MOX–Report No. 56/2014

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A uniform additive Schwarz preconditioner for the *hp*-version of Discontinuous Galerkin approximations of elliptic problems

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December 2, 2014

Abstract

In this paper we design and analyze a uniform preconditioner for a class of high order Discontinuous Galerkin schemes. The preconditioner is based on a space splitting involving the high order conforming subspace and results from the interpretation of the problem as a nearly-singular problem. We show that the proposed preconditioner exhibits spectral bounds that are uniform with respect to the discretization parameters, *i.e.*, the mesh size, the polynomial degree and the penalization coefficient. The theoretical estimates obtained are supported by several numerical simulations.

1   Introduction

In the last years, the design of efficient solution techniques for the system of equations arising from Discontinuous Galerkin (DG) discretizations of elliptic partial differential equations has become an increasingly active field of research. On the one hand, DG methods are characterized by a great versatility in treating a variety of problems and handling, for instance, non-conforming grids and *hp*-adaptive strategies. On the other hand, the main drawback of DG methods is the larger number of degrees of freedom compared to (standard) conforming discretizations. In this respect, the case of high order DG schemes is particularly representative, since the corresponding linear system of equations is very ill-conditioned: it can be proved that, for elliptic problems, the spectral condition number of the resulting stiffness matrix grows like $h^{-2}p^4$, h and p being the granularity of the underlying mesh and the polynomial approximation degree, respectively, cf. [AH11]. As a consequence, the design of effective tools for the solution of the linear system of equations arising from high order DG discretizations becomes particularly challenging.

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In the context of elliptic problems, Schwarz methods for low order DG schemes have been studied in [FK01], where overlapping and non-overlapping domain decomposition preconditioners are considered, and bounds of $O(H/\delta)$ and $O(H/h)$, respectively, are obtained for the condition number of the preconditioned operator. Here H, h and δ stand for the granularity of the coarse and fine grids and the size of the overlap, respectively. Further extensions including inexact local solvers, and the extension of two-level Schwarz methods to advection-diffusion and fourth-order problems can be found in [LT03, FK05, AA07, AA08, AA09, DS10, BBS11, AABS12]. In the field of Balancing Domain Decomposition (BDD) methods, a number of results exist in literature: exploiting a Neumann-Neumann type method, in [DGS07, DGS08] a conforming discretization is used on each subdomain combined with interior penalty method on non-conforming boundaries, thus obtaining a bound for the condition number of the resulting preconditioner of $O((1-\log(H/h))^2)$. In [DD12], using the unified framework of [ABCM02] a BDDC method is designed and analyzed for a wide range of DG methods. The *auxiliary space method* (ASM) (see *e.g.*, [Nep92, GO95b, Xu96, HX07]) is employed in the context of h-version DG methods to develop, for instance, the two-level preconditioners of [DLVZ06] and the multilevel method of [BPD08]. In both cases a stable splitting for the linear DG space is provided by a decomposition consisting of a conforming subspace and a correction, thus obtaining uniformly bounded preconditioners with respect to the mesh size.

All the previous results focus on low order (*i.e.*, linear) DG methods. In the context of preconditioning high order DG methods we mention [AH11], where a class of non-overlapping Schwarz preconditioners is introduced, and [AABP14], where a quasi-optimal (with respect to h and p) preconditioner is designed in the framework of substructuring methods for hp-Nitsche-type discretizations. A study of a BDDC scheme in the case of hp-spectral DG methods is addressed in [CPP14], where the DG framework is reduced to the conforming one via the ASM. The ASM framework is employed also in [BCPCD], where the high order conforming space is employed as auxiliary subspace, and a uniform multilevel preconditioner is designed for hp-DG spectral element methods in the case of locally varying polynomial degree. To the best of our knowledge, this preconditioner is the only uniform preconditioner designed for high order DG discretizations. We note that, in the framework of high order methods, the decomposition involving a conforming subspace was already employed in the case of a-posteriori error analysis, see for example [HSW07, BE07, ZGHS11]. In this paper, we address the issue of preconditioning high order DG methods by exploiting this kind of space splitting based on a high order conforming space and a correction. However, in our case the space decomposition is suggested by the interpretation of the high order DG scheme in terms of a nearly-singular problem, cf. [LWXZ07]. Even though the space decomposition is similar to that of [BCPCD], the preconditioner and the analysis we present differs considerably since here we employ the abstract framework of subspace correction methods provided by [XZ02]. More precisely, we are able to show that a simple pointwise Jacobi method paired with an overlapping additive Schwarz method for the conforming subspace, gives uniform convergence with respect to all the discretization parameters, *i.e.*, the mesh size, the polynomial order and the penalization coefficient appearing in the DG bilinear form.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem and the corresponding discretization through a class of symmetric DG schemes. Section 3 is devoted to few auxiliary results regarding the Gauss-Legendre-Lobatto nodes, whose properties are fundamental to prove the stability of the space decomposition proposed in Section 4. The analysis of the preconditioner is presented in Section 5 and the theoretical results are supported by the numerical simulations of Section 6.

2 Model problem and hp-DG discretization

In this section we introduce the model problem and its discretization through several Discontinuous Galerkin schemes, see also [ABCM02].

Throughout the paper, we will employ the notation $x \lesssim y$ and $x \gtrsim y$ to denote the inequalities $x \leq Cy$ and $x \geq Cy$, respectively, C being a positive constant independent of the discretization parameters. Moreover, $x \approx y$ will mean that there exist constants $C_1, C_2 > 0$ such that $C_1 y \leq x \leq C_2 y$. When needed, the constants will be written explicitly.

Given a convex polygonal/polyhedral domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$, and $f \in L^2(\Omega)$, we consider the following weak formulation of the Poisson problem with homogeneous Dirichlet boundary conditions: find $u \in V := H_0^1(\Omega)$, such that

$$\int_\Omega \nabla u \cdot \nabla v \ dx = \int_\Omega fv \ dx \qquad \forall v \in V.$$

Let $\mathcal{T}_h$ denote a conforming quasi-uniform partition of Ω into shape-regular elements κ of diameter h_κ, and set $h := \max_{\kappa \in \mathcal{T}_h} h_\kappa$. We also assume that each element $\kappa \in \mathcal{T}_h$ results from the mapping, through an affine operator F_κ, of a reference element $\hat{\kappa}$, which is the open, unit d-hypercube in $\mathbb{R}^d$, $d = 2, 3$.

We denote by $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ the set of internal and boundary faces (for $d = 2$ "face" means "edge") of $\mathcal{T}_h$, respectively, and define $\mathcal{F}_h := \mathcal{F}_h^I \cup \mathcal{F}_h^B$. We associate to any $F \in \mathcal{F}_h$ a unit vector $\mathbf{n}_F$ orthogonal to the face itself and also denote by $\mathbf{n}_{F,\kappa}$ the outward normal vector to $F \subset \partial\kappa$ with respect to κ. We observe that for any $F \in \mathcal{F}_h^B$, $\mathbf{n}_{F,\kappa} = \mathbf{n}_F$, since F belongs to a unique element. For any $F \in \mathcal{F}_h^I$, we assume $\overline{F} = \partial\overline{\kappa^+} \cap \partial\overline{\kappa^-}$, where

$$\kappa^+ := \{\kappa \in \mathcal{T}_h : F \subset \partial\kappa, \ \mathbf{n}_F \cdot \mathbf{n}_{F,\kappa} > 0\},$$
$$\kappa^- := \{\kappa \in \mathcal{T}_h : F \subset \partial\kappa, \ \mathbf{n}_F \cdot \mathbf{n}_{F,\kappa} < 0\}.$$

For regular enough vector-valued and scalar functions $\boldsymbol{\tau}$ and v, we denote by $\boldsymbol{\tau}^\pm$ and $v^\pm$ the corresponding traces taken from the interior of $\kappa^\pm$, respectively, and define the *jumps* and *averages* across the face $F \in \mathcal{F}_h^I$ as follows

$$[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}^+ \cdot \mathbf{n}_{F,\kappa^+} + \boldsymbol{\tau}^- \cdot \mathbf{n}_{F,\kappa^-}, \qquad \{\!\{\boldsymbol{\tau}\}\!\} := \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2},$$

$$[\![v]\!] := v^+ \mathbf{n}_{F,\kappa^+} + v^- \mathbf{n}_{F,\kappa^-}, \qquad \{\!\{v\}\!\} := \frac{v^+ + v^-}{2},$$

For $F \in \mathcal{F}_h^B$, the previous definitions reduce to $[\![v]\!] := v\mathbf{n}_F$ and $\{\!\{\boldsymbol{\tau}\}\!\} := \boldsymbol{\tau}$.

