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Abstract

This works gives an overview of the mathematical treatment of state-of-the-art techniques for partial differential problems where boundary data are provided only in terms of averaged quantities. A condition normally indicated as "defective boundary condition". We present and analyze several procedures by which this type of problems can be handled.

1 Introduction

In many applications of practical relevance, it could happen that only average data is available on a portion of the boundary. For instance the space average of the solution or of the stress[1].

This situation often occurs on the so-called *artificial boundaries*, i.e. portions of the boundary introduced by an artificial cut of the physical domain, as it happens, for instance, in a pipe. On such boundaries, often there are no strong physical arguments that can be used to devise suitable boundary conditions.

[1] Here with stress we mean the solution dependent quantity contained in the boundary term emerging from integration by parts when the weak formulation is derived. Depending on the problem at hand, it could represent several physical quantities, e.g. a heat flux, the elastic traction, the normal Cauchy stress, just to provide some examples. These are the quantities that are assigned when a *Neumann* (also called *natural*) boundary condition is prescribed.

In practical situations one may provide boundary information on artificial boundaries by i) the acquisition of some measurements or ii) the coupling with reduced models (typically based on the solution of another differential problem) able to give a suitable description of what happens in the cut region. However, in many contexts both techniques provide just averaged quantities. An example is *hemodynamics*, where non-invasive measurements (like Echo-Color Doppler) of blood velocity or pressure, as well as the coupling with reduced models, are often used to provide boundary information to full three-dimensional simulations [33, 11, 4, 34, 3, 32]. For the case of general hydraulic networks, see also [25], while another context where the coupling with a lumped parameter model leads to a defective condition is that of heat transfer in a pipe [16].

From the mathematical viewpoint, defective problems are not well posed since the data on the artificial boundaries are insufficient to guarantee uniqueness of the solution. Many approaches have been developed so far to fill this gap: some of them take inspiration from engineering principles and practices, others have a more mathematical foundation. In any case, suitable hypotheses are introduced in order to make the defective problems solvable.

In this review, we describe the main techniques to prescribe defective boundary conditions. To better highlight the mathematical principles behind them, we first treat the case of the Poisson equation. Then, we address the case where such strategies were originally developed, i.e. fluid-dynamics, focussing to the Stokes problem. Finally we provide some examples taken from real haemodynamic studies.

2 Defective Poisson problem

In this section, we address the simple case of a scalar Poisson problem. This will allow us to introduce all the key-points at the basis of numerical methods for the prescription of defective data.

To begin with, we consider the following defective problem on a bounded domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3, with Lipshitz boundary:

$$-\nabla \cdot (\mu \nabla u) = f \qquad \text{in } \Omega, \tag{1a}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{1b}$$

$$\int_\Sigma u \, d\Sigma = Q, \tag{1c}$$

with $\Sigma = \partial\Omega \setminus \Gamma$, $f \in L^2(\Omega)$, $Q \in \mathbb{R}$, and $\mu : \Omega \to \mathbb{R}$ bounded away from zero, i.e. $\mu \in L^\infty(\Omega)$ such that $0 < \mu_0 \leq \mu(\mathbf{x})$ for almost all $\mathbf{x} \in \Omega$ and for a suitable scalar μ_0.

Notice that in (1c) we are prescribing only the average value of u over Σ, thus a defective condition. Alternatively, we could consider the defective problem obtained by (1a)-(1b) together with

$$\int_\Sigma \mu \frac{\partial u}{\partial \mathbf{n}} \, d\Sigma = P, \tag{2}$$

with $P \in \mathbb{R}$ given, $\mathbf{n}$ the outward unit vector to Ω, and $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ the derivative normal to the boundary.

Condition (2) prescribes the average of the stress, thus it is again a defective condition. In what follows, we will refer to problems given by (1) and (1a),(1b),(2) as *mean solution* and *mean stress* problems, respectively. Of course, in both cases the solution is not unique. For this reason, suitable hypothesis should be introduced in order to find a reasonable solution of such problems. This will be discussed in the next sections. To make the exposition simpler, we will address only the case of one defective condition. The results may be readily extended to the case where defective conditions are applied to several non-overlapping parts Σ_i of $\partial\Omega$.

2.1 Empirical methods

A simplest choice to make problem (1) solvable is to select a-priori a profile of u on Σ that satisfies (1c). Thus, problem (1) is transformed into a standard Dirichlet problem,

$$-\nabla\cdot(\mu\nabla u) = f \qquad \text{in } \Omega, \tag{3a}$$
$$u = 0 \qquad \text{on } \Gamma, \tag{3b}$$
$$u = g \qquad \text{on } \Sigma, \tag{3c}$$

where $g \in H^{1/2}_{00}(\Sigma)$ satisfies

$$\int_\Sigma g\, d\Sigma = Q.$$

Now, the solution of problem (3) is clearly unique. However, such a solution is heavily influenced by the choice of the datum g. Let g be an educated guess of the "real" solution $u = g_{ex}$ on Σ, of which we actually know the average Q. Thus, the error e committed by solving (3) satisfies

$$\|e\|_{H^1(\Omega)} \leq C\|g - g_{ex}\|_{H^{1/2}(\Sigma)},$$

which of course annihilates only for $g = g_{ex}$. The fact that $\int_\Sigma (g - g_{ex})d\Sigma = 0$ does not help so much, since $\|g - g_{ex}\|_{H^{1/2}(\Sigma)}$ could still be arbitrary large. Thus, in absence of any further information about the solution at Σ, this method is rather arbitrary.

Analogously, for problem given by (1a),(1b),(2), one could think to prescribe the following Neumann condition together with (1a),(1b):

$$\mu\frac{\partial u}{\partial \mathbf{n}} = h \qquad \text{on } \Sigma,$$

with h satisfying

$$\int_\Sigma h\, d\Sigma = P.$$

Similar conclusions found for the mean solution problem hold as well in this case since the choice of h is arbitrary.

In the next subsections, we will consider four alternative strategies which are mathematically more justified.

2.2 Lagrange multiplier approach

We note that problem (1) could be equivalently written as the following constrained minimization problem: find $u \in V = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$ such that functional

$$J(v) = \frac{1}{2}\int_\Omega \mu\,(\nabla v)^2\,d\mathbf{x} - \int_\Omega f v\,d\mathbf{x} \tag{4}$$

is minimized in V under the constraint (1c).

This problem can be rewritten as an unconstrained problem by introducing the corresponding Lagrangian functional: find $u \in V$ and $\lambda \in \mathbb{R}$ such that the following Lagrangian functional

$$L(v,\xi) = J(v) + \xi\left(\int_\Sigma v\,d\Sigma - Q\right)$$

has a stationary point (in fact a saddle point) in $V \times \mathbb{R}$. The associated variational problem is: find $u \in V$ and $\lambda \in \mathbb{R}$ such that for all $(v,\xi) \in V \times \mathbb{R}$

$$(\mu\nabla u, \nabla v) + b(v,\lambda) = (f,v), \tag{5a}$$
$$b(u,\xi) = \xi Q, \tag{5b}$$

where $b(v,\xi) = \xi\int_\Sigma v d\Sigma$ and $(v,w) = \int_\Omega vw\,d\Omega$ denotes the $L^2(\Omega)$ inner product.

This formulation is the *Lagrange multiplier formulation* of the mean solution problem (1), and is in fact the extension to the defective case of the Lagrange multiplier technique to enforce Dirichlet boundary conditions proposed and analyzed, for instance, in [2].

We have the following result.

Proposition 1 *Assume that $f \in L^2(\Omega)$. Then, the problem given by* (5) *admits a unique solution $(u,\lambda) \in V \times \mathbb{R}$.*

Proof. We can use the theory illustrated, for instance, in [5]. In the case $|\Gamma| \neq 0$, thanks to bounds on μ, the bilinear form $(\mu\nabla v, \nabla w)$ is coercive and continuous with respect to the H^1-seminorm $|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$, which is in this case equivalent to the H^1 norm thanks to Poincaré inequality. The b term is a bilinear and continous form on $V \times \mathbb{R}$, indeed

$$|b(v,\xi)| \le |\xi|\int_\Sigma |v| d\Sigma \le C_\Sigma\sqrt{|\Sigma|}|\xi|\|v\|_V, \quad \forall(v,\xi) \in V \times \mathbb{R},$$

where C_Σ is the constant in the trace inequality $\|v\|_{L^2(\Sigma)} \le C_\Sigma\|v\|_V$.

