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Ghiglietti, A.; Paganoni, A.M.

MOX, Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox@mate.polimi.it http://mox.polimi.it

Statistical properties of two-color randomly reinforced urn design targeting fixed allocations

Andrea Ghiglietti[a], Anna Maria Paganoni[a]

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[a] Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano
via Bonardi 9, 20133 Milano, Italy
andrea.ghiglietti@mail.polimi.it
anna.paganoni@polimi.it

Abstract

This paper deals with the statistical properties of a response adaptive design, described in terms of a two colors urn model, targeting prespecified asymptotic allocations. Results on the rate of convergence of number of patients assigned to each treatment are proved as well as on the asymptotic behavior of the urn composition. Suitable statistics are introduced and studied to test the hypothesis on treatment's differences.
Keywords: Response adaptive designs; Clinical trials; Randomly Reinforced Urns.

1 Introduction

In this paper we focus on studying statistical properties of a response-adaptive design, described in terms of two-color urn model, able to target any fixed asymptotic allocation probability. The model considered in this work is the Modified Randomly Reinforced Urn (MRRU) presented in [4]. The generality of the mathematical setting allows this experimental design to be applied to a broad set of areas of applications. However, since urn models are usually adopted to compare two or more competitive treatments, this work will be illustrated within a clinical trial framework. In this context, adaptive designs are attractive because they aim to achieve two simultaneous goals, concerning both statistical and ethical points of view: (a) collecting evidence to determine the superior treatment, and (b) increasing the allocation of units to the superior treatment. For a complete literature review on response adaptive designs see [16, 19]. Urn models are some of the most attracting adaptive designs, since they guarantee the randomization of allocations [26]. Asymptotic results concerning urn models with an irreducible mean reinforcement matrix could be found in [5, 6, 8, 18, 26]. This irreducibility assumption is not satisfied by the Randomly Reinforced Urn (RRU) studied in [20, 24, 25], which has a diagonal mean replacement matrix.

The RRU models were introduced by [9] for binary responses, applied to the dose-finding problems in [10, 11] and then extended to the case of continuous responses by [7, 24]. An interesting property concerning RRU models is that the probability to allocate units to the best treatment converges to one as the sample size increases, that is a very attractive feature from an ethical point of view. However, because of this asymptotic behavior, RRU models are not in the large class of designs targeting a certain proportion $\eta \in (0,1)$, that usually is fixed ad hoc or computed by satisfying some optimal criteria. Hence, all the asymptotic desirable properties concerning these procedures presented in literature (for instance in [22] and [23]), are not straightforwardly fulfilled by the RRU designs. Then, in [4] a Modified Randomly Reinforced Urn design (MRRU) was introduced; this design is able to target any prespecified asymptotic allocation $\eta \in (0,1)$.

In Section 2 we describe the MRRU model, which this work is based on. Visualize an urn containing balls of two colors (red,white) that is sequentially sampled. Each time, the extracted ball is reintroduced in the urn together with a random number of balls of the same color. To fix the notation we call μ_R and μ_W the laws of the random reinforcements of red and white balls, respectively, and m_R, m_W the corresponding means. Let us call $X = (X_n)_{n\in\mathbb{N}}$ ($X_n \in \{0,1\}$, $n = 1,2,..$) the sequence of the colors sampled by the urn and $Z = (Z_n)_{n\in\mathbb{N}}$ ($Z_n \in (0,1)$, $n = 0,1,2,..$) the sequence of urn proportions before each draw. We report the main result proved in [4], concerning the almost sure convergence of the process $(Z_n)_{n\in\mathbb{N}}$ to a fixed parameter $\eta \in (0,1)$, whenever the means of the reinforcements' distributions are different. We prove that the proportion of colors sampled by the urn converges to the same limit of the urn composition. Since this proportion represents also the proportion of patients assigned to treatments, we are able to rule the asymptotic patient's allocation.

Section 3 is focused on the rate of convergence of the process $(Z_n)_{n\in\mathbb{N}}$ in the MRRU model. Important results on the asymptotic behavior of the urn proportion $(Z_n)_{n\in\mathbb{N}}$ for a RRU model were developed in [12], in the case of reinforcements with different expected values. In [12] it was proved that the rate of convergence of the process $(Z_n)_{n\in\mathbb{N}}$ to one (i.e. its limit in the case $m_R > m_W$) is equal to $1/n^\gamma$ (with $\gamma = 1 - \frac{m_W}{m_R} < 1$). Moreover, the quantity $n^\gamma(1 - Z_n)$ converges almost surely to a positive random variable, whose behavior has been studied in [17, 21]. In Theorem 3.1 of this paper it is proved that the rate of convergence of the process $(Z_n)_{n\in\mathbb{N}}$ to its limit $\eta \in (0,1)$ is $1/n$ for the MRRU model. This asymptotic result was achieved after defining a particular Markov process denoted $(\widetilde{T}_n)_{n\in\mathbb{N}}$, based on the quantities that rule the urn process. The study of stochastic properties of the process $\widetilde{T}_n$ (see Appendix and Proposition 3.1) has been crucial for proving Theorem 3.1. Moreover, Theorem 3.1 shows that the sequence $n(\eta - Z_n)$ converges in distribution to a real random variable, whose probability law is related to the unique invariant distribution π of the process $(\widetilde{T}_n)_{n\in\mathbb{N}}$.

Section 4 is dedicated to the inferential aspects concerning the design described in Section 2. We deal with a classical hypothesis test comparing the null hypothesis that reinforcement's means are equal ($m_R = m_W$) and the one-side alternative hypothesis ($m_R > m_W$). We consider different statistical tests, based either (a) on adaptive estimators of the unknown means or (b) on the urn proportion. Under the null hypothesis, the asymptotic behavior of statistics of

type (a) has been studied in many works (see for instance [23] and the bibliography therein) for adaptive designs with target allocation $\eta \in (0,1)$ and in [12] for RRU designs. Instead, asymptotic properties of statistics concerning the urn proportion in a RRU design were investigated in [1, 2, 3]. However, under the null hypothesis the asymptotic distribution of the urn proportion's limit is still unknown, except in a few particular cases. The behavior under the alternative hypothesis of statistics based on adaptive estimators of the unknown parameters has been investigated for instance in [15, 16, 27] for adaptive designs with target allocation $\eta \in (0,1)$. For RRU designs, the asymptotic properties of both types of statistics have been studied in [12]. We compare statistical properties of tests based on RRU design and tests based on the MRRU design. We conclude that, for every fixed level α, we can construct a test based on MRRU design which is asymptotically more powerful than the one based on RRU design, and proposed in [14].
In Section 5 we illustrate some simulations studies on the probability distribution π and on the statistical properties of the tests described in Section 4. To ease the comprehension the proofs concerning the process $(\widetilde{T}_n)_{n\in\mathbb{N}}$ introduced in Section 3 are postposed in Appendix.

2 The Modified Randomly Reinforced Urn Design

Consider a clinical trial with two competitive treatments, say R and W. In this section we describe a response adaptive design, presented as an urn model, able to target any fixed asymptotic allocation. This model called MRRU, introduced in [4], is a modified version of the RRU design studied in [24]. In both the cases the reinforcements are modeled as random variables following different probability distributions. In the MRRU model we modify the reinforcement scheme of the urn to asymptotically target an optimal allocation proportion. The term *target* refers to the limit of the urn proportion process. Let us consider two probability distributions μ_R and μ_W with support contained in $[\alpha_R, \beta_R]$ and $[\alpha_W, \beta_W]$ respectively, where $0 < \alpha_R \leq \beta_R < +\infty$ and $0 < \alpha_W \leq \beta_W < +\infty$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of independent uniform random variable on $(0,1)$. We will interpret μ_R and μ_W as the laws of the responses to treatment R and W, respectively. We assume that both the means $m_R = \int_{\alpha_R}^{\beta_R} x\mu_R(dx)$ and $m_W = \int_{\alpha_W}^{\beta_W} x\mu_W(dx)$ are strictly positive. Moreover,

Assumption 2.1. *At least one of these two conditions is satisfied:*

(a) *there exists a closed interval $[\alpha_0, \beta_0] \subset [\alpha_W, \beta_W]$ such that, $\forall\ x \in [\alpha_0, \beta_0]$, the measure μ_W is absolutely continuous with respect the Lebesgue measure and the derivative is strictly positive, i.e. $\exists\ \frac{\mu_W(dx)}{dx} > 0$*

(b) *there exists a closed interval $[\alpha_0, \beta_0] \subset [\alpha_R, \beta_R]$ such that, $\forall\ x \in [\alpha_0, \beta_0]$, the measure μ_R is absolutely continuous with respect the Lebesgue measure and the derivative is strictly positive, i.e. $\exists\ \frac{\mu_R(dx)}{dx} > 0$*

Consider an urn initially containing r_0 balls of color R and w_0 balls of color W. Set

$$R_0 = r_0,\ W_0 = w_0,\ D_0 = R_0 + W_0,\ Z_0 = \frac{R_0}{D_0}.$$

At time $n = 1$, a ball is sampled from the urn; its color is $X_1 = \mathbf{1}_{[0,Z_0]}(U_1)$, a random variable with Bernoulli(Z_0) distribution. Let M_1 and N_1 be two independent random variables with distribution μ_R and μ_W, respectively; assume that X_1, M_1 and N_1 are independent. Next, if the sampled ball is R, it is replaced in the urn together with $X_1 M_1$ balls of the same color if $Z_0 < \eta$, where $\eta \in (0,1)$ is a suitable parameter, otherwise the urn composition does not change; if the sampled ball is W, it is replaced in the urn together with $(1 - X_1)N_1$ balls of the same color if $Z_0 > \delta$, where $\delta < \eta \in (0,1)$ is a suitable parameter, otherwise the urn composition does not change. So we can update the urn composition in the following way

$$\begin{aligned} R_1 &= R_0 + X_1 M_1 \mathbf{1}_{[Z_0<\eta]}, \\ W_1 &= W_0 + (1 - X_1)N_1 \mathbf{1}_{[Z_0>\delta]}, \\ D_1 &= R_1 + W_1,\ Z_1 = \frac{R_1}{D_1}. \end{aligned} \tag{2.1}$$

Now iterate this sampling scheme forever. Thus, at time $n + 1$, given the sigma-field $\mathcal{F}_n$ generated by $X_1, ..., X_n, M_1, ..., M_n$ and $N_1, ..., N_n$, let $X_{n+1} = \mathbf{1}_{[0,Z_n]}(U_{n+1})$ be a Bernoulli(Z_n) random variable and, independently of $\mathcal{F}_n$ and X_{n+1}, assume that M_{n+1} and N_{n+1} are two independent random variables with distribution μ_R and μ_W, respectively. Set

$$\begin{aligned} R_{n+1} &= R_n + X_{n+1} M_{n+1} \mathbf{1}_{[Z_n<\eta]}, \\ W_{n+1} &= W_n + (1 - X_{n+1})N_{n+1} \mathbf{1}_{[Z_n>\delta]}, \\ D_{n+1} &= R_{n+1} + W_{n+1}, \\ Z_{n+1} &= \frac{R_{n+1}}{D_{n+1}}. \end{aligned} \tag{2.2}$$

We thus generate an infinite sequence $X = (X_n, n = 1, 2, ..)$ of Bernoulli random variables, with X_n representing the color of the ball sampled from the urn at time n, and a process $(Z, D) = ((Z_n, D_n), n = 0, 1, 2...)$ with values in $[0, 1] \times (0, \infty)$, where D_n represents the total number of balls in the urn before it is sampled for the $(n + 1)$-th time, and Z_n is the proportion of balls of color R; we call X the process of colors generated by the urn while (Z, D) is the process of its compositions. Let us observe that the process (Z, D) is a Markov sequence with respect to the filtration $\mathcal{F}_n$.