We now associate to the partition $\mathcal{T}_h$, the hp-Discontinuous Galerkin finite element space V_{hp} defined as

$$V_{hp} := \{v \in L^2(\Omega) : v \circ \mathsf{F}_\kappa \in \mathbb{Q}^p(\hat{\kappa}) \quad \forall\kappa \in \mathcal{T}_h\},$$

with $\mathbb{Q}^p$ denoting the space of all tensor-product polynomials on $\hat{\kappa}$ of degree $p > 1$ in each coordinate

direction. We define the lifting operators $\mathcal{R}(\boldsymbol{\tau}) := \sum_{F \in \mathcal{F}_h} r_F(\boldsymbol{\tau})$ and $\mathcal{L}(v) := \sum_{F \in \mathcal{F}_h^I} l_F(v)$, where

$$r_F : [L^2(F)]^d \to [V_{hp}]^d, \quad \int_\Omega r_F(\boldsymbol{\tau}) \cdot \boldsymbol{\eta} \; dx := -\int_F \boldsymbol{\tau} \cdot \{\!\!\{\boldsymbol{\eta}\}\!\!\} \; ds \quad \forall F \in \mathcal{F}_h.$$

$$l_F : L^2(F) \to [V_{hp}]^d, \quad \int_\Omega l_F(v) \cdot \boldsymbol{\eta} \; dx := -\int_F v [\![\boldsymbol{\eta}]\!] \; ds \qquad \forall F \in \mathcal{F}_h^I,$$

for any $\boldsymbol{\eta} \in [V_{hp}]^d$.

We then introduce the DG finite element formulation: find $u \in V_{hp}$ such that

$$\mathcal{A}(u, v) = \int_\Omega fv \; dx \quad \forall v \in V_{hp}, \tag{1}$$

with $\mathcal{A}(\cdot, \cdot) : V_{hp} \times V_{hp} \to \mathbb{R}$ defined as

$$\mathcal{A}(u, v) := \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u \cdot \nabla v \; dx + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u \cdot (\mathcal{R}([\![v]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![v]\!])) \; dx$$
$$+ \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\mathcal{R}([\![u]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![u]\!])) \cdot \nabla v \; dx + \sum_{F \in \mathcal{F}_h} \int_F \sigma [\![u]\!] \cdot [\![v]\!] ds$$
$$+ \theta \int_\Omega (\mathcal{R}([\![u]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![u]\!])) \cdot (\mathcal{R}([\![v]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![v]\!])) \; dx,$$

where $\theta = 0$ for the SIPG method of [Arn82] and $\theta = 1$ for the LDG method of [CS98]. With regard to the vector function $\boldsymbol{\beta}$, we have $\boldsymbol{\beta} = \mathbf{0}$ for the SIPG method, while $\boldsymbol{\beta} \in \mathbb{R}^d$ is a uniformly bounded (and possibly null) vector for the LDG method. The penalization function $\sigma \in L^\infty(\mathcal{F}_h)$ is defined as

$$\sigma|_F := \alpha \frac{p^2}{\min(h_{\kappa^+}, h_{\kappa^-})}, \quad F \in \mathcal{F}_h^I, \qquad \qquad \sigma|_F := \alpha \frac{p^2}{h_\kappa} \quad F \in \mathcal{F}_h^B,$$

being $\alpha \geq 1$ and $h_{\kappa^\pm}$ the diameters of the neighboring elements $\kappa^\pm \in \mathcal{T}_h$ sharing the face $F \in \mathcal{F}_h^I$.

We endow the DG space V_{hp} with the following norm

$$\|v\|_{DG}^2 := \sum_{\kappa \in \mathcal{T}_h} \|\nabla v\|_{L^2(\kappa)}^2 + \sum_{F \in \mathcal{F}_h} \|\sigma^{1/2}[\![v]\!]\|_{L^2(F)}^2,$$

and state the following result, cf. [HSS02, PS02, AH11, SW10].

Lemma 1. *The following results hold*

$$\mathcal{A}(u, v) \lesssim \|u\|_{DG} \|v\|_{DG} \quad \forall u, v \in V_{hp},$$
$$\mathcal{A}(u, u) \gtrsim \|u\|_{DG}^2 \qquad \forall u \in V_{hp}. \tag{2}$$

For the SIPG formulation coercivity holds provided the penalization coefficient α is chosen large enough.

From Lemma 1 and using the Poincarè inequality for piecewise H^1 functions of [Bre03], the following spectral bounds hold, cf. [AH11].

Lemma 2. *For any $u \in V_{hp}$ it holds that*

$$\sum_{\kappa \in \mathcal{T}_h} \|u\|_{L^2(\kappa)}^2 \lesssim \mathcal{A}(u, u) \lesssim \sum_{\kappa \in \mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \|u\|_{L^2(\kappa)}^2. \tag{3}$$

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3   Gauss-Legendre-Lobatto nodes and quadrature rule

In this section we provide some details regarding the choice of the basis functions spanning the space V_{hp} and the corresponding degrees of freedom. On the reference d-hypercube $[-1,1]^d$, we choose the basis obtained by the tensor product of the one-dimensional Lagrange polynomials on the reference interval $[-1,1]$, based on Gauss-Legendre-Lobatto (GLL) nodes. We denote by $\mathcal{N}_I(\hat{\kappa})$ ($\mathcal{N}_B(\hat{\kappa})$) the set of interior (boundary) nodes of $\hat{\kappa}$, and define $\mathcal{N}(\hat{\kappa}) := \mathcal{N}_I(\hat{\kappa}) \cup \mathcal{N}_B(\hat{\kappa})$. The analogous sets in the physical frame are denoted by $\mathcal{N}_I(\kappa)$, $\mathcal{N}_B(\kappa)$ and $\mathcal{N}(\kappa)$, where any $\xi_p \in \mathcal{N}(\kappa)$ is obtained by applying the linear mapping $\mathsf{F}_\kappa : \hat{\kappa} \to \kappa$ to the corresponding $\hat{\xi}_p \in \mathcal{N}(\hat{\kappa})$. The choice of GLL points as degrees of freedom allow us to exploit the properties of the associated quadrature rule. We recall that, given $(p+1)^d$ GLL quadrature nodes $\{\hat{\xi}_p\}$ and weights $\{\hat{\mathsf{w}}_{\xi_p}\}$, we have

$$\sum_{\hat{\xi}_p \in \mathcal{N}(\hat{\kappa})} v(\hat{\xi}_p)\hat{\mathsf{w}}_{\xi_p} = \int_{\hat{\kappa}} v \, dx \qquad \forall v \in \mathbb{Q}^{2p-1}(\hat{\kappa}),$$

which implies that

$$\sum_{\hat{\xi}_p \in \mathcal{N}(\hat{\kappa})} v(\hat{\xi}_p)^2 \hat{\mathsf{w}}_{\xi_p} \neq \int_{\hat{\kappa}} v^2 \, dx \qquad \forall v \in \mathbb{Q}^p(\hat{\kappa}).$$

However, by defining, for $v \in \mathbb{Q}^p(\hat{\kappa})$, the following norm

$$\|v\|_{0,p,\hat{\kappa}}^2 := \sum_{\xi_p \in \mathcal{N}(\hat{\kappa})} v(\hat{\xi}_p)^2 \hat{\mathsf{w}}_{\xi_p},$$

it can be proved that

$$\|v\|_{0,p,\hat{\kappa}}^2 \approx \|v\|_{L^2(\hat{\kappa})}^2, \tag{4}$$

cf. [CHQZ06, Section 5.3]. The same result holds for the physical frame κ, *i.e.*, $\|v\|_{0,p,\kappa}^2 \approx \|v\|_{L^2(\kappa)}^2$.

Considering the Lagrange basis $\{\phi_{\xi_p}\}$, $\xi_p \in \bigcup_{\kappa \in \mathcal{T}_h} \mathcal{N}(\kappa)$, we can write any $v \in V_{hp}$ as

$$v = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}(\kappa)} v(\xi_p)\phi_{\xi_p} = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}(\kappa)} v^{\xi_p}, \tag{5}$$

where we note that $v^{\xi_p} = v(\xi_p)\phi_{\xi_p}$.

