iter($A$) | 1510 | 1510 | 1511 | 1511 | 1826 | 1825 | 1824 | 1825 | 1883 | 1977 | 1973 | 1973 |

Table 7: Additive preconditioner: GMRES iteration counts. Wavenumber $k = 30, 40, 50$, number of directions $p = 13, 15, 17, 19$ ($N_S = 16$ and $1/h = 16$).

| | $k = 30$ | | | | $k = 40$ | | | | $k = 50$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H^{-1}$ \ $p$ | 13 | 15 | 17 | 19 | 13 | 15 | 17 | 19 | 13 | 15 | 17 | 19 |
| 4 | 25 | 25 | 26 | 26 | 25 | 25 | 26 | 26 | 27 | 27 | 27 | 27 |
| 8 | 17 | 17 | 19 | 20 | 16 | 17 | 18 | 19 | 16 | 16 | 17 | 18 |
| iter($A$) | 1510 | 1510 | 1511 | 1511 | 1826 | 1825 | 1824 | 1825 | 1883 | 1977 | 1973 | 1973 |

Table 8: Multiplicative preconditioner: GMRES iteration counts. Wavenumber $k = 30, 40, 50$, number of directions $p = 13, 15, 17, 19$ ($N_S = 16$ and $1/h = 16$).

the preconditioned systems, both with the additive and the multiplicative preconditioners, seem to be fairly independent of $k$, and that these preconditioners seem to be very effective in accelerating GMRES convergence.

# 5 The issue of GMRES convergence

In the following, we recall the GMRES convergence theory developed in [15, 17, 36] which provides sufficient conditions for non-stagnation of GMRES, i.e., the iterative method makes some progress in reducing the residual at each iteration step, and establishes upper bounds on the residual norm.

Given the matrix $Q \in \mathbb{C}^{N_h \times N_h}$, we define its associated field of values as

$$F(Q) = \left\{ \frac{\overline{\mathbf{x}}^T Q \mathbf{x}}{\overline{\mathbf{x}}^T \mathbf{x}}, \quad \mathbf{0} \neq \mathbf{x} \in \mathbb{C}^{N_h} \right\},$$

and denote by $\nu(F(Q))$ the distance of $F(Q)$ from the origin. The theory proposed by [15,36] states that the $\kappa$-th residual $\mathbf{r}_\kappa$ of GMRES satisfies

$$\frac{\|\mathbf{r}_\kappa\|}{\|\mathbf{r}_0\|} \leq \left(1 - \nu(F(Q))\, \nu(F(Q^{-1}))\right)^{\kappa/2}. \tag{11}$$

Denote by $H(Q)$ the hermitian part of $Q$, i.e., $H(Q) = \frac{Q+\overline{Q}^T}{2}$. If $H(Q)$ is positive definite, then $\nu(F(Q)) \geq \lambda_{\min}(H(Q))$, the minimum eigenvalue of $H(Q)$. In our case, if $A$ is the coefficient matrix of the unpreconditioned system, $H(A)$ is positive definite, since it is associated with the bilinear form $\mathcal{S}_h(\cdot,\cdot)$ (see (4)), which is a scalar product in $PW_p(\mathcal{T}_h)$. If we denote by $Q$ the left/right preconditioned matrix, i.e., $Q = P^{-1/2}A(\overline{P}^T)^{-1/2}$, with either $P = P_{ad}$ or $P = P_{mu}$, then $H(Q)$ is also positive definite. Moreover, one can bound $\nu(F(Q^{-1}))$ from below by

$$\nu(F(Q^{-1})) \geq \frac{\lambda_{\min}(H(Q))}{\|Q\|^2},$$

where $\|\cdot\|$ is the natural (complex) Euclidean matrix norm (see [30]), and write a weaker but more practical version of the bound (11):

$$\frac{\|\mathbf{r}_\kappa\|}{\|\mathbf{r}_0\|} \leq \left(1 - \frac{\lambda^2_{\min}(H(Q))}{\|Q\|^2}\right)^{\kappa/2},$$

which was firstly derived in [16, 17].