In [4] it was proved that the sequence of proportions $Z = (Z_n, n = 0, 1, 2, ...)$ of the urn process converges almost surely to the following limit

$$\lim_{n\to\infty} Z_n = \begin{cases} \eta & \text{if } m_R > m_W, \\ \\ \delta & \text{if } m_R < m_W. \end{cases}$$

In this paper we study the urn process under the hypothesis $m_R > m_W$, because the situation $m_R < m_W$ is specular. Let us notice that in this case $P(Z_n < \delta,\ \ i.o.) = 0$; then, since we will deal with asymptotic results, from now on we can assume without loss of generality $\delta = 0$.

In this section we study some interesting features of the urn process. The first result concerns the proportion of colors sampled from the urn. Here we prove that it converges to the same limit of the urn proportion Z_n.

Proposition 2.1.

$$\frac{\sum_{i=1}^{n} X_i}{n} \overset{a.s.}{\rightarrow} \eta \tag{2.3}$$

Proof. Let us denote $\xi_n = \frac{Z_{n-1}-X_n}{n}$ for any $n \geq 1$, with $\xi_0 = 0$. Then, $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of random variables adapted with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ such that

$$\sum_{i=1}^{\infty} \mathbb{E}\left[\xi_i \,|\mathcal{F}_{i-1}\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[\frac{Z_{i-1}-X_i}{i} \,\middle|\mathcal{F}_{i-1}\right] = 0$$

$$\sum_{i=1}^{\infty} \mathbb{E}\left[\xi_i^2 \,\middle|\mathcal{F}_{i-1}\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[\left(\frac{Z_{i-1}-X_i}{i}\right)^2 \,\middle|\mathcal{F}_{i-1}\right] \leq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

Applying Lemma 7 of [2] we have that $\sum \xi_n < \infty$ almost surely.
Now, we have that

$$\frac{1}{n}\sum_{i=1}^{n} Z_{i-1} - X_i = \frac{1}{n}\sum_{i=1}^{n} i\xi_i \overset{a.s}{\rightarrow} 0,$$

by using Kronecker's lemma, and so

$$\eta - \frac{\sum_{i=1}^{n} X_i}{n} = \eta - \frac{\sum_{i=1}^{n} Z_{i-1}}{n} + \frac{\sum_{i=1}^{n} Z_{i-1} - X_i}{n} \overset{a.s.}{\rightarrow} 0$$

where the first term goes to zero thanks to the Toeplitz Lemma, since Z_n converge to η almost surely. ■

The following proposition shows the rate of divergence of the total number of balls in the urn. The sequence $(D_n/n, n = 0, 1, 2, ...)$ converges almost surely to the mean of the inferior treatment.

Proposition 2.2.

$$\frac{D_n}{n} \overset{a.s.}{\rightarrow} m_W \tag{2.4}$$

Proof. Notice that

$$\begin{aligned}
&\frac{\sum_{i=1}^{n} 1-X_i}{n}\left[\frac{W_0+\sum_{i=1}^{n}(1-X_i)N_i}{\sum_{i=1}^{n} 1-X_i} - m_W\right] &=\\
&\frac{\sum_{i=1}^{n}(1-X_i)N_i}{n} - m_W\frac{\sum_{i=1}^{n} 1-X_i}{n} &=\\
&\frac{\sum_{i=1}^{n}\left[\,(1-X_i)N_i - m_W(1-X_i)\,\right]}{n} &=\\
&\frac{\sum_{i=1}^{n}(1-X_i)(N_i-m_W)}{n} &\overset{a.s.}{\rightarrow} 0
\end{aligned}$$

where the almost sure convergence to zero of the last term can be proved with the same arguments used to prove Proposition 2.1. This result implies that

$$\frac{W_0+\sum_{i=1}^{n}(1-X_i)N_i}{\sum_{i=1}^{n} 1-X_i} \overset{a.s.}{\rightarrow} m_W \tag{2.5}$$

since from Proposition 2.1 we have that $\frac{\sum_{i=1}^{n}(1-X_i)}{n} \stackrel{a.s.}{\to} 1-\eta$. Then, we have that

$$\frac{W_n}{n} \;=\; \frac{W_0 + \sum_{i=1}^{n}(1-X_i)N_i}{\sum_{i=1}^{n}(1-X_i)} \;\cdot\; \frac{\sum_{i=1}^{n}(1-X_i)}{n} \;\stackrel{a.s.}{\to}\; m_W \;\cdot\; (1-\eta)$$

Since $Z_n \stackrel{a.s.}{\to} \eta$, we get

$$\frac{R_n}{n} \;=\; \frac{W_n}{n}\frac{Z_n}{1-Z_n} \;\stackrel{a.s.}{\to}\; \mu_W(1-\eta) \;\cdot\; \frac{\eta}{1-\eta} \;=\; m_W \;\cdot\; \eta$$

Globally we obtain

$$\frac{D_n}{n} \;=\; \frac{R_n}{n} \;+\; \frac{W_n}{n} \;\stackrel{a.s.}{\to}\; m_W \;\cdot\; \eta \;+\; m_W \;\cdot\; (1-\eta) \;=\; m_W$$

■

Remark 2.1. *Notice that in a RRU model the sequence* D_n/n *converges almost surely to the mean of the superior treatment. In fact, in a RRU model, when* $m_R > m_W$*, we have that*

$$\lim_{n\to\infty}\frac{D_n}{n} \;=\; \lim_{n\to\infty}\frac{R_n}{n} \;=\; \lim_{n\to\infty}\frac{R_0 + \sum_{i=1}^{n}X_iM_i}{\sum_{i=1}^{n}X_i} \;=\; m_R \tag{2.6}$$

on a set of probability one. The result (2.6) *is proved following the same arguments of* (2.5)

Here, we show that the proportion of times the urn proportion Z_n is under the limit η converges almost surely to a quantity that depends only on the reinforcements' means m_R and m_W.

Proposition 2.3.

$$\frac{\sum_{i=1}^{n}1_{\{Z_i<\eta\}}}{n} \;\stackrel{a.s.}{\to}\; \frac{m_W}{m_R} \tag{2.7}$$

To prove Proposition 2.3 we need the following lemma

Lemma 2.1.

$$\frac{\sum_{i=1}^{n}X_{i+1}1_{\{Z_i<\eta\}}}{\sum_{i=1}^{n}1_{\{Z_i<\eta\}}} \;\stackrel{a.s.}{\to}\; \eta \tag{2.8}$$

Proof. Notice that

$$\begin{aligned}
&\frac{\sum_{i=1}^{n}1_{\{Z_{i-1}<\eta\}}}{n}\left[\,\frac{\sum_{i=1}^{n}X_i1_{\{Z_{i-1}<\eta\}}}{\sum_{i=1}^{n}1_{\{Z_{i-1}<\eta\}}} - \eta\,\right] && = \\
&\frac{\sum_{i=1}^{n}X_i1_{\{Z_{i-1}<\eta\}}}{n} \;-\; \eta\frac{\sum_{i=1}^{n}1_{\{Z_{i-1}<\eta\}}}{n} && = \\
&\frac{\sum_{i=1}^{n}\left[\;X_i1_{\{Z_{i-1}<\eta\}} \;-\; \eta1_{\{Z_{i-1}<\eta\}}\;\right]}{n} && = \\
&\frac{\sum_{i=1}^{n}[X_i1_{\{Z_{i-1}<\eta\}} - Z_{i-1}1_{\{Z_{i-1}<\eta\}}]}{n} && + \\
&\frac{\sum_{i=1}^{n}[Z_{i-1}1_{\{Z_{i-1}<\eta\}} - \eta1_{\{Z_{i-1}<\eta\}}]}{n} && \stackrel{a.s.}{\to}\; 0
\end{aligned}$$

where the almost surely convergence to zero of the last terms can be proved with the same arguments used to prove Proposition 2.1. Moreover this result implies (2.8) due to the fact that $\frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n}$ cannot be asymptotically closed to zero. This fact can be proved by contradiction: suppose that

$$P\left(\liminf_{n\to\infty} \frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n} = 0 \right) > 0. \tag{2.9}$$

But we have that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n} \geq \\
&\liminf_{n\to\infty} \frac{1}{\beta_R} \frac{R_0 + \sum_{i=1}^n X_{i+1}M_{i+1}1_{\{Z_i<\eta\}}}{\sum_{i=1}^n X_{i+1}1_{\{Z_i<\eta\}}} \cdot \frac{\sum_{i=1}^n X_{i+1}1_{\{Z_i<\eta\}}}{\sum_{i=1}^n 1_{\{Z_i<\eta\}}} \cdot \frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n} \geq \\
&\liminf_{n\to\infty} \frac{1}{\beta_R}\frac{R_n}{n} = \frac{m_W\eta}{\beta_R} > 0
\end{aligned}$$

on a set of probability one. This contradicts the assumption (2.9).

