Multigrid algorithms for hp-version Interior Penalty Discontinuous Galerkin methods on polygonal and polyhedral meshes

ANTONIETTI, P. F.; HOUSTON P.; SARTI, M.; VERANI, M.

Multigrid algorithms for hp-version Interior Penalty Discontinuous Galerkin methods on polygonal and polyhedral meshes

Paola F. Antonietti[†‡]   Marco Sarti[†§]   Marco Verani[†¶]   Paul Houston[‖]

Abstract

In this paper we analyze the convergence properties of two-level and W-cycle multigrid solvers for the numerical solution of the linear system of equations arising from hp-version symmetric interior penalty discontinuous Galerkin discretizations of second-order elliptic partial differential equations on polygonal/polyhedral meshes. We prove that the two-level method converges uniformly with respect to the granularity of the grid and the polynomial approximation degree p, provided that the number of smoothing steps, which depends on p, is chosen sufficiently large. An analogous result is obtained for the W-cycle multigrid algorithm, which is proved to be uniformly convergent with respect to the mesh size, the polynomial approximation degree, and the number of levels, provided the latter remains bounded and the number of smoothing steps is chosen sufficiently large. Numerical experiments are presented which underpin the theoretical predictions; moreover, the proposed multilevel solvers are shown to be convergent in practice, even when some of the theoretical assumptions are not fully satisfied.

1   Introduction

The original articles concerned with the construction and mathematical analysis of Discontinuous Galerkin (DG) methods date back over 50 years ago. For hyperbolic partial differential equations, in 1973 Reed & Hill, cf. [RH73], developed the first DG discretization of the neutron transport equation. Independently, DG methods were constructed for elliptic problems based on weakly enforcing Dirichlet boundary conditions; see, for example, [Aub70, Bab73, Lio68, Nit71]. In particular, we highlight the works of Nitsche [Nit71] and Baker [Bak77], which form the basis of the class of interior penalty DG methods, cf. also [Arn82, Whe78]. Since the very early work, DG methods were partially abandoned, in part due to the increase in the number of degrees of freedom compared, for instance, with their conforming counterparts. However, in the last two decades there has been a renewed interest in the field of discontinuous discretizations both from a theoretical and computational viewpoint, cf. [CKS00, HW07, Riv08, DPE12], for example. This resurgence is due to the

---

[†]MOX-Laboratory for Modeling and Scientific Computing, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy.

[‡]Email: paola.antonietti@polimi.it

[§]Email: marco.sarti@polimi.it

[¶]Email: marco.verani@polimi.it

[‖]School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom. Email: paul.houston@nottingham.ac.uk

inherent advantages offered by DG schemes, such as, for example, the limited interelement communication, which is restricted only to neighbouring elements, the local conservativity property, the simplicity in treating meshes with hanging nodes, and the development of efficient hp-adaptivity refinement strategies. Moreover, recently in [BBC$^+$14, BBC$^+$12, BBCR12, CGH14] it has been shown that the underlying DG polynomial bases may be efficiently constructed in the physical frame, without needing to map local polynomial spaces defined in a given reference/canonical frame. In this way, DG methods can easily deal with general-shaped elements, including polygonal/polyhedral elements, cf. [AGH13, BBC$^+$14, CGH14]. The flexibility of DG methods in handling general meshes has no immediate counterpart in the conforming framework, where the design of suitable finite element spaces for meshes of polygons/polyhedra is far from being a trivial task. Several examples include the Composite Finite Element Method [HS97b, HS97a], the Polygonal Finite Element Method [ST04, TS08], the Extended Finite Element Method [FB10] and the most recent Virtual Element Method [BaDVBC$^+$13].

At present, the design of solvers and preconditioners for DG discretizations on nonstandard grids lends itself to huge developments in the field of numerical analysis. Indeed, to the best of our knowledge, the only study regarding solution techniques for this class of problems is reported in [AGH14], where a non-overlapping Schwarz preconditioner for composite DG finite element methods on complicated domains is analyzed. In the current article we exploit the theoretical framework developed in [CGH14] to study the performance of a two-level and W-cycle multigrid solver. The possibility to employ general-shaped elements in the physical framework makes the choice of multilevel schemes natural. The flexibility afforded by this approach allows us to define the set of grids needed in the multigrid algorithm by agglomeration; thereby, the definition of the associated subspaces is straightforward, since inter-element continuity is not required. This property overcomes the usual difficulties encountered in the construction of agglomeration multigrid schemes in the conforming framework, where the agglomeration strategy must be followed by a proper definition of the conforming subspaces. In [CXZ98], for example, the sublevels are obtained by combining a graph based agglomeration algorithm and re-triangulations, thus resulting in a set of non-nested grids, while the associated nested subspaces are defined by introducing suitable interpolation operators. The resulting V-cycle multigrid algorithm converges uniformly with respect to the meshsize h provided that the number of levels is kept fixed.

In this paper we analyze the convergence of a two-level scheme and W-cycle multigrid method for the solution of the linear system of equations arising from the hp-version of the interior penalty DG scheme on polygonal/polyhedral meshes [CGH14], thereby, extending the theoretical framework developed in [ASV] for standard quasi-uniform meshes. Our analysis is based on the smoothing and approximation properties associated with the proposed method: the former corresponds to a Richardson iteration, whose study requires a result concerning the spectral properties of the stiffness matrix, while the latter is inherent to the interior penalty DG scheme itself and exploits the error estimates derived in [CGH14]. We show that, under suitable assumptions on the agglomerated coarse grid, the two-level method converges uniformly with respect to the granularity of the underlying partition and the polynomial approximation degree p, provided that the number of smoothing steps is chosen of order $C(p)p^2$, where $C(p)$ is a constant, which in general depends on p and the geometric properties of the grids. This implies that the generation of good quality agglomerated meshes is fundamental for the performance of the solver. We prove an analogous result for the case of W-cycle multigrid scheme. However, we remark that, in addition to the geometric assumptions on the agglomerated grids, we also require that the maximum number of element faces remains bounded. Due to the agglomeration process, the latter assumption is critical in the case of mul-

tilevel methods, which implies that our analysis is only valid if the number of levels is reasonably small. Throughout the analysis, we also track the dependence of the error reduction factor of the two solvers on the polynomial approximation order p, thereby recovering a similar result to the case when standard quasi-uniform meshes are employed, as well as the geometrical properties of the agglomerated grids.

The rest of this paper is organized as follows. In Section 2 we introduce the interior penalty DG scheme for the discretization of second-order elliptic problems on general meshes consisting of polygonal/polyhedral elements. Then in Section 3, we recall some preliminary analytical results concerning this class of schemes. In Section 4 we define the multilevel framework and introduce several technical results. We then focus first on the analysis of two-level method, followed by the extension to the W-cycle multigrid solver, where we assume that the number of levels obtained by agglomeration is kept limited. The main theoretical results are investigated through a series of numerical experiments presented in Section 5. In particular, we show that, in general, the limitation on the number of levels employed in the W-cycle multigrid solver does not seem to be restrictive in practice.

2 Model problem and discretization

We consider the weak formulation of the Poisson problem, subject to a homogeneous Dirichlet boundary condition: find $u \in V = H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega f v \; dx \qquad \forall v \in V, \tag{1}$$

with $\Omega \in \mathbb{R}^d$, $d = 2, 3$, a convex polygonal/polyhedral domain with Lipschitz boundary and f a given function in $L^2(\Omega)$.

For the sake of brevity, throughout this article, we write $x \lesssim y$ and $x \gtrsim y$ in lieu of $x \leq Cy$ and $x \geq Cy$, respectively, for a positive constant C independent of the discretization parameters. Moreover, $x \approx y$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 y \leq x \leq C_2 y$. When required, the constants will be written explicitly.

In view of the forthcoming multigrid analysis, we denote by $\{\mathcal{T}_j\}_{j=1}^J$ a sequence of partitions of the domain Ω, each of which consists of disjoint open polygonal/polyhedral elements κ of diameter h_κ, such that $\overline{\Omega} = \bigcup_{\kappa \in \mathcal{T}_j} \bar\kappa$, $j = 1, \ldots, J$. We denote the mesh size of $\mathcal{T}_j$, $j = 1, \ldots, J$, by $h_j = \max_{\kappa \in \mathcal{T}_j} h_\kappa$. To each $\mathcal{T}_j$, $j = 1, \ldots, J$, we associate the corresponding discontinuous finite element space V_j, $j = 1, \ldots, J$, defined as

$$V_j = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{P}_{p_j}(\kappa), \kappa \in \mathcal{T}_j\},$$

where $\mathcal{P}_{p_j}(\kappa)$ denotes the space of polynomials of total degree at most $p_j \geq 1$ on $\kappa \in \mathcal{T}_j$. A suitable choice of the sequences $\{\mathcal{T}_j\}_{j=1}^J$ and $\{V_j\}_{j=1}^J$ leads to the so-called h-, p-, and hp-multigrid schemes. In particular, the $h-$multigrid method is based on employing a constant polynomial approximation degree for each j, $j = 1, \ldots, J$, (i.e., $p_j = p$), on a set of nested partitions $\{\mathcal{T}_j\}_{j=1}^J$, such that the coarse level $\mathcal{T}_{j-1}$, $j = 2, \ldots, J$, is obtained by agglomeration from $\mathcal{T}_j$ in such a way that

$$h_{j-1} \lesssim h_j \leq h_{j-1} \qquad \forall j = 2, \ldots, J, \tag{2}$$

i.e., we consider a bounded variation hypothesis between subsequent levels. In the p-multigrid method, the partition is kept fixed for any j, $j = 1, \ldots, J$, while we assume that the polynomial degrees vary moderately from one level to another, *i.e.*,

$$p_{j-1} \leq p_j \lesssim p_{j-1} \qquad \forall j = 2, \ldots, J. \tag{3}$$

The hp-multigrid method is obtained by combining these two strategies. Note that with the above choices we obtain nested finite element spaces V_j, $j = 1, \ldots, J$, *i.e.*, $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_J$.

2.1 Grid assumptions

In this section, we introduce some additional notation from [CGH14], and outline some key definitions and assumptions. For any $\mathcal{T}_j$, $j = 1, \ldots, J$, when no hanging nodes/edges are included in the partition, we define the *interfaces* of the mesh $\mathcal{T}_j$ as the set of $(d-1)$-dimensional facets of the elements $\kappa \in \mathcal{T}_j$. The presence of hanging nodes/edges, on the other hand, can be handled by defining the interfaces of $\mathcal{T}_j$ as the intersection of the $(d-1)$-dimensional facets of neighboring elements. This implies that, for $d = 2$, an interface will always consist of a line segment. For $d = 3$, we assume that for each interface of an element $\kappa \in \mathcal{T}_j$, a sub-triangulation into co-planar triangles is given. We then denote by "face" a $(d-1)$-dimensional simplex (*i.e.*, a line segment for $d = 2$), which is part of the boundary of $\kappa \in \mathcal{T}_j$. As a consequence, in the two dimensional case, the face and interface of an element $\kappa \in \mathcal{T}_j$ coincide. With this notation in mind, we denote by $\mathcal{F}_j$ the set of all mesh interfaces if $d = 2$ and the set of all open triangles belonging to the sub-partition of all mesh interfaces if $d = 3$. Moreover, we have that $\mathcal{F}_j = \mathcal{F}_j^I \cup \mathcal{F}_j^B$, where $\mathcal{F}_j^I$ is the set of interior element faces of $\mathcal{T}_j$, such that $F \subseteq \partial\kappa^+ \cap \partial\kappa^-$ for any $F \in \mathcal{F}_j^I$, where $\kappa^\pm$ are two adjacent elements in $\mathcal{T}_j$. The set $\mathcal{F}_j^B$ contains the boundary element faces, *i.e.*, $F \subset \partial\Omega$ for $F \in \mathcal{F}_j^B$.

With this notation, we introduce the following assumptions on the partitions $\mathcal{T}_j$, $j = 1, \ldots, J$; in the case of the h- and hp-multigrid schemes, these assumptions must be satisfied for the meshes generated by the underlying agglomeration process.

Assumption 1. *The number of faces N_κ of any $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, is uniformly bounded,* i.e., *there exists a constant C_F such that*

$$N_\kappa \leq C_F \qquad \forall \kappa \in \mathcal{T}_j.$$

Assumption 1 is critical in our multilevel framework, because the number of faces N_κ grows with the number of levels due to the agglomeration process. As a consequence, this assumption is only satisfied if the number of levels is kept limited. However, it will be demonstrated in Section 5, that this assumption does not seem to be a limitation in practice.