We have performed some experiments on the test problem described in Section 4 with $k = 30$ and number of plane wave directions per element $p = 11, 13, 15$, on the grid with $1/h = 16$. We have compared the values of $\lambda_{\min}(H(R))$ and $\lambda_{\min}(H(R^{-1}))$ for the original matrix ($R = A$) and for the left/right preconditioned matrix $Q = P^{-1/2}A(\overline{P}^T)^{-1/2}$ ($R = Q$), with $P = P_{ad}$ constructed with $N_S = 4$ subdomains and a coarse grid with $1/H = 2$.

The results reported in Table 9 show that, while $\lambda_{\min}(H(R^{-1}))$ (which is a lower bound for $\nu(F(R^{-1}))$) is essentially the same for the unpreconditioned and the preconditioned matrices, and for all the considered values of $p$, $\lambda_{\min}(H(R))$ (which is a lower bound for $\nu(F(R))$) is definitely larger for the preconditioned matrix (and it is almost uniform in $p$). A similar

| | $p = 11$ | $p = 13$ | $p = 15$ |
|---|---|---|---|
| $\lambda_{\min}(H(A))$ | $1.3338 \cdot 10^{-5}$ | $1.0956 \cdot 10^{-7}$ | $1.0079 \cdot 10^{-9}$ |
| $\lambda_{\min}(H(A^{-1}))$ | $1.7385 \cdot 10^{-2}$ | $1.4711 \cdot 10^{-2}$ | $1.2749 \cdot 10^{-2}$ |
| $\lambda_{\min}(H(Q))$ | $5.8730 \cdot 10^{-2}$ | $4.4976 \cdot 10^{-2}$ | $3.5210 \cdot 10^{-2}$ |
| $\lambda_{\min}(H(Q^{-1}))$ | $3.1358 \cdot 10^{-2}$ | $3.1919 \cdot 10^{-2}$ | $3.1977 \cdot 10^{-2}$ |

Table 9: Values of $\lambda_{\min}(H(R))$ and $\lambda_{\min}(H(R^{-1}))$ (which are lower bound for $\nu(F(R))$ and $\nu(F(R^{-1}))$, respectively) for $R = A$ (unpreconditioned matrix) and $R = Q = P^{-1/2}A(\overline{P}^T)^{-1/2}$ (preconditioned matrix). Wavenumber $k = 30$, Cartesian mesh with $1/h = 16$; $P = P_{ad}$ constructed with $N_S = 4$ subdomains and a coarse grid with $1/H = 2$.

situation is observed when varying $k$ ($k = 30, 40, 50$); the results for $p = 15$, $1/h = 16$, $N_S = 4$ subdomains and $1/H = 2$ are reported in Table 10. Theoretical estimates of these quantities are under investigation.

| | $k = 30$ | $k = 40$ | $k = 50$ |
|---|---|---|---|
| $\lambda_{\min}(H(A))$ | $1.0079 \cdot 10^{-9}$ | $7.0787 \cdot 10^{-8}$ | $1.8717 \cdot 10^{-6}$ |
| $\lambda_{\min}(H(A^{-1}))$ | $1.2749 \cdot 10^{-2}$ | $1.2495 \cdot 10^{-2}$ | $1.3699 \cdot 10^{-2}$ |
| $\lambda_{\min}(H(Q))$ | $3.5210 \cdot 10^{-2}$ | $6.1622 \cdot 10^{-2}$ | $3.6636 \cdot 10^{-2}$ |
| $\lambda_{\min}(H(Q^{-1}))$ | $3.1977 \cdot 10^{-2}$ | $3.0694 \cdot 10^{-2}$ | $2.9770 \cdot 10^{-2}$ |

Table 10: Values of $\lambda_{\min}(H(R))$ and $\lambda_{\min}(H(R^{-1}))$ (which are lower bound for $\nu(F(R))$ and $\nu(F(R^{-1}))$, respectively) for $R = A$ (unpreconditioned matrix) and $R = Q = P^{-1/2}A(\overline{P}^T)^{-1/2}$ (preconditioned matrix). Cartesian mesh with $1/h = 16$, number of plane wave directions per element $p = 15$; $P = P_{ad}$ constructed with $N_S = 4$ subdomains and a coarse grid with $1/H = 2$.

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