Lemma 3. *For any $v \in V_{hp}$, given the decomposition (5), the following equivalence holds*

$$\|v\|_{L^2(\Omega)}^2 \approx \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}(\kappa)} \|v^{\xi_p}\|_{L^2(\kappa)}^2.$$

Proof. The proof can be restricted to the case of a single element $\kappa \in \mathcal{T}_h$. We write $v \in V_{hp}$ as in (5), and observe that

$$\|v^{\xi_p}\|_{0,p,\kappa}^2 = \sum_{\xi_p' \in \mathcal{N}(\kappa)} v^{\xi_p}(\xi_p')^2 \mathsf{w}_{\xi_p} = v^{\xi_p}(\xi_p)^2 \mathsf{w}_{\xi_p},$$

hence, by (4),

$$\|v\|_{L^2(\kappa)}^2 \approx \sum_{\xi_p \in \mathcal{N}(\kappa)} v(\xi_p)^2 \mathsf{w}_{\xi_p} = \sum_{\xi_p \in \mathcal{N}(\kappa)} v^{\xi_p}(\xi_p)^2 \mathsf{w}_{\xi_p}$$

$$= \sum_{\xi_p \in \mathcal{N}(\kappa)} \|v^{\xi_p}\|_{0,p,\kappa}^2 \approx \sum_{\xi_p \in \mathcal{N}(\kappa)} \|v^{\xi_p}\|_{L^2(\kappa)}^2,$$

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and the thesis follows summing over all $\kappa \in \mathcal{T}_h$. □

4 Space decomposition for hp-DG methods

The design of our preconditioner is based on a two-stage space decomposition: we first split the high order DG space as $V_{hp} = V_{hp}^B + V_{hp}^C$, with V_{hp}^B denoting a proper subspace of V_{hp}, to be defined later, and V_{hp}^C denoting the high order conforming subspace. As a second step, both spaces are further decomposed to build two corresponding additive Schwarz methods in each of the subspaces. The final preconditioner on V_{hp} is then obtained by combining the two subspace preconditioners. The first space splitting is suggested by the interpretation of the hp-DG formulation (1) as a nearly-singular problem. To present the motivation behind this choice, we briefly introduce the theoretical framework of [LWXZ07] regarding space decomposition methods for this class of equations. Given a finite dimensional Hilbert space V, we consider the following problem: find $u \in V$ such that

$$Au = (A_0 + \epsilon A_1)u = f, \tag{6}$$

where A_0 is symmetric and positive semi-definite and A_1 is symmetric and positive definite. As a consequence, if $\epsilon = 0$, the problem is singular, but here we are interested in the case $\epsilon > 0$ (with ϵ small), *i.e.*, (6) is nearly-singular. In general, the conditioning of problem (6) degenerates for decreasing ϵ, and this affects the performance of standard preconditioned iterative methods, unless proper initial guess are chosen. In the framework of space decomposition methods, in order to obtain a ϵ-uniform preconditioner, a key assumption on the space splitting $V_{hp} = \sum_{i=1}^{N} V_i$ is needed.

Assumption 4 ([LWXZ07]). *The decomposition $V_{hp} = \sum_{i=1}^{N} V_i$ satisfies*

$$\ker(A_0) = \sum_{i=1}^{N} (V_i \cap \ker(A_0)),$$

where $\ker(A_0)$ *is the kernel of* A_0.

We now turn to our DG framework, and show that a high order DG formulation can be indeed read as a nearly-singular problem with a suitable choice of ϵ. For the sake of simplicity, and without any loss of generality, we retrieve equation (6) working directly on a bilinear form that is spectrally equivalent to $\mathcal{A}(\cdot, \cdot)$. To this aim, let the bilinear forms $\mathcal{A}_\nabla(\cdot, \cdot)$, $\mathcal{A}_J(\cdot, \cdot)$ and $\widetilde{\mathcal{A}}(\cdot, \cdot)$ be defined as

$$\mathcal{A}_\nabla(u, v) := \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u \cdot \nabla v \, dx,$$

$$\mathcal{A}_J(u, v) := \sum_{F \in \mathcal{F}_h} \int_F [\![u]\!] \cdot [\![v]\!] \, ds,$$

$$\widetilde{\mathcal{A}}(u, v) := \mathcal{A}_\nabla(u, v) + \alpha \frac{p^2}{h} \mathcal{A}_J(u, v),$$

and let A_∇, A_J, and $\widetilde{A}$ be their corresponding operators. Clearly, A_∇ and A_J are both symmetric and positive semi-definite, and $\widetilde{A}$ is symmetric and positive definite. Moreover, thanks to Lemma 1,

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and the quasi-uniformity of the partition, the following spectral equivalence result holds

$$\mathcal{A}(u,u) \approx \|u\|_{DG}^2 \approx \widetilde{\mathcal{A}}(u,u).$$

We can then replace formulation (1) with the following equivalent problem

$$\widetilde{A}u = (A_\nabla + \frac{1}{\epsilon}A_J)u = \tilde{f}, \tag{7}$$

with $\epsilon := h/(\alpha p^2) < 1$. After some simple calculations, we can write (7) as

$$[\epsilon(A_\nabla + A_J) + (1-\epsilon)A_J]\, u = \epsilon \tilde{f},$$

which corresponds to (6) with $A_1 = A_\nabla + A_J$ and $A_0 = (1-\epsilon)A_J$. In order to obtain a suitable space splitting satisfying Assumption 4, we observe that, according to the definition above, the kernel of A_0 is given by the space of continuous polynomial functions of degree p vanishing on the boundary $\partial\Omega$. We then derive the first space decomposition

$$V_{hp} = V_{hp}^B + V_{hp}^C, \tag{8}$$

with

$$V_{hp}^B := \{v \in V_{hp} : v(\xi_p) = 0 \quad \forall \xi_p \in \bigcup_{\kappa \in \mathcal{T}_h} \mathcal{N}_{\mathrm{I}}(\kappa)\},$$

$$V_{hp}^C := \{v \in C^0(\overline{\Omega}) : v \circ \mathsf{F}_\kappa \in \mathbb{Q}^p(\hat{\kappa}) \quad \forall \kappa \in \mathcal{T}_h,\ v|_{\partial\Omega} = 0\} \subseteq H_0^1(\Omega),$$

i.e., V_{hp}^B consists of the functions in V_{hp} that are null in any degree of freedom in the interior of any $\kappa \in \mathcal{T}_h$. Moreover, we observe that $V_{hp}^B \subset V_{hp}$, and $V_{hp}^B \cap V_{hp}^C \subset V_{hp}^C$, hence Assumption 4 is satisfied by decomposition (8), which will be the basis to develop the analysis of our preconditioner for problem (1).

4.1   Technical results

In this subsection we present several results, which will be fundamental for the forthcoming analysis. We introduce a suitable interpolation operator $\mathsf{Q}_h : V_{hp} \to V_{hp}^C$, consisting of the Oswald operator, cf. [HW96, KP03, EAE04, Bur05, BE07]. For any $v \in V_{hp}$, we can define on each $\kappa \in \mathcal{T}_h$ the action of the operator Q_h, by prescribing the value of $\mathsf{Q}_h v$ in any $\xi_p \in \mathcal{N}(\kappa)$:

$$\mathsf{Q}_h v(\xi_p) := \begin{cases} 0 & \text{if } \xi_p \in \partial\Omega, \\ \dfrac{1}{\mathrm{card}(\mathcal{T}_{\xi_p})} \displaystyle\sum_{\kappa \in \mathcal{T}_{\xi_p}} v|_\kappa(\xi_p) & \text{otherwise,} \end{cases} \tag{9}$$

with $\mathcal{T}_{\xi_p} := \{\kappa' \in \mathcal{T}_h : \xi_p \in \kappa'\}$. Note that from the above definition it follows that $v - \mathsf{Q}_h v \in V_{hp}^B$, for any $v \in V_{hp}$.

In addition to the space of polynomials $\mathbb{Q}^p(\kappa)$, we define $\mathbb{Q}_0^p(\kappa)$ as

$$\mathbb{Q}_0^p(\kappa) := \{v \in \mathbb{Q}^p(\kappa) : v(\xi_p) = 0 \quad \forall \xi_p \in \mathcal{N}_{\mathrm{I}}(\kappa)\},$$

and state the following trace and inverse trace inequalities.

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Lemma 5 ([BE07, Lemma 3.1]). *The following trace and inverse trace inequalities hold*

$$\|v\|_{L^2(\partial\kappa)}^2 \lesssim \frac{p^2}{h_\kappa}\|v\|_{L^2(\kappa)}^2 \ \forall v \in \mathbb{Q}^p(\kappa), \tag{10}$$

$$\|v\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2}\|v\|_{L^2(\partial\kappa)}^2 \ \forall v \in \mathbb{Q}_0^p(\kappa). \tag{11}$$

The next result is a keypoint for the forthcoming analysis, and can be found in [BE07, Lemma 3.2]; for the sake of completeness the proof is reported in the Appendix.

Lemma 6. *For any $v \in V_{hp}$, the following estimate holds*

$$\|v - \mathsf{Q_h}v\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2}\sum_{F \in \mathcal{F}_h(\kappa)}\|[\![v]\!]\|_{L^2(F)}^2, \tag{12}$$

with $\mathcal{F}_h(\kappa) := \{F \in \mathcal{F}_h : F \cap \kappa \neq \emptyset\}$.

Thanks to Lemma 6 we can prove the following theorem.

Theorem 7. *For any $v \in V_{hp}$, it holds that*

$$\mathcal{A}(v - \mathsf{Q_h}v, v - \mathsf{Q_h}v) + \mathcal{A}(\mathsf{Q_h}v, \mathsf{Q_h}v) \lesssim \mathcal{A}(v, v), \tag{13}$$

where $\mathsf{Q_h}v \in V_{hp}^C$ is defined as in (9). Then the space decomposition defined in (8) is stable.