To prove that it satisfies the inf-sup condition it is sufficient to note that it is possible to construct a function $\phi \in H^{1/2}_{00}(\Sigma)$ so that $\int_\Sigma \phi d\Sigma = 1$. For a given $\xi \in \mathbb{R}$ we set $\phi_\xi = \xi\phi$ and find u_ξ solution of

$$\begin{aligned} &-\nabla\cdot\left(\mu\nabla u_\xi\right) = 0 && \text{in } \Omega,\\ &u_\xi = 0 && \text{on } \Gamma,\\ &u_\xi = \phi_\xi && \text{on } \Sigma. \end{aligned} \tag{6}$$

We have that $b(u_\xi,\xi)=\xi^2$ and, by standard regularity results, $\|u_\xi\|_V \le C|\xi|$ for a constant C independent of ξ. Therefore, by combining the two previous relations and taking $\beta = 1/C > 0$, we can state that for all $\xi \in \mathbb{R}$, there exists $u_\xi \in V$ satisfying

$$b(u_\xi,\xi) \ge \beta|\xi|\|u_\xi\|_V.$$

The case $\Gamma = \emptyset$, i.e. $\Sigma = \partial\Omega$, can also be treated in a standard way, by proving that the bilinear form $a(u,v) = (\mu\nabla u, \nabla v)$ is coercive on the space

$$\hat{V} = \{v \in V = H^1(\Omega) : b(v,\xi) = 0, \forall \xi \in \mathbb{R}\}.$$

Indeed, for all $v\in\hat{V}$ we may write $a(v,v) \ge \mu_0\|\nabla v\|^2_{L^2(\Omega)} = |||v|||^2$, where $|||v||| = \left(\|\nabla v\|^2_{L^2(\Omega)} + |\int_{\partial\Omega} v\, d\Omega|^2\right)^{1/2}$ is a norm equivalent to $\|v\|_{H^1(\Omega)}$. Indeed, $|||v||| \le C\|v\|_{H^1(\Omega)}$, thanks to trace inequality, so we are left to prove that there exists a constant $C>0$ so that $|||v||| \ge C\|v\|_{H^1(\Omega)}$. To show it we proceed by contradiction. Negating the statement is equivalent to say that there exists a sequence $v_n \in H^1(\Omega)$ such that $\|v_n\|_{H^1(\Omega)} = 1$ while $|||v_n||| \to 0$. Since v_n is bounded in $H^1(\Omega)$ there exists a subsequence v_{n_k} weakly converging to a $v \in H^1(\Omega)$ and such that $v_{n_k} \to v$ in $L^2(\Omega)$. For the sake of simplicity, in the sequel we will use the subscript n for the subsequence. Weak convergence implies that $\|\nabla v\|^2_{L^2(\Omega)} = \lim_{n\to\infty}(\nabla v_n, \nabla v)_{L^2(\Omega)} \le \lim_{n\to\infty}\|\nabla v_n\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$. By which

$$\|\nabla v\|_{L^2(\Omega)} \le \lim_{n\to\infty}\|\nabla v_n\|_{L^2(\Omega)}. \tag{7}$$

The hypothesis $|||v_n||| \to 0$ implies that $\|\nabla v_n\|_{L^2(\Omega)} \to 0$ thus, by (7), $\|\nabla v\|_{L^2(\Omega)} = 0$, i.e. $\|v\|_{H^1(\Omega)} = \|v\|_{L^2(\Omega)}$. The hypothesis on the norm of the elements of the sequence, the strong convergence of the subsequence in $L^2(\Omega)$ and the previous result imply $\|v\|_{L^2(\Omega)} = \lim_{n\to\infty}\|v_n\|_{L^2(\Omega)} = 1$. Now, $\|\nabla v\|_{L^2(\Omega)} = 0$, then $v = c$ where c is a constant, which is different from zero since $\|v\|_{L^2(\Omega)} = 1$. But then, since $|||v_n||| \to 0$ also implies $\lim_{n\to\infty}(\int_{\partial\Omega} v_n)^2 = (\int_{\partial\Omega} v)^2 = 0$, we have a contradiction because $(\int_{\partial\Omega} v)^2 = |\partial\Omega|^2 c^2 > 0$. □

From (5), it is easy to show that the Lagrange multiplier λ plays the role of a constant stress on Σ, i.e. the solutions u and λ satisfy

$$\lambda = -\mu\frac{\partial u}{\partial \mathbf{n}} \qquad \text{on } \Sigma.$$

Thus, this approach implicitly implies that the stress is constant on Σ. In other words, among all the possible solutions of problem (1), this technique selects the (unique) one with constant stress on Σ. We thus expect a great accuracy in those scenarios when the stress is almost constant over Σ. If we do not have further information, this technique is anyway optimal in the sense that it is the one that minimizes the energy functional (4) associated to the problem.

If we consider now a finite dimensional subspace $V_h = \mathrm{span}(\varphi_1,\ldots,\varphi_{N_h})$ approximating V, h being the mesh size, for instance a finite element space corresponding to a triangulation $\mathscr{T}_h$ of Ω [9], the Galerkin approximation of (5) leads to the following algebraic problem

$$\begin{bmatrix} A & \mathbf{b} \\ \mathbf{b}^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \lambda_h \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ Q \end{bmatrix}, \tag{8}$$

where $A_{ij} = \int_\Omega \mu\nabla\varphi_j\cdot\nabla\varphi_i d\Omega$, $b_i = \int_\Sigma \varphi_i d\Sigma$, $f_i = \int_\Omega f\varphi_i d\Omega$, for $i = 1,\ldots,N_h$, and $j = 1,\ldots,N_h$, while U_i, $i = 1,\ldots,N_h$, are the unknown coefficients (degrees of freedom) of the linear expansion of the Galerkin solution by means of the basis functions φ_i, and λ_h the discrete Lagrange multiplier.

For the numerical solution of (8), we can consider a monolithic approach where the linear system is solved e.g. by a direct or an iterative method. However, this strategy is not *modular* in the sense that we cannot exploit pre-existing codes we may have at disposal for the numerical solution of the Poisson problem. Alternatively, if $|\Gamma| \neq 0$ then A is non-singular and one could consider, like it is done in [11] for a defective Stokes problem, the *Schur complement* equation related to (8), which reads

$$\mathbf{b}^T A^{-1} \mathbf{b} \lambda = Q - \mathbf{b}^T A^{-1} \mathbf{f}. \tag{9}$$

Notice that it is, in this very simple case, just a scalar equation, whose solution requires to solve two linear systems in A. In particular, we have the following algorithm:

1. Solve the linear system $A\mathbf{U}_1 = \mathbf{f}$;
2. Compute $\lambda_1 = Q - \mathbf{b}^T \mathbf{U}_1$;
3. Solve the linear system $A\mathbf{U}_2 = \mathbf{b}$;
4. Compute $\lambda_2 = \mathbf{b}^T \mathbf{U}_2$;
5. Compute λ from (9): $\lambda = \lambda_1/\lambda_2$;
6. Compute $\mathbf{U}$ from the first of (8): $\mathbf{U} = \mathbf{U}_1 - \lambda \mathbf{U}_2$.

The previous strategy may seem more expensive than the monolithic one (we need to solve 2 linear problems instead of 1), yet, not only the matrix in (8) is of larger size, but it is also indefinite, while A is, in this case, symmetric positive definite, so more suited for efficient solvers. Moreover, with the proposed algorithm we can exploit pre-existing solvers for the Poisson problem. In particular, the first linear system (point 1.) corresponds to (1a)-(1b) with a homogeneous Neumann condition on Σ, whereas the second one (point 3.) corresponds to (1a)-(1b) with $f = 0$ and

$$\mu \frac{\partial u}{\partial \mathbf{n}} = 1 \quad \text{on } \Sigma.$$

Remark 2.1 *The previous algorithm can be extended to the case of more than one flow rate conditions (let say m), and requires the solution of $m+1$ "classical" problems [11].*

In the case $|\Gamma| = 0$ matrix A is singular and the standard Shur-complement procedure does not apply. However, since in this case $V = H^1(\Omega)$, we can take $v = 1$ in (5) to get

$$\lambda = |\partial\Omega|^{-1} \int_\Omega f \, d\Sigma. \tag{10}$$

We can then decompose the solution as $u = \mathring{u} + \overline{u}$, where $\overline{u}$ is a constant and $\mathring{u}$ is the unique solution in $H^1(\Omega) \setminus \mathbb{R} = \{\mathring{w} \in H^1(\Omega) : \int_\Omega \mathring{w} \, d\Omega = 0\}$ of

$$(\mu \nabla \mathring{u}, \nabla v) = (f, v) - (\lambda, v) \quad \forall v \in H^1(\Omega) \setminus \mathbb{R}. \tag{11}$$

Then, $\overline{u} = |\partial\Omega|^{-1}\,(Q - \int_{\partial\Omega} \mathring{u}\, d\Sigma)$.

Note that since in standard finite element approximation for this class of problems $1 \in V_h$, also λ_h can be computed by (10), while (11) can be approximated by standard means.

