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Antonietti, P.F.; Botti, M.; Parolini, N.; Pederzoli, V.; Verani, M.

MOX, Dipartimento di Matematica
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox-dmat@polimi.it https://mox.polimi.it

Polytopal Discontinuous Galerkin Discretizations of Coupled Non-Newtonian Stokes–Darcy Systems*

Paola F. Antonietti[a,1] Michele Botti[a,2] Nicola Parolini[a,3] Valentina Pederzoli[a,4] Marco Verani[a,5]

Abstract

We propose and analyze a polytopal discontinuous Galerkin method for the numerical approximation of a coupled non-Newtonian Stokes–Darcy system modeling the interaction between a non-Newtonian free-flow fluid and a non-Newtonian flow through a porous medium. Due to its geometric flexibility and arbitrary-order accuracy, the proposed discretization scheme is well-suited to configurations with complex geometries. We provide a complete *a-priori* analysis that considers shear-dependent and velocity-dependent non-Newtonian viscosity models for the free-flow and porous media regions, respectively. The well-posedness, stability, and error bounds of the method are established in the framework of generalized inf-sup theory. Error estimates are confirmed by numerical results.

1 Introduction

Coupled non-Newtonian Stokes–Darcy systems model fluid flow in interconnected free-flow and porous-medium regions. More precisely, the motion of a viscous fluid in an open region unobstructed by any porous solid matrix is modeled by the nonlinear Stokes equations in the incompressible creeping laminar regime whereas the flow within the porous medium is governed by a nonlinear Darcy-type constitutive relation that links the seepage velocity to the pressure gradient via the permeability tensor. At the interface, the two flow regimes are coupled through physically consistent transmission conditions, consisting of the Beavers–Joseph–Saffman condition [11, 47, 38], along with continuity of the normal flux and equilibrium of normal stresses across the interface. Coupled non-Newtonian Stokes–Darcy models appear in a wide range of applied science and industrial applications. Among the most relevant natural settings are subsurface flows in karst aquifers, where water circulating through underground conduits and fractures continuously interacts with the surrounding porous matrix, a system whose accurate modeling is essential for sustainable groundwater resource management and for assessing vulnerability to contamination [19, 28, 20]. In addition, biological flows, such as plasma filtration through capillary walls and fluid transport in biological tissues, provide further examples of such coupled flow phenomena. A wide range of industrial applications also falls within this modeling framework. For example, cross-flow and dead-end filtration processes are widely employed in the pharmaceutical, chemical, food processing, and aeronautical industries [36, 34, 35, 9, 43]. In these applications, the fluid typically exhibits non-Newtonian behavior, meaning that its constitutive response is governed by nonlinear laws. Unlike Newtonian fluids, the viscosity of these fluids depends

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[a] MOX–Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, Milan, 20133, Italy
[1] paola.antonietti@polimi.it
[2] michele.botti@polimi.it
[3] nicola.parolini@polimi.it
[4] valentina.pederzoli@polimi.it
[5] marco.verani@polimi.it

on the strain rate, resulting in a nonlinear relationship between shear stress and the rate of deformation. They are typically classified into two categories: shear-thinning fluids, such as blood and melted polymers, whose viscosity decreases with increasing shear rate, and shear-thickening fluids, such as cornstarch-water mixtures, whose viscosity increases with increasing shear rate. We refer, for example, to [21, 29] for a comprehensive overview of non-Newtonian fluids.

Given the applicative importance of coupled non-Newtonian Stokes and Darcy systems, many numerical methods have been proposed and analyzed in the literature for their approximate solution. In [13], the authors consider a two-phase Darcy flow in a fractured porous medium solved with a gradient discretization method. In [26], the coupling between the Navier–Stokes and a Darcy equation is discussed along with suitable interface conditions, and then it is solved using a Finite Element (FEM) approximation, while in [31], Ervin, Jenkins, and Sun developed and analyzed a Finite Element Method for a coupled nonlinear Stokes–Darcy system with generalized constitutive laws in both the free-flow and porous-medium regions, establishing rigorous error estimates for the proposed approximation. In [30], a mortar Finite Element formulation for the coupled problem is presented; the key idea is to reformulate the coupling as an interface matching problem, where an interface pressure (or Lagrange multiplier) is introduced on the Stokes–Darcy interface and approximated using a mortar finite element space. In [46], a locally conservative coupling has been studied using a mixed FEM for the Darcy flow and a Discontinuous Galerkin (DG) method for the Stokes flow, while in [39], the authors propose a new numerical method employing divergence-conforming velocity spaces. DG approximations have also been analyzed in [44, 32] for Newtonian constitutive laws, both adopting a primal formulation of the Darcy equation and mixed-order elements. In [40], Congreve and Houston propose a two-grid *hp*-version DG method for quasi-Newtonian fluid flows. A non-Newtonian Stokes–Darcy–Forchheimer model discretized via a DG method on triangular grids has been presented in [37], employing P^{l+1}/P^l discontinuous elements. The corresponding error analysis is carried out within a Lagrange multiplier framework, using P^{l+1} discontinuous elements for the multiplier. In [27] an overview of some results on the coupling of Navier–Stokes and Darcy's equations is presented. Polytopal Discontinuous Galerkin (PolyDG) methods on general polygonal and polyhedral meshes have attracted considerable attention in recent years due to their flexibility in handling complex geometries and support mesh agglomeration techniques. Such methods have been successfully developed and analyzed for (linear) Darcy flows in fractured porous media, see, e.g., [4, 5] and for (linear) Stokes flow problems, with also applications to fluid-structure interaction, see, e.g., [49, 7]. In [42], Lipnikov, Vassilev and Yotov present a PolyDG formulation for coupled, linear, Stokes–Darcy flows, while Li, Gao, Zhang and Chen, present a PolyDg method for the solution of a linear Stokes–Darcy–Darcy (bulk-fracture) model in [41]. For earlier developments of PolyDG methods for elliptic problems, we refer, e.g., to [6, 10, 17, 17]; see also the monograph [18], the review paper [3] and the references therein.

The aim of this work is to propose and analyze a PolyDG method for discretizing a coupled non-Newtonian Stokes–Darcy system, supplemented by the (physically consistent) Beaver–Joseph–Saffman interface condition, along with continuity of the normal flux and equilibrium of normal stresses across the interface. In the context of coupled Stokes-Darcy problems, it is convenient that the numerical method treats both the free flow and the porous flow equations within a unified and consistent framework, avoiding the need for different discretization strategies in the two subdomains. In particular, it is desirable that the method yields the same order of convergence in both regions, so that the accuracy of the overall coupled solution is not limited by a discrepancy in the approximation quality between the two sides of the interface [16]. In the context of the model under consideration, a PolyDG approach offers a particularly appealing numerical framework. Indeed, the discontinuous nature of the approximation space allows for local tuning of approximation parameters, such as the polynomial degree p and the mesh size h, which proves especially beneficial when dealing with non-conforming (possibly agglomerated) meshes. Furthermore, the use of polytopal meshes provides remarkable geometric flexibility, making PolyDG particularly advantageous in coupled porous-fluid problems, where the physical domain typically comprises (highly) heterogeneous subregions with irregular (or curved) interfaces. Moreover, the weak enforcement of inter-element continuity via numerical fluxes guarantees a natural embedding of the interface conditions directly in the formulation and built-in stability in advection-dominated and heterogeneous regimes [25, 45, 1]. In this paper, we first analyze the well-posedness of the continuous model, and then propose and analyze an equal-order mixed PolyDG method for its numerical approximation. On the one hand, the equal-order choice for the discontinuous discretization spaces is appealing for the practical implementation of the method, but on the other hand, it introduces two additional challenges from the analysis viewpoint: establishing the well-posedness of the discrete problem and deriving a suitable inf-sup condition for the coupled system that accounts for the pressure stabilization terms required to ensure stability when equal-order polynomials are used. To address the first challenge, we extend the results of [12] to our discrete framework. The inf-sup condition is then established by extending a generalized inf-sup condition with pressure stabilization for the Stokes equation [7] to the Darcy setting, via an approach based on space inclusions and norm ordering. A further consequence of the mixed formulation framework is that the penalty terms associated with the Darcy vector and scalar variables mirror their Stokes counterparts: the penalty for the vector variable scales as $O(h^{-1})$, while that for the scalar variable scales as $O(h)$, cf. [7]. Finally, we consider an Incomplete Interior Penalty (IIP) formulation and a simple fixed-point iteration scheme to solve the nonlinear system. For the proposed discretization, we prove its well-posedness, stability bounds, and we show that the approximation error measured in a suitable energy norm (defined as the sum of the DG norm of the Stokes' and Darcy's variables) of the error scales as $O(h^l)$, being l the polynomial order chosen for all the discrete variables.

The remainder of the manuscript is organized as follows. Section 2 presents the model problem, introduces the assumptions on the physical parameters, derives the weak formulation of the continuous problem, and establishes its well-posedness. Section 3 introduces the PolyDG discrete framework and presents the derivation of the numerical scheme. Sections 4 and 5 are devoted to the analysis of well-posedness and convergence analysis, respectively. Finally, Section 6 presents numerical convergence results for both Newtonian and non-Newtonian test cases, and in Section 7 we draw the conclusions of our work, and discuss some possible future developments.

2 Model problem and its well-posedness

In this section, we introduce the mathematical model governing the coupled shear-thinning, non-Newtonian fluid-flow system. The free-flow region, where a non-Newtonian viscous fluid moves unobstructed in an open channel, is described by the Stokes equations for incompressible creeping laminar flow, which balance viscous stresses with pressure forces while neglecting inertial effects. The flow within the adjacent porous medium is instead governed by Darcy's law, which relates the seepage velocity linearly to the pressure gradient through the permeability tensor of the medium. The two flow regimes occupy distinct but adjacent subdomains, separated by an interface, and are coupled by conditions that enforce continuity of physical quantities across the interface. Specifically, the conservation of mass across the interface is ensured by the continuity of the normal component of the velocity, while the balance of normal forces is imposed through the continuity of the normal stress. The tangential behavior at the interface is described by the Beavers–Joseph–Saffman condition, which relates the tangential component of the viscous stress in the free flow region to the tangential velocity of the free fluid at the interface, neglecting the contribution of the seepage velocity in the porous medium. On the Stokes' boundary, we consider homogeneous Dirichlet boundary conditions (no-slip condition), assuming that the velocity of the fluid is equal to the velocity of the wall. Instead, on the Darcy's boundary, we assume the normal component of the seepage velocity to be zero, meaning that there is no fluid flow across that boundary [37].

We assume that the computational domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, is divided by a planar interface Γ into two polygonal/polyhedral open subdomains Ω_S and Ω_D, such that $\overline{\Omega}_S \cup \overline{\Omega}_D = \overline{\Omega}$ and $\Omega_S \cap \Omega_D = \emptyset$, cf. Figure 1. Here, Ω_S is the free-flow region modeled by Stokes equations, and Ω_D is the porous region modeled by Darcy's law. We split the boundary of the domain in $\Gamma_S = \partial\Omega_S \cap \partial\Omega$ and $\Gamma_D = \partial\Omega_D \cap \partial\Omega$. Moreover, we define $\mathbf{n}_S$, $\mathbf{n}_D$ the unit normal vectors to Γ_S and Γ_D, respectively. We define $\mathbf{n}_\Gamma^S$ the normal unit vector to the interface Γ directed towards Ω_D, and $\mathbf{n}_\Gamma^D = -\mathbf{n}_\Gamma^S$ the unit normal vector to the interface Γ directed towards Ω_S. Finally, we define an orthogonal system of unit tangent vectors $\mathbf{t}_{\Gamma,j}$, $1 \leq j \leq d-1$ on Γ.

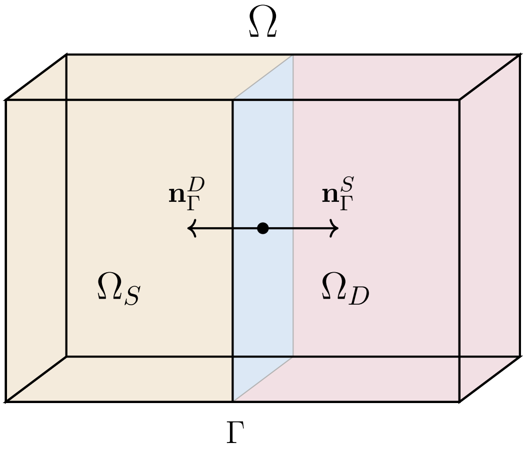


Figure 1: The computational domain.

The non-Newtonian Stokes–Darcy coupled system reads: given $\mathbf{f}_S, \mathbf{f}_D$, find $(\mathbf{u}_S, p_S)$ and $(\mathbf{u}_D, p_D)$ such that:

$$\begin{cases} -\nabla \cdot (g_S(|\mathbf{D}(\mathbf{u}^S)|)\mathbf{D}(\mathbf{u}^S) - p^S\mathbf{I}) = \mathbf{f}_S & \text{in } \Omega_S, \\ \nabla \cdot \mathbf{u}^S = 0 & \text{in } \Omega_S, \\ \mathbf{u}^S = 0 & \text{on } \Gamma_S, \\ \mathbf{K}^{-1} g_D(|\mathbf{u}^D|)\mathbf{u}^D + \nabla p^D = \mathbf{f}_D & \text{in } \Omega_D, \\ \nabla \cdot \mathbf{u}^D = 0 & \text{in } \Omega_D, \\ \mathbf{u}^D \cdot \mathbf{n}_D = 0 & \text{on } \Gamma_D, \\ \mathbf{u}^S \cdot \mathbf{n}_\Gamma^S + \mathbf{u}^D \cdot \mathbf{n}_\Gamma^D = 0 & \text{on } \Gamma, \\ p^S - (g_S(|\mathbf{D}(\mathbf{u}^S)|)\mathbf{D}(\mathbf{u}^S)\mathbf{n}_\Gamma^S) \cdot \mathbf{n}_\Gamma^S = p^D & \text{on } \Gamma, \\ \mathbf{u}^S \cdot \mathbf{t}_{\Gamma,j} + \rho(g_S(|\mathbf{D}(\mathbf{u}^S)|)\mathbf{D}(\mathbf{u}^S)\mathbf{n}_\Gamma^S) \cdot \mathbf{t}_{\Gamma,j} = 0 & \text{on } \Gamma, \text{ for all } 1 \leq j \leq d-1 \end{cases} \tag{2.1}$$

where $\mathbf{D}(\mathbf{u}) = (\nabla\mathbf{u} + \nabla^T\mathbf{u})/2$ is the symmetric gradient of the velocity $\mathbf{u}$, $g_S(|\mathbf{D}(\mathbf{u}^S)|)$ and $g_D(|\mathbf{u}^D|)$ are the velocity-dependent viscosities, $\mathbf{K}$ is a symmetric, positive definite tensor representing the permeability of the porous medium, and $\rho > 0$ is a frictional parameter. We state the following assumptions on the velocity-dependent viscosities g_S and g_D.