Proof. We observe that, from (3), the quasi-uniformity of the mesh and Lemma 6, we obtain

$$\mathcal{A}(v - \mathsf{Q_h}v, v - \mathsf{Q_h}v) \lesssim \sum_{\kappa \in \mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2}\|v - \mathsf{Q_h}v\|_{L^2(\kappa)}^2 \lesssim \alpha \sum_{\kappa \in \mathcal{T}_h}\frac{p^4}{h_\kappa^2}\frac{h_\kappa}{p^2}\sum_{F \in \mathcal{F}_h(\kappa)}\|[\![v]\!]\|_{L^2(F)}^2$$

$$\lesssim \sum_{F \in \mathcal{F}_h}\|\sigma^{1/2}[\![v]\!]\|_{L^2(F)}^2 \lesssim \mathcal{A}(v, v).$$

The upper bound (13) follows from the triangle inequality and the above estimate

$$\mathcal{A}(\mathsf{Q_h}v, \mathsf{Q_h}v) \leq \mathcal{A}(v - \mathsf{Q_h}v, v - \mathsf{Q_h}v) + \mathcal{A}(v, v) \lesssim \mathcal{A}(v, v).$$

For any $v \in V_{hp}$, we recall that $v - \mathsf{Q_h}v \in V_{hp}^B$, which implies

$$\inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \mathcal{A}(v^B, v^B) + \mathcal{A}(v^C, v^C) \leq \mathcal{A}(v - \mathsf{Q_h}v, v - \mathsf{Q_h}v) + \mathcal{A}(\mathsf{Q_h}v, \mathsf{Q_h}v) \lesssim \mathcal{A}(v, v).$$

$\square$

5 Construction and analysis of the preconditioner

In this section we introduce our preconditioner and analyze the condition number of the preconditioned system. Employing the nomenclalture of [Xu92], the preconditioner is a parallel subspace correction method (also known as additive Schwarz preconditioner, see. *e.g.*, [Lio88, Wid88, DW90]). Our construction uses a decomposition in two subspaces, cf. (8) below, and inexact subspace solvers. Each of the subspace solvers is a parallel subspace correction method itself.

5.1 Canonical representation of a parallel subspace correction method

The main ingredients needed for the analysis of the parallel subspace correction (PSC) preconditioners are suitable space splittings and the corresponding subspace solvers (see [Lio88, Wid88, DW90, Xu92, GO95a, XZ02, TW04]). In our analysis we will use the notation and the general setting from [XZ02]. We have the following abstract result.

Lemma 8 ([XZ02, Lemma 2.4]). *Let V be a Hilbert space which is decomposed as $V = \sum_{i=1}^{N} V_i$, $V_i \subset V$, $i = 1, \ldots, N$, and $T_i : V \to V_i$, $i = 1, \ldots, N$ be operators whose restrictions on V_i are symmetric and positive definite. For $T := \sum_{i=1}^{N} T_i$ the following identity holds*

$$\mathcal{A}(T^{-1}v, v) = \inf_{\substack{v_i \in V_i \\ \sum v_i = v}} \sum_{i=1}^{N} \mathcal{A}(T_i^{-1}v_i, v_i). \tag{14}$$

According to the above lemma, to show a bound on the condition number of the preconditioned system we need to show that there exist positive constants c and C such that

$$c\mathcal{A}(v, v) \leq \mathcal{A}(T^{-1}v, v) \leq C\mathcal{A}(v, v).$$

Remark 9. *In many cases we have $T_i = P_i$, $i = 1, \ldots, N$, where $P_i : V \to V_i$ are the elliptic projections defined as follows: for $v \in V$, its projection $P_i v$ is the unique element of V_i satisfying $\mathcal{A}(P_i v, v_i) := \mathcal{A}(v, v_i)$, for all $v_i \in V_i$. Note that by definition, P_i is the identity on V_i, namely, $P_i v_i = v_i = P_i^{-1} v_i$, for all $v_i \in V_i$. Hence, for $T = \sum_{i=1}^{N} P_i$, the relation (14) gives*

$$\mathcal{A}(T^{-1}v, v) = \inf_{\substack{v_i \in V_i \\ \sum v_i = v}} \sum_{i=1}^{N} \mathcal{A}(v_i, v_i). \tag{15}$$

5.2 Space splitting and subspace solvers

To fix the notation, let us point out that in what follows we use T (with subscript when necessary) to denote (sub)space solvers and preconditioners. Accordingly, P with subscript or superscript will denote elliptic projection on the corresponding subspace, which will be clear from the context.

We now define the space splitting and the corresponding subspace solvers. We recall the space decomposition from Section 4, $V_{hp} = V_{hp}^B + V_{hp}^C$, where V_{hp}^B are all functions in V_{hp} for which the degrees of freedom in the interior of any $\kappa \in \mathcal{T}_h$ vanish, and V_{hp}^C is the space of high order continuous polynomials vanishing on $\partial\Omega$. Note that $V_{hp}^B \cap V_{hp}^C \neq \{0\}$, and that V_{hp}^B contains non-smooth and oscillatory functions, while V_{hp}^C contains the smooth part of the space V_{hp}. Next, on each of these subspaces we define approximate solvers $T_B : V_{hp} \to V_{hp}^B$ and $T_C : V_{hp} \to V_{hp}^C$.
First, we decompose V_{hp}^B as follows

$$V_{hp}^B = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} V^{\xi_p}, \tag{16}$$

where

$$V^{\xi_p} := \left\{ v \in V_{hp}^B : v(\xi_p{}') = 0 \text{ for any } \xi_p{}' \in \left(\bigcup_{\kappa \in \mathcal{T}_h} \mathcal{N}_B(\kappa) \right) \setminus \{\xi_p\} \right\}.$$

The approximate solver on V_B then is a simple Jacobi method, defined as

$$T_B : V_{hp} \to V_{hp}^B, \quad T_B := \left[\sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} P^{\xi_p} \right] P_B.$$

where P_B and P^{ξ_p} are the elliptic projections on V_{hp}^B and V^{ξ_p}, respectively. Note that T_B is defined on all of V_{hp} and is also an isomorphism when restricted to V_{hp}^B, because the elliptic projection P_B and P^{ξ_p} are the identity on V_{hp}^B and V^{ξ_p}, respectively. In addition, the splitting is a direct sum, and, hence, any $v \in V_{hp}^B$ is uniquely represented as $v = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} v^{\xi_p}$, $v^{\xi_p} \in V^{\xi_p}$. Then, taking $P_i = P^{\xi_p} P_B : V_{hp} \to V^{\xi_p}$, from (15), we have

$$\mathcal{A}(T_B^{-1} v^B, v^B) = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} \mathcal{A}(v^{\xi_p}, v^{\xi_p}), \quad \forall \quad v^B \in V_{hp}^B. \tag{17}$$

Next, we introduce the preconditioner T_C on V_{hp}^C. This is the two-level overlapping additive Schwarz method introduced in [Pav92] for high order *conforming* discretizations. If we denote by N_V the number of interior vertices of $\mathcal{T}_h$, then this preconditioner corresponds to the following decomposition of V_{hp}^C:

$$V_{hp}^C = \sum_{i=0}^{N_V} V_i^C. \tag{18}$$

Here V_0^C is the (coarse) space of continuous piecewise linear functions on $\mathcal{T}_h$, and for $i = 1, \ldots, N_V$, $V_i^C := V_{hp}^C \cap H_0^1(\Omega_i)$, where Ω_i is the union of the elements sharing the i-th vertex (see Fig. 1 for a two-dimensional example). We recall that, in the case of Neumann and mixed boundary conditions, in order to obtain a uniform preconditioner, the decomposition (18) should be enriched with the subdomains associated to those vertices not lying on a Dirichlet boundary, see [Pav92] for details.
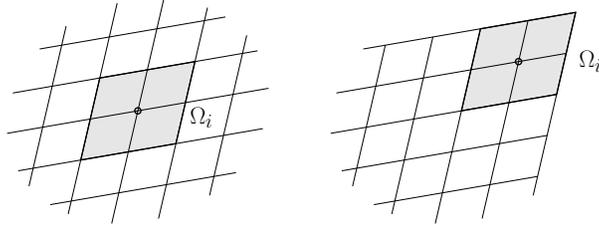
Figure 1: Examples of subdomains in a two-dimensional setting.

Then, for any V_i^C, $i = 0, \ldots, N_V$, we denote by $P_i^C : V_{hp}^C \to V_i^C$ the elliptic projections on V_i^C and define the two-level overlapping additive Schwarz operator as

$$T_C : V_{hp} \to V_{hp}^C, \quad T_C := \left[P_0^C + \sum_{i=1}^{N_V} P_i^C \right] P_C = (P_0 + P_V) P_C, \tag{19}$$

where P_C is the elliptic projection on V_{hp}^C. As in the case of V_B, we have that the restriction of T_C on V_{hp}^C is an isomorphism. In addition, from (15) with $P_i = P_i^C P_C : V_{hp} \to V_i^C$, we have

$$\mathcal{A}(T_C^{-1} v, v) = \inf_{\substack{v_i \in V_i^C \\ \sum v_i = v}} \sum_{i=0}^{N_V} \mathcal{A}(v_i, v_i). \tag{20}$$

5.3 Definition of the global preconditioner

Finally, we define the global preconditioner on V_{hp} by setting

$$T_{DG} : V_{hp} \to V_{hp}, \quad T_{DG} := T_B + T_C, \tag{21}$$

We remark that from Lemma 8, with $N = 2$, $T_1 = T_B$, $V_1 = V_{hp}^B$, $T_2 = T_C$, $V_2 = V_{hp}^C$, we have

$$\mathcal{A}(T_{DG}^{-1} v, v) = \inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \left[\mathcal{A}(T_B^{-1} v^B, v^B) + \mathcal{A}(T_C^{-1} v^C, v^C) \right]. \tag{22}$$

5.4 Condition number estimates: subspace solvers

We now show the estimates on the conditioning of the subspace solvers needed to bound the condition number of T_{DG}. The first result that we prove is on the conditioning of T_B.