We now give an alternative proof for the well posedness of (5) which gives some useful insights for the next Section. First of all we rewrite the definition of $\hat{V}$ as $\hat{V}$: $\{w \in V : \int_\Sigma w = 0\}$. We have shown in the proof of Proposition 1 that for $v \in \hat{V}$ the seminorm $|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$ is equivalent to $\|\nabla v\|_{H^1(\Omega)}$. Indeed, for a $v \in \hat{V}$ we have $|v|_{H^1(\Omega)} = |||v|||$. We can then consider the following problem: find $u \in V$ such that $\int_\Sigma u = Q$ and

$$(\mu\nabla u, \nabla v) = (f, v) \quad \forall v \in \hat{V}. \tag{12}$$

This problem can be found to be equivalent to the following differential problem: find $(u, \lambda) \in \hat{V} \times \mathbb{R}$ such that

$$\begin{aligned}
&-\nabla\cdot(\mu\nabla u) = f && \text{in } \Omega,\\
&u = 0 && \text{on } \Gamma,\\
&\int_\Sigma u\, d\Sigma = Q, &&\\
&-\mu\frac{\partial u}{\partial n} = \lambda && \text{on } \Sigma.
\end{aligned} \tag{13}$$

That is, problem (12) forces (in a weak sense) the conormal derivative $\mu\frac{\partial u}{\partial n}$ to be constant on Σ. If we have a solution of (12) we can recover λ as

$$\lambda = (f, v) - (\mu\nabla u, \nabla v) \tag{14}$$

for any $v \in V$ with $\int_\Gamma v = 1$.

Proposition 2 *Problem* (12) *is well posed, and the couple* (u, λ)*, where* λ *is obtained by* (14)*, is the unique solution of* (5)*. Moreover,* $\|u\|_{H^1(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + |Q|\right)$.

Proof. First of all $\hat{V}$ is an Hilbert subspace of $H^1(\Omega)$ and equipped with the same topology. Let $a(z,v) = (\mu\nabla z, \nabla v)$. We have already seen in the proof of Proposition 1 that the form a is bilinear, continuous and coercive $\hat{V}\times\hat{V}$. It is always possible to find a $w \in V$ such that $\int_\Sigma w = Q$ and $\|w\|_{H^1(\Omega)} \le C|Q|$. We the consider the problem: find $\hat{u} \in \hat{V}$ so that $a(\hat{u}, v) = F(v)$ for all $v \in \hat{V}$, where $F(v) = (f, v) - a(w, v)$. This is a classical elliptic problem by which well posedness is proved by standard application of Lax-Milgram Lemma and we have

$$\|\hat{u}\|_{H^1(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|w\|_{H^1(\Omega)}) \le C(\|f\|_{L^2(\Omega)} + |Q|).$$

We then set $u = \hat{u} + w$ and it is immediate to verify that u is a solution of (12), it satisfies $\int_\Gamma u d\Gamma = Q$, and it does not depend on the choice of w. Moreover,

$$\|u\|_{H^1(\Omega)} \le \|\hat{u}\|_{H^1(\Omega)} + \|w\|_{H^1(\Omega)} \le C(\|f\|_{L^2(\Omega)} + |Q|).$$

It is unique since if we have two solutions u_1 and u_2 of (12) and we set $y = u_1 - u_2$ we have $y \in \hat{V}$ and $a(y, v) = 0$ for all $v \in \hat{V}$, and this implies $y = 0$, thus $u_1 = u_2$. Now, by construction

u satisfies (5b), while, since any $v \in V$ may be written as $v = \hat{v} + c\tilde{v}$, with $\hat{v} \in \hat{V}$ and $\tilde{v} \in \tilde{V}$ and $c = \int_\Gamma v d\Gamma$, we have that

$$a(u,v) + b(v,\lambda) - (f,v) = a(u,\hat{v}) - (f,\hat{v}) + c\big(b(\tilde{v},\lambda) + a(u,\tilde{v}) - (f,\tilde{v})\big) = \\ c\big(b(\tilde{v},\lambda) + a(u,\tilde{v}) - (f,\tilde{v})\big),$$

which is zero $\forall v \in V$ if and only if $b(\tilde{v},\lambda) + a(u,\tilde{v}) - (f,\tilde{v}) = 0$. Since $b(\tilde{v},\lambda) = \lambda$ we obtain (14). So the couple (u,λ) given by the solution of (12) and λ given by (14) are solutions of problem (5).

With analogous arguments we may verify that (u,λ) solution of (5) satisfies (12) and (14). □

Remark 2.2 *One could think to apply the Lagrange multiplier approach also to the mean stress problem* (1a)-(1b)-(2) *by devising the following augmented problem*

$$(\mu\nabla u, \nabla v) + \lambda \int_\Sigma \mu \frac{\partial v}{\partial \mathbf{n}} d\Sigma = (f,v) \quad \forall v \in V,$$
$$\xi \int_\Sigma \mu \frac{\partial u}{\partial \mathbf{n}} d\Sigma = \xi P \quad \forall \xi \in \mathbb{R}.$$

However, the term $\int_\Sigma \lambda\mu \frac{\partial v}{\partial \mathbf{n}}$ is not well defined for $v \in V \subset H^1(\Omega)$ and $\lambda \in \mathbb{R}$ (unless $\Sigma = \partial\Omega$), so this formulation is, in general, not feasible. Indeed the integral should reinterpreted as a duality pairing $< \lambda, \mu \frac{\partial v}{\partial \mathbf{n}} >$ between $H^1_{00}(\Gamma)$ and its dual. Yet a non-zero constant function on Γ does not belong to $H^{1/2}_{00}(\Sigma)$. In practice, solving the stated problem numerically by means, for instance, finite elements, will give a solution that has an unwanted oscillations near the boundary of Σ, whose amplitude increases as the mesh is refined. We have similar difficulties for the mean stress problem in the the context of Stokes equations, see for instance [11].

2.3 Penalization methods

We start by observing that a way to overcome the introduction of the further unknown given by the Lagrange multiplier is to prescribe the flow rate condition (1c) not as a constrain but as a penalization. Let us consider a finite element space $V_h \subset V$. We propose to minimize at the discrete level the following functional

$$J(v_h) = \frac{1}{2}(\mu\nabla v_h, \nabla v_h) - (f, v_h) + \frac{1}{2}\gamma\left(\int_\Sigma u_h d\Sigma - Q\right)^2, \tag{16}$$

over $V_h \subset V$ and where $\gamma > 0$ is a penalization parameter. This leads to the following *penalization formulation*: find $u_h \in V_h$ such that

$$(\mu\nabla u_h, \nabla v_h) + \gamma\int_\Sigma u_h d\Sigma \int_\Sigma v_h d\Sigma = (f, v_h) + \gamma Q \int_\Sigma v_h d\Sigma \qquad \forall v_h \in V_h. \tag{17}$$

However, (17) is not consistent with (1a). It is easy to show that the truncation error is $\tau(v_h) = \int_\Sigma \mu \frac{\partial u}{\partial \mathbf{n}} v_h d\Sigma$, which of course does not go to zero when $h \to 0$.

To overcome this limitation, we adapt to the mean solution problem (1) the Nitsche penalization method introduced and analyzed in [30] for a standard Dirichlet problem, following the ideas in [42]. We remember that the Nitsche method is a strongly consistent penalization method, featuring an optimal convergence error. It consists in adding to the penalization term a consistency and, possibly, a symmetry term. The former is, as the name says, required to recover a consistent scheme, the latter is not strictly necessary but it maintains the symmetry of the original problem. To make the expressions more compact we use the notations: $a(u,v) = (\mu\nabla u, \nabla v)$, $< u,v >_\Sigma = |\Sigma|^{-1}\int_\Sigma u \int_\Sigma v$ and $|u|_\Sigma = \sqrt{< u,u >_\Sigma} = |\Sigma|^{-1/2}\,|\int_\Sigma u|$. It is evident that we have a Cauchy-Schwarz type inequality $< u,v >_\Sigma \leq |u|_\Sigma |v|_\Sigma$.

The Nitsche approximation of defective problem (1) is then: find $u_h \in V_h$ such that

$$a(u_h,v_h) + \gamma h^{-1} < u_h, v_h >_\Sigma - < \mu\frac{\partial u_h}{\partial \mathbf{n}}, v_h >_\Sigma - < \mu\frac{\partial v_h}{\partial \mathbf{n}}, u_h >_\Sigma =$$
$$(f,v_h) + \gamma h^{-1} < Q, v_h >_\Sigma - < Q, \mu\frac{\partial v_h}{\partial \mathbf{n}} >_\Sigma \qquad \forall v_h \in V_h. \quad (18)$$

We can write it in a more compact form by introducing

$$a_h(u_h,v_h) = a(u_h,v_h) + \gamma h^{-1} < u_h, v_h >_\Sigma - < \mu\frac{\partial u_h}{\partial \mathbf{n}}, v_h >_\Sigma - < \mu\frac{\partial v_h}{\partial \mathbf{n}}, u_h >_\Sigma$$

and

$$F_h(v_h) = (f,v_h) + \gamma h^{-1} < Q, v_h >_\Sigma - < Q, \mu\frac{\partial v_h}{\partial \mathbf{n}} >_\Sigma,$$

as: find $u_h \in V_h$ such that

$$a_h(u_h,v_h) = F_h(v_h) \quad \forall v_h \in V_h. \quad (19)$$

Proposition 3 *Problem* (19) *is strongly consistent with the solution provided by the Lagrange multiplier approach in* (5).