Assumption 1. We assume that $g_S(\cdot)$ and $g_D(\cdot)$ are positive and bounded, and that $g_S(|\mathbf{A}|)\mathbf{A}$ and $g_D(|\mathbf{a}|)\mathbf{a}$ are Lipschitz continuous and strongly monotone, i.e., there exist positive constants $\underline{\nu}_S, \overline{\nu}_S, \underline{\mu}_S$, $\widetilde{\mu}_S$, $\underline{\nu}_D, \overline{\nu}_D, \underline{\mu}_D$ and $\widetilde{\mu}_D$ such that, for any symmetric $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ and for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$:

$$\underline{\mu}_S \leq g_S(|\mathbf{A}|) \leq \widetilde{\mu}_S, \tag{2.2}$$

$$|g_S(|\mathbf{A}|)\mathbf{A} - g_S(|\mathbf{B}|)\mathbf{B}| \leq \overline{\nu}_S|\mathbf{A} - \mathbf{B}|, \tag{2.3}$$

$$(g_S(|\mathbf{A}|)\mathbf{A} - g_S(|\mathbf{B}|)\mathbf{B}, \mathbf{A} - \mathbf{B}) \geq \underline{\nu}_S|\mathbf{A} - \mathbf{B}|^2. \tag{2.4}$$

$$\underline{\mu}_D \leq g_D(|\mathbf{a}|) \leq \widetilde{\mu}_D, \tag{2.5}$$

$$|g_D(|\mathbf{a}|)\mathbf{a} - g_D(|\mathbf{b}|)\mathbf{b}| \leq \overline{\nu}_D|\mathbf{a} - \mathbf{b}|, \tag{2.6}$$

$$(g_D(|\mathbf{a}|)\mathbf{a} - g_D(|\mathbf{b}|)\mathbf{b}, \mathbf{a} - \mathbf{b}) \geq \underline{\nu}_D|\mathbf{a} - \mathbf{b}|^2. \tag{2.7}$$

We consider the following non-Newtonian models for the viscosity [22, 31]. We denote by $g_S(|\mathbf{D}(\mathbf{u})|)$ the dynamic viscosity of the fluid defined as:

$$g_S(|\mathbf{D}(\mathbf{u})|) = \underline{\mu}_S + (\overline{\mu}_S - \underline{\mu}_S)G(\sqrt{2}|\mathbf{D}(\mathbf{u})|) \qquad \text{where } \overline{\mu}_S > \underline{\mu}_S > 0,$$

while the effective viscosity $g_D(\cdot)$ for the non-Newtonian porous media flow is defined as:

$$g_D(|\mathbf{u}|) = \underline{\mu}_D + (\overline{\mu}_D - \underline{\mu}_D)G(\sqrt{2}|\mathbf{u}|) \qquad \text{where } \overline{\mu}_D > \underline{\mu}_D > 0.$$

The choices of $G(s)$ are listed in Table 1.

Table 1: Non-Newtonian models [37].

Model	$\mathbf{G(s)}$	
Power	s^{r-2}	$1 < r < 2$
Carreau	$(\delta + s^2)^{\frac{r-2}{2}}$	$1 \leq r < 2$, $\delta > 0$
Carreau–Yasuda	$(\delta + s^a)^{\frac{r-2}{a}}$	$a > 0,\ 1 \leq r < 2,\ \delta > 0$
Generalized cross	$(\delta + s^r)^a$	$r > 0,\ a < 0,\ ar + 1 \geq 0,\ \delta > 0$
Powell–Eyring	$\sinh(s)^{-1}/s$	

Remark 1. *If both the viscosities from $g_S(\cdot)$ and $g_D(\cdot)$ obey a Carreau–Yasuda model, i.e.:*

$$g_S(|\mathbf{D}(\mathbf{u})|) = \underline{\mu}_S + (\overline{\mu}_S - \underline{\mu}_S)(\delta + |\mathbf{D}(\mathbf{u})|^2)^{\frac{r-2}{2}} \quad \text{and} \quad g_D(|\mathbf{u}|) = \underline{\mu}_S + (\overline{\mu}_S - \underline{\mu}_S)(\delta + |\mathbf{u}|^2)^{\frac{r-2}{2}}$$

then Assumption 1 is satisfied provided that $\delta > 0$ and $r \geq 1$, cf. [37].

Finally, we assume that $k_{\min}|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi}^T\mathbf{K}\boldsymbol{\xi} \leq k_{\max}|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$.

Throughout the paper, we make use of the following notation: $a \lesssim b$ meaning that $a \leq Cb$, with C a constant independent of the discretization parameter h but that might depend on the $\overline{\mu}_S, \underline{\mu}_S, \overline{\mu}_D, \underline{\mu}_D, \overline{\nu}_S, \underline{\nu}_S, \overline{\nu}_D, \underline{\nu}_D$, the domain geometry and the polynomial degree.

We conclude this section by introducing the weak formulation of (2.1). We also extend the continuous inf-sup condition for the Stokes problem to the coupled problem, using arguments based on space inclusions. First, we define the function spaces:

$$\begin{aligned}
&\mathbf{X}^S = \{\mathbf{v}^S \in [H^1(\Omega_S)]^d : \mathbf{v}^S = 0 \text{ on } \Gamma_S\} = [H^1_{0,\Gamma_S}(\Omega_S)]^d, \qquad M^S = L^2(\Omega_S),\\
&\mathbf{X}^D = \{\mathbf{v}^D \in [L^2(\Omega_D)]^d : \nabla\cdot\mathbf{v}^D \in L^2(\Omega_D),\ \mathbf{v}^D\cdot\mathbf{n}^D_\Gamma = 0 \text{ on } \Gamma_D\}, \qquad M^D = L^2(\Omega_D),\\
&\mathbf{X} = \{(\mathbf{v}^S,\mathbf{v}^D) \in \mathbf{X}^S \times \mathbf{X}^D : \mathbf{v}^S\cdot\mathbf{n}^S_\Gamma = -\mathbf{v}^D\cdot\mathbf{n}^D_\Gamma \text{ on } \Gamma\},\\
&M = \{(q^S,q^D) \in M^S \times M^D : (q^S,1)_{\Omega_S} = -(q^D,1)_{\Omega_D}\},
\end{aligned}$$

and their norms:

$$\begin{aligned}
&\|\mathbf{v}^S\|_{\mathbf{X}^S} = \|\mathbf{v}^S\|_{H^1(\Omega_S)}, \quad \|p^S\|_{M^S} = \|p^S\|_{L^2(\Omega_S)},\\
&\|\mathbf{v}^D\|_{\mathbf{X}^D} = \|\mathbf{v}^D\|_{L^2(\Omega_D)}, \quad \|p^D\|_{M^D} = \|p^D\|_{L^2(\Omega_D)},\\
&\|\mathbf{v}\|_{\mathbf{X}} = \|\mathbf{v}^S\|_{\mathbf{X}^S} + \|\mathbf{v}^D\|_{\mathbf{X}^D}, \quad \|p\|_M = \|p^S\|_{L^2(\Omega_S)} + \|p^D\|_{L^2(\Omega_D)},
\end{aligned}$$

The weak form of (2.1) reads: find $\mathbf{u} = (\mathbf{u}^S, \mathbf{u}^D) \in \mathbf{X}$ and $p = (p^S, p^D) \in M$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) & \forall \mathbf{v} \in \mathbf{X},\\ b(\mathbf{u},q) = 0 & \forall q \in M, \end{cases} \tag{2.8}$$

where

$$
\begin{aligned}
a_S(\mathbf{u}^S,\mathbf{v}^S) &= \int_{\Omega_S} g_S(|\mathbf{D}(\mathbf{u}^S)|)\mathbf{D}(\mathbf{u}^S):\mathbf{D}(\mathbf{v}^S)\ dx + \sum_{j=1}^{d-1}\int_\Gamma \rho^{-1}(\mathbf{u}^S\cdot\mathbf{t}_{\Gamma,j})(\mathbf{v}^S\cdot\mathbf{t}_{\Gamma,j})\ ds,\\
a_D(\mathbf{u}^D,\mathbf{v}^D) &= \int_{\Omega_D} \mathbf{K}^{-1}g_D(|\mathbf{u}^D|)\mathbf{u}^D\cdot\mathbf{v}^D\ dx,\\
a(\mathbf{u},\mathbf{v}) &= a_S(\mathbf{u}^S,\mathbf{v}^S)+a_D(\mathbf{u}^D,\mathbf{v}^D),\\
b_S(\mathbf{v}^S,p^S) &= -\int_{\Omega_S} q^S\nabla\cdot\mathbf{v}^S\ dx,\\
b_D(\mathbf{v}^D,p^D) &= -\int_{\Omega_D} q^D\nabla\cdot\mathbf{v}^D\ dx,\\
b(\mathbf{v},p) &= b_S(\mathbf{v}^S,p^S)+b_D(\mathbf{v}^D,p^D),\\
(\mathbf{f},\mathbf{v}) &= \int_{\Omega_S}\mathbf{f}_S\cdot\mathbf{v}^S dx + \int_{\Omega_D}\mathbf{f}_D\cdot\mathbf{v}^D dx.
\end{aligned}
$$

2.1 Well-posedness and stability

To prove the well-posedness of problem (2.8), we first prove an inf-sup condition in the continuous setting by considering the inf-sup condition of the Stokes problem, and extending it to the coupled system.

Theorem 1 (Inf-sup condition)**.** *For all $p=(p^S,p^D)\in M$ it holds*

$$
\sup_{\mathbf{v}\in\mathbf{X},\mathbf{v}\neq\mathbf{0}} \frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{\mathbf{X}}} \gtrsim \|p\|_M. \tag{2.9}
$$

Proof. First, we observe that $[H^1_{0,\Gamma_D}(\Omega_D)]^d\subset\mathbf{X}^D$, therefore, for all $\mathbf{v}\in[H^1_{0,\partial\Omega}(\Omega)]^d$, one has $(\mathbf{v}^S,\mathbf{v}^D)=(\mathbf{v}_{|\Omega_S},\mathbf{v}_{|\Omega_D})$ is such that $\mathbf{v}^S\cdot\mathbf{n}^S_\Gamma=\mathbf{v}^D\cdot\mathbf{n}^D_\Gamma$. As a result, we have $[H^1_{0,\partial\Omega}(\Omega)]^d\subset\mathbf{X}$ which also implies that

$$
\|\cdot\|_{\mathbf{X}}\lesssim\|\cdot\|_{H^1(\Omega)}. \tag{2.10}
$$

Thus, considering the standard Stokes inf-sup condition due to the fact that $p\in M$ has zero average over Ω, it is inferred that

$$
\|p\|_{L^2(\Omega)}\lesssim \sup_{\mathbf{v}\in[H^1_{0,\partial\Omega}(\Omega)]^d}\frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{H^1(\Omega)}} \le \sup_{\mathbf{v}\in\mathbf{X},\mathbf{v}\neq 0}\frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{H^1(\Omega)}} \qquad \forall p\in M. \tag{2.11}
$$

since we are taking the superior on a larger space. Additionally, from (2.10) we get

$$
\sup_{\mathbf{v}\in\mathbf{X},\mathbf{v}\neq 0}\frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{H^1(\Omega)}} \lesssim \sup_{\mathbf{v}\in\mathbf{X},\mathbf{v}\neq 0}\frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{\mathbf{X}}} \qquad \forall p\in M. \tag{2.12}
$$

Hence, putting together (2.12) and (2.11) we obtain (2.9). □

The well-posedness of (2.8) is a consequence of (2.12) and of the boundedness and coercivity of the nonlinear form $a(\cdot,\cdot)$. We denote the kernel of the bilinear form b in (2.8) by

$$
\mathring{\mathbf{X}}=\{\mathbf{v}\in\mathbf{X}:\nabla\cdot\mathbf{v}^S=0 \text{ a.e. on } \Omega_S,\nabla\cdot\mathbf{v}^D=0 \text{ a.e. on } \Omega_D\},
$$

and its dual by $\mathring{\mathbf{X}}^*$. We define the operator $A:\mathring{\mathbf{X}}\to\mathring{\mathbf{X}}^*$ as

$$
(A(\mathbf{u}),\mathbf{v})=a(\mathbf{u},\mathbf{v}) \quad \forall\mathbf{u},\mathbf{v}\in\mathring{\mathbf{X}}. \tag{2.13}
$$

We next show the following result for the problem $A(\mathbf{u})=\mathbf{f}$ in $\mathring{\mathbf{X}}^*$, which corresponds to the restriction of (2.8) to the kernel $\mathring{\mathbf{X}}$.

Proposition 1. *Let Assumption 1 be satisfied. Then, the operator* $A : \mathring{\mathbf{X}} \to \mathring{\mathbf{X}}^*$ *is uniformly bounded, i.e.*

$$\|A(\mathbf{u})\|_{\mathring{\mathbf{X}}^*} \lesssim \|\mathbf{u}^S\|_{\mathbf{X}^S} + \|\mathbf{u}^D\|_{L^2(\Omega_D)}, \tag{2.14}$$

where the hidden constants depend on $\widetilde{\mu}_S$, $\widetilde{\mu}_D$ *and* ρ.

Proof. Owing to the definition of dual norm and (2.13), we obtain

$$\|A(\mathbf{u})\|_{\mathring{\mathbf{X}}^*} = \sup_{\mathbf{v}\in\mathring{\mathbf{X}},\mathbf{v}\neq\mathbf{0}} \frac{(A(\mathbf{u}),\mathbf{v})}{\|\mathbf{v}\|_{\mathring{\mathbf{X}}}} = \sup_{\mathbf{v}\in\mathring{\mathbf{X}},\mathbf{v}\neq\mathbf{0}} \frac{a_S(\mathbf{u}^S,\mathbf{v}^S) + a_D(\mathbf{u}^D,\mathbf{v}^D)}{\|\mathbf{v}\|_{\mathring{\mathbf{X}}}}. \tag{2.15}$$

By applying (2.2), (2.5), the Cauchy–Schwarz inequality, and the trace inequality, we obtain

$$\begin{aligned} a_S(\mathbf{u}^S,\mathbf{v}^S) &\lesssim \|\mathbf{D}(\mathbf{u}^S)\|_{L^2(\Omega_S)}\|\mathbf{D}(\mathbf{v}^S)\|_{L^2(\Omega_S)} + \|\mathbf{u}^S\|_{L^2(\Gamma)}\|\mathbf{v}^S\|_{L^2(\Gamma)} \lesssim \|\mathbf{u}^S\|_{\mathbf{X}^S}\|\mathbf{v}^S\|_{\mathbf{X}^S}, \\ a_D(\mathbf{u}^D,\mathbf{v}^D) &\lesssim \|\mathbf{u}^D\|_{L^2(\Omega_D)}\|\mathbf{v}^D\|_{L^2(\Omega_D)} \leq \|\mathbf{u}^D\|_{L^2(\Omega_D)}\|\mathbf{v}^D\|_{\mathbf{X}^D}, \end{aligned}$$

with constants depending on $\widetilde{\mu}_S, \widetilde{\mu}_D$ and ρ. Plugging these inequalities into (2.15), we have (2.14). □

Proposition 2. *Let Assumption 1 be satisfied; then, the operator* A *is coercive, i.e.,*

$$(A(\mathbf{v}),\mathbf{v}) \gtrsim \|\mathbf{v}^S\|^2_{\mathbf{X}^S} + \sum_{j=1}^{d-1} \|\mathbf{v}^S\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(\Gamma)} + \|\mathbf{v}^D\|^2_{L^2(\Omega_D)} \ \forall \mathbf{v}\in\mathring{\mathbf{X}}, \tag{2.16}$$

where the hidden constant depend on $\underline{\mu}_S, \underline{\mu}_D$, ρ, *and* k_{max}.