■

Remark 2.2. *By following the same arguments used to prove Proposition 2.1 and Lemma 2.1 it can be proved also that*

$$\frac{R_0 + \sum_{i=1}^n X_{i+1}M_{i+1}1_{\{Z_i<\eta\}}}{\sum_{i=1}^n X_{i+1}1_{\{Z_i<\eta\}}} \overset{a.s.}{\to} m_R \tag{2.10}$$

Proof. [Proof of the Proposition 2.3] Let us observe that on a set of probability one

$$\begin{aligned}
0 &= \lim_{n\to\infty} \eta - Z_n = \lim_{n\to\infty} \eta - \frac{R_n/n}{R_n/n + W_n/n} = \\
\eta &- \frac{m_R \cdot \eta \cdot \lim_{n\to\infty} \frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n}}{m_R \cdot \eta \cdot \lim_{n\to\infty} \frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n} + m_W \cdot (1-\eta)}
\end{aligned} \tag{2.11}$$

where the last equality is based on the result of Lemma 2.1. Finally, we note that the equality (2.11) holds if and only if

$$\frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n} \overset{a.s.}{\to} \frac{m_W}{m_R}$$

■

3 Asymptotic results

We want to study the asymptotic behavior of the quantity $n\cdot(\eta - Z_n)$. To do this, let us introduce a new real stochastic process $(T_n)_{n\in\mathbb{N}}$, whose features depend on the random variables ruling the urn process:

$$\begin{cases} T_0 &= \eta W_0 - (1-\eta)R_0 \\ T_{n+1} &= T_n + \eta(1-X_{n+1})\, N_{n+1} - (1-\eta)X_{i+1}\, M_{i+1}\, 1_{\{Z_n<\eta\}} \end{cases} \tag{3.1}$$

$\forall n \in \mathbb{N}$. Let us note that

$$n \cdot (\eta - Z_n) \;=\; \frac{n(\eta - Z_n)D_n}{D_n} \;=\; \frac{\eta W_n - (1-\eta)R_n}{\frac{D_n}{n}} \;=\; \frac{T_n}{\frac{D_n}{n}} \tag{3.2}$$

where $T_n = \eta W_n - (1-\eta)R_n$ satisfies the iterative equations in (3.1).
The process $(Z_n, T_n)_{n\in\mathbb{N}}$ is an homogeneous Markov sequence. Then, there exists the transition probability kernel K for the process T_n such that for any $(z_0, t_0) \in (0, \eta] \times [0, \infty) \cup (\eta, 1) \times (-\infty, 0)$ and for any $A \subset \mathbb{R}$

$$P\left(T_{n+1} \in A \mid (Z_n, T_n) = (z_0, t_0) \right) \;=\; \int_A K_{z_0}(t_0, dt)$$

The analytic form of the transition probability kernel is the following

$$\begin{aligned} K_{z_0}(t_0, dt) \;&= z_0 \; \mu_R \left(d \left(\frac{t_0 - t}{1-\eta} \right) \right) \; 1_{\{z_0 < \eta \;\wedge\; t < t_0\}} \;+ \\ & z_0 \; \delta_{t_0}(t) \; 1_{\{z_0 > \eta\}} \;\;+\;\; (1 - z_0) \; \mu_W \left(d \left(\frac{t - t_0}{\eta} \right) \right) \; 1_{\{t > t_0\}} \end{aligned} \tag{3.3}$$

If the probability measures μ_R and μ_W are absolutely continuous with respect to the Lebesgue measure, we can write as well

- $\mu_R \left(d \left(\frac{t_0 - t}{1-\eta} \right) \right) \;=\; f_R \left(\frac{t_0 - t}{1-\eta} \right) \; \frac{1}{1-\eta} \; dt$
- $\mu_W \left(d \left(\frac{t - t_0}{\eta} \right) \right) \;=\; f_W \left(\frac{t - t_0}{\eta} \right) \; \frac{1}{\eta} \; dt$

where $f_R(\cdot)$ and $f_W(\cdot)$ are the Radon Nikodym derivatives of the measures μ_R and μ_W with respect to the Lebesge measure.
Since the marginal process T_n needs to be coupled with the process Z_n to have a Markov bivariate process (T_n, Z_n), the application of many results on Markov processes in the case of continuous state space it's not straightforward. Then, we define a new auxiliary process $\widetilde{T}_n$ strictly related to T_n, in this way:

$$\begin{cases} \widetilde{T}_0 & = \eta W_0 \;-\; (1-\eta) R_0 \\ \widetilde{T}_{n+1} & = \widetilde{T}_n \;+\; \eta(1 - \widetilde{X}_{n+1}) \; N_{n+1} \;-\; (1-\eta) \widetilde{X}_{n+1} \; M_{n+1} \; 1_{\{\widetilde{T}_n > 0\}} \end{cases} \tag{3.4}$$

$\forall n \in \mathbb{N}$, where $(\widetilde{X}_n)_{n\in\mathbb{N}}$ are i.i.d. Bernoulli random variables of parameter η independent of the sequences $(M_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$. It's easy to see that $\widetilde{T}_n$ is a Markov process. In fact, the transition kernel K_η of $\widetilde{T}_n$ is independent of the quantity z_0

$$\begin{aligned} K_\eta(t_0, dt) \;&= \eta \; \mu_R \left(d \left(\frac{t_0 - t}{1-\eta} \right) \right) \; 1_{\{t_0 > 0 \;\wedge\; t < t_0\}} \;\;+\;\; \eta \; \delta_{t_0}(t) \; 1_{\{t_0 < 0\}} \;\;+ \\ & (1-\eta) \; \mu_W \left(d \left(\frac{t - t_0}{\eta} \right) \right) \; 1_{\{t > t_0\}} \end{aligned} \tag{3.5}$$

Using Assumption 2.1 we can prove (see Appendix) that the Markov process $\widetilde{T}_n$ is an aperiodic recurrent Harris chain. So, the following holds:

Proposition 3.1. *Let call π the stationary distribution of the recurrent aperiodic Harris Chain $\widetilde{T} = (\widetilde{T}_n)_{n\in\mathbb{N}}$. Then, for every $t_0 \in \mathbb{R}$, we have that*

$$\lim_{n\to\infty} \sup_{C\in\mathcal{B}(\mathbb{R})} \mid P(\widetilde{T}_n \in C \mid \widetilde{T}_0 = t_0) \; - \; \pi(A) \mid \; = \; 0 \tag{3.6}$$

Proof. The Markov process $\widetilde{T}_n$ is a recurrent aperiodic Harris Chain (see Appendix). This result implies that there exists a unique invariant distribution probability π and (3.6) holds for any t_0 such that

$$P(\; \tau^A < \infty \mid \widetilde{T}_0 = t_0 \;) \; = \; 1 \tag{3.7}$$

The thesis is proved since (3.7) holds for any $t_0 \in \mathbb{R}$ (see Appendix). ■

Now, we can state the main result

Theorem 3.1. *For any initial composition $(r_0, w_0) \in (0,\infty) \times (0,\infty)$, we have that*

$$n \cdot (\eta - Z_n) \; \xrightarrow{\mathcal{L}} \; \frac{\psi}{m_W} \tag{3.8}$$

where ψ is a real random variable with probability distribution π.

Proof. Using equation (3.2), Proposition 2.2 and Slutsky's theorem we have that it's sufficient to prove that $T_n \xrightarrow{\mathcal{L}} \psi$, where ψ is a real random variable with probability distribution π. Notice that for any interval $C \subset \mathbb{R}$

$$\begin{aligned} \mid P(T_n \in C \mid T_0 = t_0) \; - \; \pi(C) \mid \; \leq & \mid P(T_n \in C \mid T_0 = t_0) \; - \; P(\widetilde{T}_n \in C \mid \widetilde{T}_0 = t_0) \mid + \\ & \mid P(\widetilde{T}_n \in C \mid \widetilde{T}_0 = t_0) \; - \; \pi(C) \mid \end{aligned}$$

From the Proposition 3.1 we have that the second term converges to zero as long as n goes to infinity. Then, to prove the thesis we have to study the first term.

Let us take $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha_0 < \alpha < \beta < \beta_0$; then, let us introduce the set

$$B \; = \; [\; (\beta - \alpha + \alpha_0)\eta \; , \; \beta_0\eta \;] \subset \mathbb{R}$$

and the probability measure

$$\rho(C) = \frac{1}{(\beta_0 - \beta + \alpha - \alpha_0)\eta} \int_C dt$$

defined for every set $C \subset B$. Then, it is easy to see that there exists a sequence of positive numbers $(\epsilon_{z_n})_{n\in\mathbb{N}}$ such that, if $t_0 \in A$, then $K_{z_n}(t_0, C) \geq \epsilon_{z_n}\rho(C) \; \forall n \in \mathbb{N}$. By following the same procedure adopted in the proof of Proposition 5.3, a possible choice for the terms of the sequence is

$$\epsilon_{z_n} = (\beta_0 - \beta + \alpha - \alpha_0)\eta(1 - z_n) \min_{x\in(\alpha_0,\beta_0)} \left[\frac{\mu_W \, (dx)}{dx} \right]$$

Since the sequence Z_n is strictly less than one and converges to η almost surely, we have that $\epsilon := \inf_{n\in\mathbb{N}}\{\epsilon_{z_n}\} > 0$. Besides, it is trivial to see that $K_\eta(t_0, C) \geq \epsilon\rho(C)$, because $P(\bigcup_{n=1}^{\infty}\{Z_n > \eta\}) = 1$.

Let us now introduce two sequences of i.i.d. Bernoulli variables $(\xi_i)_i$ and $(\widetilde{\xi}_i)_i$, both of them with parameter ϵ. Let us define two random variables which count the number of times the processes T_n and $\widetilde{T}_n$ are in the set A

$$\nu_n := \sum_{i=1}^{n} 1_{\{T_i \in A\}} \quad \textit{and} \quad \widetilde{\nu}_n := \sum_{i=1}^{n} 1_{\{\widetilde{T}_i \in A\}}$$

Then, let us construct two sequences of stopping times

$$\begin{cases} \tau_0 = 0 \\ \tau_i := \inf\{\ n > \tau_{i-1}\ :\ \{T_n \in A\} \bigcap \{\xi_{\nu_n} = 1\}\ \}, \quad i \geq 1 \end{cases}$$

$$\begin{cases} \widetilde{\tau}_0 = 0 \\ \widetilde{\tau}_i := \inf\{\ n > \widetilde{\tau}_{i-1}\ :\ \{\widetilde{T}_n \in A\} \bigcap \{\widetilde{\xi}_{\widetilde{\nu}_n} = 1\}\ \}, \quad i \geq 1 \end{cases}$$

Naturally, the times $(\widetilde{\tau}_i)_{i\in\mathbb{N}}$ are all almost surely finite because the process $\widetilde{T}_n$ is a recurrent Harris chain. It is easy to show that also the times $(\tau_i)_{i\in\mathbb{N}}$ are almost surely finite. The procedure to prove the recurrence of the process T_n it's analogous to the one used for the process $\widetilde{T}_n$.