Proof. The solution of the Lagrange multiplier approach satisfies (using the new notation) $a(u,v) + < \lambda, v >_\Sigma - (f,v) = 0$ and $< u,v >_\Sigma = < Q,v >_\Sigma$ for all $v \in V$, thus $a(u,v_h) + < \lambda, v_h >_\Sigma - (f,v_h) = 0$ and $< u,v_h >_\Sigma = < Q,v_h >_\Sigma$ for all $v_h \in V_h$. Moreover, we have that $< \lambda, v_h >_\Sigma = - < \frac{\partial u}{\partial \mathbf{n}}, v_h >_\Sigma$, for all $v_h \in V_h$.

Consequently, $a_h(u,v_h) - F_h(v_h) = 0$ for all $v_h \in V_h$, since

$$a_h(u,v_h) - F_h(v_h) = a(u,v_h) + \gamma h^{-1} < u, v_h >_\Sigma - < \mu\frac{\partial u}{\partial \mathbf{n}}, v_h >_\Sigma - < \mu\frac{\partial v_h}{\partial \mathbf{n}}, u >_\Sigma$$
$$- (f,v_h) - \gamma h^{-1} < Q, v_h >_\Sigma + < Q, \mu\frac{\partial v_h}{\partial \mathbf{n}} >_\Sigma =$$
$$- < \lambda - \mu\frac{\partial v_h}{\partial \mathbf{n}}, v_h >_\Sigma + \gamma h^{-1} < Q - u, v_h >_\Sigma - < \mu\frac{\partial v_h}{\partial \mathbf{n}}, Q - u >_\Sigma = 0.$$

□ In particular, we have the following “orhogonality property”: $a_h(u - u_h, v_h) = 0$ for all $v_h \in V_h$.

Now, consider the following mesh dependent norm

$$\|v_h\|_h^2 = \|\sqrt{\mu}\nabla v_h\|^2_{L^2(\Omega)} + \gamma h^{-1}|v_h|_\Sigma{}^2 = a(v_h, v_h) + \gamma h^{-1}|v_h|_\Sigma. \tag{20}$$

We give, without proof, the following result

Proposition 4 *There exist two positive constants C_* and C^* such that, for any $v \in V$*

$$C_*\|v\|_{H^1(\Omega)} \leq \|v_h\|_h \leq C^* h^{-1/2}\|v\|_{H^1(\Omega)}.$$

Moreover, it is immediate to verify that $|v|_\Sigma \leq \|v\|_{L^2(\Omega)}$. We now assume that the following inverse inequalities holds, which is true, for instance, for standard Lagrangian finite elements, see, for instance [9].

Assumption 1 *There are two positive constants C_Ω and C_Σ such that for any $v_h \in V_h$*

$$h\|\nabla v_h\|_{L^2(\Omega)} \leq C_\Omega \|v_h\|_{L^2(\Omega)} \quad \textit{and} \quad h^{1/2}\|v_h\|_{L^2(\Sigma)} \leq C_\Sigma \|v_h\|_{L^2(\Omega)}, \tag{21}$$

and, consequently, since μ is bounded away from zero,

$$h\left|\mu\frac{\partial v_h}{\partial \mathbf{n}}\right|^2_\Gamma \leq \|\mu\frac{\partial v_h}{\partial \mathbf{n}}\|^2_{L^2(\Sigma)} \leq C_\Sigma^2 \mu_\Sigma (\mu\nabla v_h, \nabla v_h)^{1/2}, \tag{22}$$

where $\mu_\Sigma = \|\mu\|_{L^\infty(\Sigma)}$.

Thus, we have the following result.

Proposition 5 *If $\gamma > C_\Sigma^2\mu_\Sigma$ problem* (18) *is well posed.*

Proof. We can use Lax-Milgram Lemma. Bilinearity and continuity of a_h as well as linearity and continuity of functional F_h are easily found thanks to standard inequalities. We now prove coercivity of a_h with respect to the norm $\|v_h\|_h$ given by (20). For any $v_h \in V_h$, we have that

$$a_h(v_h, v_h) \geq (\mu\nabla v_h, \nabla v_h)^2_{L^2(\Omega)} + \gamma h^{-1}|v_h|_\Sigma{}^2 - 2 < \mu\frac{\partial v_h}{\partial \mathbf{n}}, v_h >_\Sigma.$$

Now, for any $\varepsilon > 0$, using Young's inequality and (22),

$$2 < \mu\partial v_h/\partial\mathbf{n}, v_h >_\Sigma \;\leq\; \varepsilon h|\mu\partial v_h/\partial\mathbf{n}|_\Sigma{}^2 + \frac{1}{\varepsilon h}|v_h|_\Sigma{}^2 \;\leq\; \varepsilon C_\Sigma^2\mu_\Sigma(\mu\nabla v_h, \nabla v_h) + \frac{1}{\varepsilon h}|v_h|_\Sigma{}^2,$$

that is

$$a_h(v_h, v_h) \geq (1 - \varepsilon C_\Sigma^2\mu_\Sigma)(\mu\nabla v_h, \nabla v_h) + h^{-1}(\gamma - \frac{1}{\varepsilon})|v_h|_\Sigma{}^2.$$

The desired result is obtained if $1 - \varepsilon C_\Sigma^2\mu_\Sigma > 0$ and $\gamma - \frac{1}{\varepsilon} > 0$. If $\gamma > C_\Sigma^2\mu_\Sigma$, the latter inequality is satisfied by taking $\varepsilon < \frac{1}{\mu_\Sigma C_\Sigma^2}$ and, consequently, we may find a constant $\alpha > 0$ so that $a_h(v_h, v_h) \geq \alpha\|v_h\|_h^2$. □

Proposition 6 *If u is the solution of* (5) *and u_h the solution of* (19)*, under the same conditions of Proposition 5, we have that*

$$\|u-u_h\|_h \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u-v_h\|_h,$$

where M and α are the continuity and coercivity constants of a_h.

Proof. The result is rather classical and exploits the Galerkin orhogonality proved in Proposition 3 and the results of Proposition 5. Indeed, for any $v_h \in V_h$ we have

$$\|u-u_h\|_h^2 \leq \alpha^{-1} a_h(u-u_h, u-u_h) = \alpha^{-1} a_h(u-u_h, u-v_h) \leq \alpha^{-1} M \|u-u_h\|_h \|u-v_h\|_h.$$

□ This result allows us to exploit interpolation inequalities to obtain optimal convergence rate of finite element approximations.

We consider now the defective mean stress problem (1a),(1b) and (2). In this case, in analogy with (16), we can consider the following functional to be minimized over V_h:

$$J(v_h) = \frac{1}{2}(\mu \nabla v_h, \nabla v_h) - (f, v_h) + \frac{1}{2}\gamma \left(\int_\Sigma \frac{\partial v_h}{\partial \mathbf{n}} d\Sigma - P \right)^2.$$

This leads to the following problem: find $u_h \in V_h$ such that

$$(\mu \nabla u_h, \nabla v_h) + \gamma \int_\Sigma \frac{\partial u_h}{\partial \mathbf{n}} d\Sigma \int_\Sigma \frac{\partial v_h}{\partial \mathbf{n}} d\Sigma = (f, v_h) + \gamma \frac{h}{|\Gamma|} P \int_\Sigma \frac{\partial v_h}{\partial \mathbf{n}} d\Sigma \qquad \forall v_h \in V_h.$$

Like formulation (17), the previous one is not consistent. Indeed, the truncation error is again $\tau(v_h) = \int_\Sigma \mu \frac{\partial u_h}{\partial \mathbf{n}} v_h \, d\Sigma$. If one assumes that the normal stress is constant over Σ, and thus equal to P, consistency is recovered by adding the term $P \int_\Sigma v_h \, d\Sigma$ to the right hand side of the previous formulation. For convergence one should again take γ large enough, see [22, 39].

2.4 Augmented Lagrangian formulation

It is quite natural now to consider an *augmented Lagrangian* formulation for the solution of (1). This is obtained by finding a stationary point in $V_h \times \mathbb{R}$ of the following functional:

$$L_\gamma(v_h, \xi) = \frac{1}{2} \int_\Omega \mu \, |\nabla v_h|^2 \, d\mathbf{x} - (f, v_h) + \xi \left(\int_\Sigma v_h \, d\Sigma - Q \right) + \frac{1}{2}\gamma \left(\int_\Sigma v_h \, d\Sigma - Q \right)^2,$$

which is equivalent to the following discrete formulation: find $u_h \in V_h$ and $\lambda_h \in \mathbb{R}$ such that $\forall v_h \in V_h$ and $\forall \xi \in \mathbb{R}$

$$\begin{aligned}
&(\mu \nabla u_h, \nabla v_h) + \lambda_h \int_\Sigma v_h \, d\Sigma + \gamma \int_\Sigma u_h \, d\Sigma \int_\Sigma v_h \, d\Sigma = (f, v_h) + \gamma Q \int_\Sigma v_h \, d\Sigma, \\
&\xi \int_\Sigma u \, d\Sigma = \xi Q.
\end{aligned}$$

Again, the Lagrange multiplier λ_h has the physical meaning of a (constant) stress on Σ.