Proof. Applying Korn's first inequality, the Poincaré inequality, and the fact that $\nabla\cdot\mathbf{v}^D = 0$, since $\mathbf{v} = (\mathbf{v}^S,\mathbf{v}^D)\in\mathring{\mathbf{X}}$, we get

$$\begin{aligned} (A(\mathbf{v}),\mathbf{v}) = a(\mathbf{v},\mathbf{v}) &\gtrsim \|\mathbf{D}(\mathbf{v}^S)\|^2_{L^2(\Omega_S)} + \sum_{j=1}^{d-1}\rho^{-1}\|\mathbf{v}^S\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(\Gamma)} + \|\mathbf{v}^D\|^2_{L^2(\Omega_D)} \\ &\gtrsim \|\mathbf{v}^S\|^2_{\mathbf{X}^S} + \sum_{j=1}^{d-1}\rho^{-1}\|\mathbf{v}^S\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(\Gamma)} + \|\mathbf{v}^D\|^2_{L^2(\Omega_D)}, \end{aligned}$$

and the proof is complete. □

We next show that the operator A defined as in (2.13) is strongly monotone.

Proposition 3. *Under Assumption 1 the operator* A *is strongly monotone, i.e., for all* $\mathbf{u},\mathbf{v}\in\mathring{\mathbf{X}}$,

$$(A(\mathbf{u})-A(\mathbf{v}),\mathbf{u}-\mathbf{v}) \gtrsim \|\mathbf{u}^S-\mathbf{v}^S\|^2_{\mathbf{X}^S} + \sum_{j=1}^{d-1}\|(\mathbf{u}^S-\mathbf{v}^S)\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(\Gamma)} + \|\mathbf{u}^D-\mathbf{v}^D\|^2_{L^2(\Omega_D)}, \tag{2.17}$$

where the hidden constants depend on $\underline{\nu}_S$, $\underline{\nu}_D$, ρ, *and* k_{max}.

Proof. Let $\mathbf{u} = (\mathbf{u}^S,\mathbf{u}^D), \mathbf{v} = (\mathbf{v}^S,\mathbf{v}^D)\in\mathring{\mathbf{V}}$. We write:

$$(A(\mathbf{u})-A(\mathbf{v}),\mathbf{u}-\mathbf{v}) = \underbrace{a_S(\mathbf{u}^S,\mathbf{u}^S-\mathbf{v}^S) - a_S(\mathbf{v}^S,\mathbf{u}^S-\mathbf{v}^S)}_{I_1} + \underbrace{a_D(\mathbf{u}^D,\mathbf{u}^D-\mathbf{v}^D) - a_D(\mathbf{v}^D,\mathbf{u}^D-\mathbf{v}^D)}_{I_2}.$$

By (2.4), the Korn's inequality and the Poincaré inequality, we have:

$$I_1 \gtrsim \|\mathbf{D}(\mathbf{u}^S-\mathbf{v}^S)\|^2_{L^2} + \sum_{j=1}^{d-1}\rho^{-1}\|(\mathbf{u}^S-\mathbf{v}^S)\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(\Gamma)} \gtrsim \|(\mathbf{u}^S-\mathbf{v}^S)\|^2_{\mathbf{X}^S} + \sum_{j=1}^{d-1}\rho^{-1}\|(\mathbf{u}^S-\mathbf{v}^S)\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(\Gamma)}.$$

Additionally, (2.7) implies:

$$I_2 \gtrsim \|\mathbf{u}^D-\mathbf{v}^D\|^2_{L^2(\Omega_D)}.$$

Hence, we can conclude (2.17). □

Combining the previous results, we obtain the existence and uniqueness of the solution to $A(\mathbf{u}) = \mathbf{f}$ in $\mathring{\mathbf{X}}^*$. Thanks to the inf-sup condition (2.9), we can conclude the well-posedness of problem (2.8).

We conclude the section by deriving a stability estimate. We consider problem (2.8), and we take $\mathbf{v} = \mathbf{u}$ and $q = p$. Summing up the two equations we obtain:

$$a(\mathbf{u},\mathbf{u}) = (\mathbf{f},\mathbf{u}) \lesssim \|\mathbf{f}_S\|^2_{L^2(\Omega_S)} + \|\mathbf{u}^S\|^2_{L^2(\Omega_S)} + \|\mathbf{f}_D\|^2_{L^2(\Omega_D)} + \|\mathbf{u}^D\|^2_{L^2(\Omega_D)}.$$

Due to the coercivity of the form $a(\cdot,\cdot)$, see (2.16), we obtain the following:

$$\|\mathbf{u}^S\|^2_{\mathbf{X}^S} + \sum_{j=1}^{d-1} \rho^{-1} \|\mathbf{u}^S \cdot \mathbf{t}_{\Gamma,j}\|^2_{L^2(\Gamma)} + \|\mathbf{u}^D\|^2_{\mathbf{X}^D} \lesssim \|\mathbf{f}_S\|^2_{L^2(\Omega_S)} + \|\mathbf{f}_D\|^2_{L^2(\Omega_D)}.$$

In order to prove a stability estimate for the pressure we consider the inf-sup condition (2.9)

$$\begin{aligned}\|p\|_{L^2(\Omega)} &\lesssim \sup_{\mathbf{v}\in H^1_{0,\Gamma}} \frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{\mathbf{X}}} = \sup_{\mathbf{v}\in H^1_{0,\Gamma}} \frac{(\mathbf{f},\mathbf{v}) - a(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{X}}} \\ &\lesssim \|\mathbf{u}^S\|_{\mathbf{X}^S} + \sum_{j=1}^{d-1} \rho^{-1} \|\mathbf{u}^S \cdot \mathbf{t}_{\Gamma,j}\|_{L^2(\Gamma)} + \|\mathbf{f}_S\|_{L^2(\Omega_S)} + \|\mathbf{u}^D\|_{L^2(\Omega_D)} + \|\mathbf{f}_D\|_{L^2(\Omega_D)}.\end{aligned}$$

Therefore, from the last two inequalities we derive $\|p\|_{L^2(\Omega)} \lesssim \|\mathbf{f}_S\|_{L^2(\Omega_S)} + \|\mathbf{f}_D\|_{L^2(\Omega_D)}$.

3 Discrete problem

In this section, we introduce the discrete framework of the PolyDG method, some functional analysis tools [23, 24, 25], and we introduce the discrete formulation of the problem.

We introduce a polytopal mesh partition $\mathscr{K}_h = \mathscr{K}_{h,\mathcal{S}} \cup \mathscr{K}_{h,\mathcal{D}}$, of the domain Ω, with the set of the internal and boundary faces $\mathscr{F}_h$, where $\mathscr{K}_{h,\mathcal{I}} = \{K \in \mathscr{K}_{h,\mathcal{I}} : \bar{K} \subseteq \bar{\Omega}_{\mathcal{I}}\}$, for $\mathcal{I} = \mathcal{S}, \mathcal{D}$. We further assume that $\mathscr{K}_{h,\mathcal{I}}$, for $\mathcal{I} = \mathcal{S}, \mathcal{D}$, are aligned with the subdomains $\Omega_{\mathcal{I}}$. We denote by $|K|$ the measure of the element $K \in \mathscr{K}_h$ and by h_K its diameter. Additionally, we set $h = \max_{K\in\mathscr{K}_h} h_K < 1$. We define the interface as the intersection of the $(d-1)$-dimensional facets of two neighboring elements, and we distinguish two cases. When $d = 2$, the faces are always line segments, and we denote their set as $\mathscr{F}_{h,\mathcal{I}}$, with $\mathcal{I} = \mathcal{S}, \mathcal{D}$. When $d = 3$, the faces are generic polygons; we further assume that we can decompose any such face into planar triangles. We denote the set of such triangles as $\mathscr{F}_{h,\mathcal{I}}$, with $\mathcal{I} = \mathcal{S}, \mathcal{D}$. We decompose $\mathscr{F}_{h,\mathcal{I}}$ into the set of the internal faces, $\mathscr{F}^i_{h,\mathcal{I}}$, the boundary faces, $\mathscr{F}^b_{h,\mathcal{I}}$, lying on the boundary $\partial\Omega_{\mathcal{I}} \setminus \Gamma$, and of the internal faces Γ_h, lying on the interface Γ between the subdomains $\Omega_{\mathcal{S}}$ and $\Omega_{\mathcal{D}}$. Moreover, we can split the boundary faces accordingly to the type of boundary condition imposed: $\mathscr{F}^b_{h,\mathcal{I}} = \mathscr{F}^d_{h,\mathcal{I}} \cup \mathscr{F}^n_{h,\mathcal{I}}$, where $\mathscr{F}^d_{h,\mathcal{I}}$ and $\mathscr{F}^n_{h,\mathcal{I}}$, are the boundary faces where we impose Dirichlet and Neumann boundary conditions respectively.

Assumption 2. [24] We denote by $\mathcal{H} \subset (0, +\infty)$ a countable set of meshsizes having 0 as its unique accumulation point. A family of meshes $(\mathscr{M}_h)_{h\in\mathcal{H}} = (\mathscr{K}_h, \mathscr{F}_h)_{h\in\mathcal{H}}$ is said to be regular if there exists a real number $\gamma \in (0, 1)$ independent of h such that, for all $h \in \mathcal{H}$, there exists a matching simplicial submesh, in the sense of [24], $\widetilde{\mathscr{M}_h} = (\widetilde{\mathscr{K}_h}, \widetilde{\mathscr{F}_h})$ of $\mathscr{M}_h$ that satisfies the following conditions:

1. **Shape Regularity**: for any simplex $\tau \in \widetilde{\mathscr{K}_h}$, denoting by h_τ its diameter and by r_τ its inradius, it holds $\gamma h_\tau \leq r_\tau$;
2. **Contact Regularity**: for any mesh element $K \in \mathscr{K}_h$ and any simplex $\tau \in \widetilde{\mathscr{K}_h}$ where $\widetilde{\mathscr{K}_h} = \{\tau \in \widetilde{\mathscr{K}_h} : \tau \subset K\}$ is the set of the simplices contained in K, it holds $\gamma h_K \leq h_\tau$.

From now on, we will omit the subscript h to simplify the notation. Given a polynomial degree $l \geq 0$ and a polytopal mesh $\mathscr{M} = (\mathscr{K}, \mathscr{F})$, we denote by $P^l(K)$ the space of polynomials of degree l on the element $K \in \mathscr{K}$, by $P^l(\mathscr{K})$ the set of piecewise polynomials of order l over $\mathscr{K}$. Under Assumption 2, the following trace-inverse inequality holds

$$\|v_h\|_{L^2(\partial K)} \lesssim h_K^{-\frac{1}{2}} \|v_h\|_{L^2(K)} \qquad \forall K \in \mathscr{K} \ \forall v_h \in P^l(K), \tag{3.1}$$

where the hidden constant depends on the polynomial degree l, the space dimension d, and the mesh regularity parameter γ. We define the following broken functional spaces:

$$\begin{aligned}
&\mathbf{X}_h^{\mathcal{I}} = \{\mathbf{v}_h^{\mathcal{I}} \in [L^2(\Omega_{\mathcal{I}})]^d : \mathbf{v}_h^{\mathcal{I}}|_K \in [P^l(K)]^d \ (\forall K \in \mathscr{K}_{\mathcal{I}})\},\\
&\mathbf{X}_h = \mathbf{X}_h^S \times \mathbf{X}_h^D,\\
&M_h^{\mathcal{I}} = \{q_{\mathcal{I}}^h \in M^{\mathcal{I}} : q_h^{\mathcal{I}}|_K \in P^l(K) \ (\forall K \in \mathscr{K}_{\mathcal{I}})\},\\
&M_h = \{(q_h^{\mathcal{S}}, q_h^{\mathcal{D}}) \in M_h^S \times M_h^D : (q_h^{\mathcal{S}}, 1)_{\Omega_S} + (q_h^{\mathcal{D}}, 1)_{\Omega_D} = 0\},
\end{aligned}$$

with $\mathcal{I} = S, D$. Now we introduce the average operator $\{\!\{\cdot\}\!\}$ and the jump operators $[\![\cdot]\!]$ and $[\![\cdot]\!]_{\mathbf{n}}$ [8],for the scalar and vector quantities ψ and $\boldsymbol{\varphi}$, on each internal facet $F \in \partial K_1 \cup \partial K_2$:

$$\begin{aligned}
&\{\!\{\psi\}\!\} = \frac{1}{2}(\psi|_{K_1} + \psi|_{K_2}), \quad \{\!\{\boldsymbol{\varphi}\}\!\} = \frac{1}{2}(\boldsymbol{\varphi}|_{K_1} + \boldsymbol{\varphi}|_{K_2}),\\
&[\![\psi]\!] = \psi|_{K_1} - \psi|_{K_2}, \quad [\![\boldsymbol{\varphi}]\!] = \boldsymbol{\varphi}|_{K_1} - \boldsymbol{\varphi}|_{K_2},\\
&[\![\psi]\!]_{\mathbf{n}} = \psi|_{K_1}\mathbf{n}|_{K_1} + \psi|_{K_2}\mathbf{n}|_{K_2}, \quad [\![\boldsymbol{\varphi}]\!]_{\mathbf{n}} = \boldsymbol{\varphi}|_{K_1} \cdot \mathbf{n}|_{K_1} + \boldsymbol{\varphi}|_{K_2} \cdot \mathbf{n}|_{K_2}.
\end{aligned}$$

If $F \in K_1$, with F a face on the Dirichlet boundary, we set $\{\!\{\psi\}\!\} = \psi|_{K_1}$, $\{\!\{\boldsymbol{\varphi}\}\!\} = \boldsymbol{\varphi}|_{K_1}$. Additionally, we define the trace operators for the trial functions on the faces $F \in \mathscr{F}^b$ as $[\![\psi]\!] = \psi$, $[\![\boldsymbol{\varphi}]\!] = \boldsymbol{\varphi}$, $[\![\psi]\!]_{\mathbf{n}} = \psi\mathbf{n}$ and $[\![\boldsymbol{\varphi}]\!]_{\mathbf{n}} = \boldsymbol{\varphi} \cdot \mathbf{n}$.