Lemma 10. *Let T_B denote the Jacobi preconditioner defined in (5.2). Then there exist two positive constants C_1^J and C_2^J, independent of the granularity of the mesh h, the polynomial approximation degree p and the penalization coefficient α, such that*

$$\mathcal{A}(T_B^{-1} v^B, v^B) \geq \mathsf{C}_1^J \mathcal{A}(v^B, v^B) \quad \forall v^B \in V_{hp}^B \tag{23}$$

$$\mathcal{A}(T_B^{-1}(v - \mathsf{Q_h} v), v - \mathsf{Q_h} v) \leq \mathsf{C}_2^J \mathcal{A}(v - \mathsf{Q_h} v, v - \mathsf{Q_h} v) \quad \forall v \in V_{hp}, \tag{24}$$

with $\mathsf{Q_h} v$ defined in (9).

Proof. We refer to the space decomposition (16) and write

$$v^B = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} v^{\xi_p}.$$

For the lower bound (23), we employ the eigenvalue estimate (3) and Lemma 3, thus obtaining

$$\mathcal{A}(v^B, v^B) \lesssim \sum_{\kappa \in \mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \|v^B\|_{L^2(\kappa)}^2 \lesssim \sum_{\kappa \in \mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} \|v^{\xi_p}\|_{L^2(\kappa)}^2.$$

We now observe that for any $\xi_p \in \mathcal{N}_B(\kappa)$, $v^{\xi_p} \in \mathbb{Q}_0^p(\kappa)$, and we can thus apply the inverse trace inequality (11) to obtain

$$\mathcal{A}(v^B, v^B) \lesssim \sum_{\kappa \in \mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} \|v^{\xi_p}\|_{L^2(\kappa)}^2 \lesssim \sum_{\kappa \in \mathcal{T}_h} \alpha \frac{p^2}{h_\kappa} \sum_{\xi_p \in \mathcal{N}_B(\kappa)} \|v^{\xi_p}\|_{L^2(\partial \kappa)}^2.$$

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Noting that $\|v^{\xi_p}\|^2_{L^2(\partial\kappa)} = \|[\![v^{\xi_p}]\!]\|^2_{L^2(\partial\kappa)}$, it follows that

$$
\begin{aligned}
\mathcal{A}(v^B, v^B) &\lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha\frac{p^2}{h_\kappa} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|v^{\xi_p}\|^2_{L^2(\partial\kappa)} \lesssim \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|\sigma^{1/2}[\![v^{\xi_p}]\!]\|^2_{L^2(\partial\kappa)} \\
&\lesssim \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|v^{\xi_p}\|^2_{DG},
\end{aligned}
$$

and the thesis follows from the coercivity bound (2) and (17).

With regard to the upper bound (24), for the sake of simplicity we denote $w = (I - \mathsf{Q_h})v$, and observe that $w = (I - \mathsf{Q_h})w$. Since $w \in V^B_{hp}$, we write

$$
w = \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} w^{\xi_p},
$$

and, from (17),

$$
\mathcal{A}(T_B^{-1}w, w) = \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \mathcal{A}(w^{\xi_p}, w^{\xi_p}).
$$

Applying again the estimate (3) and Lemma 3, we obtain

$$
\begin{aligned}
\sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \mathcal{A}(w^{\xi_p}, w^{\xi_p}) &\lesssim \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \alpha\frac{p^4}{h_\kappa^2} \|w^{\xi_p}\|^2_{L^2(\kappa)} \lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha\frac{p^4}{h_\kappa^2} \|w\|^2_{L^2(\kappa)} \\
&\lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha\frac{p^4}{h_\kappa^2} \|(I - \mathsf{Q_h})w\|^2_{L^2(\kappa)} \lesssim \mathcal{A}(w, w),
\end{aligned}
$$

where the last steps follows from Lemma 6 and the quasi-uniformity of the mesh. $\qquad\square$

For the analysis of the additive preconditioner T_C given in (19), we need several preliminary results (see [Pav92] for additional details). First of all, given the decomposition

$$
v = v_0 + \sum_{i=1}^{N_V} v_i \qquad \forall v \in V^C_{hp},\ v_0 \in V^C_0,\ v_i \in V^C_i, \tag{25}
$$

we define the coarse function v_0 as the L^2-projection on the space V^C_0, i.e., $v_0 := \mathcal{I}_0 v$ with $\mathcal{I}_0 v$ satisfying

$$
\|v - \mathcal{I}_0 v\|^2_{L^2(\Omega)} \lesssim h^2 |v|^2_{H^1(\Omega)}, \tag{26}
$$

$$
|\mathcal{I}_0 v|^2_{H^1(\Omega)} \lesssim |v|^2_{H^1(\Omega)}, \tag{27}
$$

for any $v \in H^1_0(\Omega)$. For any $i = 1, \ldots, N_V$, the functions v_i appearing in (25) are defined as

$$
v_i := I_p(\theta_i(v - v_0)),
$$

where θ_i is a proper partition of unity and I_p is an interpolation operator, described in the following.

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For any Ω_i, $i = 1, \dots, N_V$, the partition of unity θ_i is such that $\theta_i \in V_{h1}^C$ and it can be defined by prescribing its values at the vertices $\{\mathsf{v}\}$ belonging to $\overline{\Omega}_i$, and imposing it to be zero on $\Omega \setminus \overline{\Omega}_i$, see Fig. 2 for $d = 2$. More precisely,

$$\theta_i(\mathsf{v}) = \begin{cases} 1 & \text{if } \mathsf{v} \text{ is the internal vertex or } \mathcal{F}_{\mathsf{v}} \subset \mathcal{F}_h^B, \\ 0 & \text{otherwise,} \end{cases}$$

with $\mathcal{F}_{\mathsf{v}} := \{F \in \mathcal{F}_h, F \subseteq \partial\Omega_i : \mathsf{v} \in F\}$.
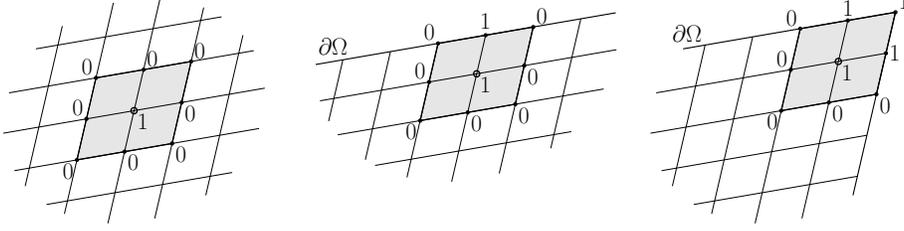
Figure 2: Values of the partition of unity θ_i for $d = 2$.

It follows that:

$$\operatorname{supp}(\theta_i) = \Omega_i, \qquad 0 \leq \theta_i \leq 1, \qquad \sum_{i=1}^{N_V} \theta_i = 1, \qquad |\nabla\theta_i| \lesssim \frac{1}{h}. \tag{28}$$

As interpolation operator I_p, we make use of the operator defined in [Pav92]: setting $z := v - v_0$, we define

$$I_p(\theta_i z)(\xi_p) = (\theta_i z)(\xi_p) \quad \forall \xi_p \in \mathcal{N}(\kappa), \forall \kappa \in \Omega_i. \tag{29}$$

Notice that, despite defined locally, $I_p(\theta_i z)$ belongs to V_i^C since the interelement continuity is guaranteed by the fact that the $(p + 1)^{d-1}$ GLL points on a face uniquely determine a tensor product polynomial of degree p defined on that face. The following result holds.

Lemma 11 ([Pav92, Lemma 3.1, Lemma 3.3]). *The interpolation operator* $I_p : \mathbb{Q}^{p+1}(\hat{\kappa}) \to \mathbb{Q}^p(\hat{\kappa})$, *defined in* (29), *is bounded uniformly in the* H^1 *seminorm, i.e.,*

$$|I_p(u)|_{H^1(\hat{\kappa})} \lesssim |u|_{H^1(\hat{\kappa})} \qquad \forall u \in \mathbb{Q}^{p+1}(\hat{\kappa}). \tag{30}$$

Once the partition of unity and the interpolation operator are defined, we are able to complete the analysis of T_C. In analogy to Lemma 10, which is based on (17), we now use (20) and the above auxiliary results to show the following lemma.