For its numerical solution, we can consider the following *Uzawa method*:
Given $\gamma > 0$, a stopping tolerance $\tau > 0$, a parameter $\rho > 0$, and $\lambda_h^{(0)} \in \mathbb{R}$, for $k = 1, 2, \ldots$:

1. Find $u_h^{(k)}$ solution of

$$\left(\mu \nabla u_h^{(k)}, \nabla v_h\right) + \lambda_h^{(k)} \int_\Sigma v_h \, d\Sigma + \gamma \int_\Sigma u_h^{(k)} \, d\Sigma \int_\Sigma v_h \, d\Sigma = (f, v_h) + \gamma Q \int_\Sigma v_h \, d\Sigma, \, \forall v_h \in V_h;$$

2. Update the Lagrange multiplier:

$$\lambda_h^{(k+1)} = \lambda_h^{(k)} + \rho \left(Q - \int_\Sigma u_h^{(k)} \, d\Sigma \right);$$

3. Stop if $|\lambda_h^{(k+1)} = \lambda_h^{(k)}| \leq \tau$.

The convergence of the previous method is guaranteed for $0 < \rho_0 \leq \rho \leq 2\gamma$, for a suitable ρ_0, see [15].

The previous method allows, unlike the penalization ones, to prescribe condition (1c) strongly, for any $\gamma > 0$. The presence of the penalization term improves convergence of the Uzawa method with respect to when it is applied to the classical Lagrange multiplier approach. Of course, this method is not of particular interest in case of a single flow rate, since, as highlighted in Section 2.2, the solution in this case is achieved in two steps with the Schur complement approach. However, in the case of defective conditions applied to several portion of the boundary, the algorithm based on the Schur complement requires $m+1$ solutions of a Poisson problem, see Remark 2.1. Thus, the Uzawa algorithm could be competitive if it allows a satisfactory convergence in less than $m+1$ iterations. We do not report here the extension of the algorithm to the case of more than one flow rate conditions, since it is straightforward.

2.5 Methods based on control theory

The last strategy we present could be considered as the dual of the Lagrange multiplier approach. Indeed, in this case the flow rate condition (1c) is used to build the functional to be minimized, while the differential problem (1a),(1b) defines the constraint. This gives rise to the following *optimal control* problem: given $\alpha \geq 0$, find $z \in \mathbb{R}$ such that

$$z = \mathrm{argmin}_{s \in \mathbb{R}} J(v(s), s) = \frac{1}{2} \left(\int_\Sigma v(s) d\Sigma - Q \right)^2 + \frac{\alpha}{2} (s - z_0)^2, \tag{24}$$

where $v = v(s) \in V$ satisfies

$$(\mu \nabla v(s), \nabla \psi) = (f, \psi) + \int_\Sigma s \psi \, d\Sigma \quad \forall \psi \in V. \tag{25}$$

This corresponds to the weak form of the following differential problem

$$-\nabla\cdot(\mu\nabla v) = f \qquad \text{in } \Omega, \tag{26a}$$

$$v = 0 \qquad \text{on } \Gamma, \tag{26b}$$

$$\mu\frac{\partial v}{\partial \mathbf{n}} = s \qquad \text{on } \Sigma. \tag{26c}$$

The solution u is then recovered by setting $u = v(z)$. The term involving the parameter α is a *Tikhonov regularization* term [8] and z_0 a reference value. Notice also that z assumes the same meaning of the Lagrange multiplier λ. This is a control problem with control on the Neumann boundary and boundary observations on the same portion of the boundary.

If $|\Gamma| > 0$, $\alpha > 0$, $f \in L^2(\Omega)$ and for $\partial\Omega$ with Liptshitz boundary, the map $s \to v(s) : \mathbb{R} \to V$ is linear and continuous and the functional $J : \mathscr{C} \to \mathbb{R}$ defined in (24) is convex, coercive and differentiable. This implies that the previous constrained minimization problem admits a unique solution.

Proceeding us usual in control theory for PDEs [27, 36, 19], problem (24)-(26) is equivalent to the following *first order optimality* conditions (also referred to as *Karush-Kuhn-Tucker* (KKT) conditions): find $z \in \mathbb{R}$, $u \in V$ and $\lambda_u \in V$ such that

$$\text{State pbl}: \quad (\mu\nabla u, \nabla v) + z\int_\Sigma v\, d\Sigma = (f, v), \tag{27a}$$

$$\text{Adj pbl}: \quad (\mu\nabla v, \nabla\lambda_u) + \int_\Sigma u\, d\Sigma \int_\Sigma v\, d\Sigma = Q\int_\Sigma v\, d\Sigma, \tag{27b}$$

$$\text{Opt. cond}: \quad \int_\Sigma \lambda_u\, d\Sigma + \alpha(z - z_0) = 0, \tag{27c}$$

for all $v \in V$ and where λ_u is the solution of the adjoint problem. The optimality condition corresponds to setting to zero the Frechèt derivative of $J'(s)$.

For the numerical solutions of the previous problem, a monolithic approach could be considered, which corresponds to solve a linear system of the form

$$\begin{bmatrix} A_{uu} & 0 & \mathbf{a}_{uz} \\ A_{u\lambda} & A_{\lambda\lambda} & 0 \\ 0 & A_{z\lambda} & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \Lambda \\ z_h \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_\lambda \\ \alpha z_0 \end{bmatrix},$$

where the various term derive, for instance, from a Galerkin discretization of (27).

However, this implies loss of modularity and the need to solve a relatively large system. Alternatively, one could consider a descent algorithm, the simplest one being a steepest descent, which gives rise to:

Given $z_h^{(0)} \in \mathbb{R}$, a suitable relaxation parameter $\beta_k > 0$, and a tolerance $\tau > 0$, for $k = 1, \ldots$:

1. Solve
$$A_{uu}\mathbf{U}^{(k)} = \mathbf{F}_u - \mathbf{a}_{uz}z_h^{(k-1)};$$

2. Solve

$$A_{\lambda\lambda}\Lambda^{(k)} = \mathbf{F}_\lambda - A_{\lambda u}\mathbf{U}^{(k)};$$

3. Update the control variable

$$z_h^{(k)} = z_h^{(k-1)} - \beta_k \left(A_{z\lambda}\Lambda^{(k)} + \alpha z_h^{(k)} \right);$$

4. Stop if $|z^{(k)} - z^{(k-1)}| \leq \tau$.

The first step is equivalent to solve the Poisson problem with Neumann data $z^{(k-1)}$, the second problem is again a Poisson problem with Neumann data $\int_\Sigma u^{(k)}\, d\Sigma$. Therefore, they can be both tackled with standard solvers. Other, more efficient solvers for the control problem may be found, for instance in [31].

It may seem that this method is less efficient than the other ones, yet it has the advantage of being rather flexible for more general problems. For instance, it may be used to implement defective Robin conditions $\int_\Sigma (u + \beta\mu\partial u/\partial\mathbf{n})\, d\Sigma = Q$ (which however may be implemented also by the Nitsche's penalization approach, see [14]).

3 Defective boundary condition for Stokes/Navier Stokes

We now briefly describe some extensions of the proposed techniques to the Stokes equations, which form the basis for the application to Navier-Stokes. Indeed, defective boundary problems have been originally studied in the context of fluid-dynamics [7, 21], in particular in hemodynamics where often the measures or the coupling with reduced models provide only average data on the artificial sections [11].

For the sake of exposition we consider the following steady Stokes problem (all the strategies reported can be extended to the case of unsteady Navier-Stokes). Let the velocity $\mathbf{u}$ and pressure p be solution of:

$$-\mu\triangle\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{28a}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{28b}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma, \tag{28c}$$

$$\int_\Sigma \mathbf{u}\cdot\mathbf{n}\, d\Sigma = Q, \tag{28d}$$

where the notation introduced Section 2 has been used. The previous is a *flow rate* defective problem, where only the average of normal component of the velocity is known. This is a typical situation when clinical measures are known in hemodynamics or geometrically reduced models are coupled at the artificial sections [34].

Alternative to the flow rate (28d), the following mean stress condition could be prescribed on the artificial sections:

$$\int_\Sigma (-p\mathbf{n} + \mu\nabla\mathbf{u}\,\mathbf{n})\, d\Sigma = -|\Sigma| P, \tag{29}$$

We refer to the defective problem give by (28a)-(28b)-(28c)-(29) as *mean stress* problem. Often, the viscosity term is neglected in condition (29) since it is negligible on artificial section with respect to the pressure. In this case we have

$$\int_{\Sigma} p\, d\Sigma = |\Sigma| P, \tag{30}$$

and we refer to the corresponding defective condition as *mean pressure* condition.

In the following subsections, we review the most classical approaches proposed so far for the two defective problems introduced above.