We define the following penalty functions: $\sigma_{\mathcal{I}} : \mathscr{F} \to \mathbb{R}$ and $\xi_{\mathcal{I}} : \mathscr{F}^i \to \mathbb{R}$ as:

$$\sigma_{\mathcal{I}}\big|_F = \begin{cases} \gamma_{\mathcal{I}}^v \max\limits_{K^+,K^-} \left\{\frac{l^2}{h_K}\right\} & F \in \mathcal{F}_{\mathcal{I}}^i,\\ \gamma_{\mathcal{I}}^v \dfrac{l^2}{h_K} & F \in \mathcal{F}_{\mathcal{I}}^b \cup \Gamma, \end{cases} \qquad \xi_{\mathcal{I}}\big|_F = \gamma_{\mathcal{I}}^p \min_{K^+,K^-}\left\{\frac{h_K}{l}\right\} \qquad F \in \mathcal{F}_{\mathcal{I}}^i, \tag{3.2}$$

with $\mathcal{I} = S, D, \Gamma$ and $\gamma_{\mathcal{I}}^v$ and $\gamma_{\mathcal{I}}^p$ are user-dependent penalty parameters.

Now, we define the following discrete forms:

$$\begin{aligned}
a_h^S(\mathbf{u}_h^{\mathcal{S}}, \mathbf{v}_h^{\mathcal{S}}) = &\sum_{K \in \mathscr{K}_{\mathcal{S}}} \int_K g_S(|\mathbf{D}(\mathbf{u}_h^{\mathcal{S}})|)\mathbf{D}(\mathbf{u}_h^{\mathcal{S}}) : \mathbf{D}(\mathbf{v}_h^{\mathcal{S}}) \, dx \ - \sum_{F \in \mathscr{F}_{\mathcal{S}}} \int_F \{\!\{g_S(|\mathbf{D}(\mathbf{u}_h^{\mathcal{S}})|)\mathbf{D}(\mathbf{u}_h^{\mathcal{S}})\mathbf{n}\}\!\} \cdot [\![\mathbf{v}_h^{\mathcal{S}}]\!] \, ds\\
&+ \sum_{j=1}^{d-1} \sum_{F \in \Gamma} \int_F \rho^{-1}(\mathbf{u}_h^{\mathcal{S}} \cdot \mathbf{t}_{\Gamma,j})(\mathbf{v}_h^{\mathcal{S}} \cdot \mathbf{t}_{\Gamma,j}) \, ds \ + \sum_{F \in \mathscr{F}_{\mathcal{S}}} \int_F \sigma_S [\![\mathbf{u}_h^{\mathcal{S}}]\!] \cdot [\![\mathbf{v}_h^{\mathcal{S}}]\!] \, ds,\\
a_h^D(\mathbf{u}_h^{\mathcal{D}}, \mathbf{v}_h^{\mathcal{D}}) = &\sum_{K \in \mathscr{K}_{\mathcal{D}}} \int_K \mathbf{K}^{-1} g_D(|\mathbf{u}_h^{\mathcal{D}}|)\mathbf{u}_h^{\mathcal{D}} \cdot \mathbf{v}_h^{\mathcal{D}} \, dx + \sum_{F \in \mathscr{F}_{\mathcal{D}}^i} \int_F \sigma_D [\![\mathbf{u}_h^{\mathcal{D}}]\!]_{\mathbf{n}} [\![\mathbf{v}_h^{\mathcal{D}}]\!]_{\mathbf{n}} \, ds.
\end{aligned}$$

Since in the analysis we need to impose the continuity of the normal jump of the velocities on Γ, we add a stabilization term:

$$a_h^{\Gamma}(\mathbf{u}_h, \mathbf{v}_h) = \sum_{F \in \Gamma} \int_F \sigma_{\Gamma} [\![\mathbf{u}_h]\!]_{\mathbf{n}} [\![\mathbf{v}_h]\!]_{\mathbf{n}} \, ds,$$

where $\mathbf{u}_h = (\mathbf{u}_h^{\mathcal{S}}, \mathbf{u}_h^{\mathcal{D}})$. Hence, we set

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = a_h^S(\mathbf{u}_h^{\mathcal{S}}, \mathbf{v}_h^{\mathcal{S}}) + a_h^D(\mathbf{u}_h^{\mathcal{D}}, \mathbf{v}_h^{\mathcal{D}}) + a_h^{\Gamma}(\mathbf{u}_h, \mathbf{v}_h).$$

We also define the following bilinear forms

$$\begin{aligned}
b_h^S(\mathbf{v}_h^{\mathcal{S}}, p_h^{\mathcal{S}}) &= -\sum_{K\in\mathscr{K}_{\mathcal{S}}} \int_K p_h^{\mathcal{S}} \nabla\cdot \mathbf{v}_h^{\mathcal{S}}\ dx\ + \sum_{F\in\mathscr{F}_{\mathcal{S}}} \int_F \{\!\{p_h^{\mathcal{S}}\}\!\} [\![\mathbf{v}_h^{\mathcal{S}}]\!]_{\mathbf{n}}\ ds,\\
b_h^D(\mathbf{v}_h^{\mathcal{D}}, p_h^{\mathcal{D}}) &= -\sum_{K\in\mathscr{K}_{\mathcal{D}}} \int_K p_h^{\mathcal{D}} \nabla\cdot \mathbf{v}_h^{\mathcal{D}}\ dx\ + \sum_{F\in\mathscr{F}_{\mathcal{D}}} \int_F \{\!\{p_h^{\mathcal{D}}\}\!\} [\![\mathbf{v}_h^{\mathcal{D}}]\!]_{\mathbf{n}}\ ds,\\
b_h^\Gamma(\mathbf{v}_h, p_h) &= \sum_{F\in\Gamma} \int_F p_h^{\mathcal{D}} [\![\mathbf{v}_h]\!]_{\mathbf{n}} ds.
\end{aligned}$$

and set $b_h(\mathbf{v}_h, p_h)$ as

$$b_h(\mathbf{v}_h, p_h) = b_h^S(\mathbf{v}_h^{\mathcal{S}}, p_h^{\mathcal{S}}) + b_h^D(\mathbf{v}_h^{\mathcal{D}}, p_h^{\mathcal{D}}) + b_h^\Gamma(\mathbf{v}_h, p_h),$$

where $p_h = (p_h^{\mathcal{S}}, p_h^{\mathcal{D}})$. Furthermore, we add the following stabilization term on the pressure:

$$s_h(p_h, q_h) = \sum_{F\in\mathscr{F}_{\mathcal{S}}^i} \int_F \xi_S [\![p_h^{\mathcal{S}}]\!]_{\mathbf{n}} [\![q_h^{\mathcal{S}}]\!]_{\mathbf{n}} ds + \sum_{F\in\mathscr{F}_{\mathcal{D}}^i} \int_F \xi_D [\![p_h^{\mathcal{D}}]\!]_{\mathbf{n}} [\![q_h^{\mathcal{D}}]\!]_{\mathbf{n}} ds + \sum_{F\in\Gamma} \int_F \xi_\Gamma [\![p_h]\!]_{\mathbf{n}} [\![q_h]\!]_{\mathbf{n}} ds.$$

Finally, we define the following forcing term:

$$F_h(\mathbf{v}_h) = (\mathbf{f}_S, \mathbf{v}_h^{\mathcal{S}})_{\Omega_S} + (\mathbf{f}_D, \mathbf{v}_h^{\mathcal{D}})_{\Omega_D}.$$

The discrete problem reads: find $\mathbf{u}_h = (\mathbf{u}_h^{\mathcal{S}}, \mathbf{u}_h^{\mathcal{D}}) \in \mathbf{X}_h$ and $p_h = (p_h^{\mathcal{S}}, p_h^{\mathcal{D}}) \in M_h$ such that:

$$\begin{cases} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = F(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{X}_h,\\ b_h(\mathbf{u}_h, q_h) - s_h(p_h, q_h) = 0 & \forall q_h \in M_h. \end{cases} \tag{3.3}$$

4 Well-posedness of the discrete problem

In this section, we analyze the well-posedness of the discrete problem (3.3). First, we define some useful notation, prove some properties of the bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, and we show how we can extend the inf-sup condition for the Stokes problem to the coupled problem. Then we prove the existence and uniqueness of the solution, showing that the form associated with the complete problem is weakly continuous and maps linearly into a coercive space. Finally, we show a continuous dependence of the problem on the data.

Now, we introduce the discrete norms that are used in the *a-priori* analysis of the PolyDG scheme.

Definition 1 (Discrete norms)**.** For all $\mathbf{u}_h \in \mathbf{X}_h$ and $p_h \in M_h$ we define

$$\begin{aligned}
|\!|\!|(\mathbf{u}_h, p_h)|\!|\!|_E &= |\!|\!|\mathbf{u}_h|\!|\!|_{\mathbf{X}_h} + |\!|\!|p_h|\!|\!|_{M_h},\\
|\!|\!|\mathbf{u}_h|\!|\!|_{\mathbf{X}_h} &= \|\mathbf{u}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S} + \|\mathbf{u}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} + \left(\sum_{F\in\Gamma} \sigma_\Gamma \|[\![\mathbf{u}_h]\!]_{\mathbf{n}}\|^2_{L^2(F)}\right)^{\frac{1}{2}},\\
|\!|\!|p_h|\!|\!|_{M_h} &= \|p_h^{\mathcal{S}}\|_{M_h^S} + \|p_h^{\mathcal{D}}\|_{M_h^D} + \left(\sum_{F\in\Gamma} \xi_\Gamma \|[\![p_h]\!]_{\mathbf{n}}\|^2_{L^2(F)}\right)^{\frac{1}{2}},
\end{aligned}$$

where

$$\|\mathbf{u}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S} = \|\mathbf{u}_h^{\mathcal{S}}\|_{dG} + \left(\sum_{j=1}^{d-1}\sum_{F\in\Gamma}\rho^{-1}\|\mathbf{u}_h^{\mathcal{S}}\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(F)}\right)^{\frac{1}{2}},$$

$$\|\mathbf{u}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} = \|\mathbf{u}_h^{\mathcal{D}}\|_{L^2(\Omega_D)} + \left(\sum_{F\in\mathscr{F}_{\mathcal{D}}^i}\sigma_D\|[\![\mathbf{u}_h^{\mathcal{D}}]\!]_{\mathbf{n}}\|^2_{L^2(F)}\right)^{\frac{1}{2}},$$

$$\|\mathbf{u}_h^{\mathcal{I}}\|_{dG} = \|\nabla_h\mathbf{u}_h^{\mathcal{I}}\|_{L^2(\Omega_I)} + \left(\sum_{F\in\mathscr{F}_{\mathcal{I}}}\sigma_{\mathcal{I}}\|[\![\mathbf{u}_h^{\mathcal{I}}]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}},$$

$$\|p_h^{\mathcal{I}}\|_{M_h^I} = \|p_h^{\mathcal{I}}\|_{L^2(\Omega_I)} + \left(\sum_{F\in\mathscr{F}_{\mathcal{I}}^i}\xi_{\mathcal{I}}\|[\![p_h^{\mathcal{I}}]\!]_{\mathbf{n}}\|^2_{L^2(F)}\right)^{\frac{1}{2}}, \qquad \text{with } \mathcal{I}=\{S,D\},$$

having denoted with ∇_h the piecewise broken gradient operator. Additionally, we define:

$$\begin{aligned}|\!|\!|\mathbf{u}_h|\!|\!|_{dG} &= \|\mathbf{u}_h^{\mathcal{S}}\|_{dG} + \|\mathbf{u}_h^{\mathcal{D}}\|_{dG} + \left(\sum_{F\in\Gamma}\sigma_\Gamma\|[\![\mathbf{u}_h]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}},\\ |\!|\!|\mathbf{u}_h|\!|\!|_{div} &= |\!|\!|\mathbf{u}_h|\!|\!|_{\mathbf{X}_h} + \|\nabla_h\cdot\mathbf{u}_h^{\mathcal{D}}\|_{L^2(\Omega_D)}.\end{aligned}$$

4.1 Properties of the coupling bilinear form

Here, we prove that $b_h(\cdot,\cdot)$ is inf-sup stable and continuous. In particular, we extend the inf-sup condition for the Stokes problem on polygonal grids [7] to the non-Newtonian Stokes–Darcy coupled system (3.3).

We now prove the following generalized inf-sup condition:

Theorem 2. *For all $p_h \in M_h$, it holds*

$$\sup_{\mathbf{v}_h\in\mathbf{X}_h,\mathbf{v}_h\neq\mathbf{0}}\frac{b_h(\mathbf{v}_h,p_h)}{|\!|\!|\mathbf{v}_h|\!|\!|_{div}} + |p_h|_J \gtrsim \|p_h\|_{L^2(\Omega)}, \tag{4.1}$$

where $|p_h|_J^2 = s_h(p_h,p_h)$.