Assumption 2. *For any $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, we assume that*

$$h_\kappa^d \geq |\kappa| \gtrsim h_\kappa^d,$$

with $d = 2, 3$.

Assumption 3. *Let $\mathcal{T}_j^\sharp = \{\mathcal{K}\}$, $j = 1, \ldots, J$, denote a covering of Ω consisting of shape-regular d-dimensional simplices $\mathcal{K}$. We assume that, for any $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, there exists $\mathcal{K} \in \mathcal{T}_j^\sharp$ such that $\kappa \subset \mathcal{K}$ and*

$$\mathrm{card}\left\{ \kappa' \in \mathcal{T}_j : \kappa' \cap \mathcal{K} \neq \emptyset, \ \mathcal{K} \in \mathcal{T}_j^\sharp \ \ such \ that \ \ \kappa \subset \mathcal{K} \right\} \lesssim 1.$$

Consequently, for each pair κ, $\mathcal{K} \in \mathcal{T}_j^\sharp$, with $\kappa \subset \mathcal{K}$,

$$\mathrm{diam}(\mathcal{K}) \lesssim h_\kappa.$$

To keep the notation as simple as possible we will assume that our decompositions are quasi-uniform.

Assumption 4. *We assume that the mesh size h_j, $j = 1, \ldots, J$, is such that*

$$h_j \approx \min_{\kappa \in \mathcal{T}_j} h_\kappa.$$

We remark that the above assumption can be weakened and, according to [CGH14], only a local bounded variation property is needed for our theoretical analysis; see Remark 23 below for details. We will also need the following definitions.

Definition 5. *For each $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, we denote by $\mathcal{F}_\kappa^\flat$ the set of all possible d-simplices contained in κ and having at least one face in common with κ. Moreover, we denote by $\kappa_F^\flat$, an element in $\mathcal{F}_\kappa^\flat$ sharing a face F with $\kappa \in \mathcal{T}_j$.*

Definition 6. *For any $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, we denote by T_κ the family of all possible sub-tessellations $\mathcal{T}_\kappa$ of κ consisting of d-simplices τ, such that $\bar{\kappa} = \bigcup_{\tau \in \mathcal{T}_\kappa} \bar{\tau}$, and write h_τ to denote the diameter of $\tau \in \mathcal{T}_\kappa$.*

2.2 DG formulation

The definition of the proceeding DG method is based on employing suitable jump and average operators. To this end, for (sufficiently smooth) vector-valued and scalar functions $\boldsymbol{\tau}$ and v, respectively, we define jumps and averages across $F \in \mathcal{F}_j$, $j = 1, \ldots, J$, as follows:

$$[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^-, \quad \{\!\{\boldsymbol{\tau}\}\!\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}, \qquad F \in \mathcal{F}_j^I,$$

$$[\![v]\!] = v^+ \mathbf{n}^+ + v^- \mathbf{n}^-, \qquad \{\!\{v\}\!\} = \frac{v^+ + v^-}{2}, \qquad F \in \mathcal{F}_j^I,$$

$$[\![v]\!] = v^+ \mathbf{n}^+, \qquad\qquad \{\!\{\boldsymbol{\tau}\}\!\} = \boldsymbol{\tau}^+, \qquad\qquad F \in \mathcal{F}_j^B,$$

where $v^\pm$ and $\boldsymbol{\tau}^\pm$ denote the traces of v and $\boldsymbol{\tau}$ on F taken from the interior of $\kappa^\pm$, respectively, and $\mathbf{n}^\pm$ the outward unit normal vectors to $\partial \kappa^\pm$, respectively, cf. [ABCM02].

On any level j, $j = 1, \ldots, J$, we consider the bilinear form $\mathcal{A}_j(\cdot, \cdot) : V_j \times V_j \to \mathbb{R}$, corresponding to the symmetric interior penalty DG method, defined by

$$\mathcal{A}_j(u, v) = \sum_{\kappa \in \mathcal{T}_j} \int_\kappa \nabla u \cdot \nabla v \; dx - \sum_{F \in \mathcal{F}_j} \int_F \left(\{\!\{\nabla u\}\!\} \cdot [\![v]\!] + [\![u]\!] \cdot \{\!\{\nabla v\}\!\} \right) \; ds$$

$$+ \sum_{F \in \mathcal{F}_j} \int_F \sigma_j [\![u]\!] \cdot [\![v]\!] \; ds, \qquad (4)$$

where $\sigma_j \in L^\infty(\mathcal{F}_j)$ denotes the interior penalty stabilization function. To define the stabilization function σ_j, $j = 1, \ldots, J$, we first introduce an inverse inequality for polygonal/polyhedral elements; to this end, we recall the following definition.

Definition 7. *Let $\widetilde{\mathcal{T}}_j$, $j = 1, \ldots, J$, denote the subset of elements $\kappa \in \mathcal{T}_j$, such that each polygonal/polyhedral element $\kappa \in \widetilde{\mathcal{T}}_j$ can be covered by at most $m_{\mathcal{T}_j}$ shape-regular d-simplices K_i, $i = 1, \ldots, m_{\mathcal{T}_j}$, such that*

$$\operatorname{dist}(\kappa, \partial K_i) \lesssim \frac{\operatorname{diam}(K_i)}{p_j^2},$$

and

$$|K_i| \gtrsim c_{as}|\kappa|,$$

for all $i = 1, \ldots, m_{\mathcal{T}_j}$.

The following inverse inequality for general-shaped elements is derived in [CGH14, Lemma 4.4].

Lemma 8. *Let $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, be a polygonal/polyhedral element, and let $F \subset \partial\kappa$ be one of its faces, and $\widetilde{\mathcal{T}}_j$ be defined according to Definition 7. Then, for each $v \in \mathcal{P}_{p_j}(\kappa)$, we have*

$$\|v\|_{L^2(F)}^2 \le C_{INV}(p_j, \kappa, F) \frac{p_j^2|F|}{|\kappa|} \|v\|_{L^2(\kappa)}^2, \tag{5}$$

with

$$C_{INV}(p_j, \kappa, F) = C_{\mathrm{inv}} \begin{cases} \min\left\{\dfrac{|\kappa|}{\sup_{\kappa_F^\flat \subset \kappa}|\kappa_F^\flat|}, p_j^{2d}\right\}, & \text{if } \kappa \in \widetilde{\mathcal{T}}_j, \\[2.5ex] \dfrac{|\kappa|}{\sup_{\kappa_F^\flat \subset \kappa}|\kappa_F^\flat|}, & \text{if } \kappa \in \mathcal{T}_j \setminus \widetilde{\mathcal{T}}_j, \end{cases}$$

and $\kappa_F^\flat \in \mathcal{F}_\kappa^\flat$ as in Definition 5. The positive constant C_{inv} is independent of $|\kappa|/\sup_{\kappa_F^\flat \in \kappa}|\kappa_F^\flat|$, p_j and v.

The interior penalty stabilization function $\sigma_j : \mathcal{F}_j \to \mathbb{R}^+$ is then given by

$$\sigma_j(x) = \begin{cases} C_\sigma^j \displaystyle\max_{\kappa \in \{\kappa^+, \kappa^-\}} \left\{ C_{INV}(p_j, \kappa, F) \dfrac{p_j^2|F|}{|\kappa|} \right\}, & x \in F, \ F \in \mathcal{F}_j^I, \ F \subset \partial\kappa^+ \cap \partial\kappa^-, \\[2.5ex] C_\sigma^j C_{INV}(p_j, \kappa, F) \dfrac{p_j^2|F|}{|\kappa|}, & x \in F, \ F \in \mathcal{F}_j^B, \ F \subset \partial\kappa^+ \cap \partial\Omega, \end{cases}$$

with $C_\sigma^j > 0$ independent of p_j, $|F|$ and $|\kappa|$, cf. [CGH14].

In this article, we develop two-level and W-cycle multigrid schemes to compute the solution of the following problem on the finest level J: find $u_J \in V_J$ such that

$$\mathcal{A}_J(u_J, v_J) = \int_\Omega f v_J \ dx \quad \forall v_J \in V_J. \tag{6}$$

3 Preliminary results

We first endow the finite element spaces V_j, $j = 1, \ldots, J$, with the following DG norm:

$$\|w\|_{DG,j}^2 = \sum_{\kappa \in \mathcal{T}_j} \int_\kappa |\nabla w|^2 \ dx + \sum_{F \in \mathcal{F}_j} \int_F \sigma_j |[\![w]\!]|^2 \ ds.$$

The well–posed of the DG formulation is established in the following lemma.

Lemma 9. *The following continuity and coercivity bounds, respectively, hold*

$$\mathcal{A}_j(u,v) \leq \mathsf{C}_{\mathsf{cont}} \|u\|_{DG,j} \|v\|_{DG,j} \quad \forall u, v \in V_j,$$
$$\mathcal{A}_j(u,u) \geq \mathsf{C}_{\mathsf{coerc}} \|u\|_{DG,j}^2 \qquad \forall u \in V_j, \tag{7}$$

where $\mathsf{C}_{\mathsf{cont}}$ and $\mathsf{C}_{\mathsf{coerc}}$ are positive constants, independent of the discretization parameters, provided that $C_\sigma^j > C_F$, $j = 1, \ldots, J$.

Proof. See [CGH14]. □

The proceeding error estimates are based on the following approximation result, which is a simplified version of the analogous bound presented in [CGH14, Proof of Theorem 5.2]. To this end, we define $\mathcal{E} : H^s(\Omega) \to H^s(\mathbb{R}^d)$, $s \in \mathbb{N}_0$, such that $\mathcal{E}v|_\Omega = v$ to denote the extension operator presented in Stein [Ste70].

Lemma 10. *Assuming Assumptions 1, 2, 3, and 4 hold, let $v|_\kappa \in H^k(\kappa)$, $k > d/2$, such that $\mathcal{E}v|_\mathcal{K} \in H^k(\mathcal{K})$, for each $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, where $\kappa \subset \mathcal{K}$, $\mathcal{K} \in \mathcal{T}_j^\sharp$. Then there exists a projection operator $\widetilde{\Pi}_j : L^2(\Omega) \to V_j$ such that*

$$\|v - \widetilde{\Pi}_j v\|_{DG,j} \lesssim \mathsf{C}_{\mathsf{interp}}(p_j) \frac{h_j^{s-1}}{p_j^{k-1}} \|v\|_{H^k(\Omega)}, \tag{8}$$

where

$$\mathsf{C}_{\mathsf{interp}}^2(p_j) = \max_{\kappa \in \mathcal{T}_j} \left(1 + p_j \sum_{F \subset \partial\kappa} C_{INV}(p_j, F) C_m(p_j, F) \right),$$

with $s = \min\{p_j + 1, k\}$, and

$$C_{INV}(p_j, F) = \begin{cases} \max_{\kappa \in \{\kappa^+, \kappa^-\}} C_{INV}(p_j, \kappa, F), & F \in \mathcal{F}_j^I, F \subset \partial\kappa^+ \cap \partial\kappa^-, \\ C_{INV}(p_j, \kappa, F), & F \in \mathcal{F}_j^B, F \subset \partial\Omega \cap \partial\kappa. \end{cases}$$

Analogously,

$$C_m(p_j, F) = \begin{cases} \max_{\kappa \in \{\kappa^+, \kappa^-\}} C_m(p_j, \kappa, F), & F \in \mathcal{F}_j^I, F \subset \partial\kappa^+ \cap \partial\kappa^-, \\ C_m(p_j, \kappa, F), & F \in \mathcal{F}_j^B, F \subset \partial\Omega \cap \partial\kappa, \end{cases}$$

with

$$C_m(p_j, \kappa, F) = \min \left\{ \frac{h_\kappa^d}{\sup_{\kappa_F^\flat \subset \kappa} |\kappa_F^\flat|}, \frac{1}{p_j^{1-d}} \right\}.$$

We point that, as for Lemma 10 stated above, any bound derived under the validity of Assumption 1 will necessarily lead to a dependence on C_F in the resulting constant. Next, we state error bounds for the underlying interior penalty DG scheme in terms of both the DG and $L^2(\Omega)$-norms, cf. [CGH14].