3.1 Empirical methods

The most used strategy in the engineering community to prescribe the flow rate condition (28d) is to select a priori a velocity profile $\mathbf{g}$ such that $\int_{\Sigma} \mathbf{g} \cdot \mathbf{n}\, d\Sigma = Q$ and then prescribe the Dirichlet condition

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Sigma.$$

Classical choices for circular sections are the parabolic one, which works well for example in the carotids [6], the flat one, which is quite often used in the aorta [28], and the one based on the Womersley solution [20]. However, in practical situations the artificial sections are not circular, and a suitable morphing is needed [20]. In any case, the choice of the velocity profile in general influences the numerical solution and introduces an error inside the computational domain. In particular, it is known from the computational practice that the flow fully develops after a characteristic distance from the section. This means that inside the region identified by this length, an error due to the wrong choice of the velocity profile is associated to the solution, whereas outside this region the solution could be considered accettable. This is the reason why in the engineering practice, the computational domain is extended at the section at hand of a length which is comparable with the characteristic length needed to the flow to fully develop. This characteristic length is known to increase for increasing Reynolds number Re [35]. In particular, for steady flows in a cylindrical domain, its value can be approximated by $0.058\, D\, Re$ (D being the diameter of the inlet section) [41]. In [38], it has been proved that the error features an exponential decay with respect to the distance from the section where the arbitrary profile is prescribed, with a constant which increases with Re.

Regarding the mean stress problem (28a)-(28b)-(28c)-(29), a classical empirical approach consists in selecting a constant stress aligned with the normal direction [14], i.e.

$$-p\mathbf{n} + \mu \nabla \mathbf{u}\, \mathbf{n} = -P\mathbf{n}.$$

This assumption is in general acceptable for example in hemodynamics, where the pressure mainly changes along the axial direction. The previous Neumann condition has been proposed also to treat the mean pressure problem (28a)-(28b)-(28c)-(30) [21]. However, in this case the corresponding weak formulation is not consistent with the defective condition [14]. To recover a consistent approximation, the *curl-curl* formulation of the Stokes problem should be considered since the corresponding natural condition is the pressure [7, 37].

3.2 Lagrange multiplier approach

The Lagrange multiplier approach for the flow rate problem (28) has been introduced in [11]. Following the idea reported in Section 2.2, the following augmented formulation is obtained: find $\mathbf{u} \in [V]^d$, $p \in Q = L^2(\Omega)$ and $\lambda \in \mathbb{R}$ such that for all $(v, q, \xi) \in [V]^d \times Q \times \mathbb{R}$,

$$(\mu \nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + b(\mathbf{v} \cdot \mathbf{n}, \lambda) = (\mathbf{f}, \mathbf{v}) \tag{31a}$$

$$(q, \nabla \cdot \mathbf{u}) = 0 \tag{31b}$$

$$b(\mathbf{u} \cdot \mathbf{n}, \xi) = \xi Q, \tag{31c}$$

where the notation is the same of Section 2.2. An inf-sup condition holds true for the previous augmented Stokes problem [37]. Again, the Lagrange multiplier has the physical meaning of constant normal stress on Σ [11].

Notice that the flow rate condition (28d) does not provide any information (neither defective) on the tangential velocity. To close the augmented problem, boundary conditions on the tangential velocity or stress should be considered. This choice is given by the choice of the functional space $[V]^d$. In particular, if its functions are not constrained in the tangential direction on Σ, a homogeneous Neumann condition is implicitely assumed for the tangential stress; otherwise, if they vanish in the tangential direction, they imply Dirichlet condition on the tangential velocity.

For its numerical solution, we can consider either a monolithic approach or an algorithm similar to that presented in Section 2.2 based on the Schur complement equation [11]. Analogously to the Poisson case, this algorithm is modular and consists in the solution of two Stokes problems with Neumann conditions on Σ. The extension to the Navier-Stokes case has been obtained in [37]. In both the cases, for m flow rate conditions this algorithm relies on the solution of $m+1$ Stokes/Navier-Stokes problems. For this reason, in [38] an inexact splitting algorithm has been proposed to save computational time, consisting in the solution of just 1 Stokes/Navier-Stokes problem, where however an error near to Σ is introduced. The authors noticed that since the error is introduced by solving a null flow rate problem arising from the splitting by means of a homogeneous Dirichlet condition, the error is the smallest one provided by any empirical approach, since the Reynolds number at the section at hand is zero.

The extension of the Lagrange multiplier approach to the case of compliant walls has been addressed in [13], whereas the case of quasi-Newtonian fluid has been analyzed in [10], where an error analysis for the numerical approximation is also given.

3.3 Penalization methods

The Nitche's approach reported in Sect. 2.3 may be extended to the Stokes problem.

In [42], a consistent penalization method for the mean flux problem as been designed. It reads: find $\mathbf{u}_h \in [V_h]^d$ and $p_h \in Q_h$, such that for all $\mathbf{v}_h \in [V_h]^d$ and $q_h \in Q_h$,

$$
\begin{aligned}
(\mu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) + \gamma\int_\Sigma \mathbf{u}_h\cdot\mathbf{n}\,d\Sigma\int_\Sigma \mathbf{v}_h\cdot\mathbf{n}\,d\Sigma - \frac{1}{|\Sigma|}\int_\Sigma \mu\nabla\mathbf{u}_h\cdot\mathbf{n}\,d\Sigma\int_\Sigma \mathbf{v}_h\cdot\mathbf{n}\,d\Sigma \\
-\frac{1}{|\Sigma|}\int_\Sigma \mathbf{u}_h\cdot\mathbf{n}\,d\Sigma\int_\Sigma \mu\nabla\mathbf{v}_h\cdot\mathbf{n}\,d\Sigma + \frac{1}{|\Sigma|}\int_\Sigma p_h\,d\Sigma\int_\Sigma \mathbf{v}_h\cdot\mathbf{n}\,d\Sigma = \\
(\mathbf{f},\mathbf{v}_h) + \gamma Q\int_\Sigma \mathbf{v}_h\cdot\mathbf{n}\,d\Sigma - \frac{1}{|\Sigma|}Q\int_\Sigma \mu\nabla\mathbf{v}_h\cdot\mathbf{n}\,d\Sigma, \\
-(q_h, \nabla\cdot\mathbf{u}_h) + \frac{1}{|\Sigma|}\int_\Sigma q_h\,d\Sigma\int_\Sigma \mathbf{u}_h\cdot\mathbf{n}\,d\Sigma = \frac{1}{|\Sigma|}Q\int_\Sigma q_h\,d\Sigma.
\end{aligned}
$$

In [42], it has been proved that, if it exists a constant c such that $\mu\nabla\mathbf{u}\,\mathbf{n} - p\mathbf{n} = c\mathbf{n}$ on Σ, then the previous formulation is consistent with (28) and that if $\gamma = \widehat{\gamma}/(h|\Sigma|)$ with $\widehat{\gamma}$ large enough, the solution is unique. The arguments are similar to those illustrated for the Poisson problem.

Referring to the notation introduced in Section 2, the algebraic problem related to the Galerkin approximation of the previous Nitsche formulation is

$$
\begin{bmatrix} A^Q & (B^Q)^T \\ B^Q & 0 \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^Q \\ \mathbf{G}^Q \end{bmatrix},
$$

where $A^Q_{ij} = (\mu\nabla\varphi_j, \varphi_i) + \gamma\int_\Sigma \varphi_j\cdot\mathbf{n}\,d\Sigma\int_\Sigma \varphi_i\cdot\mathbf{n}\,d\Sigma - \frac{1}{|\Sigma|}\int_\Sigma \mu\nabla\varphi_j\cdot\mathbf{n}\,d\Sigma\int_\Sigma \varphi_i\cdot\mathbf{n}\,d\Sigma - \frac{1}{|\Sigma|}\int_\Sigma \varphi_i\cdot\mathbf{n}\,d\Sigma\int_\Sigma \mu\nabla\varphi_j\cdot\mathbf{n}\,d\Sigma$, $B^Q_{kj} = -(\psi_k, \nabla\cdot\varphi_j) + \frac{1}{|\Sigma|}\int_\Sigma \psi_k\,d\Sigma\int_\Sigma \varphi_j\cdot\mathbf{n}\,d\Sigma$, $\mathbf{P}$ collects the pressure unknowns, $F^Q_i = (\mathbf{f}, \varphi_i) + \gamma Q\int_\Sigma \varphi_i\cdot\mathbf{n}\,d\Sigma - \frac{1}{|\Sigma|}Q\int_\Sigma \mu\nabla\varphi_i\cdot\mathbf{n}\,d\Sigma$, $G^Q_k = \frac{1}{|\Sigma|}Q\int_\Sigma \psi_k\,d\Sigma$ and where ψ_k are the basis function for the pressure approximation. The previous linear system preserves the saddle-point nature of the classical Stokes problem.