Proof. This proof is divided into two parts. First, we first establish the discrete inf-sup condition for the coupled problem with respect to the $|\!|\!|\cdot|\!|\!|_{dG}$ norm of the velocity $\mathbf{v}_h$ [7] under a weighted norm on the facets comprising the interface Γ. To this end, we consider the limiting case in which the entire fluid domain is governed by the Stokes equations, and demonstrate that the inf-sup condition holds for a general weighted norm. Then we show how the inf-sup condition for the Stokes problem can be extended to the coupled case, using arguments involving inequalities among norms. We now proceed by exposing the first part of the proof, introducing the following weighted average operator:

$$\{\!\{\phi\}\!\}_\omega = \omega\phi_1 + (1-\omega)\phi_2,$$

and we take

$$\omega = \begin{cases}\frac{1}{2} & \text{on } \mathscr{F}_{\mathcal{S}}\cup\mathscr{F}_{\mathcal{D}}\\ 0 & \text{on } \Gamma\end{cases}.$$

With this definition of average, we can still apply the following equality, c.f. [8]:

$$[\![\phi\psi]\!] = [\![\phi]\!]\{\!\{\psi\}\!\}_\omega + [\![\psi]\!]\{\!\{\phi\}\!\}_{\bar{\omega}},$$

where $\bar{\omega} = 1 - \omega$. Indeed,

$$\llbracket \phi \rrbracket \{\!\{ \psi \}\!\}_\omega + \llbracket \psi \rrbracket \{\!\{ \phi \}\!\}_{\bar{\omega}} = (\phi_1 - \phi_2)(\omega \psi_1 + (1-\omega)\psi_2) + (\psi_1 - \psi_2)((1-\omega)\phi_1 + \omega \phi_2)$$
$$= \psi_1 \phi_1 - \psi_2 \phi_2 = \llbracket \phi \psi \rrbracket.$$

Reasoning as in [7], we recall that at the continuous level for any $q_h \in M_h \subset L^2_0(\Omega)$ there exists $\mathbf{v}^{q_h} \in \mathbf{X}$ such that:

$$\nabla \cdot \mathbf{v}^{q_h} = q_h \quad \|\mathbf{v}^{q_h}\|_{\mathbf{X}} \lesssim \|q_h\|_{L^2(\Omega)}.$$

Applying an element-wise integration by part, using $\llbracket \mathbf{v}^{q_h} \rrbracket = 0$ for any $F \in \mathscr{F}_h$, observing that $\nabla q_h \in \mathbf{X}_h$ and considering the global polynomial interpolation operator $\mathbf{\Pi}^l : \mathbf{X} \to \mathbf{X}_h$ of [7] we get

$$\begin{aligned}
\|q_h\|^2_{L^2(\Omega)} &= \int_\Omega q_h \nabla \cdot \mathbf{v}^{q_h} = -\int_\Omega \nabla q_h \cdot \mathbf{v}^{q_h} + \sum_{F \in \mathscr{F}_h} \int_F \llbracket q_h \rrbracket \{\!\{ \mathbf{v}^{q_h} \}\!\} \\
&= -\int_\Omega \nabla q_h \cdot \mathbf{\Pi}^l \mathbf{v}^{q_h} + \int_\Omega \nabla q_h \cdot (\mathbf{\Pi}^l \mathbf{v}^{q_h} - \mathbf{v}^{q_h}) + \sum_{F \in \mathscr{F}_h} \int_F \llbracket q_h \rrbracket \{\!\{ \mathbf{v}^{q_h} \}\!\}_\omega \\
&= \int_\Omega q_h \nabla \cdot \mathbf{\Pi}^l \mathbf{v}^{q_h} - \sum_{F \in \mathscr{F}_h} \int_F \llbracket q_h \rrbracket \{\!\{ \mathbf{\Pi}^l \mathbf{v}^{q_h} \}\!\}_\omega - \sum_{F \in \mathscr{F}_h} \int_F \{\!\{ q_h \}\!\}_{\bar{\omega}} \llbracket \mathbf{\Pi}^l \mathbf{v}^{q_h} \rrbracket \\
&\quad + \sum_{F \in \mathscr{F}_h} \int_F \llbracket q_h \rrbracket \{\!\{ \mathbf{v}^{q_h} \}\!\}_\omega + \int_\Omega \nabla q_h \cdot (\mathbf{\Pi}^l \mathbf{v}^{q_h} - \mathbf{v}^{q_h}) \\
&= -b^h(\mathbf{\Pi}^l \mathbf{v}^{q_h}, q_h) + \int_\Omega \nabla q_h \cdot (\mathbf{\Pi}^l \mathbf{v}^{q_h} - \mathbf{v}^{q_h}) + \sum_{F \in \mathscr{F}_h} \int_F \llbracket q_h \rrbracket \{\!\{ (\mathbf{v}^{q_h} - \mathbf{\Pi}^l \mathbf{v}^{q_h}) \}\!\}_\omega.
\end{aligned}$$

Now the proof follows exactly the steps reported on [7] to obtain

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h, \mathbf{v}_h \neq \mathbf{0}} \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{dG}} + |p_h|_J \gtrsim \|p_h\|_{L^2(\Omega)}. \tag{4.2}$$

Now, we proceed with the second part of the proof starting from (4.2), and recall that

$$\|\llbracket \mathbf{v}_h^{\mathcal{D}} \rrbracket\|_{L^2(F)} \geq \|\llbracket \mathbf{v}_h^{\mathcal{D}} \rrbracket_{\mathbf{n}}\|_{L^2(F)}, \qquad \text{and} \qquad \|\nabla_h \mathbf{v}_h\|_{L^2(\Omega)} \geq \|\nabla_h \cdot \mathbf{v}_h\|_{L^2(\Omega)}.$$

Additionally, by a discrete Poincarè inequality (cf. [25, Corollary 5.4] and [14, Theorem 1.8]), we have

$$\|\mathbf{v}_h^{\mathcal{D}}\|_{dG} \gtrsim \|\mathbf{v}_h^{\mathcal{D}}\|_{L^2(\Omega_D)},$$

and, as a result, we infer

$$\|\mathbf{v}_h^{\mathcal{D}}\|_{dG} \gtrsim \|\mathbf{v}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} + \|\nabla_h \cdot \mathbf{v}_h^{\mathcal{D}}\|_{L^2(\Omega_D)}. \tag{4.3}$$

Next, we apply the global discrete trace inequality $\|\mathbf{v}_h^{\mathcal{S}}\|_{L^2(\partial\Omega_S)} \lesssim \|\mathbf{v}_h^{\mathcal{S}}\|_{dG}$, which is a consequence of [24, Theorem 6.7], to write

$$\sum_{j=1}^{d-1} \sum_{F \in \Gamma} \rho^{-1} \|\mathbf{v}_h^{\mathcal{S}} \cdot \mathbf{t}_{\Gamma,j}\|^2_{L^2(F)} \lesssim \rho^{-1} \|\mathbf{v}_h^{\mathcal{S}}\|^2_{L^2(\Gamma)} \lesssim \rho^{-1} \|\mathbf{v}_h^{\mathcal{S}}\|^2_{L^2(\partial\Omega_S)} \lesssim \rho^{-1} \|\mathbf{v}_h^{\mathcal{S}}\|^2_{dG}.$$

As a result, owing to the definition of the $\|\cdot\|_{\mathbf{X}_h^S}$-norm, we have

$$\|\mathbf{v}_h^{\mathcal{S}}\|_{dG} \gtrsim \|\mathbf{v}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S}. \tag{4.4}$$

Hence, applying (4.3) and (4.4) we obtain:

$$\|\mathbf{v}_h\|_{dG} \gtrsim \|\mathbf{v}_h\|_{div} \Rightarrow \frac{1}{\|\mathbf{v}_h\|_{div}} \gtrsim \frac{1}{\|\mathbf{v}_h\|_{dG}}.$$

Finally, from (4.2), we can conclude that:

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h, \mathbf{v}_h \neq 0} \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{div}} + |p_h|_J \gtrsim \sup_{\mathbf{v}_h \in \mathbf{X}_h, \mathbf{v}_h \neq 0} \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{dG}} + |p_h|_J \gtrsim \|p_h\|_{L^2(\Omega)}.$$

□

We next state the following results.

Corollary 1. *For all $p_h \in M_h$ it holds*

$$\sup_{\mathbf{v}_h \in X_h, \mathbf{v}_h \neq 0} \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} + |p_h|_J \gtrsim \|p_h\|_{L^2(\Omega)}, \tag{4.5}$$

where $|p_h|_J^2 = s_h(p_h, p_h)$.

Proof. Starting from Theorem 2, we observe that, by definition: $\|\mathbf{v}_h\|_{div} \gtrsim \|\mathbf{v}_h\|_{\mathbf{X}_h}$, which implies that:

$$\frac{1}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \gtrsim \frac{1}{\|\mathbf{v}_h\|_{div}}.$$

Therefore,

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h, \mathbf{v}_h \neq 0} \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} + |p_h|_J \gtrsim \sup_{\mathbf{v}_h \in \mathbf{X}_h, \mathbf{v}_h \neq 0} \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{div}} + |p_h|_J \gtrsim \|p_h\|_{L^2(\Omega)},$$

and the proof is complete. □

The next lemma establishes the existence of a generalized right-inverse of the discrete divergence operator.

Lemma 1. *For each $p_h \in M_h$, there exists $\boldsymbol{\xi}_h \in \mathbf{X}_h$ such that*

$$\|p_h\|^2_{L^2(\Omega)} \lesssim b_h(\boldsymbol{\xi}_h, p_h) + s_h(p_h, p_h) \qquad \text{and} \qquad \|\boldsymbol{\xi}_h\|_{\mathbf{X}_h} \leq \|p_h\|_{L^2(\Omega)}. \tag{4.6}$$

Proof. The discrete inf-sup inequality of Corollary 1 implies, for all $p_h \in M_h$, the existence of $\overline{\mathbf{v}}_h \in \mathbf{X}_h$ such that

$$\frac{b_h(\overline{\mathbf{v}}_h, p_h)}{\|\overline{\mathbf{v}}_h\|_{\mathbf{X}_h}} + |p_h|_J \gtrsim \|p_h\|_{L^2(\Omega)}. \tag{4.7}$$

Then, we take $\boldsymbol{\xi}_h = \frac{\|p_h\|_{L^2(\Omega)} \overline{\mathbf{v}}_h}{\|\overline{\mathbf{v}}_h\|_{\mathbf{X}_h}}$ and observe that, by the definition of $\boldsymbol{\xi}_h$, we have $\|\boldsymbol{\xi}_h\|_{\mathbf{X}_h} \leq \|p_h\|_{L^2(\Omega)}$. To prove the first inequality in (4.6), we observe that

$$b_h(\boldsymbol{\xi}_h, p_h) + |p_h|_J^2 = \left(\frac{b_h(\overline{\mathbf{v}}_h, p_h)}{\|\overline{\mathbf{v}}_h\|_{\mathbf{X}_h}} + |p_h|_J \right) \|p_h\|_{L^2(\Omega)} - |p_h|_J \|p_h\|_{L^2(\Omega)} + |p_h|_J^2.$$

Therefore, applying (4.7) and the Young inequality we obtain

$$\begin{aligned} b_h(\boldsymbol{\xi}_h, p_h) + |p_h|_J^2 &\gtrsim \|p_h\|^2_{L^2(\Omega)} - |p_h|_J \|p_h\|_{L^2(\Omega)} + |p_h|_J^2 \\ &\gtrsim \frac{\|p_h\|^2_{L^2(\Omega)} + |p_h|_J^2}{2} \gtrsim \|p_h\|^2_{L^2(\Omega)}. \end{aligned}$$

□

The continuity of the bilinear form $b_h(\cdot, \cdot)$ is established in the following lemma; its proof is reported in Appendix A.

Lemma 2. *The bilinear form $b_h(\mathbf{v}_h, p_h) = b_h^S(\mathbf{v}_h^{\mathcal{S}}, p_h^{\mathcal{S}}) + b_h^D(\mathbf{v}_h^{\mathcal{D}}, p_h^{\mathcal{D}}) + b_h^\Gamma(\mathbf{v}_h, p_h)$ is continuous, i.e.*

$$|b_h(\mathbf{v}_h, p_h)| \lesssim \|\mathbf{v}_h\|_{div} \|p_h\|_{M_h} \qquad \forall \mathbf{v}_h \in \mathbf{X}_h, \forall p_h \in M_h.$$

4.2 Well-posedness of the discrete problem

We start by proving the continuity and monotonicity of $a_h(\cdot,\cdot)$ in the following lemmas, whose proofs can be found in Appendix B.

Lemma 3. *Let Assumption 1 and Assumption 2 be satisfied. Then,*

$$|a_h(\mathbf{u}_h, \mathbf{w}_h) - a_h(\mathbf{v}_h, \mathbf{w}_h)| \lesssim \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{X}_h} \|\mathbf{w}_h\|_{\mathbf{X}_h} \qquad \forall \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{X}_h. \tag{4.8}$$

Lemma 4. *Let Assumption 1 and Assumption 2 be satisfied. Then for sufficiently large σ_S, $a_h(\cdot,\cdot)$ is monotone, i.e.,*

$$(a_h(\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h) - a_h(\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h)) \gtrsim \|\mathbf{u}_h - \mathbf{v}_h\|^2_{\mathbf{X}_h} \qquad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h.$$

We rewrite problem (3.3) in the form

$$\mathcal{A}_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = F((\mathbf{v}_h, q_h)), \tag{4.9}$$

where

$$\mathcal{A}_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) - b_h(\mathbf{u}_h, q_h) + s_h(p_h, q_h). \tag{4.10}$$

To prove the well-posedness of (4.9), we refer to [12] (Theorem 4), and we need to prove that the form $\mathcal{A}_h$ is linearly mapped coercive, in the sense of [12].

Lemma 5. *Let Assumption 1 and Assumption 2 be satisfied. The form $\mathcal{A}_h(\cdot,\cdot)$ defined as* (4.10) *is linearly mapped coercive.*

Proof. We need to prove that for any $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times M_h$

$$\mathcal{A}_h((\mathbf{u}_h, p_h), \Phi(\mathbf{u}_h, p_h)) \gtrsim \|(\mathbf{u}_h, p_h)\|_E{}^2,$$

with Φ a bijection. We recall that the inf-sup condition (4.5) implies Lemma 1. Now, we define Φ as:

$$\Phi(\mathbf{u}_h, p_h) = (\alpha \mathbf{u}_h + \beta \boldsymbol{\xi}_h, \alpha p_h),$$

with α and β to be appropriately chosen. By definition $\Phi(\mathbf{u}_h, p_h)$ is linear. To prove that is a bijection we simply need to prove that is an injection, i.e. $\Phi(\mathbf{u}_h, p_h) = (\mathbf{0}, 0) \Leftrightarrow (\mathbf{u}_h, p_h) = (\mathbf{0}, 0)$. On the one hand if $\Phi(\mathbf{u}_h, p_h) = (\mathbf{0}, 0)$ this implies,

$$\begin{aligned} &\alpha \mathbf{u}_h + \beta \boldsymbol{\xi}_h = \mathbf{0}, \\ &\alpha p_h = 0, \end{aligned}$$

From this we derive that $\alpha p_h = 0 \Rightarrow p_h = 0$. Condition (4.6) implies that also $\boldsymbol{\xi}_h = 0$. From this we have $\alpha \mathbf{u}_h + \beta \boldsymbol{\xi}_h = \mathbf{0} \Rightarrow \mathbf{u}_h = \mathbf{0}$. The inverse implication is trivial. Now, we write:

$$\begin{aligned} \mathcal{A}_h((\mathbf{u}_h, p_h), \Phi(\mathbf{u}_h, p_h)) &= \mathcal{A}_h((\mathbf{u}_h, p_h), (\alpha \mathbf{u}_h + \beta \boldsymbol{\xi}_h, \alpha p_h)) \\ &= a_h(\mathbf{u}_h, \alpha \mathbf{u}_h + \beta \boldsymbol{\xi}_h) + b_h(\alpha \mathbf{u}_h + \beta \boldsymbol{\xi}_h, p_h) - b_h(\mathbf{u}_h, \alpha p_h) + s_h(p_h, \alpha p_h) \\ &= \alpha a_h(\mathbf{u}_h, \mathbf{u}_h) + \beta a_h(\mathbf{u}_h, \boldsymbol{\xi}_h) + \alpha b_h(\mathbf{u}_h, p_h) \\ &+ \beta b_h(\boldsymbol{\xi}_h, p_h) - \alpha b_h(\mathbf{u}_h, p_h) + \alpha s_h(p_h, p_h) \\ &= \alpha a_h(\mathbf{u}_h, \mathbf{u}_h) + \beta a_h(\mathbf{u}_h, \boldsymbol{\xi}_h) + \beta b_h(\boldsymbol{\xi}_h, p_h) + \alpha s_h(p_h, p_h). \end{aligned} \tag{4.11}$$

We study the different terms separately. From Lemma 4 we recover:

$$\alpha a_h(\mathbf{u}_h, \mathbf{u}_h) \gtrsim \alpha \|\mathbf{u}_h\|_{\mathbf{X}_h}{}^2.$$

From Lemma 3 and Young's inequality we recover:

$$\beta(a_h(\mathbf{u}_h, \boldsymbol{\xi}_h)) \gtrsim -\beta \|\mathbf{u}_h\|_{\mathbf{X}_h} \|\xi_h\|_{\mathbf{X}_h} \gtrsim -\beta \left(\frac{\|\mathbf{u}_h\|_{\mathbf{X}_h}{}^2}{2\epsilon} + \epsilon \frac{\|\xi_h\|_{\mathbf{X}_h}{}^2}{2} \right),$$

with Young's constant $\epsilon > 0$ to be properly chosen. Furthermore, using the inf-sup condition (4.6) we get:

$$\beta b_h(\boldsymbol{\xi}_h, p_h) \geq \beta \|p_h\|^2_{L^2(\Omega)} - \beta |p_h|^2_J.$$

Considering the last inequalities, choosing $\epsilon \simeq \frac{\beta}{\alpha}$, and considering (4.6), equation (4.11) becomes:

$$\mathcal{A}_h((\mathbf{u}_h, p_h), \Phi(\mathbf{u}_h, p_h)) \gtrsim \frac{\alpha}{2} \|\mathbf{u}_h\|_{\mathbf{X}_h}{}^2 + \left[-\frac{(\beta)^2}{2\alpha} + \beta \right] \|p_h\|^2_{L^2(\Omega)} + (\alpha - \beta)|p_h|^2_J.$$

Choosing $\alpha \geq \beta$ we obtain:

$$\mathcal{A}_h((\mathbf{u}_h, p_h), \Phi(\mathbf{u}_h, p_h)) \gtrsim \beta \|(\mathbf{u}_h, p_h)\|^2_E,$$

and the proof is complete. □

4.3 A-priori stability estimate

In this section, we derive an a-priori stability estimate for the discrete solution $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times M_h$ of problem (3.3).