Lemma 12. *Let* T_C *denote the two-level overlapping additive Schwarz preconditioner defined in* (19). *Then there exist two positive constants* C_1^C *and* C_2^C, *independent of the discretization parameters, i.e., the granularity of the mesh* h *and the polynomial approximation degree* p, *such that*

$$\mathcal{A}(T_C^{-1}v, v) \geq \mathsf{C}_1^C \mathcal{A}(v, v) \tag{31}$$

$$\mathcal{A}(T_C^{-1}v, v) \leq \mathsf{C}_2^C \mathcal{A}(v, v), \tag{32}$$

for any $v \in V_{hp}^C$.

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Proof. We first prove the lower bound (31), and given the decomposition (25), we can write

$$\mathcal{A}(v,v) = \sum_{i,j=0}^{N_V} \mathcal{A}(v_i, v_j) \lesssim \mathcal{A}(v_0, v_0) + \sum_{i,j=1}^{N_V} \mathcal{A}(v_i, v_j).$$

We now note that $\mathcal{A}(v_i, v_j) \neq 0$ only if $i = j$ and $\Omega_i \cap \Omega_j \neq \emptyset$, and since each Ω_i is overlapped by a limited number of neighboring subdomains, we conclude that

$$\mathcal{A}(v,v) \lesssim \mathcal{A}(v_0, v_0) + \sum_{i,j=1}^{N_V} \mathcal{A}(v_i, v_j) \lesssim \mathcal{A}(v_0, v_0) + \sum_{i=1}^{N_V} \mathcal{A}(v_i, v_i).$$

Inequality (31) follows from the bound above and (20), denoting with C_1^C the hidden constant.

Note that, from (20), the upper bound (32) is proved provided the following inequality holds

$$\sum_{i=0}^{N_V} \mathcal{A}(v_i, v_i) \leq \mathsf{C}_2^C \mathcal{A}(v,v) \qquad \forall v \in V_{hp}^C. \tag{33}$$

We recall that $v_0 = \mathcal{I}_0 v$, and from (27) it follows that

$$\mathcal{A}(v_0, v_0) = \mathcal{A}(\mathcal{I}_0 v, \mathcal{I}_0 v) \lesssim \mathcal{A}(v,v). \tag{34}$$

For $i = 1, \ldots, N_V$, we have $v_i = I_p(\theta_i z)$, with $z = v - v_0$, and by (30), we obtain

$$|v_i|_{H^1(\kappa')}^2 \lesssim |\theta_i z|_{H^1(\kappa')}^2 \lesssim \sum_{j=1}^{d} \left\| \frac{\partial \theta_i}{\partial x_j} z + \theta_i \frac{\partial z}{\partial x_j} \right\|_{L^2(\kappa')}^2,$$

for any $\kappa' \in \Omega_i$. By (28) it holds that

$$|\nabla \theta_i| \lesssim \frac{1}{h}, \qquad \|\theta_i\|_{L^\infty} \leq 1,$$

hence,

$$|v_i|_{H^1(\kappa')}^2 \lesssim \frac{1}{h^2} \|z\|_{L^2(\kappa')}^2 + \sum_{j=1}^{d} \|\frac{\partial z}{\partial x_j}\|_{L^2(\kappa')}^2 \lesssim \frac{1}{h^2} \|v - v_0\|_{L^2(\kappa')}^2 + |v - v_0|_{H^1(\kappa')}^2.$$

On any element κ', a limited number of components v_i are different from zero (at most four for $d = 2$, and eight for $d = 3$), which implies that we can sum over all the components v_i, $i = 1, \ldots, N_V$, and then over all the elements, thus obtaining

$$\sum_{i=1}^{N_V} |v_i|_{H^1(\Omega)}^2 \lesssim \frac{1}{h^2} \|v - v_0\|_{L^2(\Omega)}^2 + |v - v_0|_{H^1(\Omega)}^2 \lesssim |v|_{H^1(\Omega)}^2,$$

where the last step follows from (26) and (27). The addition of the above result and (34), gives (33), denoting with C_2^C the resulting hidden constant. $\qquad\square$

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5.5 Condition number estimates: global preconditioner

We are now ready to prove the main result of the paper regarding the condition number of the preconditioned problem.

Theorem 13. *Let T_{DG} be defined as in (21). Then, for any $v \in V_{hp}$, it holds that*

$$\mathcal{A}(v, v) \lesssim \mathcal{A}(T_{DG}^{-1}v, v) \lesssim \mathcal{A}(v, v), \tag{35}$$

where the hidden constants are independent of the discretization parameters, i.e.*, the mesh size h, the polynomial approximation degree p, and the penalization coefficient α.*

Proof. To prove the upper bound, we first consider the identity (22). Recalling that, by definition (9), $v - Q_h v \in V_{hp}^B$, for any $v \in V_{hp}$, we obtain

$$
\begin{aligned}
\mathcal{A}(T_{DG}^{-1}v, v) &= \inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \left[\mathcal{A}(T_B^{-1}v^B, v^B) + \mathcal{A}(T_C^{-1}v^C, v^C) \right] \\
&\leq \mathcal{A}(T_B^{-1}(v - Q_h v), v - Q_h v) + \mathcal{A}(T_C^{-1}Q_h v, Q_h v).
\end{aligned}
$$

From the bounds (24) and (32) for $Q_h v$, it follows that

$$
\begin{aligned}
\mathcal{A}(T_{DG}^{-1}v, v) &\leq \mathcal{A}(T_B^{-1}(v - Q_h v), v - Q_h v) + \mathcal{A}(T_C^{-1}Q_h v, Q_h v) \\
&\lesssim \mathcal{A}(v - Q_h v, v - Q_h v) + \mathcal{A}(Q_h v, Q_h v) \lesssim \mathcal{A}(v, v),
\end{aligned}
$$

where the last step follows from (13). The lower bound follows from (22), the bounds (23) and (31), and a triangle inequality

$$
\begin{aligned}
\mathcal{A}(T_{DG}^{-1}v, v) &= \inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \left[\mathcal{A}(T_B^{-1}v^B, v^B) + \mathcal{A}(T_C^{-1}v^C, v^C) \right] \\
&\gtrsim \inf_{v^C \in V_{hp}^C} \left[\mathcal{A}(v^B, v^B) + \mathcal{A}(v^C, v^C) \right] \gtrsim \mathcal{A}(v, v).
\end{aligned}
$$

$\square$

6 Numerical experiments

In this section we present some numerical tests to verify the theoretical estimates provided in Lemma 10, Lemma 12 and Theorem 13. We consider problem (1) in the two dimensional case with $\Omega = (-1, 1)^2$ and SIPG and LDG discretizations. For the first experiment, we set $h = 0.0625$, the penalization parameter $\alpha = 10$ and $\boldsymbol{\beta} = \mathbf{1}$ for the LDG method. In Table 1, we show the numerical evaluation of the constants C_1^J and C_2^J of Lemma 10 and C_1^C and C_2^C of Lemma 12, as a function of the polynomial order employed in the discretization: the constants are independent of p, as expected from theory. With regard to the constants C_1^C and C_2^C, we observe that the values are the same for both the SIPG and LDG methods, since the preconditioner on the conforming subspace reduces to the same operator regardless of the DG scheme employed.

Table 1: Left and middle: numerical evaluation of the constants C_1^J and C_2^J of Lemma 10 as a function of p for the SIPG and LDG methods; right: numerical evaluation of the constants C_1^C and C_2^C of Lemma 12 as a function of p

| | SIPG ($\alpha = 10$, $\boldsymbol{\beta} = \mathbf{0}$) | | LDG ($\alpha = 10$, $\boldsymbol{\beta} = \mathbf{1}$) | | | |
	C_1^J	C_2^J	C_1^J	C_2^J	C_1^C	C_2^C
$p = 2$	0.4036	3.0084	0.3844	3.6393	0.2500	1.1606
$p = 3$	0.4343	2.9133	0.4129	3.3232	0.2500	1.0742
$p = 4$	0.4502	2.8304	0.4298	3.1487	0.2500	1.0934
$p = 5$	0.4605	2.7633	0.4410	3.0321	0.2500	1.0820
$p = 6$	0.4674	2.7088	0.4489	2.9467	0.2500	1.0854

Table 2 shows a comparison of the spectral condition number of the original system ($\mathsf{K}(A)$) and of the preconditioned one ($\mathsf{K}(T_{DG})$). While the former grows as p^4, cf. [AH11], the latter is constant with p, as stated in (35). The theoretical results are further confirmed by the number of iterations N_{iter}^{PCG} and N_{iter}^{CG} of the Preconditioned Conjugate Gradient (PCG) and the Conjugate Gradient (CG), respectively, needed to reduce the initial relative residual of a factor of 10^{-8}.