Theorem 11. *Assume Assumptions 1, 2, 3, and 4 hold. We denote by $u_j \in V_j$, $j = 1, \ldots, J$, the DG solution of problem* (6) *posed on level j, i.e.,*

$$\mathcal{A}_j(u_j, v_j) = \int_\Omega f v_j \, dx \quad \forall v_j \in V_j.$$

If the solution u of (1) *satisfies $u|_\kappa \in H^k(\kappa)$, $k > 1 + d/2$, such that $\mathcal{E}u|_\mathcal{K} \in H^k(\mathcal{K})$, for each $\kappa \in \mathcal{T}_j$, $j = 1, \ldots, J$, where $\kappa \subset \mathcal{K}$, $\mathcal{K} \in \mathcal{T}_j^\sharp$, then the following results hold*

$$\|u - u_j\|_{DG,j} \lesssim \mathsf{G}(p_j) \frac{h_j^{(s-1)}}{p_j^{(k-1)}} \|u\|_{H^k(\Omega)}, \tag{9}$$

$$\|u - u_j\|_{L^2(\Omega)} \lesssim \mathsf{C}_{\mathsf{L}^2}(p_j) \frac{h_j^s}{p_j^k} \|u\|_{H^k(\Omega)}, \tag{10}$$

where

$$\mathsf{G}^2(p_j) = 1 + \max_{\kappa \in \mathcal{T}_j} \mathcal{G}_\kappa(F, C_{INV}, C_m, p_j), \qquad \mathsf{C}_{\mathsf{L}^2}(p_j) = \mathsf{G}(p_j) \mathsf{C}_{\mathsf{interp}}(p_j),$$

and

$$\mathcal{G}_\kappa(F, C_{INV}, C_m, p_j) = \frac{1}{p_j} \sum_{F \subset \partial\kappa} \frac{C_m(p_j, F)}{C_{INV}(p_J, F)}$$

$$+ p_j \sum_{F \subset \partial\kappa} C_{INV}(p_j, F) C_m(p_j, F),$$

with $s = \min\{p_j + 1, k\}$, $p_j \geq 1$, and $\mathsf{C}_{\mathsf{interp}}(p_j)$, $C_m(p_j, F)$ and $C_{INV}(p_j, F)$ defined as in Lemma 10.

Proof. The error bound (9) stems from the general result derived in [CGH14, Theorem 5.2]. We now proceed with the proof of the bound on the $L^2(\Omega)$-norm of the error, cf. (10). To this end, we employ a standard duality argument: let $w \in V$, be the solution of the problem

$$\int_\Omega \nabla w \cdot \nabla v \, dx = \int_\Omega (u - u_j) v \, dx \qquad \forall v \in V,$$

$j = 1, \ldots, J$. Exploiting a standard elliptic regularity assumption, we note that

$$\|w\|_{H^2(\Omega)} \lesssim \|u - u_j\|_{L^2(\Omega)}.$$

According to Galerkin orthogonality, we immediately obtain

$$\|u - u_j\|_{L^2(\Omega)}^2 = \mathcal{A}_j(u - u_j, w)$$
$$= \mathcal{A}_j(u - u_j, w - w_I)$$
$$\lesssim \|u - u_j\|_{DG,j} \|w - w_I\|_{DG,j}$$

for all $w_I \in V_j$. Hence, selecting $w_I = \tilde{\Pi}_j w$, by (8) we get

$$\|w - w_I\|_{DG,j} \lesssim \mathsf{C}_{\mathsf{interp}}(p_j) \frac{h_j}{p_j} \|w\|_{H^2(\Omega)} \lesssim \mathsf{C}_{\mathsf{interp}}(p_j) \frac{h_j}{p_j} \|u - u_j\|_{L^2(\Omega)},$$

which together with (9) gives the desired result. $\qquad \square$

We also need to introduce an appropriate inverse inequality; to this end, we first recall the following result, cf. [Geo08, Lemma 3.7].

Lemma 12. *Let K be a shape-regular simplex. Then for any $v \in \mathcal{P}_p(K)$ there exists a simplex $\hat{\kappa} \subset K$, with the same shape as K and faces parallel to the faces of K, with $\mathrm{dist}(\partial\hat{\kappa}, \partial K) < C_{as}\mathrm{diam}(K)/p^2$, for some constant $C_{as} > 0$, independent of v, K and p, such that*

$$\|v\|_{L^2(\hat{\kappa})} \geq \frac{1}{2}\|v\|_{L^2(K)}.$$

With the above result we now prove the following lemma.

Lemma 13. *For any $v \in V_j$, $j = 1, \ldots, J$, the following inverse estimate holds*

$$\|\nabla u\|_{L^2(\kappa)}^2 \lesssim C_I(p_j, \kappa)p_j^4 h_\kappa^{-2}\|u\|_{L^2(\kappa)}^2,$$

with

$$C_I(p_j, \kappa) = \begin{cases} \min\left\{\dfrac{h_\kappa^2}{\sup_{\mathcal{T}_\kappa \in \mathsf{T}_\kappa} \min_{\tau \in \mathcal{T}_\kappa} h_\tau^2}, p_j^{2d}\right\}, & \text{if } \kappa \in \widetilde{\mathcal{T}}_j, \\ \dfrac{h_\kappa^2}{\sup_{\mathcal{T}_\kappa \in \mathsf{T}_\kappa} \min_{\tau \in \mathcal{T}_\kappa} h_\tau^2}, & \text{if } \kappa \in \mathcal{T}_j \setminus \widetilde{\mathcal{T}}_j, \end{cases}$$

Proof. For $\kappa \in \mathcal{T}_j \setminus \tilde{\mathcal{T}}_j$, we recall the family T_κ of sub-tessellations $\mathcal{T}_\kappa$ of κ defined in Definition 6. We then have, by standard inverse estimates on simplicial elements, that

$$\|\nabla u\|_{L^2(\kappa)}^2 = \sum_{\tau \in \mathcal{T}_\kappa} \|\nabla u\|_{L^2(\tau)}^2 \lesssim p_j^4 \sum_{\tau \in \mathcal{T}_\kappa} h_\tau^{-2}\|u\|_{L^2(\tau)}^2 \lesssim \frac{p_j^4}{\min_{\tau \in \mathcal{T}_\kappa} h_\tau^2}\|u\|_{L^2(\kappa)}^2.$$

In order to obtain a sharp bound, we take the supremum over all $\mathcal{T}_\kappa \in \mathsf{T}_\kappa$, namely,

$$\|\nabla u\|_{L^2(\kappa)}^2 \lesssim p_j^4 h_\kappa^{-2} \frac{h_\kappa^2}{\sup_{\mathcal{T}_\kappa \in \mathsf{T}_\kappa} \min_{\tau \in \mathcal{T}_\kappa} h_\tau^2}\|u\|_{L^2(\kappa)}^2. \tag{11}$$

For $\kappa \in \tilde{\mathcal{T}}_j$, we consider the covering of κ by shape-regular simplices K_i, $i = 1, \ldots, m_{\mathcal{T}_j}$, such that

$$|K_i| \gtrsim |\kappa|, \tag{12}$$

see Definition 7. We recall the following inverse estimates on the simplex K_i, cf. [Sch98, Geo08],

$$\|\nabla u\|_{L^\infty(K_i)}^2 \lesssim p_j^4 h_{K_i}^{-2}\|u\|_{L^\infty(K_i)}^2, \tag{13}$$
$$\|u\|_{L^\infty(K_i)}^2 \lesssim p_j^{2d}|K_i|^{-1}\|u\|_{L^2(K_i)}^2, \tag{14}$$

for any $u \in \mathcal{P}_{p_j}(K_i)$. By exploiting the covering of $\kappa \subset \bigcup_{i=1}^{m_{\mathcal{T}_j}} K_i$ and the bounds (13) and (14), we obtain

$$\|\nabla u\|_{L^2(\kappa)}^2 \lesssim |\kappa|\|\nabla u\|_{L^\infty(\kappa)}^2 \lesssim |\kappa| \sum_{i=1}^{m_{\mathcal{T}_j}} \|\nabla u\|_{L^\infty(K_i)}^2$$

$$\lesssim |\kappa|p_j^4 \sum_{i=1}^{m_{\mathcal{T}_j}} h_{K_i}^{-2}\|u\|_{L^\infty(K_i)}^2 \lesssim |\kappa|p_j^{4+2d} \sum_{i=1}^{m_{\mathcal{T}_j}} \frac{h_{K_i}^{-2}}{|K_i|}\|u\|_{L^2(K_i)}^2. \tag{15}$$

9

We now define $\hat{\kappa}_i \subset K_i$ to denote the simplex relative to K_i as outlined in Lemma 12. Hence, utilizing Lemma 12 and Definition 7, gives

$$\frac{1}{4}\|u\|^2_{L^2(K_i)} \leq \|u\|^2_{L^2(\hat{\kappa}_i)} \leq \|u\|^2_{L^2(K_j \cap \kappa)}, \tag{16}$$

since $\hat{\kappa}_i \subset \kappa$, and hence $\hat{\kappa}_i \subset K_i \cap \kappa \subset K_i$, cf. [CGH14]. Substituting (16) into (15), and employing inequality (12) gives

$$\|\nabla u\|^2_{L^2(\kappa)} \lesssim p_j^{4+2d} \sum_{i=1}^{m_{\mathcal{T}_j}} h_{K_i}^{-2} \|u\|^2_{L^2(K_i \cap \kappa)} \lesssim p_j^{4+2d} h_\kappa^{-2} \|u\|^2_{L^2(\kappa)}, \tag{17}$$

where, given (12), we assume that $h_{K_i} \gtrsim h_\kappa$. We then take the minimum between (11) and (17) to deduce the desired result. $\qquad\square$

The inverse estimate presented in Lemma 13 is fundamental to the proof of the following upper bound on the maximum eigenvalue of $\mathcal{A}_j(\cdot, \cdot)$. We remind that the analogous result on standard grids can be found in [AH11].

Theorem 14. *Given that Assumptions 1, 2 and 4 hold, then for any $u \in V_j$, $j = 1, \ldots, J$, we have that*

$$\mathcal{A}_j(u, u) \lesssim \mathsf{C}_{\mathsf{eig}}(p_j) \frac{p_j^4}{h_j^2} \|u\|^2_{L^2(\Omega)},$$

where $\mathsf{C}_{\mathsf{eig}}(p_j) = \mathsf{C}_{\mathsf{I}}(p_j) + C_\sigma^j \mathrm{C}_{INV}^2(p_j)$, $C_{INV}(p_j) = \max_{\kappa \in \mathcal{T}_j} C_{INV}(p_j, \kappa, F)$, and $\mathsf{C}_{\mathsf{I}}(p_j) = \max_{\kappa \in \mathcal{T}_j} \mathsf{C}_{\mathsf{I}}(p_j, \kappa)$.

Proof. Given the continuity of the bilinear forms $\mathcal{A}_j(\cdot, \cdot)$ stated in Lemma 9, we restrict ourselves to estimate the two terms involved in the DG norm. The local contributions of the H^1 seminorm can be simply bounded by applying Lemma 13 and the quasi-uniformity of the partition, *i.e.*,

$$\sum_{\kappa \in \mathcal{T}_j} |u|^2_{H^1(\kappa)} \lesssim \sum_{\kappa \in \mathcal{T}_j} \mathsf{C}_{\mathsf{I}}(p_j, \kappa) p_j^4 h_\kappa^{-2} \|u\|^2_{L^2(\kappa)} \lesssim \mathsf{C}_{\mathsf{I}}(p_j) \frac{p_j^4}{h_j^2} \|u\|^2_{L^2(\Omega)}.$$

For the norm of the jump across $F \in \mathcal{F}_j^I$, with $F \subset \partial\kappa^+ \cap \partial\kappa^-$, we employ the inverse inequality (5); thereby,

$$\|\sigma_j^{1/2} [\![u]\!]\|^2_{L^2(F)} = \int_F \sigma_j |[\![u]\!]|^2 \, ds \lesssim \int_F \sigma_j |u^+|^2 \, ds + \int_F \sigma_j |u^-|^2 \, ds$$

$$\lesssim \sigma_j \left(C_{INV}(p_j, \kappa^+, F) \frac{|F|}{|\kappa^+|} p_j^2 \|u\|^2_{L^2(\kappa^+)} \right.$$

$$\left. + C_{INV}(p_j, \kappa^-, F) \frac{|F|}{|\kappa^-|} p_j^2 \|u\|^2_{L^2(\kappa^-)} \right)$$

$$\lesssim C_\sigma^j p_j^4 \left(\max_{\kappa \in \{\kappa^+, \kappa^-\}} C_{INV}(p_j, \kappa, F) \frac{|F|}{|\kappa|} \right)^2 (\|u\|^2_{L^2(\kappa^+)} + \|u\|^2_{L^2(\kappa^-)}).$$

Summing over the internal faces and employing Assumptions 1, 2, and 4 gives

$$
\sum_{F \in \mathcal{F}_j^I} \| \sigma_j^{1/2} [\![u]\!] \|_{L^2(F)}^2 \lesssim \sum_{\kappa \in \mathcal{T}_j} \sum_{F \subset \partial \kappa} C_\sigma^j p_j^4 \left(\max_{\kappa \in \{\kappa^+, \kappa^-\}} C_{INV}(p_j, \kappa, F) \frac{|F|}{|\kappa|} \right)^2 \| u \|_{L^2(\kappa)}^2
$$

$$
\lesssim C_\sigma^j \mathrm{C}_{INV}^2(p_j) p_j^4 \sum_{\kappa \in \mathcal{T}_j} \frac{h_\kappa^{2(d-1)}}{h_\kappa^{2d}} \| u \|_{L^2(\kappa)}^2
$$

$$
\lesssim C_\sigma^j \mathrm{C}_{INV}^2(p_j) \frac{p_j^4}{h_j^2} \| u \|_{L^2(\Omega)}^2.
$$

An analogous result also holds for boundary faces; the statement of the theorem now follows immediately. $\qquad \square$

The theoretical results derived in this section form the basis of the analysis of the proposed multigrid algorithms presented in the following section.