Proposition 4. *Let Assumption 1 and Assumption 2 be satisfied. Moreover, assume that that the penalty parameters σ_S, σ_D and σ_Γ are large enough. Let $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times M_h$ be the unique discrete solution of* (3.3). *Then,*

$$\|\mathbf{u}_h\|^2_{\mathbf{X}_h} + \|\nabla_h \cdot \mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(\Omega_D)} + \|p_h\|^2_{M_h} \lesssim \|\mathbf{f}_S\|^2_{L^2(\Omega_S)} + \|\mathbf{f}_D\|^2_{L^2(\Omega_D)}.$$

Proof. We consider the discrete problem (3.3) and we sum the two equations:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) - b_h(\mathbf{u}_h, q_h) + s_h(p_h, q_h) = F(\mathbf{v}_h).$$

Taking $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = p_h$ we obtain:

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{u}_h) + s_h(p_h, q_h) = F(\mathbf{u}_h) &\leq \|\mathbf{f}_S\|_{L^2(\Omega_S)} \|\mathbf{u}_h^{\mathcal{S}}\|_{L^2(\Omega_S)} + \|\mathbf{f}_D\|_{L^2(\Omega_D)} \|\mathbf{u}_h^{\mathcal{D}}\|_{L^2(\Omega_D)} \\ &\lesssim \|\mathbf{f}_S\|^2_{L^2(\Omega_S)} + \|\mathbf{u}_h^{\mathcal{S}}\|^2_{L^2(\Omega_S)} + \|\mathbf{f}_D\|^2_{L^2(\Omega_D)} + \|\mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(\Omega_D)}. \end{aligned}$$

Finally, we get:

$$\|\mathbf{u}_h\|^2_{\mathbf{X}_h} + |p_h|^2_J \lesssim \|\mathbf{f}_S\|^2_{L^2(\Omega_S)} + \|\mathbf{f}_D\|^2_{L^2(\Omega_D)}. \tag{4.12}$$

Now we study the bound of $\|\nabla_h \cdot \mathbf{u}_h^{\mathcal{D}}\|_{L^2(\Omega_D)}$. To do so we consider the second equation of (3.3) and test it against $\tilde{q}_h = (0, \nabla \cdot \mathbf{u}_h^{\mathcal{D}})$, to obtain

$$-b_h(\mathbf{u}_h, \tilde{q}_h) + s_h(p_h, \tilde{q}_h) = 0.$$

From the previous identity we can derive the following:

$$\begin{aligned} \|\nabla \cdot \mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(\Omega_D)} \lesssim &\sum_{F \in \mathscr{F}_{\mathcal{D}}} \sigma_D \|[\![\mathbf{u}_h^{\mathcal{D}}]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{F \in \Gamma} \sigma_\Gamma \|[\![\mathbf{u}_h]\!]_{\mathbf{n}}\|^2_{L^2(F)} \\ &+ \sum_{F \in \mathscr{F}^i_{\mathcal{D}}} \xi_D \|[\![p_h^{\mathcal{D}}]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{F \in \Gamma} \xi_\Gamma \|[\![p_h]\!]_{\mathbf{n}}\|^2_{L^2(F)}. \end{aligned} \tag{4.13}$$

The complete derivation of the estimate can be found in Appendix C. Now, we consider the discrete inf-sup condition (4.5) and we observe that:

$$b_h(\mathbf{v}_h, p_h) = F(\mathbf{v}_h) - a_h(\mathbf{u}_h, \mathbf{v}_h) \lesssim \|\mathbf{f}_{\mathcal{S}}\|_{L^2(\Omega_S)} \|\mathbf{v}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S} + \|\mathbf{f}_{\mathcal{D}}\|_{L^2(\Omega_D)} \|\mathbf{v}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} + \|\mathbf{u}_h\|_{\mathbf{X}_h} \|\mathbf{v}_h\|_{\mathbf{X}_h}.$$

Then, we recover:

$$\|p_h\|_{L^2(\Omega)} \lesssim \sup_{\mathbf{v}_h \in X_h} \frac{\|\mathbf{f}_{\mathcal{S}}\|_{L^2(\Omega_S)} \|\mathbf{v}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S} + \|\mathbf{f}_{\mathcal{D}}\|_{L^2(\Omega_D)} \|\mathbf{v}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} + \|\mathbf{u}_h\|_{\mathbf{X}_h} \|\mathbf{v}_h\|_{\mathbf{X}_h}}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} + |p_h|_J. \tag{4.14}$$

Now we observe that summing up Equations (4.12), (4.14) and (4.13), we obtain the following estimate:

$$\|\mathbf{u}_h\|^2_{\mathbf{X}_h} + \|\nabla_h \cdot \mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(\Omega_D)} + \|p_h\|^2_{M_h} \lesssim \|\mathbf{f}_S\|^2_{L^2(\Omega_S)} + \|\mathbf{f}_D\|^2_{L^2(\Omega_D)},$$

and the proof is complete. □

5 Error estimates

In this section we prove a bound for $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{X}_h}$ and $\|p-p_h\|_{M_h}$. By applying the triangular inequality, we only need to find a bound for $\|\mathbf{u}_h-\mathbf{u}_I\|_{\mathbf{X}_h}$ and $\|p_h-p_I\|_{M_h}$ where $\mathbf{u}_I$ and p_I are suitable interpolation of the continuous solution.

First, we introduce the following definition for the element-wise L^2-orthogonal projection: $\pi_h^{0,l}$: $L^1(\Omega)\to P^l(\mathscr{K})$ defined such that: for all $v\in L^1(\Omega)$ and all $K\in\mathscr{K}$

$$(\pi_h^{0,l}v)_{|K}=\pi_K^{0,l}v,\qquad \text{with}\quad \int_K(\pi_K^{0,l}v)\,w_h=\int_K v\,w_h,\quad \forall w_h\in P^l(K).$$

We also recall the following result from [24]:

Theorem 3 (Approximation properties of the L^2-orthogonal projector)**.** *Let Assumption 2 be satisfied. Let a polynomial degree $l\ge 0$, an integer $s\in\{0,...,l+1\}$ and a real number $p\in[1,\infty]$ be given. Then, for any K element or face of $\mathscr{M}$, all $v\in W^{s,p}(K)$, and all $m\in\{0,...,s\}$,*

$$|v-\pi_K^{0,l}v|_{W^{m,p}(K)}\lesssim h_K^{s-m}|v|_{W^{s,p}(K)}.$$

Moreover, if $s\ge 1$, for all $K\in\mathscr{K}$, all $v\in W^{s,p}(K)$, all $F\in\mathscr{F}_K$, and all $m\in\{0,...,s-1\}$, it holds that

$$h_K^{\frac{1}{p}}|v-\pi_K^{0,l}v|_{W^{m,p}(F)}\lesssim h_K^{s-m}|v|_{W^{s,p}(K)}.$$

To simplify the notation, we define the interpolation errors and the discrete errors: $\mathbf{e}_I^u=\mathbf{u}_I-\mathbf{u}$ and $e_I^p=p_I-p$ are the velocity and pressure interpolation errors, and $\mathbf{e}_h^u=\mathbf{u}_h-\mathbf{u}_I$ and $e_h^p=p_h-p_I$ are the discrete errors, where $(\mathbf{u},p)$ is the solution of the continuous problem and $(\mathbf{u}_h,p_h)$ is the solution of the discrete problem. Additionally, we define: $\mathbf{e}^u=\mathbf{u}_h-\mathbf{u}$ and $e^p=p_h-p$, and we observe that $\mathbf{e}^u=\mathbf{e}_h^u+\mathbf{e}_I^u$ and $e^p=e_h^p+e_I^p$.

Now, we state the following error estimate theorem.

Theorem 4. *Let Assumption 1 and Assumption 2 be satisfied. Let the mesh size be such that $h\lesssim\rho$ and assume that the penalty parameters* (3.2) *are large enough. Additionally, we assume that the solutions to* (2.8) *satisfy*

$$\mathbf{u}_\mathcal{S}\in[H^{t_S+1}(\Omega_\mathcal{S})]^d,\ \mathbf{u}_\mathcal{D}\in[H^{t_D+1}(\Omega_\mathcal{D})]^d,\ p_\mathcal{S}\in H^{m_S+1}(\Omega_\mathcal{S})\ \text{ and }\ p_\mathcal{D}\in H^{m_D+1}(\Omega_\mathcal{D}).$$

Then, it holds:

$$\begin{aligned}|\!|\!|(\mathbf{e}_h^u,e_h^p)|\!|\!|_E^2\lesssim & h^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}(\Omega_S)}+h^{2t_D}|\mathbf{u}^D|^2_{H^{t_D+1}(\Omega_D)}+h^{2t}|\mathbf{u}|^2_{H^{t+1}(\Omega)}\\ &+h^{2m_S+2}|p^\mathcal{S}|_{H^{m_S+1}(\Omega_\mathcal{S})}+h^{2m_D+2}|p^\mathcal{D}|_{H^{m_D+1}(\Omega_\mathcal{D})},\end{aligned}$$

where t_S and t_D are the polynomial degrees for the Stokes' and Darcy's velocity fields, m_S and m_D are the polynomial degrees of the Stokes and Darcy's pressures, resepctively, and $t=\min\{t_\mathcal{S},t_\mathcal{D}\}$.

Remark 2. *We observe that the error bounds are consistent with the error convergence theory for the Discontinuous Galerkin method applied to the Stokes equation [45].*

Proof. We observe that the following consistency property holds

$$\begin{cases} a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)=(\mathbf{f},\mathbf{v}_h) & \forall\mathbf{v}_h\in\mathbf{X}_h,\\ -b_h(\mathbf{u},q_h)+s_h(p,q_h)=0 & \forall q_h\in M_h.\end{cases}\tag{5.1}$$

assuming that the continuous solutions $(\mathbf{u},p)$ are smooth enough. Now, we subtract (5.1) from (3.3) and we obtain the following error equations

$$\begin{cases} a_h(\mathbf{u}_h,\mathbf{v}_h)-a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,e^p)=0 & \forall\mathbf{v}_h\in\mathbf{X}_h,\\ \qquad\qquad -b_h(\mathbf{e}^u,q_h)+s_h(e^p,q_h)=0 & \forall q_h\in M_h.\end{cases}\tag{5.2}$$

Using the interpolation and discrete error definitions, system (5.2) becomes:

$$\begin{cases} a_h(\mathbf{u}_h,\mathbf{v}_h)-a_h(\mathbf{u}_I,\mathbf{v}_h)+a_h(\mathbf{u}_I,\mathbf{v}_h)-a_h(\mathbf{u},\mathbf{v}_h) \\ \qquad\qquad +b_h(\mathbf{v}_h,e_h^p)+b_h(\mathbf{v}_h,e_I^p)=0 \qquad \forall \mathbf{v}_h\in\mathbf{X}_h, \\ -b_h(\mathbf{e}_h^u,q_h)-b_h(\mathbf{e}_I^u,q_h)+s_h(e_h^p,q_h)+s_h(e_I^p,q_h)=0 \qquad \forall q_h\in M_h. \end{cases} \tag{5.3}$$

Then, we test against $\mathbf{v}_h=\mathbf{e}_h^u$ and $q_h=e_h^p$, to obtain

$$\begin{cases} a_h(\mathbf{u}_h,\mathbf{e}_h^u)-a_h(\mathbf{u}_I,\mathbf{e}_h^u)+b_h(\mathbf{e}_h^u,e_h^p)=a_h(\mathbf{u},\mathbf{e}_h^u)-a_h(\mathbf{u}_I,\mathbf{e}_h^u)-b_h(\mathbf{e}_h^u,e_I^p) \\ -b_h(\mathbf{e}_h^u,e_h^p)+s_h(e_h^p,e_h^p)=b_h(\mathbf{e}_I^u,e_h^p)-s_h(e_I^p,e_h^p) \end{cases}$$

Summing up the two equations above, we obtain

$$a_h(\mathbf{u}_h,\mathbf{e}_h^u)-a_h(\mathbf{u}_I,\mathbf{e}_h^u)+s_h(e_h^p,e_h^p)=\underbrace{a_h(\mathbf{u},\mathbf{e}_h^u)-a_h(\mathbf{u}_I,\mathbf{e}_h^u)}_{\mathbf{R_1}}\underbrace{-b_h(e_I^p,\mathbf{e}_h^u)}_{\mathbf{R_2}}+\underbrace{b_h(\mathbf{e}_I^u,e_h^p)}_{\mathbf{R_3}}\underbrace{-s_h(e_I^p,e_h^p)}_{\mathbf{R_4}}. \tag{5.4}$$

Now, for the monotonicity of $a_h(\cdot,\cdot)$, see Lemma 4, we have:

$$|\!|\!|\mathbf{e}_h^u|\!|\!|_{\mathbf{X}_h}^2+|e_h^p|_J^2\lesssim \mathbf{R_1}+\mathbf{R_2}+\mathbf{R_3}+\mathbf{R_4}.$$