Table 2: Condition number of the unpreconditioned ($\mathsf{K}(A)$) and preconditioned ($\mathsf{K}(T_{DG})$) linear systems of equations and corresponding CG (N_{iter}^{CG}) and PCG (N_{iter}^{PCG}) iteration counts as a function of p for the SIPG and LDG methods

| | SIPG ($\alpha = 10$, $\boldsymbol{\beta} = \mathbf{0}$) | | | | LDG ($\alpha = 10$, $\boldsymbol{\beta} = \mathbf{1}$) | | | |
	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}
$p = 2$	$5.26 \cdot 10^3$	284	14.26	27	$8.88 \cdot 10^3$	392	35.02	36
$p = 3$	$1.52 \cdot 10^4$	450	14.22	25	$2.29 \cdot 10^4$	556	38.29	31
$p = 4$	$3.38 \cdot 10^4$	684	14.72	26	$4.89 \cdot 10^4$	851	37.74	33
$p = 5$	$6.27 \cdot 10^4$	919	15.35	24	$8.83 \cdot 10^4$	1137	38.37	30
$p = 6$	$1.05 \cdot 10^5$	1200	15.98	25	$1.45 \cdot 10^5$	1482	42.65	32

The second numerical experiment aims at verifying the uniformity of the proposed preconditioner with respect to the penalization coefficient α. In this case, we consider the same test case presented above, but we now fix the polynomial approximation degree $p = 2$ and increase α. The numerical data obtained are reported in Table 3: as done before, we compare the spectral condition numbers of the unpreconditioned and preconditioned systems and the iteration counts of the CG and PCG methods. As predicted from theory, while $\mathsf{K}(A)$ grows like α, the values of $\mathsf{K}(T_{DG})$ are constant.

Table 3: Condition number of the unpreconditioned ($\mathsf{K}(A)$) and preconditioned ($\mathsf{K}(T_{DG})$) linear systems of equations and corresponding CG (N_{iter}^{CG}) and PCG (N_{iter}^{PCG}) iteration counts as a function of α for the SIPG and LDG methods

	SIPG ($p=2$, $\boldsymbol{\beta}=\mathbf{0}$)				LDG ($p=2$, $\boldsymbol{\beta}=\mathbf{1}$)			
	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}
$\alpha=2$	$1.04\cdot 10^3$	137	12.66	28	$4.55\cdot 10^3$	297	62.54	47
$\alpha=5$	$2.62\cdot 10^3$	205	13.02	28	$6.17\cdot 10^3$	338	41.94	39
$\alpha=10$	$5.26\cdot 10^3$	284	14.26	27	$8.88\cdot 10^3$	392	35.02	36
$\alpha=10^2$	$5.41\cdot 10^4$	690	15.73	28	$5.78\cdot 10^4$	717	29.32	31
$\alpha=10^3$	$5.44\cdot 10^5$	1116	15.90	28	$5.47\cdot 10^5$	1142	28.92	30
$\alpha=10^4$	$5.44\cdot 10^6$	1509	15.91	28	$5.44\cdot 10^6$	1518	28.89	30

A   Proof of Lemma 6

We first introduce some additional notation. For any $\kappa \in \mathcal{T}_h$, we define $\partial\kappa_{d-1}$ as the set of $(d-1)$-dimensional affine varieties in $\partial\kappa$, and $\partial_\ell\kappa$, $\ell \in \{d-2,\ldots,0\}$, as the set obtained as the intersection of two distinct elements in $\partial_{\ell+1}\kappa$. We observe that $\partial\kappa_{d-1} \subset \mathcal{F}_h$ for any $\kappa \in \mathcal{T}_h$. The set of nodes of each element κ can be further decomposed as

$$\mathcal{N}(\kappa) = \mathcal{N}_{\mathrm{I}}(\kappa) \cup \mathcal{N}_{\mathrm{B}}(\kappa) = \mathcal{N}_{\mathrm{I}}(\kappa) \cup \bigcup_{\ell=0}^{d-1} \mathcal{V}_\ell, \qquad (36)$$

with $\mathcal{V}_\ell$, $\ell \in \{d-1,\ldots,0\}$, representing the set of interior nodes of $\partial\kappa_\ell$ (see Fig. 3).
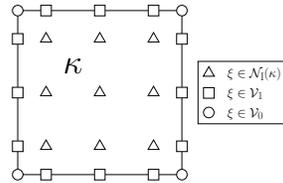
Figure 3: Distribution of the nodes $\xi_p \in \mathcal{N}(\kappa)$ for $p=4$, $d=2$.

In analogy to $\mathbb{Q}_0^p(\kappa)$, we define $\mathbb{Q}_0^p(\partial\kappa_\ell)$, $\ell \in \{d-1,\ldots,1\}$, as

$$\mathbb{Q}_0^p(\partial\kappa_\ell) := \{v \in \mathbb{Q}^p(\partial\kappa_\ell) : v(\xi_p) = 0 \quad \forall \xi_p \in \mathcal{V}_\ell\},$$

and remark that a corresponding form of the trace and inverse trace inequalities of Lemma 5 can be obtained on $\mathbb{Q}^p(\kappa_\ell)$ and $\mathbb{Q}_0^p(\partial\kappa_\ell)$.

The proof of Lemma 6 can be found in [BE07, Lemma 3.2]. Here we reproduce the same steps, with only minor changes, mainly regarding the notation.
We denote $w = (v - \mathsf{Q}_\mathsf{h}v)|_\kappa$ and observe that, according to (9), it holds

$$w(\xi_p) = 0 \qquad \forall \xi_p \in \mathcal{N}_{\mathrm{I}}(\kappa).$$

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Given the set of nodes $\xi_p \in \mathcal{N}(\kappa)$ and the associated Lagrangian nodal basis functions $\{\phi_{\xi_p}\}$, we can write

$$w = \sum_{\xi_p \in \mathcal{N}_{\mathrm{B}}(\kappa)} w(\xi_p) \phi_{\xi_p},$$

cf. (5). From the decomposition (36), it follows that

$$w = \sum_{\xi_p \in \mathcal{V}_{d-1}} w(\xi_p) \phi_{\xi_p} + \sum_{\ell=0}^{d-2} \sum_{\xi_p \in \mathcal{V}_\ell} w(\xi_p) \phi_{\xi_p} = \sum_{\xi_p \in \mathcal{V}_{d-1}} w(\xi_p) \phi_{\xi_p} + \sum_{\ell=0}^{d-2} \mathsf{r}_\ell, \tag{37}$$

where for any $\ell \in \{0, \ldots, d-2\}$ we have

$$\mathsf{r}_\ell \in \mathbb{Q}_0^p(\kappa), \qquad \mathsf{r}_\ell \in \mathbb{Q}_0^p(\partial \kappa_l) \qquad \forall l \in \{\ell+1, \ldots, d-1\}.$$

Let us introduce $\mathcal{V}_{d-1,F}$ as the set of interior nodes of $F \subset \partial \kappa$. For any $\xi_p \in \mathcal{V}_{d-1,F}$, by (9), we have that

$$w(\xi_p) = \boldsymbol{\gamma}_F \cdot [\![v]\!](\xi_p), \qquad \boldsymbol{\gamma}_F := \begin{cases} \mathbf{n}_{F,\kappa} & \text{if } F \subset \partial \Omega, \\ \dfrac{1}{2} \mathbf{n}_{F,\kappa} & \text{otherwise.} \end{cases}$$

We the above notation, we have

$$\sum_{\xi_p \in \mathcal{V}_{d-1}} w(\xi_p) \phi_{\xi_p} = \sum_{F \subset \partial \kappa} \psi_F, \quad \text{where} \quad \psi_F := \boldsymbol{\gamma}_F \cdot \sum_{\xi_p \in \mathcal{V}_{d-1,F}} [\![v]\!](\xi_p) \phi_{\xi_p}$$

We next observe that $\psi_F(\xi_p) = 0$ for any $\xi_p \in \mathcal{N}_{\mathrm{I}}(\kappa)$, *i.e.*, $\psi_F \in \mathbb{Q}_0^p(\kappa)$, and $\psi_F(\xi_p) = 0$ also for any $\xi_p \in \partial \kappa \setminus F$. We can then apply the inverse trace inequality (11), thus obtaining

$$\|\psi_F\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2} \|\psi_F\|_{L^2(\partial \kappa)}^2 \lesssim \frac{h_\kappa}{p^2} \|\psi_F\|_{L^2(F)}.$$

Recalling (37), it follows

$$\|w\|_{L^2(\kappa)}^2 \lesssim \sum_{F \subset \partial \kappa} \frac{h_\kappa}{p^2} \|\psi_F\|_{L^2(F)}^2 + \sum_{\ell=0}^{d-2} \|\mathsf{r}_\ell\|_{L^2(\kappa)}^2. \tag{38}$$