4 Two-level and W-cycle multigrid algorithms

The forthcoming analysis is based on the classical multigrid theoretical framework already employed in [ASV] for high-order DG schemes on standard quasi-uniform meshes. The two key ingredients in the construction of our proposed multigrid schemes are the inter-grid transfer operators and the smoothing scheme. The prolongation operator connecting the space V_{j-1} to V_j, $j = 2, \ldots, J$, is denoted by $I_{j-1}^j : V_{j-1} \to V_j$, while its adjoint with respect to the $L^2(\Omega)$-inner product $(\cdot, \cdot)$ is the restriction operator $I_j^{j-1} : V_j \to V_{j-1}$:

$$
(I_{j-1}^j v, w) = (v, I_j^{j-1} w) \qquad \forall v \in V_{j-1}, w \in V_j.
$$

As a smoothing scheme, we choose a Richardson iteration, whose operator is defined as:

$$
B_j = \Lambda_j \mathrm{Id}_j, \tag{18}
$$

with Id_j the identity operator on level V_j, and $\Lambda_j \in \mathbb{R}$ is an upper bound for the spectral radius of the operator $A_j : V_j \to V_j$, defined as

$$
(A_j u, v) = \mathcal{A}_j(u, v) \quad \forall u, v \in V_j, \quad j = 1, \ldots, J. \tag{19}
$$

For the definition of the solvers, we first address the two-level method. Given the following problem

$$
A_J u_J = f_J,
$$

with $A_J : V_J \to V_J$ defined according to (19), and $f_J \in V_J$ such that

$$
(f_J, v) = \int_\Omega f v \, dx \quad \forall v \in V_J,
$$

in Algorithm 1 we outline the two-level cycle, where $\mathsf{MG}_{2\mathsf{lvl}}(z_0, m_1, m_2)$ denotes the approximate solution obtained after one iteration, with initial guess z_0 and m_1, m_2 pre- and post-smoothing steps, respectively.

Algorithm 1 Two-level scheme

Pre-smoothing:
for $i = 1, \dots, m_1$ **do**
$\quad z^{(i)} = z^{(i-1)} + B_J^{-1}(f_J - A_J z^{(i-1)});$
end for

Coarse grid correction:
$r_{J-1} = I_J^{J-1}(f_J - A_J z^{(m_1)});$
$e_{J-1} = A_{J-1}^{-1} r_{J-1};$
$z^{(m_1+1)} = z^{(m_1)} + I_{J-1}^J e_{J-1};$

Post-smoothing:
for $i = m_1 + 2, \dots, m_1 + m_2 + 1$ **do**
$\quad z^{(i)} = z^{(i-1)} + B_J^{-1}(f_J - A_J z^{(i-1)});$
end for

$\mathsf{MG}_{2\mathsf{lvl}}(z_0, m_1, m_2) = z^{(m_1+m_2+1)}.$

As a multilevel extension of Algorithm 1, we consider a standard W-cycle scheme. On level j, we consider

$$A_j z = g,$$

for a given $g \in V_j$. The approximate solution obtained by applying the j-th level iteration to the above linear system, with initial guess z_0 and m_1, m_2 number of pre- and post-smoothing steps, respectively, is denoted by $\mathsf{MG}_{\mathcal{W}}(j, g, z_0, m_1, m_2)$. On the coarsest level $j = 1$, the corresponding subproblem is solved based on employing a direct method, *i.e.*,

$$\mathsf{MG}_{\mathcal{W}}(1, g, z_0, m_1, m_2) = A_1^{-1} g,$$

while for $j > 1$ we apply the recursive procedure outlined in Algorithm 2. We observe that Algorithm 1 can be considered as a special case of Algorithm 2, corresponding to $J = 2$.

4.1   Convergence analysis of the two-level method

We first define the following norms based on the operator A_j, $j = 1, \dots, J$,

$$\|v\|_{s,j} = \sqrt{(A_j^s v, v)_j} \qquad \forall s \in \mathbb{R}, \ v \in V_j, \quad j = 1, \dots, J.$$

Hence,

$$\|v\|_{1,j}^2 = (A_j v, v)_j = \mathcal{A}_j(v, v), \quad \|v\|_{0,j}^2 = (v, v)_j = \|v\|_{L^2(\Omega)}^2 \quad \forall v \in V_j.$$

For the proceeding analysis, we need to introduce some additional hypotheses on the agglomerated meshes employed both within the two-level method studied in this section, as well as the W-cycle multigrid algorithm analyzed in Section 4.2. To this end, for any $F \in \mathcal{F}_j \cap \mathcal{F}_{j-1}$, $j = 2, \dots, J$, we denote by $\kappa_j^\pm$ and $\kappa_{j-1}^\pm$ the neighboring elements sharing the face F in $\mathcal{T}_j$ and $\mathcal{T}_{j-1}$, respectively. It is trivial to see that $\kappa_j^\pm \subset \kappa_{j-1}^\pm$, since the grids are nested. We then assume that, there exists

---

Algorithm 2 Multigrid W-cycle scheme

Pre-smoothing:
for $i = 1, \ldots, m_1$ **do**
$\quad z^{(i)} = z^{(i-1)} + B_j^{-1}(g - A_j z^{(i-1)});$
end for

Coarse grid correction:
$r_{k-1} = I_j^{j-1}(g - A_j z^{(m_1)});$
$\overline{e}_{j-1} = \mathsf{MG}_{\mathcal{W}}(j-1, r_{j-1}, 0, m_1, m_2);$
$e_{j-1} = \mathsf{MG}_{\mathcal{W}}(j-1, r_{j-1}, \overline{e}_{j-1}, m_1, m_2);$
$z^{(m_1+1)} = z^{(m_1)} + I_{j-1}^j e_{j-1};$

Post-smoothing:
for $i = m_1 + 2, \ldots, m_1 + m_2 + 1$ **do**
$\quad z^{(i)} = z^{(i-1)} + B_j^{-1}(g - A_j z^{(i-1)});$
end for

$\mathsf{MG}_{\mathcal{W}}(j, g, z_0, m_1, m_2) = z^{(m_1+m_2+1)}.$

$\Theta > 0$ such that

$$1 < \frac{|\kappa_{j-1}^{\pm}|}{|\kappa_j^{\pm}|} \leq \Theta \quad \forall F \in \mathcal{F}_j \cap \mathcal{F}_{j-1}, \tag{20}$$

and

$$\frac{|\kappa_j^{\pm}|}{\sup_{\kappa_F^{\flat} \in \kappa_j^{\pm}} |\kappa_F^{\flat}|} \approx \frac{|\kappa_{j-1}^{\pm}|}{\sup_{\kappa_F^{\flat} \in \kappa_{j-1}^{\pm}} |\kappa_F^{\flat}|}, \tag{21}$$

which implies, together with (3), that

$$C_{INV}(p_j, \kappa_j^{\pm}, F) \approx C_{INV,j-1}(p_{j-1}, \kappa_{j-1}^{\pm}, F),$$

for any $F \in \mathcal{F}_j \cap \mathcal{F}_{j-1}$, $j = 2, \ldots, J$.

Remark 15. *The above assumption is satisfied if the agglomeration algorithm preserves the shape-regularity of the elements. In Figure 1, we show two examples of possible macroelements: the agglomerate on the left is not suitable to guarantee assumption* (21) *due to the presence of a dominant dimension, while the element on the right can be considered appropriate. Moreover, we note that the fulfilment of the above geometric assumptions* (20) *and* (21) *can be considered a good criterion in evaluating the quality of the agglomerated grids employed in the multigrid algorithm.*

In order to undertake the convergence analysis of the two-level solver outlined in Algorithm 1, we follow the approach developed in [ASV]. We then provide an estimate based on the error propagation operator, which is defined as

$$\mathbb{E}_{m_1,m_2}^{2\mathrm{lvl}} v = G_J^{m_2}(\mathrm{Id}_J - I_{J-1}^J P_J^{J-1}) G_J^{m_1}, \tag{22}$$

with $G_J = \mathrm{Id}_J - B_J^{-1} A_J$, and the operator $P_J^{J-1} : V_J \to V_{J-1}$ defined as

$$\mathcal{A}_{J-1}(P_J^{J-1} v, w) = \mathcal{A}_J(v, I_{J-1}^J w) \qquad \forall v \in V_J, w \in V_{J-1}. \tag{23}$$
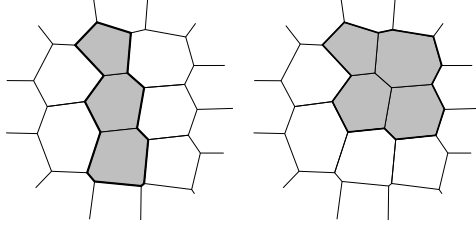
Figure 1: Examples of agglomerated elements.

We now study separately the *smoothing property* and the *approximation property*. Before proceeding with the analysis, we first observe that by Theorem 14, we can bound Λ_j, $j = 1, \ldots, J$, in (18) as follows

$$\Lambda_j \lesssim \mathsf{C}_{\mathsf{eig}}(p_j) \frac{p_j^4}{h_j^2}.$$

The last result is employed to prove the smoothing property in the next lemma; see [ASV, Lemma 4.3] for the proof.

Lemma 16 (Smoothing property). *For any $v \in V_j$, $j = 1, \ldots, J$, we have*

$$
\begin{aligned}
&\|G_j^m v\|_{1,j} \leq \|v\|_{1,j}, \\
&\|G_j^m v\|_{s,j} \lesssim \mathsf{C}_{\mathsf{eig}}(p_j)^{(s-t)/2} p_j^{2(s-t)} h_j^{t-s} (1+m)^{(t-s)/2} \|v\|_{t,j}
\end{aligned}
\tag{24}
$$

for $0 \leq t < s \leq 2$ and $m \in \mathbb{N} \setminus \{0\}$.

The *approximation property* results by exploiting the $L^2(\Omega)$ error estimates stated in (10) on levels J and $J-1$.