From the discrete inf-sup condition (4.1) and the first equation in (5.3) it is inferred that

$$\begin{aligned} \|e_h^p\|_{L^2(\Omega)} &\lesssim \sup_{\mathbf{v}_h\in\mathbf{X}_h,\mathbf{v}_h\neq\mathbf{0}} \frac{b_h(\mathbf{v}_h,e_h^p)}{|\!|\!|\mathbf{v}_h|\!|\!|_{div}}+|e_h^p|_J \\ &= \sup_{\mathbf{v}_h\in\mathbf{X}_h,\mathbf{v}_h\neq\mathbf{0}} \frac{a_h(\mathbf{u},\mathbf{v}_h)-a_h(\mathbf{u}_I,\mathbf{v}_h)-b_h(\mathbf{v}_h,e_I^p)-a_h(\mathbf{u}_h,\mathbf{v}_h)+a_h(\mathbf{u}_I,\mathbf{v}_h)}{|\!|\!|\mathbf{v}_h|\!|\!|_{div}}+|e_h^p|_J \\ &\lesssim |\!|\!|\mathbf{e}_I^u|\!|\!|_{\mathbf{X}_h}+|\!|\!|\mathbf{e}_h^u|\!|\!|_{div}+\|e_I^p\|_{L^2(\Omega)}+|e_h^p|_J. \end{aligned} \tag{5.5}$$

Putting together (5.4), (5.5), and proceeding as in the stability estimates, taking into account that we can bound the norm of the divergence of $\mathbf{e}_h^u$ as in (4.13), we obtain that:

$$|\!|\!|\mathbf{e}_h^u|\!|\!|_{div}^2+|\!|\!|e_h^p|\!|\!|_{M_h}^2\lesssim \mathbf{R_1}+\mathbf{R_2}+\mathbf{R_3}+\mathbf{R_4}+|\!|\!|\mathbf{e}_I^u|\!|\!|_{\mathbf{X}_h}^2+\|e_I^p\|_{L^2(\Omega)}. \tag{5.6}$$

Now, we study the bound of $|\!|\!|\mathbf{e}_I^u|\!|\!|_{\mathbf{X}_h}^2$ and $\|e_I^p\|_{L^2(\Omega)}$. Referring to Theorem 3, we conclude that:

$$|\!|\!|\mathbf{e}_I^u|\!|\!|_{\mathbf{X}_h}^2\lesssim h^{2t_{\mathcal{S}}}|\mathbf{u}^{\mathcal{S}}|^2_{H^{t_{\mathcal{S}}+1}(\Omega_{\mathcal{S}})}+h^{2t_{\mathcal{D}}}|\mathbf{u}_{\mathcal{D}}|^2_{H^{t_{\mathcal{D}}+1}(\Omega_{\mathcal{D}})}+h^{2t}|\mathbf{u}|^2_{H^{t+1}(\Omega)},$$

and

$$\|e_I^p\|_{L^2(\Omega)}\lesssim h_K^{2(m_{\mathcal{S}}+1)}|p^{\mathcal{S}}|^2_{H^{m_{\mathcal{S}}+1}(\Omega_{\mathcal{S}})}+h_K^{2(m_{\mathcal{D}}+1)}|p^{\mathcal{D}}|^2_{H^{m_{\mathcal{D}}+1}(\Omega_{\mathcal{D}})}.$$

Now we analyze the terms $\mathbf{R_1},\mathbf{R_2},\mathbf{R_3}$ and $\mathbf{R_4}$ separately. For $\mathbf{R_1}$, we set

$$\begin{aligned} \mathbf{R_1} = & \underbrace{\sum_{K\in\mathscr{K}_{\mathcal{S}}}\int_K (g_S(|\mathbf{D}(\mathbf{u}^S)|)\mathbf{D}(\mathbf{u}^S)-g_S(|\mathbf{D}(\mathbf{u}_I^S)|)\mathbf{D}(\mathbf{u}_I^S)):\mathbf{D}(\mathbf{e}_h^{u^{\mathcal{S}}})dx}_{I_1} \\ & \underbrace{-\sum_{F\in\mathscr{F}_{\mathcal{S}}}\int_F \{\!\{g_S(|\mathbf{D}(\mathbf{u}^S)|)\mathbf{D}(\mathbf{u}^S)\mathbf{n}-g_S(|\mathbf{D}(\mathbf{u}_I^S)|)\mathbf{D}(\mathbf{u}_I^S)\mathbf{n}\}\!\}\cdot[\![\mathbf{e}_h^{u^{\mathcal{S}}}]\!]ds}_{I_2} \\ & +\underbrace{\sum_{j=1}^{d-1}\sum_{F\in\Gamma}\int_F \rho^{-1}(\mathbf{e}_h^{u^{\mathcal{S}}}\cdot\mathbf{t}_{\Gamma,j})(\mathbf{e}_I^{u^{S}}\cdot\mathbf{t}_{\Gamma,j})ds}_{I_3}+\underbrace{\sum_{K\in\mathscr{K}_{\mathcal{D}}}\int_K[(\mathbf{K}^{-1}(g_D(|\mathbf{u}^D|)\mathbf{u}^D-g_D(|\mathbf{u}_I^D|)\mathbf{u}_I^D)]\cdot(\mathbf{e}_h^{u^{\mathcal{D}}})dx}_{I_4} \\ & +\underbrace{\sum_{F\in\mathscr{F}_{\mathcal{S}}}\int_F\sigma_S[\![\mathbf{e}_I^{u^S}]\!]\cdot[\![\mathbf{e}_h^{u^{\mathcal{S}}}]\!]ds+\sum_{F\in\mathscr{F}_{\mathcal{D}}}\int_F\sigma_D[\![\mathbf{e}_I^{u^D}]\!]_{\mathbf{n}}[\![\mathbf{e}_h^{u^{\mathcal{D}}}]\!]_{\mathbf{n}}ds+\sum_{F\in\Gamma}\int_F\sigma_\Gamma[\![\mathbf{e}_I^u]\!]_{\mathbf{n}}[\![\mathbf{e}_h^u]\!]_{\mathbf{n}}ds,}_{I_5}. \end{aligned}$$

First, we consider I_1 and we apply the Lipschitz continuity of $g_S(|\cdot|)\cdot$, (2.3), Young's inequality and the approximation property of the L^2-interpolation, see (3), to obtain

$$I_1 \lesssim h^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}(\Omega_S)} + \|\nabla_h \mathbf{e}_h^{u^S}\|^2_{L^2(\Omega_S)}.$$

For the bound of I_2, we proceed as before and we use also the trace inverse inequality to recover the norm of the interpolant on Ω_S, i.e.

$$I_2 \lesssim h^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}(\Omega_S)} + \sum_{F\in\mathscr{F}_S} h_K^{-1}\|[\![\mathbf{e}_h^{u^S}]\!]\|_{L^2(F)}.$$

Now, we consider the interface terms and employ Young's inequality, the trace inequality, and the approximation properties of the L^2-interpolation to obtain

$$I_3 \lesssim \sum_{K:\partial K\cap\Gamma\neq\emptyset} h_K^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}(K)} + \sum_{F\in\Gamma}\rho^{-1}\|\mathbf{e}_h^{u^S}\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(F)} \qquad \forall j=1,...,d-1.$$

For the terms relative to Ω_D, as before, we apply the trace-inverse inequality, Young's inequality and the approximation properties of the interpolation. Additionally, we apply the Lipschitz continuity of $g_D(|\cdot|)\cdot$, see (2.6), to obtain

$$I_4 \lesssim h^{2(t_D+1)}|\mathbf{u}^D|^2_{H^{t_D+1}(\Omega_D)} + \|\mathbf{e}_h^{u^D}\|_{L^2(\Omega_D)}.$$

Finally, considering the last term contributing to $\mathbf{R_1}$, we have

$$\begin{aligned}I_5 &\lesssim h_K^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}} + h_K^{2t_D}|\mathbf{u}^D|^2_{H^{t_D+1}}(\Omega_D) + h_K^{2t}|\mathbf{u}|^2_{H^{t+1}(\Omega)}\\ &+ \sum_{F\in\mathscr{F}_S}\sigma_S\|[\![\mathbf{e}_h^{u^S}]\!]\|^2_{L^2(F)} + \sum_{F\in\mathscr{F}_D}\sigma_D\|[\![\mathbf{e}_h^{u^D}]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{F\in\Gamma}\sigma_\Gamma\|[\![\mathbf{e}_h^{u}]\!]_{\mathbf{n}}\|^2_{L^2(F)}.\end{aligned}$$

Now, we proceed to study the other terms of (5.6). For $\mathbf{R_2}, \mathbf{R_3}$ and $\mathbf{R_4}$ we apply the Cauchy-Schwarz and Young's inequalities for the volume integrals to obtain

$$\begin{aligned}\mathbf{R_2} &\lesssim h^{2m_S+2}|p^S|^2_{H^{m_S+1}(\Omega_S)} + h_K^{2m_D+2}|p^D|^2_{H^{m_D+1}(\Omega_D)} + \|\nabla_h\cdot\mathbf{e}_h^{u^S}\|^2_{L^2(\Omega_S)} + \|\nabla_h\cdot\mathbf{e}_h^{u^D}\|^2_{L^2(\Omega_D)}\\ &+ \sum_{F\in\mathscr{F}_S} h_K^{-1}\|[\![\mathbf{e}_h^{u}]\!]\|^2_{L^2(F)} + \sum_{F\in\mathscr{F}_D} h_K^{-1}\|[\![\mathbf{e}_h^{u^D}]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{F\in\Gamma} h_K^{-1}\|[\![\mathbf{e}_h^{u}]\!]_{\mathbf{n}}\|^2_{L^2(F)},\\ \mathbf{R_3} &\lesssim h^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}(\Omega_S)} + h_K^{2t_D}|\mathbf{u}^D|^2_{H^{t_D+1}(\Omega_D)} + 2(\|e_h^{p^S}\|^2_{L^2(\Omega_S)} + \|e_h^{p^D}\|^2_{L^2(\Omega_D)}),\\ \mathbf{R_4} &\lesssim h^{2(m_S+1)}|p^S|^2_{H^{m_S+1}(\Omega_S)} + h^{2(m_D+1)}|p^D|^2_{H^{m_D+1}(\Omega_D)} + \sum_{F\in\mathscr{F}_S^i}\xi_S\|[\![e_h^{p^S}]\!]\|^2_{L^2(F)}\\ &+ \sum_{F\in\mathscr{F}_D^i}\xi_D\|[\![e_h^{p^D}]\!]\|^2_{L^2(F)} + \sum_{F\in\Gamma}\xi_\Gamma\|[\![e_h^{p}]\!]\|^2_{L^2(F)}.\end{aligned}$$

Finally, putting together all the previous bounds we obtain:

$$\begin{aligned}|\!|\!|\mathbf{e}_h^u|\!|\!|^2_{div} + |\!|\!|e_h^p|\!|\!|^2_{M_h} \lesssim & h^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}(\Omega_S)} + h^{2t_D}|\mathbf{u}^D|^2_{H^{t_D+1}(\Omega_D)} + h^{2t}|\mathbf{u}|^2_{H^{t+1}(\Omega)}\\ &+ h^{2m_S+2}|p^S|_{H^{m_S+1}(\Omega_S)} + h^{2m_D+2}|p^D|_{H^{m_D+1}(\Omega_D)},\end{aligned}$$

which implies

$$\begin{aligned}|\!|\!|(\mathbf{e}_h^u, e_h^p)|\!|\!|^2_E \lesssim & h^{2t_S}|\mathbf{u}^S|^2_{H^{t_S+1}(\Omega_S)} + h^{2t_D}|\mathbf{u}^D|^2_{H^{t_D+1}(\Omega_D)} + h^{2t}|\mathbf{u}|^2_{H^{t+1}(\Omega)}\\ &+ h^{2m_S+2}|p^S|_{H^{m_S+1}(\Omega_S)} + h^{2m_D+2}|p^D|_{H^{m_D+1}(\Omega_D)},\end{aligned}$$

with $t = \min\{t_S, t_D\}$, and the proof is completed. □

6 Numerical results

In this section, we present numerical experiments to verify the error bounds established in the previous section and to assess the practical performance of the proposed discretization. The numerical implementations are carried out in the open-source Lymph MATLAB library [2], implementing the PolyDG method for multi-physics in $2D$ domains.

6.1 Test case 1: linear constitutive law (Newtonian fluids)

We consider a numerical convergence test for a Newtonian fluid, on a square domain $\Omega = (-1,1) \times (-1,1)$ with an interface $\Gamma = \{x = 0\}$. On the left part of the domain, $\Omega_S = (-1,0) \times (-1,1)$, we solve the Stokes equation, while on the right part of the domain, $\Omega_D = (0,1) \times (-1,1)$, we solve the Darcy equation. We consider the manufactured solution:

$$\mathbf{u}_{\mathcal{S}}^{ex}(\mathbf{x}) = \begin{bmatrix} x^3\pi\cos(\pi y)\sin(\pi y) \\ -\frac{3}{2}x^2\sin(\pi y)^2 \end{bmatrix}, \quad p_{\mathcal{S}}^{ex}(\mathbf{x}) = e^y\cos(\pi x),$$

$$\mathbf{u}_{\mathcal{D}}^{ex}(\mathbf{x}) = \begin{bmatrix} x^2\pi\cos(\pi y)\sin(\pi y) \\ -x\sin(\pi y)^2 \end{bmatrix}, \quad p_{\mathcal{D}}^{ex}(\mathbf{x}) = e^y\cos(\pi x) - 6\pi x^2\cos(\pi y)\sin(\pi y).$$

The problem is solved with polynomial degree $l = 2, 3, 4$ and on a sequence of successively finer polygonal meshes with diameter $h = 0.2485, 0.1811, 0.1301, 0.0902, 0.0640, 0.0457, 0.0325, 0.0228$. In Figure 2a we report the computed error estimates in the energy norm $\|\!|(\cdot,\cdot)|\!\|_E$ and visually highlight the square root of the numerical order of convergence. The results are in agreement with Theorem 4 as the error goes to zero with the predicted algebraic rate h^l, as h goes to zero.