Let us imagine that when the process (either T_n or $\widetilde{T}_n$) is in the set A, we flip a Bernoulli with parameter ϵ: if it comes up one, the process evolves by using the probability law $\rho(dt)$; otherwise, if it comes up zero, the process moves according to the modified transition kernel $\frac{K_\eta(t_0,dt)-\epsilon\rho(dt)}{1-\epsilon}$. The sequences ξ_n and $\widetilde{\xi}_n$ represent the outcomes of the Bernoulli trials when the process is in A. Let us denote as λ_{τ_i} and $\widetilde{\lambda}_{\widetilde{\tau}_i}$ the probability measures of the random variables T_{τ_i} and $\widetilde{T}_{\widetilde{\tau}_i}$ respectively, when both the processes start from the same initial point $t_0 \in \mathbb{R}$. Hence, we have that

$$\int_A \lambda_{\tau_i}(dt) K_{z_{\tau_i}}(t,C) \ = \ \int_A \lambda_{\tau_i}(dt)\rho(C) \ = \ \rho(C) \ = \ \int_A \widetilde{\lambda}_{\widetilde{\tau}_i}(dt)\rho(C) \ = \ \int_A \widetilde{\lambda}_{\widetilde{\tau}_i}(dt) K_\eta(t,C)$$

for any $C \in \mathcal{B}(\mathbb{R})$.

By comparing the transition kernels of the processes T_n and $\widetilde{T}_n$ we have that

$$\begin{aligned} &|K_{z_n}(t_0,dt) - K_\eta(t_0,dt)| \ = \ |(z_n-\eta)\ \mu_R\left(d\left(\frac{t_0-t}{1-\eta}\right)\right)\ 1_{\{t_0>0\ \wedge\ t<t_0\}}\ + \\ &(z_n-\eta)\ \delta_{t_0}(t)\ 1_{\{t_0<0\}}\ -\ (z_n-\eta)\ \mu_W\left(d\left(\frac{t-t_0}{\eta}\right)\right)\ 1_{\{t>t_0\}}|\ \leq \\ &\omega_n\eta\ \mu_R\left(d\left(\frac{t_0-t}{1-\eta}\right)\right)\ 1_{\{t_0>0\ \wedge\ t<t_0\}}\ +\ \omega_n\eta\ \delta_{t_0}(t)\ 1_{\{t_0<0\}}\ + \\ &\omega_n(1-\eta)\ \mu_W\left(d\left(\frac{t-t_0}{\eta}\right)\right)\ 1_{\{t>t_0\}}\ =\ \omega_n K_\eta(t_0,dt) \end{aligned}$$

for any $\omega_n \geq \frac{|z_n-\eta|}{\min\{\eta;1-\eta\}}$. Therefore, since Z_n converge to η a.s., there exists a sequence $(\omega_n)_{n\in\mathbb{N}}$, going to zero as n goes to infinity, such that for any $t_0 \in \mathbb{R}$

$$|\ K_{z_n}(t_0,dt) - K_\eta(t_0,dt)\ |\ \leq\ \omega_n K_\eta(t_0,dt)$$

By using this inequality, for any integer $k, n, n_0 \in \mathbb{N}$, any $t_0 \in \mathbb{R}$ and any set $C \in \mathcal{B}(\mathbb{R})$, we can obtain

$$
\begin{aligned}
&\mid P(T_{\tau_n+k} \in C \mid T_0 = t_0) \ - \ P(\widetilde{T}_{\widetilde{\tau}_{n_0}+k} \in C \mid \widetilde{T}_0 = t_0) \mid \ = \\
&\left| \int_A \lambda_{\tau_n}(ds_0) \cdot P(T_{\tau_n+k} \in C \mid T_{\tau_n} = s_0) \ - \ \int_A \widetilde{\lambda}_{\widetilde{\tau}_{n_0}}(ds_0) \cdot P(\widetilde{T}_{\widetilde{\tau}_{n_0}+k} \in C \mid \widetilde{T}_{\tau_{n_0}} = s_0) \right| \ = \\
&\left| \int_A \int_{\mathbb{R}^{k-1}} \int_C \left(\lambda_{\tau_n}(ds_0) \prod_{j=1}^{k} \ K_{z_{\tau_n+j-1}}(s_{j-1}, ds_j) \right) \ - \ \int_A \int_{\mathbb{R}^{k-1}} \int_C \left(\widetilde{\lambda}_{\widetilde{\tau}_{n_0}}(ds_0) \prod_{j=1}^{k} \ K_{\eta}(s_{j-1}, ds_j) \right) \right| \ = \\
&\left| \int_{\mathbb{R}^{k-1}} \int_C \left(\rho(ds_1) \prod_{j=2}^{k} \ K_{z_{\tau_n+j-1}}(s_{j-1}, ds_j) \right) \ - \ \int_{\mathbb{R}^{k-1}} \int_C \left(\rho(ds_1) \prod_{j=2}^{k} \ K_{\eta}(s_{j-1}, ds_j) \right) \right| \ \leq \\
&\max \left\{ \int_{\mathbb{R}^{k-1}} \int_C \left(\rho(ds_1) \prod_{j=2}^{k} \ (1+\omega_{\tau_n+j-1}) K_{\eta}(s_{j-1}, ds_j) \right) \ - \ \int_{\mathbb{R}^{k-1}} \int_C \left(\rho(ds_1) \prod_{j=2}^{k} \ K_{\eta}(s_{j-1}, ds_j) \right) \ ; \right. \\
&\qquad \left. \int_{\mathbb{R}^{k-1}} \int_C \left(\rho(ds_1) \prod_{j=2}^{k} \ K_{\eta}(s_{j-1}, ds_j) \right) \ - \ \int_{\mathbb{R}^{k-1}} \int_C \left(\rho(ds_1) \prod_{j=2}^{k} \ (1+\omega_{\tau_n+j-1}) K_{\eta}(s_{j-1}, ds_j) \right) \right\} \\
&\sum_{j=2}^{k} \binom{k-1}{j-1} \left[\max_{i \in \{1,2,\ldots,k-1\}} (\omega_{\tau_n+i}) \right]^{j-1} \int_{\mathbb{R}^{k-1}} \int_C \left(\rho(dt_1) \prod_{j=2}^{k} K_{\eta}(s_{j-1}, ds_j) \right) \ \leq \\
&\left(2^{k-1} - 1 \right) \max_{i \in \{1,2,\ldots,k-1\}} (\omega_{\tau_n+i})
\end{aligned}
$$

Therefore, we can prove that, for every $k, n_0 \in \mathbb{N}$,

$$
\lim_{n\to\infty} \sup_{C \in \mathcal{B}(\mathbb{R})} \mid P(T_{\tau_n+k} \in C \mid T_0 = t_0) \ - \ P(\widetilde{T}_{\widetilde{\tau}_{n_0}+k} \in C \mid \widetilde{T}_0 = t_0) \mid \ = \ 0
$$

Let define the stopping time

$$
\tau_n^* := \sup \{ \ \tau_i \leq n \ , \ i \in \mathbb{N} \ \}
$$

We have

$$
\begin{aligned}
&\lim_{n\to\infty} \sup_{C \in \mathcal{B}(\mathbb{R})} \mid P(T_n \in C \mid T_0 = t_0) \ - \ \ P(\widetilde{T}_n \in C \mid \widetilde{T}_0 = t_0) \mid \ \leq \\
&\lim_{n\to\infty} \sup_{C \in \mathcal{B}(\mathbb{R})} \mid P(T_{\tau_n^*+(n-\tau_n^*)} \in C \mid T_0 = t_0) \ - \ P(\widetilde{T}_{\widetilde{\tau}_m+(n-\tau_n^*)} \in C \mid \widetilde{T}_0 = t_0) \mid + \\
&\lim_{n\to\infty} \sup_{C \in \mathcal{B}(\mathbb{R})} \mid P(\widetilde{T}_{\widetilde{\tau}_m+(n-\tau_n^*)} \in C \mid \widetilde{T}_0 = t_0) \ - \ P(\widetilde{T}_n \in C \mid \widetilde{T}_0 = t_0) \mid \ = \ 0
\end{aligned}
$$

where the second term converges to zero if we let $m = m_n$ goes to infinity as n increase, since $P(\widetilde{T}_n \in C | \widetilde{T}_0 = t_0)$ is a Cauchy sequence. ∎

4 Testing hypothesis

In this section we focus on the inferential aspects concerning the MRRU design. Let us introduce the classical hypothesis test aiming at comparing the means of two distributions $\mu_R,\ \mu_W$

$$H_0:\ m_R - m_W = 0 \qquad vs \qquad H_1:\ m_R - m_W > 0. \tag{4.1}$$

We approach to the statistical problem (4.1) considering first a no-adaptive design, and then the MRRU model. Let $(M_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ be i.i.d. sequences of random variables with distribution μ_R and μ_W, respectively. For a fixed design with sample sizes n_R and n_W, the usual test statistics is

$$\zeta_0 \;=\; \frac{\overline{M}_{n_R} - \overline{N}_{n_W}}{\sqrt{\frac{s_R^2}{n_R} + \frac{s_W^2}{n_W}}} \tag{4.2}$$

where $\overline{M}_{n_R}$ and $\overline{N}_{n_W}$ are the sample means and s_R^2 and s_W^2 are consistent estimators of the variances. When the no-adaptive design allows both the sample sizes n_R and n_W goes to infinity, by the central limit theorem we have that, under the null hypothesis, ζ_0 converges in distribution to a standard normal variable. Then, fixing a significance level $\alpha \in (0,1)$, we define

$$R_\alpha \;=\; \{\zeta_0 > z_\alpha\} \tag{4.3}$$

as the critical region asymptotically of level α, with z_α as the α-percentage point of the standard gaussian distribution. Now, let us assume that the rate of divergence of the sample sizes is such that $\frac{n_R}{n_R+n_W} \to \eta$, for some $\eta \in (0,1)$. Then, the power of the test defined in (4.3) can be approximated, for large n_R and n_W, as

$$P\left(Z + \sqrt{n}\frac{m_R - m_W}{\sqrt{\frac{\sigma_R^2}{\eta} + \frac{\sigma_W^2}{1-\eta}}} \;>\; z_\alpha \right), \tag{4.4}$$

where Z is a gaussian standard random variable.