Lemma 17 (Approximation property). *For any $v \in V_J$, the following inequality holds*

$$\|(\mathrm{Id}_J - I_{J-1}^J P_J^{J-1})v\|_{0,J} \lesssim \mathsf{C}_{\mathsf{L}^2}(p_J) \frac{h_{J-1}^2}{p_{J-1}^2} \|v\|_{2,J}.
\tag{25}$$

Proof. For any $v \in V_J$, we consider the following equality

$$
\begin{aligned}
\|(\mathrm{Id}_J - I_{J-1}^J P_J^{J-1})v\|_{0,J} &= \|(\mathrm{Id}_J - I_{J-1}^J P_J^{J-1})v\|_{L^2(\Omega)} \\
&= \sup_{0 \neq \phi \in L^2(\Omega)} \frac{\int_\Omega \phi (\mathrm{Id}_J - I_{J-1}^J P_J^{J-1})v \; dx}{\|\phi\|_{L^2(\Omega)}}.
\end{aligned}
\tag{26}
$$

Next, we consider the solution η of the following problem

$$\int_\Omega \nabla \eta \cdot \nabla v \; dx = \int_\Omega \phi v \; dx \quad \forall v \in V,$$

for $\phi \in L^2(\Omega)$, and let $\eta_J \in V_J$ and $\eta_{J-1} \in V_{J-1}$ be the corresponding DG approximations in V_J and V_{J-1}, respectively, given by

$$
\begin{aligned}
\mathcal{A}_J(\eta_J, v) &= \int_\Omega \phi v \; dx \quad \forall v \in V_J, \\
\mathcal{A}_{J-1}(\eta_{J-1}, v) &= \int_\Omega \phi v \; dx \quad \forall v \in V_{J-1}.
\end{aligned}
\tag{27}
$$

14

By Theorem 11 and the hypotheses (2) and (3), we deduce that

$$\|\eta - \eta_J\|_{L^2(\Omega)} \lesssim C_{L^2}(p_J) \frac{h_{J-1}^2}{p_{J-1}^2} \|\eta\|_{H^2(\Omega)},$$

$$\|\eta - \eta_{J-1}\|_{L^2(\Omega)} \lesssim C_{L^2}(p_{J-1}) \frac{h_{J-1}^2}{p_{J-1}^2} \|\eta\|_{H^2(\Omega)},$$

and from a standard elliptic regularity assumption, it follows that

$$\|\eta - \eta_J\|_{L^2(\Omega)} \lesssim C_{L^2}(p_J) \frac{h_{J-1}^2}{p_{J-1}^2} \|\phi\|_{L^2(\Omega)},$$

$$\|\eta - \eta_{J-1}\|_{L^2(\Omega)} \lesssim C_{L^2}(p_{J-1}) \frac{h_{J-1}^2}{p_{J-1}^2} \|\phi\|_{L^2(\Omega)}, \tag{28}$$

Recalling the definition of P_J^{J-1}, cf. (23), and (27), for any $w \in V_{J-1}$, we get

$$\mathcal{A}_{J-1}(P_J^{J-1}\eta_J, w) = \mathcal{A}_J(\eta_J, I_{J-1}^J w) = \mathcal{A}_J(\eta_J, w) = \int_\Omega \phi w \, dx = \mathcal{A}_{J-1}(\eta_{J-1}, w).$$

Hence,

$$\eta_{J-1} = P_J^{J-1}\eta_J. \tag{29}$$

According to [ASV, Lemma 4.1], the following generalized Cauchy-Schwarz inequality holds

$$\mathcal{A}_J(v, w) \leq \|v\|_{0,J} \|w\|_{2,J}, \tag{30}$$

for any $v, w \in V_J$. Next, we observe that by Assumption 2, hypothesis (21) and (3) we can state the following results for any $F \in \mathcal{F}_J \cap \mathcal{F}_{J-1}$

$$C_m(p_J, F) \approx C_m(p_{J-1}, F), \qquad C_{INV}(p_J, F) \approx C_{INV}(p_{J-1}, F),$$
$$C_{interp}(p_J) \approx C_{interp}(p_{J-1}), \qquad \mathcal{G}(p_J) \approx \mathcal{G}(p_{J-1}),$$

which implies that

$$C_{L^2}(p_J) \approx C_{L^2}(p_{J-1}).$$

We now employ (27) and the definition of P_J^{J-1} in (23), followed by (29), the Cauchy-Schwarz inequality (30) and the error estimates (28), to get

$$\int_\Omega \phi (\mathrm{Id}_J - I_{J-1}^J P_J^{J-1}) v \, dx = \mathcal{A}_J(\eta_J, v) - \mathcal{A}_J(\eta_J, I_{J-1}^J P_J^{J-1} v)$$

$$= \mathcal{A}_J(\eta_J, v) - \mathcal{A}_{J-1}(P_J^{J-1}\eta_J, P_J^{J-1} v)$$

$$= \mathcal{A}_J(\eta_J, v) - \mathcal{A}_{J-1}(\eta_{J-1}, P_J^{J-1} v)$$

$$= \mathcal{A}_J(\eta_J - I_{J-1}^J \eta_{J-1}, v)$$

$$\leq \|\eta_J - \eta_{J-1}\|_{0,J} \|v\|_{2,J}$$

$$\leq (\|\eta_J - \eta\|_{L^2(\Omega)} + \|\eta_{J-1} - \eta\|_{L^2(\Omega)}) \|v\|_{2,J}$$

$$\lesssim (C_{L^2}(p_J) + C_{L^2}(p_{J-1})) \frac{h_{J-1}^2}{p_{J-1}^2} \|\phi\|_{L^2(\Omega)} \|v\|_{2,J}$$

$$\lesssim C_{L^2}(p_J) \frac{h_{J-1}^2}{p_{J-1}^2} \|\phi\|_{L^2(\Omega)} \|v\|_{2,J}. \tag{31}$$

15

Substituting (31) into (26) gives the desired result. □

The convergence result for the two-level method, involving the error propagation operator $\mathbb{E}_{m_1,m_2}^{2\text{lvl}}$ defined in (22), is obtained by combining Lemma 16 and Lemma 17.

Theorem 18. *There exists a positive constant $C_{2\text{lvl}}$ independent of the mesh size and the polynomial approximation degree, such that*

$$\||\mathbb{E}_{m_1,m_2}^{2\text{lvl}}v\||_{1,J} \leq \mathsf{C}_{2\text{lvl}}\Sigma_J\||v\||_{1,J}, \tag{32}$$

for any $v \in V_J$, with

$$\Sigma_J = \tilde{\mathsf{C}}(p_J)\frac{p_J^2}{(1+m_1)^{1/2}(1+m_2)^{1/2}},$$

where $\tilde{\mathsf{C}}(p_J) = \mathsf{C}_{\text{eig}}(p_J)\mathsf{C}_{\mathsf{L}^2}(p_J)$. Therefore, the two-level method converges uniformly provided the number of pre- and post-smoothing steps satisfy

$$(1+m_1)^{1/2}(1+m_2)^{1/2} \geq \chi\tilde{\mathsf{C}}(p_J)p_J^2,$$

for a positive constant $\chi > C_{2\text{lvl}}$.

Proof. The statement of the theorem follows in a straightforward manner by applying the smoothing property (24) twice, the approximation property (25) and exploiting the bounded variation assumptions (2) and (3). □

We observe that the rate of convergence is independent of the mesh size, but depends on p_J, as in the case of standard quasi-uniform meshes. Moreover, a dependence on p_J is also hidden in $\tilde{\mathsf{C}}(p_J)$, which also involves the geometric properties of the partitions. As a consequence, a good quality agglomerated coarse grid is fundamental to guarantee the uniformity of the solver.

4.2 Convergence of the W-cycle multigrid algorithm

As mentioned at the beginning of Section 2, we recall that in the multilevel case, Assumption 1 represents a critical issue. In the following analysis, we then assume that the number of levels is limited, in such a way that the number of interfaces on each level can be bounded by a constant C_F that does not lead to an excessive over-penalization due to the penalization parameter C_σ^j, $j = 1, \ldots, J$.

To proceed, we first need to establish the equivalence between DG norms on subsequent grid levels. We point out that, in contrast to the case of standard quasi-uniform grids presented in [ASV], such an equivalence result does not follow in a straightforward manner; indeed, here we need to exploit the hypotheses (20) and (21) introduced in the previous section. Under these assumptions, the proof of the following result follows immediately.

Lemma 19. *Assuming (21) holds, then for any $v \in V_{j-1}$, $j = 2, \ldots, J$, we have that*

$$\|v\|_{DG,j} \leq \mathsf{C}_{\text{equiv}}\|v\|_{DG,j-1}, \tag{33}$$

where $\mathsf{C}_{\text{equiv}} = \mathsf{C}_{\text{equiv}}(\Theta)$, in general, depends on the quality of the agglomerated grids.

16

Lemma 19 is essential to deduce the stability of the operators I_{j-1}^j and P_j^{j-1}, $j = 2, \ldots, J$. In particular, we state the following bounds.

Lemma 20. *There exists a positive constant* $\mathsf{C}_{\mathsf{stab}}$, *independent of the mesh size, the polynomial approximation degree and the level* j, $j = 2, \ldots, J$, *such that*

$$\|\!|I_{j-1}^j v|\!\|_{1,j} \leq \mathsf{C}_{\mathsf{stab}} \|\!|v|\!\|_{1,j-1} \qquad\qquad \forall v \in V_{j-1}, \tag{34}$$

$$\|\!|P_j^{j-1} v|\!\|_{1,j-1} \leq \mathsf{C}_{\mathsf{stab}} \|\!|v|\!\|_{1,j} \qquad\qquad \forall v \in V_j. \tag{35}$$

The proof of Lemma 20 is based on employing inequality (33); for details, see [ASV, Lemma 4.6].

Remark 21. *We stress that the constant* $\mathsf{C}_{\mathsf{stab}}$ *depends on* $\mathsf{C}_{\mathsf{equiv}}$ *in* (33), *which means that the quality of the agglomerated meshes plays a crucial role in keeping this constant bounded, thus resulting in the uniformity with respect to the mesh size and the number of levels as shown in Theorem 22 below.*

The error propagation operator associated to Algorithm 2 is defined as

$$\begin{cases} \mathbb{E}_{1,m_1,m_2} v &= 0 \\ \mathbb{E}_{j,m_1,m_2} v &= G_j^{m_2}(\mathrm{Id}_j - I_{j-1}^j(\mathrm{Id}_j - \mathbb{E}_{j-1,m_1,m_2}^2)P_j^{j-1})G_j^{m_1} v, \ \ j = 2, \ldots, J, \end{cases} \tag{36}$$

where $G_j = \mathrm{Id}_j - B_j^{-1}A_j$ and P_j^{j-1} is defined analogously to (23), cf. [Hac85, Bra93]. Then the convergence estimate for the W-cycle multigrid scheme follows from Theorem 18 and the stability estimates (34) and (35).

Theorem 22. *Let* C_{2lvl} *and* C_{stab} *be defined as in Theorem 18 and Lemma 20, respectively. Then, there exists a constant* $\widehat{\mathsf{C}} > \mathsf{C}_{\mathsf{2lvl}}$, *independent of the mesh size, the polynomial approximation degree and the level* j, $j = 1, \ldots, J$, *such that, if the number of pre- and post-smoothing steps satisfy*

$$(m_1 + 1)^{1/2}(m_2 + 1)^{1/2} \geq \begin{cases} p_j^2 \tilde{\mathsf{C}}(p_j) \dfrac{C_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2}{\widehat{\mathsf{C}} - \mathsf{C}_{\mathsf{2lvl}}} & \text{if } \tilde{\mathsf{C}}(p_{j-1}) \leq \tilde{\mathsf{C}}(p_j), \\[4mm] p_j^2 \dfrac{\tilde{\mathsf{C}}(p_{j-1})^2}{\tilde{\mathsf{C}}(p_j)} \dfrac{C_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2}{\widehat{\mathsf{C}} - \mathsf{C}_{\mathsf{2lvl}}} & \text{otherwise,} \end{cases} \tag{37}$$

then

$$\|\!|\mathbb{E}_{j,m_1,m_2} v|\!\|_{1,j} \leq \widehat{\mathsf{C}} \Sigma_j \|\!|v|\!\|_{1,j} \quad \forall v \in V_j, \tag{38}$$

with

$$\Sigma_j = \tilde{\mathsf{C}}(p_j) \frac{p_j^2}{(1+m_1)^{1/2}(1+m_2)^{1/2}}. \tag{39}$$

Proof. The proof follows the derivation of the analogous result presented in [ASV, Theorem 4.7]. For $j = 1$, the statement of the theorem trivially holds. For $j > 1$, by an induction hypothesis, we assume that (38) holds for $j - 1$. By the definition of the error propagation operator $\mathbb{E}_{j,m_1,m_2} v$ in (36), it follows that

$$\|\!|\mathbb{E}_{j,m_1,m_2} v|\!\|_{1,j} \leq \|\!|G_j^{m_2}(\mathrm{Id}_j - I_{j-1}^j P_j^{j-1})G_j^{m_1} v|\!\|_{1,j}$$

$$+ \|\!|G_j^{m_2} I_{j-1}^j \mathbb{E}_{j-1,m_1,m_2}^2 P_j^{j-1} G_j^{m_1} v|\!\|_{1,j}.$$