A Nitsche formulation has been proposed for the mean stress problem in [39]. The corresponding weak formulation reads: find $\mathbf{u}_h \in [V_h]^d$ and $p_h \in Q_h$, such that for all $\mathbf{v}_h \in [V_h]^d$ and $q_h \in Q_h$,

$$
\begin{aligned}
(\mu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) - \gamma\int_\Sigma \mu\nabla\mathbf{u}_h\cdot\mathbf{n}\,d\Sigma\int_\Sigma \mu\nabla\mathbf{v}_h\cdot\mathbf{n}\,d\Sigma \\
+\frac{1}{|\Sigma|}\int_\Sigma p_h\,d\Sigma\int_\Sigma \nabla\mathbf{v}_h\mathbf{n}\,d\Sigma = (\mathbf{f},\mathbf{v}_h) - P\int_\Sigma \mathbf{v}_h\cdot\mathbf{n}\,d\Sigma + \gamma P|\Sigma|\int_\Sigma \mu\nabla\mathbf{v}_h\cdot\mathbf{n}\,d\Sigma, \\
-(q_h, \nabla\cdot\mathbf{u}_h) - \gamma\int_\Sigma p_h\,d\Sigma\int_\Sigma q_h\,d\Sigma + \gamma\int_\Sigma \nabla\mathbf{u}_h\cdot\mathbf{n}\,d\Sigma\int_\Sigma q_h\,d\Sigma = -\gamma P|\Sigma|\int_\Sigma q_h\,d\Sigma.
\end{aligned}
$$

In [39], it has been proved that, if it exists a constant c such that $\mu\nabla\mathbf{u}\,\mathbf{n} - p\mathbf{n} = c\mathbf{n}$ on Σ, then the previous formulation is consistent with (28a)-(28b)-(28c)-(29) and that if $\gamma = \widehat{\gamma}h/|\Sigma|$ with $\widehat{\gamma}$ large enough, we have again a unique solution. The corresponding algebraic problem related to the Galerkin approximation of the Nitsche formulation reads

$$
\begin{bmatrix} A^P & (B^P)^T \\ B^P & C^P \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^P \\ \mathbf{G}^P \end{bmatrix},
$$

where $A^P_{ij} = (\mu\nabla\varphi_j, \nabla\varphi_i) - \gamma\int_\Sigma \mu\nabla\varphi_j\cdot\mathbf{n}\,d\Sigma\int_\Sigma \mu\nabla\varphi_i\cdot\mathbf{n}\,d\Sigma$, $B^P_{kj} = -(\psi_k, \nabla\cdot\varphi_j) + \frac{1}{|\Sigma|}\int_\Sigma \psi_k\,d\Sigma\int_\Sigma \nabla\varphi_j\mathbf{n}\,d\Sigma$, $C^P_{kl} = -\gamma\int_\Sigma \psi_l\,d\Sigma\int_\Sigma \psi_k\,d\Sigma$, $F^P_j = (\mathbf{f}, \varphi_j) - P\int_\Sigma \varphi_j\cdot\mathbf{n}\,d\Sigma + \gamma P|\Sigma|\int_\Sigma \mu\nabla\varphi_j\cdot\mathbf{n}\,d\Sigma$ and $G^P_k = -\gamma P|\Sigma|\int_\Sigma \psi_k\,d\Sigma$.

An alternative formulation consistent with the mean pressure problem (28a)-(28b)-(28c)-(30) has been proposed in [39].

3.4 Augmented Lagrangian formulation

Following the idea reported in Section 2.4, we can introduce also for the flow rate problem (28) an augmented Lagrangian fomultion where both a Lagrange multiplier and a penalization term are introduced. This allows to prescribe strongly the flow rate condition (28d) and to improve the convergence in an Uzawa-like algorithm, see Section 2.4. In particular, this formulation reads: find $\mathbf{u}_h \in [V_h]^d$, $p_h \in Q_h$ and $\lambda_h \in \mathbb{R}$ such that $\forall \mathbf{v}_h \in [V_h]^d$, $q_h \in Q_h$ and $\xi \in \mathbb{R}$,

$$\begin{aligned}
&(\mu \nabla u_h, \nabla v_h) - (p_h, \nabla \cdot \mathbf{v}_h) + \lambda_h \int_\Sigma \mathbf{v}_h \cdot \mathbf{n}\, d\Sigma + \gamma \int_\Sigma \mathbf{u}_h \cdot \mathbf{n}\, d\Sigma \int_\Sigma \mathbf{v}_h \cdot \mathbf{n}\, d\Sigma \\
&\qquad = (\mathbf{f}, \mathbf{v}_h) + \gamma Q \int_\Sigma \mathbf{v}_h \cdot \mathbf{n}\, d\Sigma, \\
&(q_h, \nabla \cdot \mathbf{u}_h) = 0, \\
&\xi \int_\Sigma u_h\, d\Sigma = \xi Q.
\end{aligned}$$

3.5 Methods based on control

As observed in Section 2.5, these techniques are based on minimizing a functional related to the flow rate condition (28d) under the constrain given by the Stokes problem. In analogy of what observed in Section 2.5, the control variable z is here the constant normal component of the normal stress [12]

$$-p\mathbf{n} + \nabla \mathbf{u}\, \mathbf{n} = z\mathbf{n} \qquad \text{on } \Sigma.$$

Referring to the notation of Section 2, this leads to the following first order optimality conditions: find $z \in \mathbb{R}$, $\mathbf{u} \in [V]^d$, $p \in Q$, $\lambda_u \in [V]^d$ and $\lambda_p \in Q$ such that

$$\begin{aligned}
\text{State pbl :} \quad & (\mu \nabla u, \nabla v) - (p, \nabla \cdot \mathbf{v}) + z \int_\Sigma \mathbf{v} \cdot \mathbf{n}\, d\Sigma = (\mathbf{f}, \mathbf{v}), \\
& (q, \nabla \cdot \mathbf{u}) = 0 \\
\text{Adj pbl :} \quad & (\mu \nabla v, \nabla \lambda_u) - (\lambda_p, \nabla \cdot \mathbf{v}) + \int_\Sigma \mathbf{u} \cdot \mathbf{n}\, d\Sigma \int_\Sigma \mathbf{v} \cdot \mathbf{n}\, d\Sigma = Q \int_\Sigma \mathbf{v} \cdot \mathbf{n}\, d\Sigma, \\
& (q, \nabla \cdot \lambda_u) = 0 \\
\text{Opt. cond :} \quad & \int_\Sigma \lambda_u \cdot \mathbf{n}\, d\Sigma + \alpha(z - z_0) = 0,
\end{aligned}$$

for all $\mathbf{v} \in [V]^d$ and $q \in Q$. Existence and unicity of the solution under the constraint that the normal stress is constant and aligned along the normal component are provided in [12].

In [24, 17, 18], the complete normal stress is chosen as control variable $\mathbf{z}$

$$-p\mathbf{n} + \nabla \mathbf{u}\, \mathbf{n} = \mathbf{z} \qquad \text{on } \Sigma.$$

This allows one to treat also cases where the normal stress is not supposed to be aligned with the axial direction, e.g. for not orthogonal artificial sections. Existence of a solution is provided [24]. Alternatively, the value of the velocity $\mathbf{u}$ on Σ could be used as control variables $\mathbf{z}$, see [24].

The optimal control approach has been proposed also for the mean pressure problem (28a)-(28b)-(28c)-(30) in [12]. In this case, the control variable is set equal to the flow rate or to the complete normal stress on Σ, see also [24] for the latter case.

The case of fluid problem in compliant vessels has been addressed in [13].

4 Some applications to hemodynamics

We present here two examples of applications in heamodynamics on real geometries reconstructed from radiological images acquired at Ospedale Ca' Granda - Policlinico di Milano, Italy. For both numerical experiments, we have considered the incompressible Navier-Stokes equations in rigid domains and a flow rate condition (28d) at the inlet. For the prescription of the flow rate condition, we have used the Lagrange multipliers approach presented in Section 3.2 and the algorithm based on the Schur complement equation introduced in [37] for Navier-Stokes equations. We also used $P2-P1$ Finite Elements and the Backward Difference Formula of order 2 (BDF2) for the space and time discretization, respectively.

The results have been obtained with the parallel Finite Element library *LIFEV* developed at MOX - Politecnico di Milano, INRIA - Paris, CMCS - EPF Lausanne, and Emory University - Atlanta (*www.lifev.org*). The linear system arising at each time step has been solved with GMRes preconditioned with an Additive Shwartz preconditioner.

4.1 The case of a stenotic carotid

In the first numerical experiment, we consider a stenotic carotid due to the presence of an atheromasic plaque at the bifurcation. We prescribed the flow rates depicted in Figure 1, left, at the inlet (Common Carotid Artery, CCA) and at one of the two outlets, namely at the Internal Carotid Artery (ICA). As a comparison, we considered also the case where a parabolic profile fitting the flow rate is used instead of the Lagrange multipliers approach.

In Figure 2 we observe that the numerical result obtained when a parabolic profile is prescribed blows up. This is due to the swirling nature of velocity pattern in the ICA, induced by the stenosis, which is not able to fit the parabolic profile prescribed at the outlet. Instead, the Lagrange multiplier approach works well, adjusting the velocity profile at the ICA so that the mathing with the inner velocity is stable.