Now, let us consider an adaptive design described in term of an urn model. Let us denote $N_R(n)$ and $N_W(n)$ as the sample sizes after the firsts n draws, $\overline{M}(n)$ and $\overline{N}(n)$ the corresponding sample means and $s_R^2(n)$ and $s_W^2(n)$ the adaptive consistent estimators. Plugging in (4.2) the corresponding adaptive quantities, we obtain the statistics

$$\zeta_0(n) \;=\; \frac{\overline{M}(n) - \overline{N}(n)}{\sqrt{\frac{s_R^2(n)}{N_R(n)} + \frac{s_W^2(n)}{N_W(n)}}} \tag{4.5}$$

Using Proposition 3.1 of [4] and Slutsky's Theorem, it can be deduced from the no-adaptive case that for the MRRU model, if $m_R = m_W$, the statistics $\zeta_0(n)$ converges to a standard normal variable. Hence, the critical region (4.3) still defines a test asymptotically of level α. Moreover, calling η the limit of the urn proportion Z_n under the alternative hypothesis, the power of the test defined in (4.3) can be approximated, for large n, as (4.4).

Remark 4.1. *The behavior of the statistics ζ_0 defined in* (4.5) *in the case of RRU model was studied in [12]. In that paper, the asymptotic normality of $\zeta_0(n)$ under the null hypothesis was proved; then* (4.3) *defines a test of asymptotic level α also in the RRU case. However, under the alternative hypothesis $\zeta_0(n)$*

converges to a mixture of gaussian distributions, where the mixing variable φ^2 is a strictly positive random variable such that

$$\frac{N_W(n)}{n^{m_W/m_R}} \overset{a.s.}{\to} \varphi^2 \tag{4.6}$$

Therefore, it follows that in the RRU case the power of the test defined in (4.3) *can be approximated, for large n, as*

$$P\left(Z + n^{\frac{m_W}{2m_R}} \varphi \frac{m_R - m_W}{\sigma_W} \ > \ z_\alpha \right), \tag{4.7}$$

where Z is a gaussian standard random variable independent of φ.

A different test statistics based on the urn proportion of a RRU model has been investigated in [13]. Let us denote as $c_\alpha^{(0,1)}$ the α-percentage point of the distribution of the limiting proportion Z_∞ under the null hypothesis in a RRU model. Then, the critical region

$$\{Z_n > c_\alpha^{(0,1)}\} \tag{4.8}$$

defines a test asymptotically of level α. As explained in [13], the power of this test can be approximated, for large n, as

$$P\left(\varphi^2 \ < \ (1 - c_\alpha^{(0,1)}) \frac{m_R}{m_W} n^{1 - \frac{m_W}{m_R}} \right) \tag{4.9}$$

where φ^2 is the random quantity defined in (4.6).
Now, we consider the statistics Z_n as the urn proportion of a MRRU model, with parameters δ and η. Let us denote as $c_\alpha^{(\delta,\eta)}$ the α-percentage point of the distribution of the limiting proportion Z_∞ when the mean responses are equal. Then, the critical region

$$\{Z_n > c_\alpha^{(\delta,\eta)}\} \tag{4.10}$$

defines a test asymptotically of level α. Under the alternative hypothesis, the asymptotic behavior of the proportion Z_n is shown in Theorem 3.1. The power of the test $\{Z_n > c_\alpha^{(\delta,\eta)}\}$ can be approximated, for large n, as

$$P\left(\psi \ < \ (\eta - c_\alpha^{(\delta,\eta)}) m_W n \right) \tag{4.11}$$

where ψ is the random quantity defined in Theorem 3.1.

5 Simulation study

This section is dedicated to presenting the simulation studies aim at exploring the asymptotic behavior of the urn proportion Z_n. In this section, all the urns are simulated with the following choice of parameters: $\delta = 0.2$ and $\eta = 0.8$. Further studies based on changing the values of δ or η can be of great interest, but this wasn't the real purpose of the paper.

Initially, we focus on supporting the convergence result shown in Theorem 3.1. The reinforcement distributions μ_R and μ_W are chosen to be gaussians, with means set to $m_R = 10$ and $m_W = 5$ respectively. The variances are assumed to be equal and fixed at $\sigma_R^2 = \sigma_W^2 = 1$. Theorem 3.1 shows that, when $m_R > m_W$, the quantity $n(\eta - Z_n)m_W$ converges in distribution to a random variable ψ, whose probability law is π. Through some simulations, we compute the empirical distribution of $n(\eta - Z_n)m_W$ for $n = 10^2$ and $n = 10^4$. The corresponding histograms are presented in Figure 1.

In proposition 3.1 it was proved that the probability measure π is the unique invariant distribution of the process $(\widetilde{T}_n)_{n\in\mathbb{N}}$. This means π is the unique solution of the functional equation

$$\int_{\mathbb{R}} K_\eta(x, dy)\pi(dx) = \pi(dy) \tag{5.1}$$

where K_η is the transition kernel of the process $\widetilde{T}_n$ defined in (3.5). Taking the discrete version of (5.1) we compute the density of the measure π, which is superimposed on both the histograms in Figure 1. The quite perfect agreement betweeen the empirical distribution of $n(\eta - Z_n)m_W$ and the discrete estimation of π gave to the authors the impetus to prove the convergence result described in Theorem 3.1.

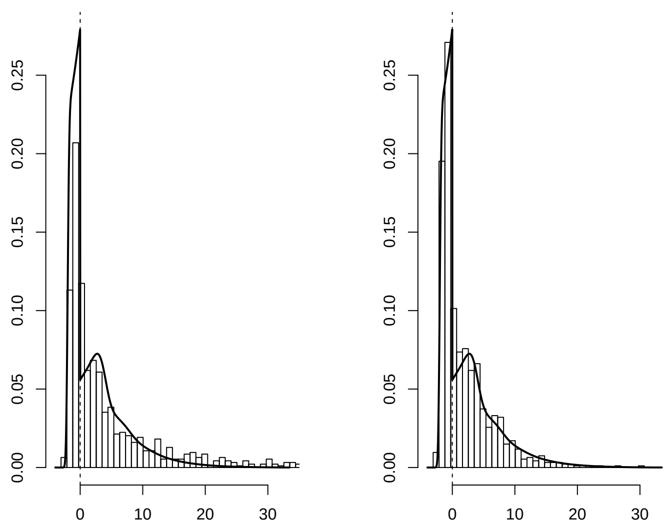

Figure 1: Histograms of ψ obtained simulating the empirical distribution of $n(\eta - Z_n)m_W$ for large n, with superimposed the density of ψ obtained by numerically solving the discrete version of (5.1). Left panel: $n = 10^2$. Right panel: $n = 10^4$.

The simulation study also encouraged the authors to prove some further theoretical results. The first one we present is related to an easy expression for a quantile of the probability law of the limiting variable ψ. In general, the asymptotic distribution of the quantity $n(\eta - Z_n)$ depends on the value η and on the reinforcements distributions μ_R and μ_W. Nevertheless, the following proposition state that 0 is always the $\frac{m_W}{m_R}$-percentage point of the distribution π, regardless η or the types of distributions involved.

Proposition 5.1.

$$P(\psi > 0) \;=\; \frac{m_W}{m_R} \tag{5.2}$$

Proof. Since $P(Z_n < \eta) = P(T_n > 0)$ we know that $P(Z_n < \eta)$ is a convergent sequence. In particular

$$\lim_{n\to\infty} P(Z_n < \eta) \;=\; P(\psi > 0) \;=\; \pi([0,\infty))$$

Therefore, by using the dominated convergence theorem, the Toeplitz Lemma and Proposition 2.3, we obtain

$$P(\psi > 0) \;=\; \lim_{n\to\infty} P(Z_n < \eta) \;=\; \lim_{n\to\infty} \frac{\sum_{i=1}^n P(Z_i < \eta)}{n} \;=$$
$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}[1_{\{Z_i<\eta\}}]}{n} \;=\; \mathbb{E}\left[\lim_{n\to\infty} \frac{\sum_{i=1}^n 1_{\{Z_i<\eta\}}}{n}\right] \;=\; \mathbb{E}\left[\frac{m_W}{m_R}\right] \;=\; \frac{m_W}{m_R}$$

■

Another interesting result, that came out from the simulation analysis, concerns the correspondence between the asymptotic distribution of Z_n and a linear transformation of the reinforcement laws. This property is explained in the following proposition

Proposition 5.2. *Let Z_n and $\widehat{Z}_n$ be the urn proportions of two MRRU models with reinforcements distributions (μ_R, μ_W) and $(\widehat{\mu}_R, \widehat{\mu}_W)$ respectively. Assume that there exists $c > 0$ such that, for any $a, b \in \mathbb{R}$ with $a < b$*

$$\begin{cases} \widehat{\mu}_R(\ (a,b)\ ) & = \mu_R(\ (ca,cb)\ ) \\ \widehat{\mu}_W(\ (a,b)\ ) & = \mu_W(\ (ca,cb)\ ) \end{cases} \tag{5.3}$$

i.e. $\widehat{M}_n \overset{\mathcal{L}}{=} c\cdot M_n$ *and* $\widehat{N}_n \overset{\mathcal{L}}{=} c\cdot N_n$ *for any* $n \in \mathbb{N}$.
Then, for any $a, b \in \mathbb{R}$ with $a < b$, we have

$$\widehat{\pi}(\ (a,b)\ ) \;=\; \pi(\ (c\cdot a, c\cdot b)\ ) \tag{5.4}$$

i.e. $\widehat{\psi} \overset{\mathcal{L}}{=} c\cdot\psi$.