17

The first term corresponds to a two-level method between level j and $j-1$. We now observe that the smoothing property of Lemma 16 and the approximation property of Lemma 17 can be extended to any level V_j, $j = 2, \ldots, J$, and we therefore have, by Theorem 18, that

$$\|G_j^{m_2}(\mathrm{Id}_j - I_{j-1}^j P_j^{j-1})G_j^{m_1}v\|_{1,j} \leq \mathsf{C}_{2\mathsf{lvl}}\Sigma_j\|v\|_{1,j}.$$

The bound on the second term is obtained by applying the smoothing property (24) for $j = 2, \ldots, J$, the stability estimates (34) and (35) and the induction hypothesis; thereby, we get

$$\|G_j^{m_2}I_{j-1}^j \mathbb{E}_{j-1,m_1,m_2}^2 P_j^{j-1}G_j^{m_1}v\|_{1,j} \leq \mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2\Sigma_{j-1}^2\|v\|_{1,j}.$$

We then obtain

$$\|\mathbb{E}_{j,m_1,m_2}v\|_{1,j} \leq \left(\mathsf{C}_{2\mathsf{lvl}}\Sigma_j + \mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2\Sigma_{j-1}^2\right)\|v\|_{1,j}.$$

We now bound Σ_{j-1} with Σ_j as follows: if $\tilde{\mathsf{C}}(p_{j-1}) \leq \tilde{\mathsf{C}}(p_j)$, then

$$\Sigma_{j-1}^2 = \tilde{\mathsf{C}}(p_{j-1})^2 \frac{p_{j-1}^4}{(1+m_1)(1+m_2)} \leq \tilde{\mathsf{C}}(p_j)^2 \frac{p_j^4}{(1+m_1)(1+m_2)} \tag{40}$$

$$= \tilde{\mathsf{C}}(p_j) \frac{p_j^2}{(1+m_1)^{1/2}(1+m_2)^{1/2}}\Sigma_j.$$

Otherwise,

$$\Sigma_{j-1}^2 = \tilde{\mathsf{C}}(p_{j-1})^2 \frac{p_{j-1}^4}{(1+m_1)(1+m_2)} \leq \frac{\tilde{\mathsf{C}}(p_{j-1})^2}{\tilde{\mathsf{C}}(p_j)} \frac{p_j^4}{(1+m_1)(1+m_2)} \tag{41}$$

$$\leq \frac{\tilde{\mathsf{C}}(p_{j-1})^2}{\tilde{\mathsf{C}}(p_j)} \frac{p_j^2}{(1+m_1)^{1/2}(1+m_2)^{1/2}}\Sigma_j.$$

For $\tilde{\mathsf{C}}(p_{j-1}) \leq \tilde{\mathsf{C}}(p_j)$ by (40), we have that

$$\mathsf{C}_{2\mathsf{lvl}}\Sigma_j + \mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2\Sigma_{j-1}^2 \leq \left(\mathsf{C}_{2\mathsf{lvl}} + \mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2\tilde{\mathsf{C}}(p_j)\frac{p_j^2}{(1+m_1)^{1/2}(1+m_2)^{1/2}}\right)\Sigma_j.$$

We then observe that if m_1 and m_2 are such that

$$(1+m_1)^{1/2}(1+m_2)^{1/2} \geq p_j^2\tilde{\mathsf{C}}(p_j)\frac{C_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2}{\widehat{\mathsf{C}} - \mathsf{C}_{2\mathsf{lvl}}},$$

it follows that

$$\mathsf{C}_{2\mathsf{lvl}}\Sigma_j + \mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2\Sigma_{j-1}^2 \leq \widehat{\mathsf{C}}\Sigma_j.$$

For $\tilde{\mathsf{C}}(p_{j-1}) > \tilde{\mathsf{C}}(p_j)$, starting from (41) and following the same steps, we deduce the statement of the theorem, provided m_1 and m_2 are such that

$$(1+m_1)^{1/2}(1+m_2)^{1/2} \geq p_j^2\frac{\tilde{\mathsf{C}}(p_{j-1})^2}{\tilde{\mathsf{C}}(p_j)}\frac{C_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2}{\widehat{\mathsf{C}} - \mathsf{C}_{2\mathsf{lvl}}}.$$

$\square$

18

As in the two-level case, inequality (38) implies that the convergence of the method is guaranteed if the number of smoothing steps is chosen sufficiently large, cf. (37). Moreover, compared to the case of standard quasi-uniform grids, cf. [ASV], the bound (37) on the number of smoothing steps involves a strong dependence on the geometrical properties of the underlying agglomerated meshes, which in principle, could lead to restrictive conditions on the hierarchy of grids employed. However, we remark that, in practice, the numerical simulations indicate that the proposed multigrid algorithms converge uniformly, even when low quality agglomerated grids are employed; moreover, an increase in the polynomial order does not seem to require a higher number of smoothing steps to obtain a convergent iteration, cf. Section 5 for details.

Remark 23. *Whenever the agglomerated grids are not quasi-uniform,* i.e., *Assumption 4 is not satisfied, Theorem 18 and Theorem 22 still hold. More precisely, we need to introduce the ratio θ_j between the maximum and minimum element size on level j*

$$\theta_j = \frac{\max_{\kappa \in \mathcal{T}_j} h_\kappa}{\min_{\kappa \in \mathcal{T}_j} h_\kappa}, \quad j = 1, \dots, J.$$

Moreover, we assume that there exists a constant $\mathsf{C}_{\mathsf{mesh}}$, independent of the granularity of the mesh, such that

$$\theta_j \le \mathsf{C}_{\mathsf{mesh}}, \quad j = 1, \dots, J.$$

Then the results in Theorem 18 and Theorem 22 hold with

$$\Sigma_j = \theta_j^2 \tilde{\mathsf{C}}(p_j) \frac{p_j^2}{(1+m_1)^{1/2}(1+m_2)^{1/2}},$$

cf. (39), and the bound (37) is modified as follows

$$(1+m_1)^{1/2}(1+m_2)^{1/2} \ge \begin{cases} \dfrac{\mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2}{\widehat{\mathsf{C}} - \mathsf{C}_{\mathsf{2lvl}}} \dfrac{\mathsf{C}_{\mathsf{mesh}}^4}{\theta_j^2} \tilde{\mathsf{C}}(p_j) p_j^2 & \text{if } \tilde{\mathsf{C}}(p_{j-1}) \le \tilde{\mathsf{C}}(p_j), \\[3mm] \dfrac{C_{\mathsf{stab}}^2 \widehat{\mathsf{C}}^2}{\widehat{\mathsf{C}} - \mathsf{C}_{\mathsf{2lvl}}} \dfrac{\mathsf{C}_{\mathsf{mesh}}^4}{\theta_j^2} \dfrac{\tilde{\mathsf{C}}(p_{j-1})^2}{\tilde{\mathsf{C}}(p_j)} p_j^2 & \text{otherwise.} \end{cases} \quad (42)$$

Remark 24. *We recall that in Theorem 22 and Remark 23, in order to guarantee the convergence of the method, we require a lower bound on the number of smoothing steps, cf. (37) and (42). In fact, for $\tilde{\mathsf{C}}(p_{j-1}) \le \tilde{\mathsf{C}}(p_j)$, we obtain*

$$\widehat{\mathsf{C}} \Sigma_j = \widehat{\mathsf{C}} \theta_j^2 \tilde{\mathsf{C}}(p_j) \frac{p_j^2}{(1+m_1)^{1/2}(1+m_2)^{1/2}} \le \frac{\widehat{\mathsf{C}} - \mathsf{C}_{\mathsf{2lvl}}}{\mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}} \frac{\theta_j^4}{\mathsf{C}_{\mathsf{mesh}}^4} \frac{\tilde{\mathsf{C}}(p_j) p_j^2}{\tilde{\mathsf{C}}(p_j) p_j^2}$$

$$\le \frac{\widehat{\mathsf{C}} - \mathsf{C}_{\mathsf{2lvl}}}{\mathsf{C}_{\mathsf{stab}}^2 \widehat{\mathsf{C}}} \frac{\mathsf{C}_{\mathsf{mesh}}^4}{\mathsf{C}_{\mathsf{mesh}}^4} < 1.$$

An analogous result can be obtained for $\tilde{\mathsf{C}}(p_{j-1}) > \tilde{\mathsf{C}}(p_j)$. Moreover, we note that we have considered the general case of (42), since (37) can be regarded as a particular case.
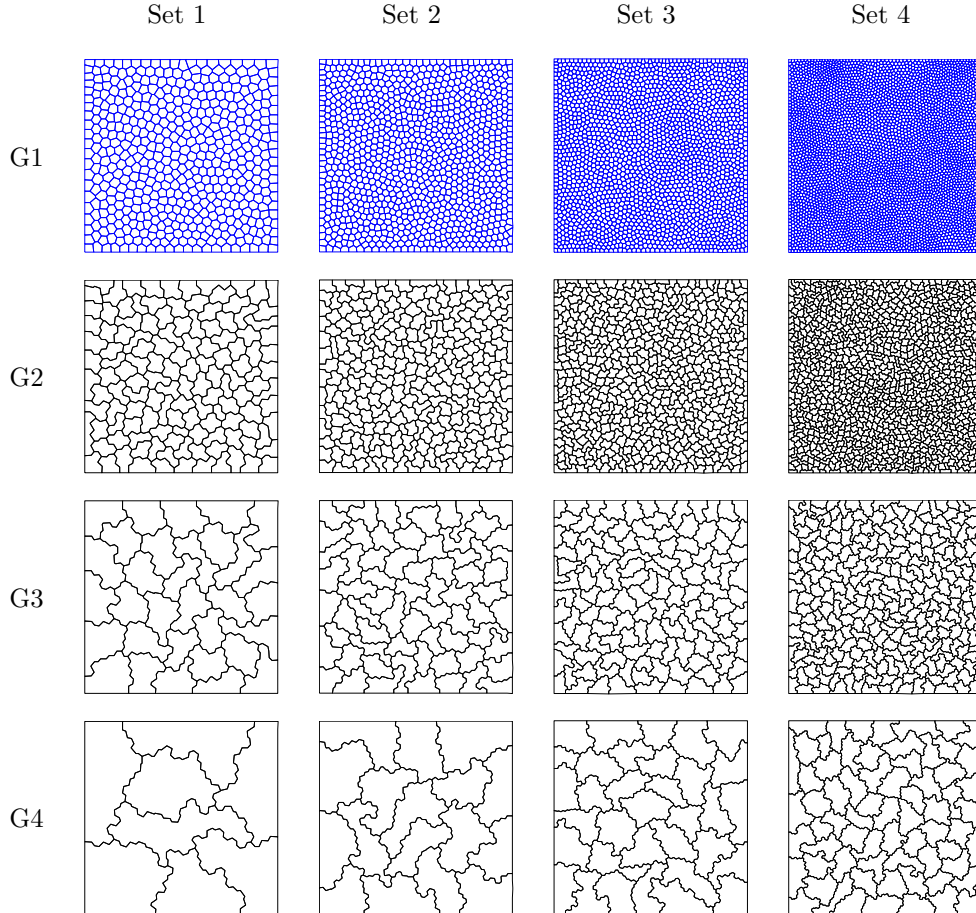
Figure 2: Sets of nested grids employed for numerical simulations.