In order to bound the terms on the right hand side of (38), we proceed by considering $d = 1, 2, 3$ as separate cases. For $d = 1$, inequality (38) reduces to

$$\|w\|_{L^2(\kappa)}^2 \lesssim \sum_{F \subset \partial \kappa} \frac{h_\kappa}{p^2} \|\psi_F\|_{L^2(F)}^2, \tag{39}$$

and we observe that by the definition of ψ_F it holds that $\psi_F|_F = \boldsymbol{\gamma}_F \cdot [\![v]\!]$. As a consequence, (39) implies (12). For $d = 2$, (38) reduces to

$$\|w\|_{L^2(\kappa)}^2 \lesssim \sum_{F \subset \partial \kappa} \frac{h_\kappa}{p^2} \|\psi_F\|_{L^2(F)}^2 + \|\mathsf{r}_0\|_{L^2(\kappa)}^2.$$

18

First of all, we recall that the function ψ_F is equal to zero on ∂F and coincides with $\boldsymbol{\gamma}_F \cdot [\![v]\!]$ on any $\xi_p \in \mathcal{V}_{d-1,F}$, which means that $\psi_F - \boldsymbol{\gamma}_F \cdot [\![v]\!] \in \mathbb{Q}_0^p(F)$. By applying the inverse trace inequality (11) and the trace inequality (10) , we get

$$\|\psi_F - \boldsymbol{\gamma}_F \cdot [\![v]\!]\|_{L^2(F)}^2 \lesssim \frac{h_F}{p^2}\|[\![v]\!]\|_{L^2(\partial F)}^2 \lesssim \frac{h_F}{p^2}\frac{p^2}{h_F}\|[\![v]\!]\|_{L^2(F)}^2 \lesssim \|[\![v]\!]\|_{L^2(F)}^2. \tag{40}$$

From (40) and the triangle inequality, it follows

$$\|\psi_F\|_{L^2(F)}^2 \lesssim \|[\![v]\!]\|_{L^2(F)}^2. \tag{41}$$

We next estimate the term $\|r_0\|_{L^2(2)}$. To this aim, we recall that $r_0 \in \mathbb{Q}_0^p(\kappa)$ and $r_0 \in \mathbb{Q}_0^p(F)$. This allows us to apply the inverse trace inequality (11) twice, thus obtaining

$$\|r_0\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2}\sum_{F \subset \partial\kappa}\|r_0\|_{L^2(F)}^2 \lesssim \frac{h_\kappa}{p^2}\sum_{F \subset \partial\kappa}\frac{h_F}{p^2}\|r_0\|_{L^2(\partial F)}^2.$$

Moreover, we note that, for $d = 2$, ∂F is given only by two nodes and for $\xi_p \in \partial F$, a simple calculation leads to

$$r_0(\xi_p) = \sum_{F' \in \mathcal{F}_{\xi_p}}\boldsymbol{\eta}_{F,F'}(\xi_p) \cdot [\![v]\!](\xi_p), \tag{42}$$

where $\mathcal{F}_{\xi_p} := \{F' \in \mathcal{F}_h : \xi_p \in F'\}$ and

$$\boldsymbol{\eta}_{F,F'}(\xi_p) := \begin{cases} \pm\dfrac{1}{2}\mathbf{n}_{F'} & \text{if } \xi_p \in \partial\Omega, \\[2mm] \pm\dfrac{3}{8}\mathbf{n}_{F'} & \text{if } F' \subset \partial\kappa \setminus \partial\Omega \text{ and } \xi_p \notin \partial\Omega, \\[2mm] \pm\dfrac{1}{8}\mathbf{n}_{F'} & \text{otherwise.} \end{cases}$$

We then have

$$\|r_0\|_{L^2(\partial F)}^2 = \sum_{\xi_p \in \partial F}\sum_{F' \in \mathcal{F}_{\xi_p}}|\boldsymbol{\eta}_{F,F'}(\xi_p) \cdot [\![v]\!](\xi_p)|^2 \lesssim \sum_{\xi_p \in \partial F}\sum_{F' \in \mathcal{F}_{\xi_p}}|[\![v]\!](\xi_p)|^2$$

$$\lesssim \sum_{\xi_p \in \partial F}\sum_{F' \in \mathcal{F}_{\xi_p}}\frac{p^2}{h_{F'}}\|[\![v]\!]\|_{L^2(F')}^2 \lesssim \sum_{F':F \cap F' \neq \emptyset}\frac{p^2}{h_{F'}}\|[\![v]\!]\|_{L^2(F')}^2,$$

where the second step follows by the trace inequality (10). Finally we obtain

$$\|r_0\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2}\sum_{F \subset \partial\kappa}\frac{h_F}{p^2}\sum_{F':F \cap F' \neq \emptyset}\frac{p^2}{h_{F'}}\|[\![v]\!]\|_{L^2(F')}^2 \tag{43}$$

$$\lesssim \frac{h_\kappa}{p^2}\sum_{F \in \mathcal{F}_h(\kappa)}\|[\![v]\!]\|_{L^2(F)}^2.$$

By combining (41) and (43) the desired result follows. Finally, for $d = 3$, inequality (38) reduces to

$$\|w\|_{L^2(\kappa)}^2 \lesssim \sum_{F \subset \partial\kappa}\frac{h_\kappa}{p^2}\|\psi_F\|_{L^2(F)}^2 + \|r_0\|_{L^2(\kappa)}^2 + \|r_1\|_{L^2(\kappa)}^2.$$

19

The first two terms on the right hand side can be bounded reasoning as before. To estimate the last term on the right hand side, we first observe that $r_1 \in \mathbb{Q}_0^p(\kappa)$ and $r_1 \in \mathbb{Q}_0^p(F)$, and therefore we can apply again (11) twice and obtain

$$\|r_1\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2} \sum_{F \subset \partial\kappa} \|r_1\|_{L^2(F)}^2 \lesssim \frac{h_\kappa}{p^2} \sum_{F \subset \partial\kappa} \frac{h_F}{p^2} \sum_{E \subset \partial F} \|r_1\|_{L^2(E)}^2. \tag{44}$$

In analogy to the estimate regarding r_0, cf. (42), the following result can be proved

$$r_1|_E = \sum_{\xi_p \in \mathcal{V}_{1,E}} w(\xi_p)\phi_{\xi_p} = \sum_{\xi_p \in \mathcal{V}_{1,E}} \sum_{F' \in \mathcal{F}_{\xi_p}} \boldsymbol{\eta}_{F,F'}(\xi_p) [\![v]\!](\xi_p)\phi_{\xi_p},$$

being $\mathcal{V}_{1,E}$ the set of interior nodes of the edge $E \subset \partial F$ and $\mathcal{F}_E := \{F \in \mathcal{F}_h : E \subset F\}$. We then write

$$\|r_1\|_{L^2(E)}^2 \lesssim \sum_{F' \in \mathcal{F}_E} \left\| \sum_{\xi_p \in \mathcal{V}_{1,E}} [\![v]\!](\xi_p)\phi_{\xi_p} \right\|_{L^2(E)}^2$$

$$\lesssim \sum_{F' \in \mathcal{F}_E} \|[\![v]\!]\|_{L^2(E)}^2 + \left\| \sum_{\xi_p \in \mathcal{V}_{1,E}} [\![v]\!](\xi_p)\phi_{\xi_p} - [\![v]\!] \right\|_{L^2(E)}^2,$$

and observe that

$$\sum_{\xi_p \in \mathcal{V}_{1,E}} [\![v]\!](\xi_p)\phi_{\xi_p} - [\![v]\!] \in \mathbb{Q}_0^p(E), \qquad \sum_{\xi_p \in \mathcal{V}_{1,E}} [\![v]\!](\xi_p)\phi_{\xi_p} = 0 \text{ on } \partial E,$$

which implies by (11) and (10)

$$\left\| \sum_{\xi_p \in \mathcal{V}_{1,E}} [\![v]\!](\xi_p)\phi_{\xi_p} - [\![v]\!] \right\|_{L^2(E)}^2 \lesssim \frac{h_E}{p^2} \sum_{\xi_p \in \partial E} |[\![v]\!](\xi_p)|^2 \lesssim \|[\![v]\!]\|_{L^2(E)}^2,$$

hence,

$$\|r_1\|_{L^2(E)}^2 \lesssim \sum_{F' \in \mathcal{F}_E} \|[\![v]\!]\|_{L^2(E)}^2. \tag{45}$$

From (44), (45) and (10), we finally obtain

$$\|r_1\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2} \sum_{F \subset \partial\kappa} \frac{h_F}{p^2} \sum_{E \subset \partial F} \|r_1\|_{L^2(E)}^2$$

$$\lesssim \frac{h_\kappa}{p^2} \sum_{F \subset \partial\kappa} \frac{h_F}{p^2} \sum_{E \subset \partial F} \sum_{F' \in \mathcal{F}_E} \|[\![v]\!]\|_{L^2(E)}^2$$

$$\lesssim \frac{h_\kappa}{p^2} \sum_{F \in \mathcal{F}_h(\kappa)} \|[\![v]\!]\|_{L^2(F)}^2.$$

which combined with the analogous result for r_0 and the bound on the norm of ψ_F, gives the thesis.

Acknowledgements

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