Proof. Let us call the initial compositions of the two urn processes as (r_0, w_0) and $(\widehat{r}_0, \widehat{w}_0)$. The proof will be based on the particular choice $\widehat{r}_0 = c\cdot r_0$ and $\widehat{w}_0 = c\cdot w_0$. However, since from Proposition 3.1 the invariant distribution π is independent of the initial composition, the generality of the result still holds.
For any $n \geq 1$, by conditioning to the event $\{(\widehat{T}_n, \widehat{Z}_n) = (c\cdot T_n, Z_n)\}$, we have that

$$\begin{aligned} \widehat{T}_{n+1} \;&=\; \widehat{T}_n \;+\; \eta(1-\widehat{X}_{n+1})\widehat{N}_{n+1} \;-\; (1-\eta)\widehat{X}_{n+1}1_{\{\widehat{T}_n>0\}}\widehat{M}_{n+1} \;= \\ &=\; c\cdot T_n \;+\; \eta(1-\widehat{X}_{n+1})\widehat{N}_{n+1} \;-\; (1-\eta)\widehat{X}_{n+1}1_{\{T_n>0\}}\widehat{M}_{n+1} \;\overset{\mathcal{L}}{=} \\ &\overset{\mathcal{L}}{=}\; c\cdot T_n \;+\; \eta(1-X_n)c\cdot N_{n+1} \;-\; (1-\eta)X_{n+1}1_{\{T_n>0\}}c\cdot M_{n+1} \;=\; c\cdot T_{n+1} \end{aligned} \tag{5.5}$$

$$
\begin{aligned}
\widehat{Z}_{n+1} &= \frac{\widehat{R}_{n+1}}{\widehat{R}_{n+1}+\widehat{W}_{n+1}} = \\
&= \frac{\widehat{R}_n+\widehat{X}_{n+1}\widehat{M}_{n+1}}{\widehat{R}_n+\widehat{W}_n+\widehat{X}_{n+1}\widehat{M}_{n+1}+(1-\widehat{X}_{n+1})\widehat{N}_{n+1}} = \\
&= \frac{c\cdot R_n+\widehat{X}_{n+1}\widehat{M}_{n+1}}{c\cdot R_n+c\cdot W_n+\widehat{X}_{n+1}\widehat{M}_{n+1}+(1-\widehat{X}_{n+1})\widehat{N}_{n+1}} \overset{\mathcal{L}}{=} \\
&\overset{\mathcal{L}}{=} \frac{R_n+X_{n+1}c\cdot M_{n+1}}{R_n+W_n+X_{n+1}c\cdot M_{n+1}+(1-X_{n+1})c\cdot N_{n+1}} = Z_{n+1}
\end{aligned} \tag{5.6}
$$

For ease of notation, let us denote $\lambda_{(T_n,Z_n)}$ and $\lambda_{(\widehat{T}_n,\widehat{Z}_n)}$ as the bivariate laws of the couple of random variables (T_n, Z_n) and $(\widehat{T}_n, \widehat{Z}_n)$ respectively. Then, let us notice that the equivalence of the initial compositions of the two processes Z_n and $\widehat{Z}_n$ implies that the event $\{(\widehat{T}_0, \widehat{Z}_0) = (c\cdot T_0, Z_0)\}$ has probability one. Hence, for any $n \geq 1$, we have

$$
\begin{aligned}
\lambda_{(\widehat{T}_n,\widehat{Z}_n)} &= \int_{\mathbb{R}^{n-1}\times(0,1)^{n-1}} \lambda_{(\widehat{T}_n,\widehat{Z}_n)|(\widehat{T}_{n-1},\widehat{Z}_{n-1})}\cdot\lambda_{(\widehat{T}_{n-1},\widehat{Z}_{n-1})|(\widehat{T}_{n-2},\widehat{Z}_{n-2})}\cdots\lambda_{(\widehat{T}_1,\widehat{Z}_1)|(\widehat{T}_0,\widehat{Z}_0)} = \\
&= \int_{\mathbb{R}^{n-1}\times(0,1)^{n-1}} \lambda_{(cT_n,Z_n)|(cT_{n-1},Z_{n-1})}\cdot\lambda_{(cT_{n-1},Z_{n-1})|(cT_{n-2},Z_{n-2})}\cdots\lambda_{(cT_1,Z_1)|(cT_0,Z_0)} = \\
&= \lambda_{(cT_n,Z_n))}
\end{aligned}
$$

The thesis is proved since the equivalence $\lambda_{(\widehat{T}_n,\widehat{Z}_n)} = \lambda_{(c\cdot T_n,Z_n)}$ implies that $\widehat{\pi} = \pi$. ■

The assumption (5.3) implies also that $\widehat{m}_R = c\cdot m_R$ and $\widehat{m}_W = c\cdot m_W$. Then, from Theorem 3.1 we deduce the equivalence between the asymptotic laws of Z_n and $\widehat{Z}_n$. Propositions 5.1 and 5.2 suggest that urn processes with the same reinforcement means ratio present also similar asymptotic behavior. For this reason, we prefer to use the ratio $\frac{m_R}{m_W}$ as parameter measuring the means' distance, instead of the usual mean difference $m_R - m_W$.

Here we present some simulations concerning the hypothesis test (4.1). In particular, we focus on comparing the power of the tests defined in (4.8) and (4.10). The empirical power is computed using $n = 10^4$ subject, in correspondence of different values of the ratio $\frac{m_R}{m_W}$. The empirical power functions are reported in Figure 2. As shown in Figure 2, the MRRU design constructs a test more powerful then the one based on the RRU design with the sample size, for any choice of the reinforcement means. Although this property makes the MRRU design very attractive, the RRU model has the advantage that, with the same sample size, it allocates less subject to the inferior treatment. Hence, what is really interesting is studing the power functions of the tests (4.8) and (4.10), in correspondence of a different values of N_W, i.e. the number of subjects assigned to the inferior treatment. We compute the empirical power functions for $N_W = 20, 50, 100, 500$ and we report the graphics in Figure 3. From the analysis of the power functions in Figure 3, different considerations can be done depending on the size of the ratio $\frac{m_R}{m_W}$. For high values of $\frac{m_R}{m_W}$ the power of the tests (4.8) and (4.10) are very similar. When the ratio $\frac{m_R}{m_W}$ is small the power of the test based on MRRU design seems to be considerable greater, for any value of N_W.

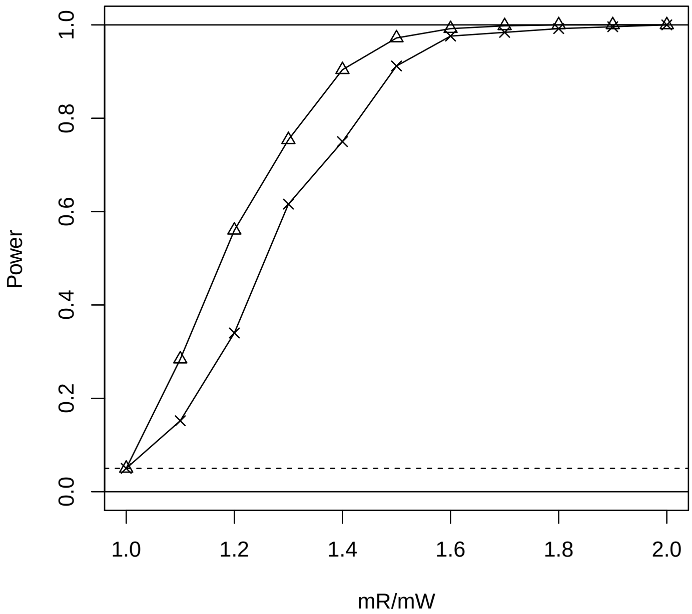

Figure 2: The empirical power functions of test (4.8) (line with crosses) and of test (4.10) (line with triangles).

Appendix

In the following we assume, without loss of generality, that condition (a) of the Assumption 2.1 is satisfied; the symmetric case (b) is straightforward.

Lemma 5.1. *For any $t_0 \in \mathbb{R}$, there exists $\bar{t} > t_0$ such that*

$$\forall\ t > \bar{t}, \quad \forall\ \epsilon > 0, \qquad P\left(\bigcup_{k=1}^{\infty} \left\{ \widetilde{T}_k \in [t, t+\epsilon] \right\} \ \mid \widetilde{T}_0 = t_0 \right) \ > \ 0 \quad (5.7)$$

Proof. Let us take $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha_0 < \alpha < \beta < \beta_0$. At first, notice that if $t \in (t_0 + \alpha\eta, t_0 + \beta\eta)$, then

$$P\left(\ \widetilde{T}_1 \in (t, t+dt) \mid \widetilde{T}_0 = t_0 \ \right) \ = \ (1-\eta)\ \mu_W \left(d \left(\frac{t-t_0}{\eta} \right) \right) \ > \ 0$$

since $\frac{t-t_0}{\eta} \in (\alpha, \beta)$.

For the same reason, for any $k \in \mathbb{N}$, we have that if $t \in (t_0 + k\alpha\eta, t_0 + k\beta\eta)$, then

$$P\left(\ \widetilde{T}_k \in (t, t+dt) \mid \widetilde{T}_0 = t_0 \ \right) \ \geq \ (1-\eta)^k\ \mu_W \left(d \left(\frac{t-t_0}{k\eta} \right) \right)^k \ > \ 0$$

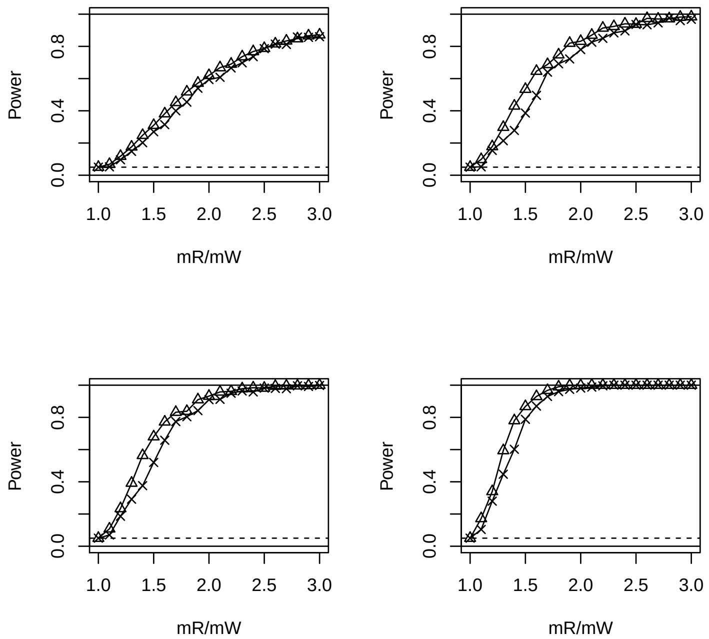

Figure 3: The empirical power functions of test (4.8) (line with crosses) and of test (4.10) (line with triangles). Top left panel: $N_W = 20$. Top right panel: $N_W = 50$. Bottom left panel: $N_W = 100$. Bottom right panel: $N_W = 20$.