5   Numerical results

In this section we present several numerical simulations to verify the theoretical estimates provided in Theorem 18 and Theorem 22 in the case of a two dimensional problem on the unit square $\Omega = (0,1)^2$. For the numerical tests, we consider the sets of meshes shown in Figure 2. The initial polygonal element meshes are generated using the software package `PolyMesher` [TPPM12], and consist of 512 (Set 1), 1024 (Set 2), 2048 (Set 3) and 4096 (Set 4) elements. Each initial grid is then subsequently coarsened in order to obtain a sequence of nested partitions by employing the software package MGridGen [MK01a, MK01b]. Before testing the performance of the two-level and W-cycle multigrid solvers presented in Algorithm 1 and Algorithm 2, respectively, we first address the issue of the choice of the penalization coefficient C_σ^j in (4). According to Lemma 9, the bilinear form $\mathcal{A}_j(\cdot,\cdot)$ is coercive provided that $C_\sigma^j > C_F$, with C_F an upper bound for the maximum number of element interfaces in the partition $\mathcal{T}_j$; see Assumption 1. In Table 1, we report the coercivity constant $\mathsf{C}_{\mathsf{coerc}}$ of (7) for a fixed value of $C_\sigma^j \equiv C_\sigma = 10$ for $j = 1, \ldots, 4$. We observe that, despite

	Set 1	Set 2	Set 3	Set4
G1	0.7385	0.7375	0.7370	0.7364
G2	0.7624	0.7564	0.7559	0.7545
G3	0.7827	0.7818	0.7720	0.7611
G4	0.8153	0.8054	0.8001	0.7827

Table 1: Value of the coercivity constant $\mathsf{C}_{\mathsf{coerc}}$ for the sets of grids considered in Figure 2 with $C_\sigma^j = C_\sigma = 10$, $j = 1, \dots, 4$.
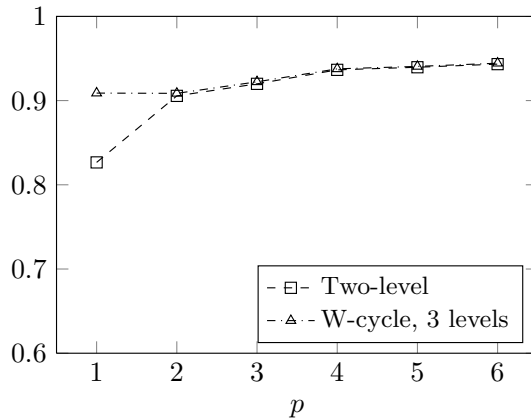

Figure 3: Estimates of $C_{2\mathrm{lvl}}\Sigma_J$ and $\widehat{\mathsf{C}}\Sigma_3$ in (32) and (38), respectively, as a function of p, and $m_1 = m_2 = m = 2p^2$.

the increase in the value of C_F from grid G1 to grid G4, the bilinear form is uniformly coercive for a constant value of the penalization coefficient, which in general does not satisfy the theoretical assumption. As a consequence, in the following, we set $C_\sigma^j \equiv C_\sigma = 10$ for $j = 1, \dots, 4$.

We now consider the grids in Set 1, and numerically evaluate the constant $C_{2\mathrm{lvl}}\Sigma_J$, $J = 2$, in Theorem 18 and the constant $\widehat{\mathsf{C}}\Sigma_3$ in Theorem 22, for the h-version of the two solvers, based on selecting $m_1 = m_2 = m = 2p^2$, cf. Figure 3. Here, we observe that $C_{2\mathrm{lvl}}\Sigma_2$ and $\widehat{\mathsf{C}}\Sigma_3$ are roughly (asymptotically) constant, as the polynomial degree p increases; thereby, this implies that $\tilde{\mathsf{C}}(p_J)$, $J = 2, 3$, respectively, is approximately $\mathcal{O}(1)$, as p increases.

Next, we investigate the performance of the two-level and W-cycle multigrid schemes in terms of the convergence factor

$$\rho = \exp\left(\frac{1}{N} \ln \frac{\|\mathbf{r}_N\|_2}{\|\mathbf{r}_0\|_2} \right),$$

where N denotes the number of iterations required to attain convergence up to a (relative) tolerance of 10^{-8} and $\mathbf{r}_N$ and $\mathbf{r}_0$ are the final and initial residual vectors, respectively. In Table 2, we report the iteration counts and the convergence factor (in parenthesis), needed to attain convergence of the h-version of the two-level (TL) method and W-cycle multigrid scheme (with 3 and 4 levels), as a function of the number of elements (given by the choice of different grid sets), and the number of smoothing steps ($m_1 = m_2 = m$). Here, we have fixed the polynomial approximation order on each level $p_j \equiv p = 1$. We first observe that, although the agglomerated grids, in general, do not necessarily strictly satisfy Assumptions 1 and 2, the number of iterations, for fixed m, does not

	Set 1			Set 2		
	TL	W-cycle		TL	W-cycle	
		3 lvl	4 lvl		3 lvl	4 lvl
$m = 3$	133 (0.87)	160 (0.89)	167 (0.90)	121 (0.86)	191 (0.91)	188 (0.91)
$m = 5$	95 (0.82)	113 (0.85)	113 (0.85)	88 (0.81)	121 (0.86)	125 (0.86)
$m = 8$	72 (0.77)	82 (0.80)	81 (0.80)	67 (0.76)	86 (0.81)	88 (0.81)
$m = 12$	57 (0.72)	63 (0.74)	62 (0.74)	54 (0.71)	65 (0.75)	67 (0.76)
$m = 16$	49 (0.68)	52 (0.70)	51 (0.69)	46 (0.67)	55 (0.71)	56 (0.72)
$m = 20$	44 (0.65)	45 (0.66)	44 (0.66)	40 (0.63)	48 (0.68)	49 (0.68)
	$N_{iter}^{CG} = 445$			$N_{iter}^{CG} = 633$		
	Set 3			Set 4		
	TL	W-cycle		TL	W-cycle	
		3 lvl	4 lvl		3 lvl	4 lvl
$m = 3$	140 (0.88)	188 (0.91)	192 (0.91)	162 (0.89)	198 (0.91)	198 (0.91)
$m = 5$	99 (0.83)	124 (0.86)	128 (0.87)	112 (0.85)	131 (0.87)	131 (0.87)
$m = 8$	74 (0.78)	89 (0.81)	91 (0.82)	83 (0.80)	94 (0.82)	94 (0.82)
$m = 12$	58 (0.73)	68 (0.76)	69 (0.76)	65 (0.75)	73 (0.77)	72 (0.77)
$m = 16$	49 (0.68)	56 (0.72)	57 (0.72)	55 (0.71)	61 (0.74)	61 (0.74)
$m = 20$	43 (0.65)	48 (0.68)	49 (0.68)	49 (0.68)	53 (0.71)	53 (0.70)
	$N_{iter}^{CG} = 946$			$N_{iter}^{CG} = 1234$		

Table 2: Iteration counts and converge factor (in parenthesis) of the h-version of the two-level and W-cycle solvers and iteration counts of the CG method as a function of m ($C_\sigma^j \equiv C_\sigma = 10$, $p = 1$).

	Set 1			Set 2			Set 3		
	TL	W-cycle		TL	W-cycle		TL	W-cycle	
		3 lvl	4 lvl		3 lvl	4 lvl		3 lvl	4 lvl
$m = 3$	1281	1334	1342	1168	1272	1362	1230	1379	1391
$m = 5$	816	832	839	737	790	844	774	852	860
$m = 8$	546	551	561	487	517	551	513	555	557
$m = 12$	388	394	400	343	363	385	362	387	384
$m = 16$	305	312	316	268	284	299	284	301	296
$m = 20$	254	261	263	222	235	246	235	249	242
	$N_{iter}^{CG} = 1954$			$N_{iter}^{CG} = 2809$			$N_{iter}^{CG} = 4174$		

Table 3: Iteration counts of the h-version of the two-level and W-cycle solvers and the CG method as a function of m and the number of levels ($C_\sigma^j \equiv C_\sigma = 10$, $p = 3$).

significantly increase with the number of elements in the underlying mesh; moreover, for the W-cycle solver, the number of iterations remains bounded with the number of levels. As expected, the convergence is faster for larger values of m and the solvers are convergent provided the number of smoothing steps is sufficiently large. For each grid, we have also reported the iteration counts N_{iter}^{CG} for the Conjugate Gradient (CG) method, which shows that the two proposed solvers outperform the CG scheme in terms of the number of iterations required to attain convergence, even when a small number of smoothing steps are employed. Table 3 presents analogous results for the first three sets of meshes, in the case when $p = 3$. Here, we observe that, as expected, the convergence factor increases, but the increase in p does not require an increase in the minimal number of smoothing steps needed to ensure that the underlying multilevel solvers are convergent.

	TL	W-cycle		N_{iter}^{CG}
		3 lvl	4 lvl	
$p = 1$	88	121	125	633
$p = 2$	357	434	443	1701
$p = 3$	737	790	844	2809
$p = 4$	958	1093	1184	4574
$p = 5$	876	1096	1201	6796

Table 4: Iteration counts of the h-version of the two-level and W-cycle solvers and the CG method as a function of p and the number of levels ($C_\sigma^j \equiv C_\sigma = 10$, $m = 5$).

	$p = 2$	$p = 3$		$p = 4$			$p = 5$		
	TL	TL	W-cycle	TL	W-cycle		TL	W-cycle	
			3 lvl		3 lvl	4 lvl		3 lvl	4 lvl
$m = 12$	334	631	1528	860	1028	1051	890	1197	1418
$m = 14$	292	550	607	748	889	908	772	1033	1220
$m = 16$	261	489	538	663	784	800	683	910	1071
$m = 18$	236	441	483	597	703	716	614	814	955
$m = 20$	216	402	439	543	637	649	558	737	862
	$N_{iter}^{CG} = 1701$	$N_{iter}^{CG} = 2809$		$N_{iter}^{CG} = 4574$			$N_{iter}^{CG} = 6796$		

Table 5: Iteration counts of the hp-version of the two-level and W-cycle solvers and the CG method as a function of m and the number of levels ($C_\sigma^j \equiv C_\sigma = 10$).

A more exhaustive investigation of the effect of increasing p is reported in Table 4, where we consider a fine grid of 1024 elements and the corresponding agglomerated meshes (Set 2 in Figure 2). We observe that, even though both multilevel solvers converge for a fixed value of m, with increasing p, the number of iterations required to attain convergence increases as p grows. However, the two-level and W-cycle multigrid solvers still employ less iterations, than the number required by the CG method. In Table 5, we report the number of iterations of the hp-version of the two solvers as a function of the number of smoothing steps and the number of levels for varying p. Here, $p = p_J$ denotes the polynomial approximation degree on the finest level V_J, while, because of the hp-approach, the polynomial order is decreased from the finest level to the coarser ones in such a way that $p_{j-1} = p_j - 1$. We observe that the introduction of the hp-multigrid is detrimental for the convergence of the method, since the minimum number of smoothing steps needed to obtain a convergent method increases with respect to the h-version. Then, we can conclude that, as the p-version of the method does not exhibit uniform convergence with respect to the polynomial order, the hp-approach turns out to be not very effective for problems resulting from high-order discretizations.

As a second numerical test, we consider the same problem discretized by the interior penalty DG method of Section 2 on an initial mesh of triangles, thus reproducing a more common scenario in the framework of finite element discretizations. In analogy to the case of polygonal meshes, we consider four sets of nested grids obtained by agglomeration, cf. Figure 4. The sets considered derive from initial meshes of 528 (Triangle Set 1), 1086 (Triangle Set 2), 2198 (Triangle Set 3) and 4318 (Triangle Set 4) elements. In Table 6, we show the iteration counts needed to attain convergence with respect to a fixed tolerance of 10^{-8} (*i.e.*, the number of elements) and the number of smoothing steps of the h-version of the two-level and W-cycle multigrid
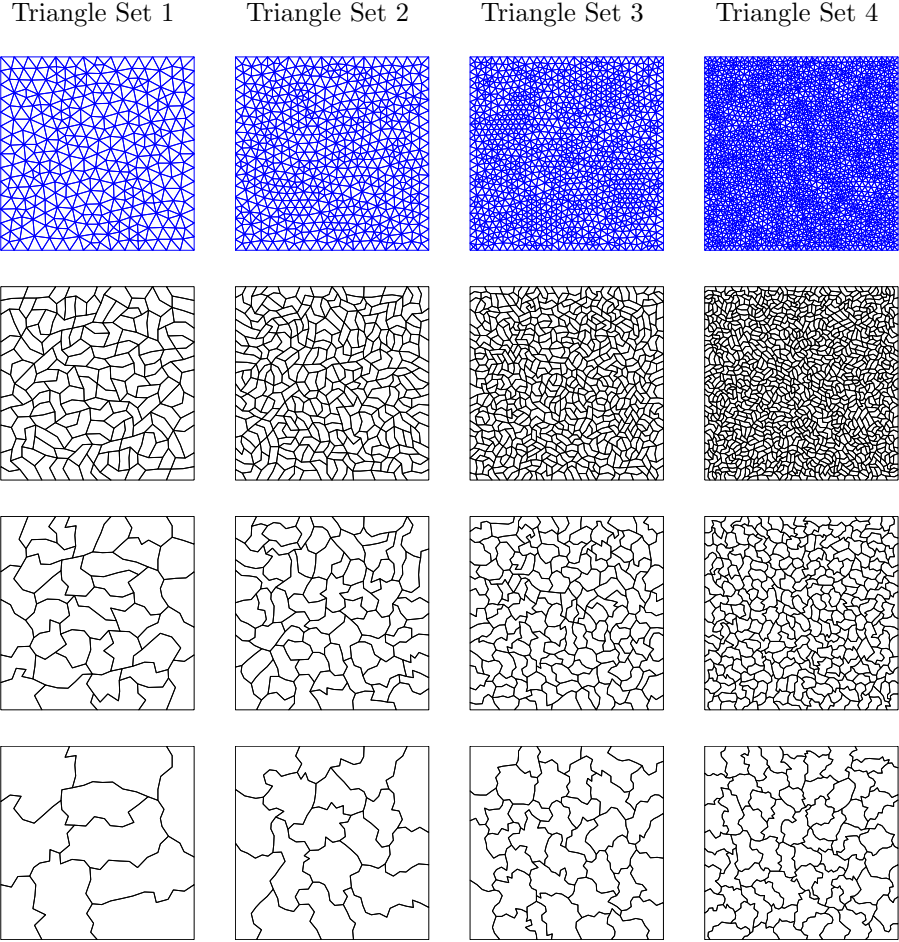
Figure 4: Sets of nested grids employed for numerical simulations.

solvers, with $p_j = p = 1$. We recall that, as in the previous numerical test, we have considered $C_\sigma^j \equiv C_\sigma = 10$, for each j. The results are similar to the case of initial polygonal meshes, with uniform convergence with respect to the granularity of the mesh, and in the case of the W-cycle solver, also with respect to the number of levels. We again attain improved performance, compared to the standard CG method, in terms of the number of iterations required to attain convergence.