4.2 The case of an aortic abdominal aneurysm

In the second numerical experiment, we consider a blood flow simulation in an aortic abdominal aneurysm (AAA). We prescribed the flow rates depicted in Figure 1, right,

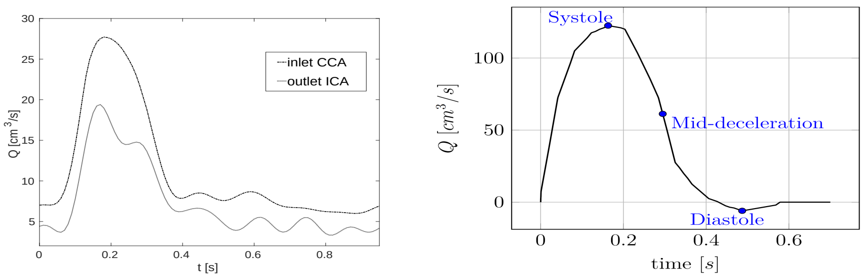

Figure 1: Flow waveforms prescribed at the inlet. Left: carotid simulation. Right: AAA simulation.

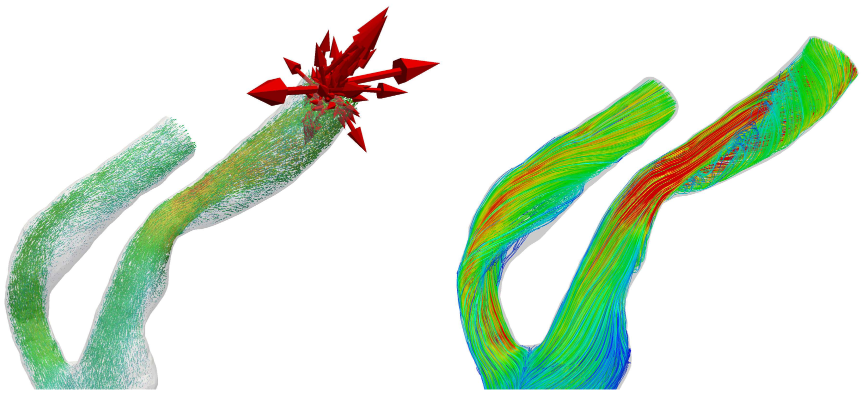

Figure 2: Blood velocity field in the case of a parabolic profile at the ICA (left) and of Lagrange multiplier approach at the ICA (right). .

at the inlet, and homogeneous Neumann conditions at the two outlets. The specific geometry related to this pathology, characterized by a sudden change of diameter, allows transition to turbulence effect to develop [1, 23, 26]. In particular, transitional flow may appear in AAA during the late systolic and diastolic phases, and are usually localized in the distal end of the AAA sac.

We consider here a *large eddy simulation* (LES) model for the description of such effects. In particular, we consider the eddy viscosity *σ- model* [29] (see [40] for further details) and two meshes: *mesh I* (about $1.1 \cdot 10^6$ dofs for the velocity and $3.7 \cdot 10^5$ for the pressure) and *mesh II* (about $2.0 \cdot 10^6$ dofs for the velocity and $6.8 \cdot 10^5$ for the pressure).

In Figure 3, we report the results at four time instants for three different simulations, namely: i) a no model simulation (i.e. without LES) in mesh I; ii) a LES model simulation in mesh I; iii) a LES model simulation in mesh II. In Figure 4 we report for the same specific cases the vorticity patterns.

From these results, we observe some differences between the results obtained with LES and "no-model" simulations, whereas a good agreement between those obtained with LES a two different meshes, highlighting that probably LES simulation with mesh

I is enough to have accurate results. Also in this case we have used the Lagrange multiplier approach to impose mean conditions at the inlet sections, proving that the method works well also in presence of turbulent models.

5 Conclusions

With this work we wanted to give an introductory overview of techniques to apply defective conditions in problems governed by partial differential equations. We have illustrated various possibilities. We can conclude that for the mean flow problem the Lagrange multiplier approach has proved to be very effective. The Nitsche type penalization has the advantage of avoiding an additional saddle point problem, and is more flexible since it can accommodate also mean stress condition (and in fact also defective conditions of Robin type), and is a valid alternative. The control approach is rather interesting, but also rather costly, and, so far, has not found much use in practical applications. However it may be advantageous if the constraints to be imposed are more complex than the standard ones.

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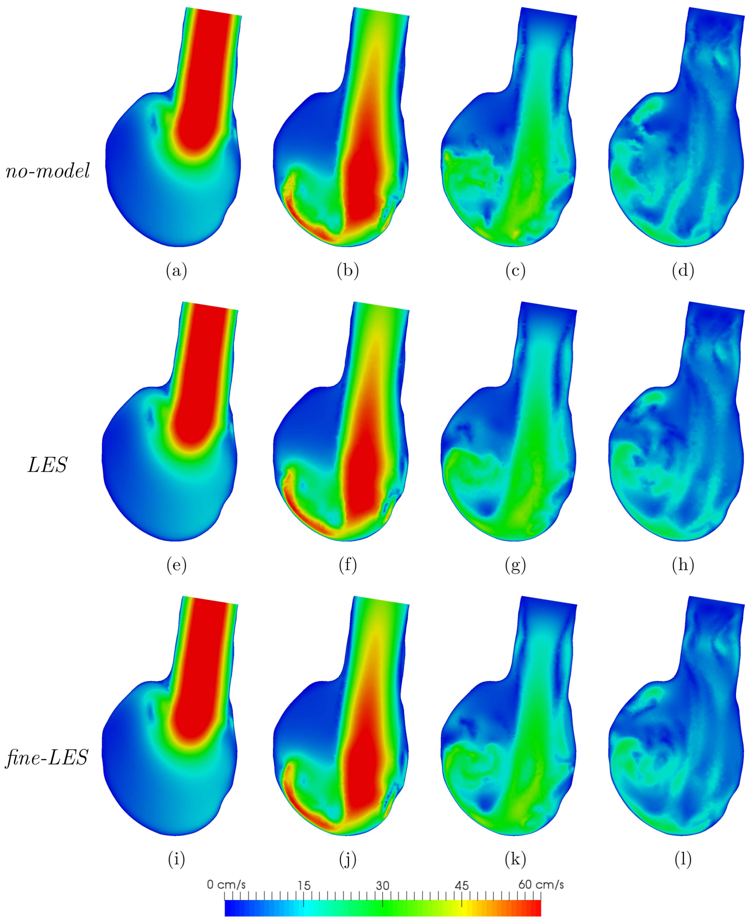

Figure 3: Velocity pattern at four different instants, from left to right: systole $t = 0.16$ s, mid-deceleration $t = 0.29$ s, early diastole $t = 0.40$ s, late diastole $t = 0.49$ s. Top: no-model simulation; Middle: LES simulation; Bottom: LES simulation on a finer mesh.

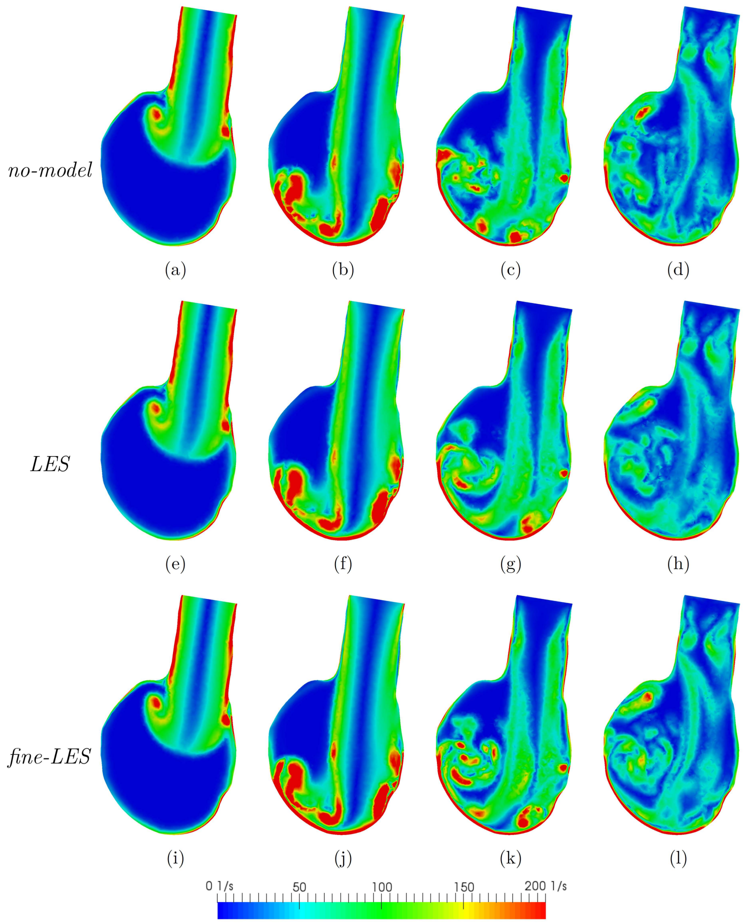

Figure 4: Vorticity pattern at four different instants, from left to right: systole $t = 0.16$ s, mid-deceleration $t = 0.29$ s, early diastole $t = 0.40$ s, late diastole $t = 0.49$ s. Top: no-model simulation; Middle: LES simulation; Bottom: LES simulation on a finer mesh.

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