Let us introduce the sequence of sets $(A_k)_k$ such that

$$A_k \;=\; \begin{cases} (\ t_0 + (k-1)\beta\eta\ ,\ t_0 + k\alpha\eta\ ) & \text{if } k < \frac{\beta}{\beta-\alpha}, \\ \emptyset & \text{otherwise .} \end{cases}$$

for $k \geq 1$. Then, for any $n \in \mathbb{N}$, we have that if

$$t \;\in\; (\ t_0\ ,\ t_0\ +\ n\beta\eta\ )\ /\ \bigcup_{k=1}^{n} A_k,$$

then

$$t \;\in\; \bigcup_{k=1}^{n} (\ t_0 + k\alpha\eta\ ,\ t_0 + k\beta\eta\ ),$$

and

$$P\left(\bigcup_{k=1}^{n} \{\widetilde{T}_k \in (t, t+dt)\} \mid \widetilde{T}_0 = t_0 \right) \;\geq\; (1-\eta)^{n_0}\ \mu_W \left(d\left(\frac{t-t_0}{n_0\eta}\right)\right)^{n_0} \;>\; 0,$$

where we choose

$$n_0 = \left[\frac{t-t_0}{\beta\eta}\right] + 1$$

Therefore, a sufficient condition for $P\left(\bigcup_{k=1}^{\infty}\left\{\widetilde{T}_k \in [t, t+\epsilon]\right\} | \widetilde{T}_0 = t_0\right) > 0$ is

$$t \ \in \ (t_0, \infty) \ / \bigcup_{k=1}^{[\frac{\beta}{\beta+\alpha}]} (\ t_0 + (k-1)\beta\eta \ , \ t_0 + k\alpha\eta \ ),$$

so the thesis holds for any $\bar{t} \ \geq \ t_0 + \left[\frac{\beta}{\beta-\alpha}\right] \alpha\eta$. ■

Proposition 5.3. *The Markov process* $\widetilde{T} = (\widetilde{T}_n)_{n\in\mathbb{N}}$ *on the state space* $\mathbb{R}$ *is a Harris Chain.*

Proof. Let us start reminding that the Markov process $\widetilde{T}_n$ on the state space $\mathbb{R}$ is a *Harris chain* if there exist $A, B \subset \mathbb{R}$, a constant $\epsilon > 0$ and a probability measure ρ with $\rho(B) = 1$, such that

(a) If $\tau^A := inf\{n \geq 0 \ : \ \widetilde{T}_n \in A\}$, then $P(\tau^A < \infty \mid \widetilde{T}_0 = t_0) > 0$ for any $t_0 \in \mathbb{R}$.

(b) If $t_0 \in A$ and $C \subset B$, then $K_\eta(t_0, C) \geq \epsilon\rho(C)$.

Let us prove the condition (a). Let $A = [0, (\beta-\alpha)\eta]$.

- First case: $t_0 \in [0, (\beta-\alpha)\eta]$

The condition (a) is trivial, since $P(\tau_A = 0 \mid \widetilde{T}_0 = t_0 \in A) = 1$.

- Second case: $t_0 > (\beta-\alpha)\eta$

We fix $\bar{t} \ \geq \ t_0 + \left[\frac{\beta}{\beta-\alpha}\right] \alpha\eta$ and we define $\bar{n} \in \mathbb{N}$, $I \subset \mathbb{R}$ as follows

$$\bar{n} \ = \ \left[\frac{\bar{t}}{(1-\eta)x_0}\right] + 1,$$

$$I \ = \ [\ \bar{n}(1-\eta)x_0 \ , \ \bar{n}(1-\eta)x_0 + (\beta-\alpha)\eta \ ] \,,$$

where $x_0 \in [\alpha_R, \beta_R]$ is chosen such that, for every $\epsilon > 0$, $\mu_R([x_0, x_0+\epsilon]) > 0$. Fixing $\widetilde{t} \in I$, we have from the previous lemma that for every $\zeta > 0$

$$P\left(\bigcup_{k=1}^{\infty}\left\{\widetilde{T}_k \in [\widetilde{t}, \widetilde{t}+\zeta]\right\} \ | \ \widetilde{T}_0 = t_0\right) \ > \ 0,$$

since $\widetilde{t} \geq \bar{n}(1-\eta)x_0 \geq \bar{t}$. Then, let fix ζ small enough, such that $\widetilde{t}+\zeta \in I$. Let

$$\widetilde{n} := inf\left\{ n \geq 1 \ : \ P\left(\bigcup_{k=1}^{n}\left\{\widetilde{T}_k \in [\widetilde{t}, \widetilde{t}+\zeta]\right\} | \ \widetilde{T}_0 = t_0 > (\beta-\alpha)\eta\right) > 0 \right\}$$

We can write

$$P(\ \tau^A < \infty \mid \widetilde{T}_0 = t_0\ ) \ \geq \ P(\ \widetilde{T}_{\widetilde{n}+\bar{n}} \in (0, (\beta-\alpha)\eta) \mid \widetilde{T}_0 = t_0\ ) \ \geq$$

$$P(\ \widetilde{T}_{\widetilde{n}+\bar{n}} \in (0, (\beta-\alpha)\eta) \mid \widetilde{T}_{\widetilde{n}} \in [\widetilde{t}, \widetilde{t}+\zeta]\ ) \ \cdot \ P(\ \widetilde{T}_{\widetilde{n}} \in [\widetilde{t}, \widetilde{t}+\zeta] \mid \widetilde{T}_0 = t_0\ )$$

We have already proved that the second term of this product is strictly positive, so we focus on the first term. Let us call

$$\widetilde{t}_{min} := \arg \min_{t \in [\widetilde{t}, \widetilde{t}+\zeta]} P(\ \widetilde{T}_{\widetilde{n}+\bar{n}} \in (0, (\beta-\alpha)\eta) \mid \widetilde{T}_{\widetilde{n}} = t\ )$$

we have

$$P(\ \widetilde{T}_{\widetilde{n}+\bar{n}} \in (0,(\beta-\alpha)\eta) \mid \widetilde{T}_{\widetilde{n}} \in [\widetilde{t}, \widetilde{t}+\zeta]\ ) \ \geq\ P(\ \widetilde{T}_{\widetilde{n}+\bar{n}} \in (0,(\beta-\alpha)\eta) \mid \widetilde{T}_{\widetilde{n}} = \widetilde{t}_{min}\ ) \quad \geq$$

$$\prod_{s=1}^{\bar{n}} K_\eta\left(\ \widetilde{t}_{min} - (s-1)(1-\eta)x_0 \quad , \quad [\widetilde{t}_{min} - s(1-\eta)x_0; \widetilde{t}_{min} - s(1-\eta)x_0 + dt]\ \right) \quad =$$

$$(\ \eta \cdot \mu_R(dx_0)\ )^{\bar{n}} \ >\ 0$$

because $\widetilde{t}_{min} - \bar{n}(1-\eta)x_0 \in (0, (\beta-\alpha)\eta)$.

- Third case: $t_0 < 0$

We fix $\bar{t} \ \geq\ \max\left\{\ t_0 + \left[\frac{\beta}{\beta-\alpha}\right]\alpha\eta\ ;\ 0\ \right\}$ and then we follow the same strategy used in the second case ($t_0 > (\beta-\alpha)\eta$).

Let us prove the condition (b) Let

$$B \ =\ [\ (\beta - \alpha + \alpha_0)\eta\ ,\ \beta_0\eta\ ] \subset \mathbb{R}$$

and the probability measure

$$\rho(C) = \frac{1}{(\beta_0 - \beta + \alpha - \alpha_0)\eta} \int_C dt$$

for any set $C \subset B$. For every $t_0 \in A$,

$$K_\eta(t_0, C)\ \geq\ \int_C (1-\eta)\, \mu_W\left(d\left(\frac{t-t_0}{\eta}\right)\right)\ \geq\ (1-\eta)\int_C \min_{(t_0,t)\in A\times B}\left[\frac{\mu_W\left(d\left(\frac{t-t_0}{\eta}\right)\right)}{dt}\right]\ dt$$

$$=\ (1-\eta)\int_C \min_{x\in(\alpha_0,\beta_0)}\left[\frac{\mu_W\,(dx)}{dx}\right]\ dt$$

Now if we define

$$\epsilon = (\beta_0 - \beta + \alpha - \alpha_0)\eta(1-\eta) \min_{x\in(\alpha_0,\beta_0)}\left[\frac{\mu_W\,(dx)}{dx}\right]$$

we obtain

$$K_\eta(t_0, C)\ \geq\ \epsilon\ \cdot\ \frac{1}{(\beta_0 - \beta + \alpha - \alpha_0)\eta}\int_C dt\ =\ \epsilon\ \cdot\ \rho(C)$$

∎

In what follows, for any interval $I \subset \mathbb{R}$, we will refer to $(\tau_i^I)_i$ as the sequence of stopping times

$$\begin{cases} \tau_0^I = 0 \\ \tau_i^I := inf\left\{\ n > \tau_{i-1}^I\ :\ \widetilde{T}_n \in I\ \right\}, \quad i \geq 1 \end{cases}$$

For ease of notation, we will denote τ^I as τ_1^I.

Proposition 5.4. *The Harris chain* $\widetilde{T} = (\widetilde{T}_{n\in\mathbb{N}}$ *on the state space* $\mathbb{R}$ *is recurrent* .

Proof. Let us remind that $\widetilde{T}_n$ is recurrent if $P(\tau^A < \infty \mid \widetilde{T}_0 \in A) = 1$, for any initial probability distribution $\widetilde{\lambda}_0$, where $\tau^A := inf\{n \geq 1 \ : \ \widetilde{T}_n \in A\}$. In particular, we are able to prove a stronger property, that is $P(\tau^A < \infty \mid \widetilde{T}_0 = t_0) = 1$ for any $t_0 \in \mathbb{R}$, which implies the condition we need.