	Triangle Set 1			Triangle Set 2		
	TL	W-cycle		TL	W-cycle	
		3 lvl	4 lvl		3 lvl	4 lvl
$m = 4$	246 (0.90)	258 (0.90)	262 (0.90)	282 (0.89)	291 (0.90)	292 (0.90)
$m = 6$	177 (0.87)	185 (0.87)	188 (0.87)	199 (0.86)	205 (0.87)	204 (0.87)
$m = 10$	120 (0.81)	125 (0.82)	127 (0.82)	133 (0.81)	136 (0.82)	136 (0.82)
$m = 14$	94 (0.77)	98 (0.78)	99 (0.78)	104 (0.77)	106 (0.78)	106 (0.78)
$m = 18$	79 (0.74)	82 (0.74)	83 (0.74)	87 (0.74)	89 (0.75)	89 (0.75)
	$N_{iter}^{CG} = 551$			$N_{iter}^{CG} = 771$		
	Triangle Set 3			Triangle Set 4		
	TL	W-cycle		TL	W-cycle	
		3 lvl	4 lvl		3 lvl	4 lvl
$m = 4$	328 (0.90)	333 (0.91)	329 (0.90)	421 (0.91)	425 (0.91)	422 (0.91)
$m = 6$	231 (0.87)	234 (0.88)	232 (0.87)	292 (0.88)	293 (0.89)	292 (0.89)
$m = 10$	153 (0.82)	154 (0.83)	153 (0.82)	190 (0.83)	191 (0.84)	189 (0.84)
$m = 14$	118 (0.78)	119 (0.79)	118 (0.78)	145 (0.79)	148 (0.80)	146 (0.80)
$m = 18$	98 (0.75)	99 (0.75)	98 (0.75)	120 (0.76)	123 (0.77)	122 (0.77)
	$N_{iter}^{CG} = 1145$			$N_{iter}^{CG} = 1630$		

Table 6: Iteration counts and converge factor (in parenthesis) of the h-version of the two-level and W-cycle solvers and iteration counts of the CG method as a function of m ($C_\sigma^j \equiv C_\sigma = 10$, $p = 1$). Starting mesh of triangles.

References

[ABCM02]   D. N. Arnold, Fr. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2001/2002.

[AGH13]   P. F. Antonietti, S. Giani, and P. Houston. *hp*–Version composite discontinuous Galerkin methods for elliptic problems on complicated domains. *SIAM J. Sci. Comput.*, 35(3):A1417–A1439, 2013.

[AGH14]   P. F. Antonietti, S. Giani, and P. Houston. Domain decomposition preconditioners for Discontinuous Galerkin methods for elliptic problems on complicated domains. *J. Sci. Comput.*, 60(1):203–227, 2014.

[AH11]   P. F. Antonietti and P. Houston. A class of domain decomposition preconditioners for *hp*-discontinuous Galerkin finite element methods. *J. Sci. Comput.*, 46(1):124–149, 2011.

[Arn82]   D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[ASV]   P. F. Antonietti, M. Sarti, and M. Verani. Multigrid algorithms for *hp*-Discontinuous Galerkin discretizations of elliptic problems. *SIAM J. Numer. Anal.* To appear.

[Aub70]   J.P. Aubin. Approximation des problèmes aux limites non homogènes pour des
             opérateurs non linéaires. *J. Math. Anal. Appl.*, 30:510–521, 1970.

[Bab73]   I. Babuška. The finite element method with penalty. *Math. Comp.*, 27(122):221–228,
             1973.

[BaDVBC+13] L. Beirão Da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini, and A. Russo.
             Basic principles of virtual element methods. *Mathematical Models and Methods in
             Applied Sciences*, 23(01):199–214, 2013.

[Bak77]   G. A. Baker. Finite element methods for elliptic equations using nonconforming
             elements. *Math. Comp.*, 31(137):45–59, 1977.

[BBC+12]   F. Bassi, L. Botti, A. Colombo, D. A. Di Pietro, and P. Tesini. On the flexibility
             of agglomeration based physical space discontinuous Galerkin discretizations. *J.
             Comput. Phys.*, 231(1):45–65, 2012.

[BBC+14]   F. Bassi, L. Botti, A. Colombo, F. Brezzi, and G. Manzini. Agglomeration-based
             physical frame dg discretizations: An attempt to be mesh free. *Math. Models
             Methods Appl. Sci.*, 24(8):1495–1539, 2014.

[BBCR12]   F. Bassi, L. Botti, A. Colombo, and S. Rebay. Agglomeration based discontinuous
             Galerkin discretization of the Euler and Navier-Stokes equations. *Comput. & Fluids*,
             61:77–85, 2012.

[Bra93]   J.H. Bramble. *Multigrid Methods*. Number 294 in Pitman Research Notes in Math-
             ematics Series. Longman Scientific & Technical, Harlow, UK, 1993.

[CGH14]   A. Cangiani, E. H. Georgoulis, and P. Houston. *hp*-Version discontinuous Galerkin
             methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci.*,
             24(10):2009–2041, 2014.

[CKS00]   B. Cockburn, G. E. Karniadakis, and C.-W. Shu, editors. *Discontinuous Galerkin
             methods*, Springer-Verlag, Berlin, 2000. Theory, computation and applications.
             Papers from the 1st International Symposium held in Newport, RI, May 24-26,
             1999.

[CXZ98]   T. F. Chan, J. Xu, and L. Zikatanov. An agglomeration multigrid method for
             unstructured grids. In *Domain decomposition methods, 10 (Boulder, CO, 1997)*,
             volume 218 of *Contemp. Math.*, pages 67–81. Amer. Math. Soc., Providence, RI,
             1998.

[DPE12]   D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin meth-
             ods*, volume 69 of *Mathématiques & Applications (Berlin) [Mathematics & Appli-
             cations]*. Springer, Heidelberg, 2012.

[FB10]   T.-P. Fries and T. Belytschko. The extended/generalized finite element method: An
             overview of the method and its applications. *International Journal for Numerical
             Methods in Engineering*, 84(3):253–304, 2010.

[Geo08]   E. H. Georgoulis. Inverse-type estimates on *hp*-finite element spaces and applications. *Math. Comp.*, 77(261):201–219 (electronic), 2008.

[Hac85]   W. Hackbusch. *Multi-grid methods and applications*, volume 4 of *Springer series in computational mathematics*. Springer, Berlin, 1985.

[HS97a]   W. Hackbusch and S.A. Sauter. Composite finite elements for problems containing small geometric details. Part II: Implementation and numerical results. *Comput. Visual Sci.*, 1(4):15–25, 1997.

[HS97b]   W. Hackbusch and S.A. Sauter. Composite finite elements for the approximation of PDEs on domains with complicated micro-structures. *Numer. Math.*, 75(4):447–472, 1997.

[HW07]   J. S. Hesthaven and T. Warburton. *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*. Springer Publishing Company, Incorporated, 1st edition, 2007.

[Lio68]   J.-L. Lions. Problèmes aux limites non homogènes à donées irrégulières: Une méthode d'approximation. In *Numerical Analysis of Partial Differential Equations (C.I.M.E. 2 Ciclo, Ispra, 1967), Edizioni Cremonese, Rome*, pages 283–292. 1968.

[MK01a]   I. Moulitsas and G. Karypis. *Mgridgen/Parmgridgen Serial/Parallel Library for Generating Coarse Grids for Multigrid Methods*. University of Minnesota, Department of Computer Science/Army HPC Research Center, 2001. Available at: `www-users.cs.umn.edu/~moulitsa/software.html`.

[MK01b]   I. Moulitsas and G. Karypis. Multilevel algorithms for generating coarse grids for multigrid methods,. In *Supercomputing 2001 Conference Proceedings*, 2001.

[Nit71]   J. Nitsche. Über ein Variationsprinzip zur Lösung von Dirichlet Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abh. Math. Sem. Uni. Hamburg*, 36:9–15, 1971.

[RH73]   W.H. Reed and T.R. Hill. Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-479, Los Alamos Scientific Laboratory, 1973.

[Riv08]   B. Rivière. *Discontinuous Galerkin methods for solving elliptic and parabolic equations*, volume 35 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. Theory and implementation.

[Sch98]   C. Schwab. *p- and hp-finite element methods*. Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, 1998. Theory and applications in solid and fluid mechanics.

[ST04]   N. Sukumar and A. Tabarraei. Conforming polygonal finite elements. *International Journal for Numerical Methods in Engineering*, 61(12):2045–2066, 2004.

[Ste70]   E.M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton, University Press, Princeton, N.J., 1970.

[TPPM12]   C. Talischi, G.H. Paulino, A. Pereira, and I.F.M. Menezes. Polymesher: A general-purpose mesh generator for polygonal elements written in matlab. *Structural and Multidisciplinary Optimization*, 45(3):309–328, 2012.

[TS08]   A. Tabarraei and N. Sukumar.   Extended finite element method on polygonal and quadtree meshes. *Computer Methods in Applied Mechanics and Engineering*, 197(5):425–438, 2008.

[Whe78]   M. F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM J. Numer. Anal.*, 15(1):152–161, 1978.

MOX Technical Reports, last issues

**Dipartimento di Matematica "F. Brioschi",
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)**

55/2014 ANTONIETTI, P. F.; HOUSTON P.; SARTI, M.; VERANI, M.
Multigrid algorithms for hp-version Interior Penalty Discontinuous Galerkin methods on polygonal and polyhedral meshes

54/2014 FERRARIO, E.; PINI, A.
Uncertainties in renewable energy generation systems: functional data analysis, monte carlo simulation, and fuzzy interval analysis

53/2014 IEVA, F.; PAGANONI, A.M., PIETRABISSA, T.
Dynamic clustering of hazard functions: an application to disease progression in chronic heart failure

52/2014 DEDE , L..; QUARTERONI, A.; S. ZHU, S.
Isogeometric analysis and proper orthogonal decomposition for parabolic problems

51/2014 DASSI, F.; PEROTTO, S.; FORMAGGIA, L.
A priori anisotropic mesh adaptation on implicitly defined surfaces

50/2014 BARTEZZAGHI, A.; CREMONESI, M.; PAROLINI, N.; PEREGO, U.
An explicit dynamics GPU structural solver for thin shell finite elements

49/2014 D. BONOMI, C. VERGARA, E. FAGGIANO, M. STEVANELLA, C. CONTI, A. REDAELLI, G. PUPPINI ET AL
Influence of the aortic valve leaflets on the fluid-dynamics in aorta in presence of a normally functioning bicuspid valve

48/2014 PENTA, R; AMBROSI, D; SHIPLEY, R.
Effective governing equations for poroelastic growing media

47/2014 PENTA, R; AMBROSI, D; QUARTERONI, A.
Multiscale homogenization for fluid and drug transport in vascularized malignant tissues

46/2014 PENTA, R; AMBROSI, D.
The role of the microvascular tortuosity in tumor transport phenomena