Let

- I be the closed interval defined as

$$I := [-(1-\eta)\beta_R, 0],$$

- c be the constant defined as

$$c := \min_{t\in I} P\left(\tau^A < \infty \mid \widetilde{T}_0 = t\right)$$

  c is strictly positive because, the process $\widetilde{T}_n$ is an Harris chain and so $P(\tau^A < \infty \mid \widetilde{T}_0 = t_0) > 0 \ \forall t_0 \in \mathbb{R}$,

- $\widetilde{n}$ be the integer defined as

$$\widetilde{n} := \ inf \left\{ \ n \geq 1 \ : \ \min_{x\in I} P\left(\bigcup_{k=1}^{\widetilde{n}} \{\widetilde{T}_k \in A\} \mid \widetilde{T}_0 = x\right) \geq \frac{c}{2} \ \right\}$$

Now, we focus on proving that the stopping times $(\tau_i^I)_i$ are almost surely finite:

$$P\left(\tau^I = \infty \mid \widetilde{T}_0 = t_0\right) = 0 \tag{5.8}$$

(a) First case: $t_0 \in (0, \infty)$

Looking at the transition kernels (3.3) and (3.5) of the processes T_n and $\widetilde{T}_n$ respectively, we note that for any $t_0 \in (0, \infty)$, $P(\widetilde{T}_1 \leq T_1 \mid \widetilde{T}_0 = T_0 = t_0) \ = \ 1$. This implies that

$$P(\widetilde{T}_1 > 0 \mid \widetilde{T}_0 = t_0) \ \leq \ P(T_1 > 0 \mid T_0 = t_0) \tag{5.9}$$

Then, we have that

$$\begin{aligned} P\left(\tau^I = \infty \mid \widetilde{T}_0 = t_0\right) \quad & = P\left(\tau^{(-\infty,0)} = \infty \mid \widetilde{T}_0 = t_0\right) \quad = \\ P\left(\bigcap_{n=1}^{\infty} \left\{\widetilde{T}_n > 0\right\} \mid \widetilde{T}_0 = t_0\right) \quad & \leq P\left(\bigcap_{n=1}^{\infty} \{T_n > 0\} \mid T_0 = t_0\right) \quad = 0 \end{aligned}$$

where the passage from $\widetilde{T}_n$ to T_n is due to the relation (5.9) and the latest probability is equal to zero because $P(T_n < 0 \ \ i.o. \mid T_0 = t_0) = P(Z_n > \eta \ \ i.o. \mid T_0 = t_0) = 1$ for any $t_0 \in \mathbb{R}$.

(b) Second case: $t_0 \in (-\infty, 0]$

Looking at the transition kernels (3.3) and (3.5) we have that for any $t_0 \in (-\infty, 0]$,

$$P(\widetilde{T}_1 < 0 \mid \widetilde{T}_0 = t_0) \;\leq\; P(T_1 < 0 \mid T_0 = t_0) \tag{5.10}$$

and following the same arguments of the case (a) this leads to

$$P\left(\tau^{(0,\infty)} = \infty \mid \widetilde{T}_0 = t_0\right) \;=\; 0 \tag{5.11}$$

Hence, we have

$$\begin{aligned}
&P\left(\tau^{I} = \infty \mid \widetilde{T}_0 = t_0\right) && = \\
&P\left(\tau^{I} = \infty \mid \{\tau^{(0,\infty)} < \infty\}\bigcap\{\widetilde{T}_0 = t_0\}\right) && = \\
&P\left(\bigcap_{n=1}^{\infty}\{\widetilde{T}_n \notin I\} \mid \{\tau^{(0,\infty)} < \infty\}\bigcap\{\widetilde{T}_0 = t_0\}\right) && \leq \\
&P\left(\bigcap_{n=\tau^{(0,\infty)}+1}^{\infty}\{\widetilde{T}_n \notin I\} \mid \{\tau^{(0,\infty)} < \infty\}\bigcap\{\widetilde{T}_0 = t_0\}\right) && \leq \\
&\sup_{x\in(0,\infty)} P\left(\bigcap_{n=1}^{\infty}\{\widetilde{T}_n \notin I\} \mid \widetilde{T}_0 = x\right) && = \\
&\sup_{x\in(0,\infty)} P\left(\tau^{I} = \infty \mid \widetilde{T}_0 = x\right) && = 0
\end{aligned}$$

since from the case (a) we have that $\forall t_0 > 0,\ P(\tau^I = \infty \mid \widetilde{T}_0 = t_0) = 0$. Therefore, we conclude that $P(\bigcap_{i=1}^{\infty} \tau_i^I < \infty \mid \widetilde{T}_0 = t_0) = 1$, which means $(\tau_i^I)_i$ is sequence of stopping times almost surely finite.

Then, let us define the sequence of stopping times

$$\begin{cases} \tau_0 = 0 \\ \tau_i := inf\left\{\; n > \tau_{i-1} + \widetilde{n} \;:\; \widetilde{T}_n \in I \;\right\}, \quad i \geq 1 \end{cases}$$

Since $\bigcup_{n=1}^{\infty} \tau_n \subset \bigcup_{n=1}^{\infty} \tau_n^I$, the stopping times $(\tau_n, n = 0, 1, 2, ..)$ are almost surely finite.

Therefore, for any $t_0 \in \mathbb{R}$ we have that

$$
\begin{aligned}
& P\left(\tau^A = \infty \mid \widetilde{T}_0 = t_0\right) \;=\; P\left(\bigcap_{n=1}^{\infty} \{T_n \notin A\} \mid \widetilde{T}_0 = t_0\right) && \leq \\
& P\left(\bigcap_{i=0}^{\infty} \bigcap_{n=\tau_i+1}^{\tau_i+\widetilde{n}} \{\widetilde{T}_n \notin A\} \mid \widetilde{T}_0 = t_0\right) && = \\
& \prod_{i=1}^{\infty} P\left(\bigcap_{n=\tau_i+1}^{\tau_i+\widetilde{n}} \{\widetilde{T}_n \notin A\} \mid \bigcap_{j=0}^{i-1} \bigcap_{n=\tau_j+1}^{\tau_j+\widetilde{n}} \{\widetilde{T}_n \notin A\}\right) && = \\
& \prod_{i=1}^{\infty} \left[1 - P\left(\bigcup_{n=\tau_i+1}^{\tau_i+\widetilde{n}} \{\widetilde{T}_n \in A\} \mid \bigcap_{j=0}^{i-1} \bigcap_{n=\tau_j+1}^{\tau_j+\widetilde{n}} \{\widetilde{T}_n \notin A\}\right)\right] && = \\
& \prod_{i=1}^{\infty} \left[1 - \int_I P\left(\bigcup_{n=\tau_i+1}^{\tau_i+\widetilde{n}} \{\widetilde{T}_n \in A\} \mid \widetilde{T}_{\tau_i} = x\right) P\left(\widetilde{T}_{\tau_i} = dx \mid \bigcap_{j=0}^{i-1} \bigcap_{n=\tau_j+1}^{\tau_j+\widetilde{n}} \{\widetilde{T}_n \notin A\}\right)\right] && = \\
& \prod_{i=1}^{\infty} \left[1 - \int_I P\left(\bigcup_{n=1}^{\widetilde{n}} \{\widetilde{T}_n \in A\} \mid \widetilde{T}_0 = x\right) P\left(\widetilde{T}_{\tau_i} = dx \mid \bigcap_{j=0}^{i-1} \bigcap_{n=\tau_j+1}^{\tau_j+\widetilde{n}} \{\widetilde{T}_n \notin A\}\right)\right] && \leq \\
& \prod_{i=1}^{\infty} \left[1 - \min_{x \in I} P\left(\bigcup_{n=1}^{\bar{n}} \{\widetilde{T}_n \in A\} \mid \widetilde{T}_0 = x\right)\right] && \leq \\
& \prod_{i=1}^{\infty} \left[1 - \frac{c}{2}\right] && = 0
\end{aligned}
$$

and so the thesis is proved. ■

Proposition 5.5. *The recurrent Harris Chain $\widetilde{T} = (\widetilde{T}_n)_{n \in \mathbb{N}}$ on the state space $\mathbb{R}$ is aperiodic.*

Proof. The recurrent Harris chain $\widetilde{T}_n$ is aperiodic if there exists $n_0 \in \mathbb{N}$ such that $P(\widetilde{T}_n \in A \mid \widetilde{T}_0 \in A) > 0$, for any integer $n \geq n_0$ and for any distribution law $\widetilde{\lambda}_0$ on $\widetilde{T}_0$.
Let define the stopping time $\tau_1^{A^-}$ as follows

$$
\tau^{A^-} := \inf\left\{n > \tau^{(-\infty,0)} \;:\; \widetilde{T}_n \in A\right\} \tag{5.12}
$$

This stopping time is almost surely finite. In fact, since $P(\tau^{(-\infty,0)} < \infty | \widetilde{T}_0 = t_0) = 1$ for any $t_0 \in \mathbb{R}$, we have that

$$
\begin{aligned}
& P\left(\tau^{A^-} < \infty \mid \widetilde{T}_0 \in A\right) \;=\; P\left(\tau^{A^-} < \infty \mid \{\tau^{(-\infty,0)} < \infty\} \bigcap \{\widetilde{T}_0 \in A\}\right) \;= \\
& P\left(\bigcup_{n=\tau^{(-\infty,0)}}^{\infty} \{\widetilde{T}_n \in A\} \mid \{\tau^{(-\infty,0)} < \infty\} \bigcap \{\widetilde{T}_0 \in A\}\right) \;\geq \\
& \min_{x \in (-\infty,0)} P\left(\bigcup_{n=0}^{\infty} \{\widetilde{T}_n \in A\} \mid \widetilde{T}_0 = x\right) \;=\; \min_{x \in (-\infty,0)} P\left(\tau^A < \infty \mid \widetilde{T}_0 = x\right) \;=\; 1
\end{aligned}
$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $P(\tau^{A^-} = n_0 \mid \widetilde{T}_0 \in A) > 0$. We notice also that

$$P\left(\widetilde{T}_{n_0} \in A \mid \widetilde{T}_0 \in A\right) \;\geq\; P\left(\{\widetilde{T}_{n_0} \in A\} \bigcap \{\tau^{A^-} = n_0\} \mid \widetilde{T}_0 \in A\right) \;=$$
$$P\left(\widetilde{T}_{n_0} \in A \mid \{\tau^{A^-} = n_0\} \bigcap \{\widetilde{T}_0 \in A\}\right) \;\cdot\; P\left(\tau^{A^-} = n_0 \mid \widetilde{T}_0 \in A\right) \;=$$
$$P\left(\widetilde{T}_{\tau^{A^-}} \in A \mid \widetilde{T}_0 \in A\right) \;\cdot\; P\left(\tau^{A^-} = n_0 \mid \widetilde{T}_0 \in A\right) \;=\; P\left(\tau^{A^-} = n_0 \mid \widetilde{T}_0 \in A\right) \;>\; 0$$

Then, for every $n \geq n_0$, we have

$$P\left(\widetilde{T}_n \in A \mid \widetilde{T}_0 \in A\right) \;\geq\; P\left(\tau^{A^-} = n \mid \widetilde{T}_0 \in A\right) \;\geq\; \eta^{n-n_0} \cdot P\left(\tau^{A^-} = n_0 \mid \widetilde{T}_0 \in A\right) \;>\; 0$$

and so the thesis is proved. ■

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Dipartimento di Matematica "F. Brioschi",
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