6.2 Test case 2: Carreau constitutive law

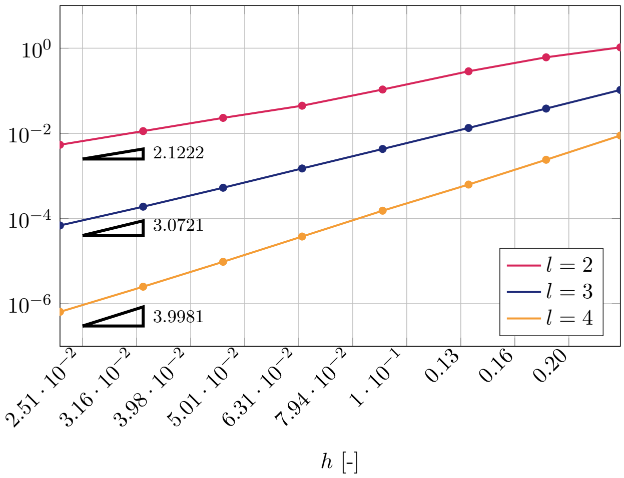
We next present a numerical convergence test for a non-Newtonian fluid, with viscosity modeled following the Carreau constitutive law. We consider a domain defined as in Section 6.1. We took a similar manufactured solution:

$$\mathbf{u}_{\mathcal{S}}^{ex}(\mathbf{x}) = \begin{bmatrix} x^3\pi\cos(\pi y)\sin(\pi y) \\ -\frac{3}{2}x^2\sin(\pi y)^2 \end{bmatrix}, \quad p_{\mathcal{S}}^{ex}(\mathbf{x}) = e^y\cos(\pi x),$$
$$\mathbf{u}_{\mathcal{D}}^{ex}(\mathbf{x}) = \begin{bmatrix} x^2\pi\cos(\pi y)\sin(\pi y) \\ -x\sin(\pi y)^2 \end{bmatrix}, \quad p_{\mathcal{D}}^{ex}(\mathbf{x}) = e^y\cos(\pi x) - (g_{\mathcal{S}}(|\mathbf{D}(\mathbf{u}_h^{\mathcal{S}})|)\mathbf{D}(\mathbf{u}_h^{\mathcal{S}})\mathbf{n})\mathbf{n},$$

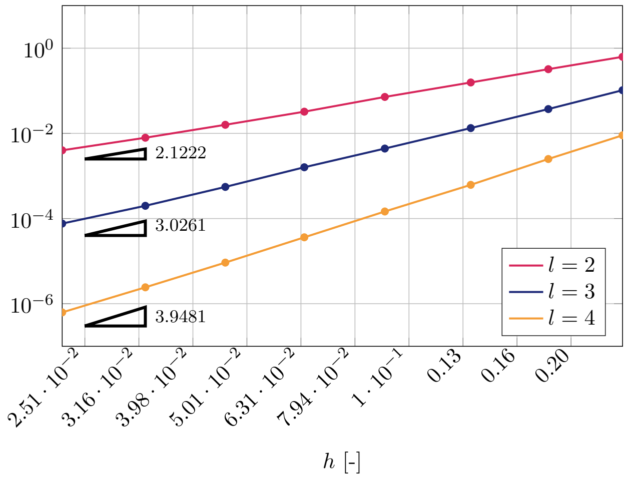
assuming the following constitutive equations

$$g_{\mathcal{S}}(|\mathbf{D}(\mathbf{u}_h^{\mathcal{S}})|) = \underline{\nu_{\mathcal{S}}} + (\overline{\nu_{\mathcal{S}}} - \underline{\nu_{\mathcal{S}}})\left(1 + \frac{1}{2}|\mathbf{D}(\mathbf{u}_h^{\mathcal{S}})|^2\right)^{-\frac{1}{4}},$$
$$g_{\mathcal{D}}(|\mathbf{u}_h^{\mathcal{S}}|) = \underline{\nu_{\mathcal{D}}} + (\overline{\nu_{\mathcal{D}}} - \underline{\nu_{\mathcal{D}}})\left(1 + \frac{1}{2}|\mathbf{u}_h^{\mathcal{D}}|^2\right)^{-\frac{1}{4}},$$

with $\underline{\nu_{\mathcal{S}}} = \underline{\nu_{\mathcal{D}}} = 0.001$ and $\overline{\nu_{\mathcal{S}}} = \overline{\nu_{\mathcal{D}}} = 0.5$ [37]. The forcing terms and the boundary conditions are then set accordingly. The problem is solved with $l = 2, 3, 4$ on the same sequence of meshes of Section 6.1. For the solution of the non-Newtonian system, we adopt a fixed-point iteration scheme with tolerance 10^{-10}. In Figure 2b we report the computed error estimates of the energy norm $\|\!|(\cdot,\cdot)|\!\|_E$ and visually highlight the root mean square of the numerical order of convergence. Again, the results are in agreement with the theory defined in Section 5 as the error goes to zero with the predicted algebraic rate h^l, as h goes to zero.



(a) Test case 1: Computed errors $\|(\mathbf{e}^u, e^p)\|_E$ and the root mean square of their numerical order of convergence.



(b) Test case 2: Computed errors $\|(\mathbf{e}^u, e^p)\|_E$ and the root mean square of their numerical order of convergence.

7 Conclusions

We have developed and analyzed a polytopal discontinuous Galerkin method for the numerical approximation of a coupled non-Newtonian Stokes-Darcy system describing the interaction between a non-Newtonian free-flow fluid and a non-Newtonian porous-medium flow. A complete well-posedness analysis has been established for the continuous problem. Within the framework of generalized inf-sup theory, we also prove the well-posedness, stability, and convergence of the proposed numerical method, and derived corresponding error estimates. The theoretical estimates were validated through two numerical convergence tests using manufactured solutions: the first assuming a linear (Newtonian) viscosity, and the second assuming a nonlinear (non-Newtonian) viscosity following the Carreau model, which is commonly used for shear-thinning non-Newtonian fluids. A natural extension of this work would be to incorporate more general nonlinear viscosity laws, such as a Forchheimer term in the porous medium model, as considered in [37], used to predict high velocity flow in porous media [33], or a Carreau–Yasuda model [48], employed in hemodynamic applications [15]. Further directions include the analysis of the non-Newtonian coupled problem under more general mesh assumptions, allowing an *hp* analysis that also accounts for the dependence on the polynomial degree.

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A Appendix

In this section, we provide a complete proof of the continuity of $b_h(\cdot,\cdot)$, i.e.,

$$|b_h(\mathbf{v}_h, p_h)| \lesssim \|\!|\mathbf{v}_h|\!\|_{div} \|\!|p_h|\!\|_{M_h} \qquad \forall \mathbf{v}_h \in \mathbf{X}_h, \forall p_h \in M_h.$$

Proof. From the definitions of the norms given in Definition (1) we immediately have

$$\begin{aligned}
|b_h^\Gamma(\mathbf{v}_h, p_h)| &= \sum_{F\in\Gamma}\int_F |p_h^{\mathcal{D}}[\![\mathbf{v}_h]\!]_{\mathbf{n}}|ds \le \sum_{F\in\Gamma}\|p_h^{\mathcal{D}}\|_{L^2(F)}\|[\![\mathbf{v}_h]\!]_{\mathbf{n}}\|_{L^2(F)} \\
&= \sum_{F\in\Gamma}\|p_h^{\mathcal{D}}\|_{L^2(F)}\|[\![\mathbf{v}_h]\!]_{\mathbf{n}}\|_{L^2(F)}\left(\frac{h_K}{h_K}\right)^{\frac{1}{2}} \le \sum_{K\in\mathscr{K}:\partial K\cap\Gamma\neq\emptyset} C_\gamma \|\mathbf{v}_h\|_{\mathbf{X}_h}\|p_h^{\mathcal{D}}\|_{L^2(K)} \\
&\lesssim \|\mathbf{v}_h\|_{\mathbf{X}_h}\|p_h^{\mathcal{D}}\|_{L^2(\Omega)}, \\
|b_h^S(\mathbf{v}_h^{\mathcal{S}}, p_h^{\mathcal{S}})| &= \sum_{K\in\mathscr{K}_{\mathcal{S}}}\int_K |p_h^{\mathcal{S}}\nabla\cdot\mathbf{v}_h^{\mathcal{S}}|ds + \sum_{F\in\mathscr{F}_{\mathcal{S}}}\int_F |\{\!\{p_h^{\mathcal{S}}\}\!\}[\![\mathbf{v}_h^{\mathcal{S}}]\!]_{\mathbf{n}}|ds \\
&\le \|p_h^{\mathcal{S}}\|_{L^2(\Omega_S)}\|\nabla\cdot\mathbf{v}_h^{\mathcal{S}}\|_{L^2(\Omega_S)} + \sum_{F\in\mathscr{F}_{\mathcal{S}}}\|\{\!\{p_h^{\mathcal{S}}\}\!\}\|_{L^2(F)}\|[\![\mathbf{v}_h^{\mathcal{S}}]\!]_{\mathbf{n}}\|_{L^2(F)}\left(\frac{h_K}{h_K}\right)^{\frac{1}{2}} \\
&\lesssim \|p_h^{\mathcal{S}}\|_{M_h^S}\|\mathbf{v}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S} + \|p_h^{\mathcal{S}}\|_{M_h^S}\|\mathbf{v}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S} \lesssim \|p_h^{\mathcal{S}}\|_{M_h^S}\|\mathbf{v}_h^{\mathcal{S}}\|_{\mathbf{X}_h^S}, \\
|b_h^D(\mathbf{v}_h^{\mathcal{D}}, p_h^{\mathcal{D}})| &= \sum_{K\in\mathscr{K}_{\mathcal{D}}}\int_K |p_h^{\mathcal{D}}\nabla\cdot\mathbf{v}_h^{\mathcal{D}}|ds + \sum_{F\in\mathscr{F}_{\mathcal{D}}}\int_F |\{\!\{p_h^{\mathcal{D}}\}\!\}[\![\mathbf{v}_h^{\mathcal{D}}]\!]_{\mathbf{n}}|ds \\
&\le \|p_h^{\mathcal{D}}\|_{L^2(\Omega_D)}\|\nabla\cdot\mathbf{v}_h^{\mathcal{D}}\|_{L^2(\Omega_D)} + \sum_{F\in\mathscr{F}_{\mathcal{D}}}\|\{\!\{p_h^{\mathcal{D}}\}\!\}\|_{L^2(F)}\|[\![\mathbf{v}_h^{\mathcal{D}}]\!]_{\mathbf{n}}\|_{L^2(F)}\left(\frac{h_K}{h_K}\right)^{\frac{1}{2}} \\
&\lesssim \|p_h^{\mathcal{D}}\|_{M_h^D}(\|\mathbf{v}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} + \|\nabla_h\mathbf{v}_h^{\mathcal{D}}\|_{L^2(\Omega_D)}) + \|p_h^{\mathcal{D}}\|_{M_h^D}\|\mathbf{v}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} \\
&\lesssim \|p_h^{\mathcal{D}}\|_{M_h^D}(\|\mathbf{v}_h^{\mathcal{D}}\|_{\mathbf{X}_h^D} + \|\nabla_h\mathbf{v}_h^{\mathcal{D}}\|_{L^2(\Omega_D)}).
\end{aligned}$$

Collecting the above bounds, we obtain

$$|b_h(\mathbf{v}_h, p_h)| \lesssim \|\mathbf{v}_h\|_{div}\|p_h\|_{M_h}.$$

□

B Appendix

In this section, we prove the continuity and the monotonicity of $a_h(\cdot,\cdot)$, i.e.,

$$|a_h(\mathbf{u}_h, \mathbf{w}_h) - a_h(\mathbf{v}_h, \mathbf{w}_h)| \lesssim \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{X}_h}\|\mathbf{w}_h\|_{\mathbf{X}_h} \qquad \forall \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{X}_h,$$

and

$$(a_h(\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h) - a_h(\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h)) \gtrsim \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{X}_h}^2 \qquad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h.$$

First, we prove the continuity.

Proof. We use the Cauchy-Schwarz inequality, the discrete trace inequality (3.1), the inverse inequalities, eq. (2.3) and eq. (2.6):

$$\left|\sum_{K\in\mathscr{K}_\mathcal{S}}\int_K (g_S(|\mathbf{D}(\mathbf{u}_h^\mathcal{S})|)\mathbf{D}(\mathbf{u}_h^\mathcal{S}) - g_S(|\mathbf{D}(\mathbf{v}_h^\mathcal{S})|)\mathbf{D}(\mathbf{v}_h^\mathcal{S})) : \mathbf{D}(\mathbf{w}_h^\mathcal{S})\, dx\right| \begin{aligned}[t] &\leq \overline{\nu}_S \|\mathbf{D}(\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S})\|_{L^2(\Omega_S)}\|\mathbf{D}(\mathbf{w}_h^\mathcal{S})\|_{L^2(\Omega_S)} \\ &\lesssim \|\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S}\|_{\mathbf{X}_h^S}\|\mathbf{w}_h^\mathcal{S}\|_{\mathbf{X}_h^S}, \end{aligned}$$

$$\begin{aligned} \left|\sum_{F\in\mathscr{F}_\mathcal{S}}\int_F \{\!\{(g_S(|\mathbf{D}(\mathbf{u}_h^\mathcal{S})|)\mathbf{D}(\mathbf{u}_h^\mathcal{S})\mathbf{n}_S - g_S(|\mathbf{D}(\mathbf{v}_h^\mathcal{S})|)\mathbf{D}(\mathbf{v}_h^\mathcal{S})\mathbf{n})\}\!\} \cdot [\![\mathbf{w}_h^\mathcal{S}]\!]\, ds\right| &\lesssim \\ &\lesssim \sum_{K\in\mathscr{K}_\mathcal{S}} \|\nabla(\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S})\|_{L^2(K)}\|\mathbf{w}_h^\mathcal{S}\|_{L^2(K)} \\ &\lesssim \|\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S}\|_{\mathbf{X}_h^S}\|\mathbf{w}_h^\mathcal{S}\|_{\mathbf{X}_h^S}, \end{aligned}$$

$$\begin{aligned} \left|\sum_{F\in\mathscr{F}_\mathcal{S}}\int_F \frac{\sigma_S}{h_K}[\![\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S}]\!]\cdot[\![\mathbf{w}_h^\mathcal{S}]\!]\, ds\right| &\leq \left(\sum_{F\in\mathscr{F}_\mathcal{S}} \frac{\sigma_S}{h_K}\|[\![\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S}]\!]\|_{L^2(F)}^2\right)^{\frac{1}{2}} \left(\sum_{F\in\mathscr{F}_\mathcal{S}} \frac{\sigma_S}{h_K}\|[\![\mathbf{w}_h^\mathcal{S}]\!]\|_{L^2(F)}^2\right)^{\frac{1}{2}} \\ &\leq \|\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S}\|_{\mathbf{X}_h^S}\|\mathbf{w}_h^\mathcal{S}\|_{\mathbf{X}_h^S}, \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{d-1}\left|\sum_{F\in\Gamma}\int_F \rho^{-1}((\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S})\cdot\mathbf{t}_{\Gamma,j})(\mathbf{w}_h^\mathcal{S}\cdot\mathbf{t}_{\Gamma,j})\, ds\right| &\leq \\ &\leq \sum_{j=1}^{d-1}\left(\sum_{F\in\Gamma}\rho^{-1}\|(\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S})\cdot\mathbf{t}_{\Gamma,j}\|_{L^2(F)}^2\right)^{\frac{1}{2}} \left(\sum_{F\in\Gamma}\rho^{-1}\|\mathbf{w}_h^\mathcal{S}\cdot\mathbf{t}_{\Gamma,j}\|_{L^2(F)}^2\right)^{\frac{1}{2}}, \end{aligned}$$

$$\left|\sum_{K\in\mathscr{K}_\mathcal{D}}\int_K (\mathbf{K}^{-1}g_D(|\mathbf{u}_h^\mathcal{D}|)\mathbf{u}_h^\mathcal{D} - \mathbf{K}^{-1}g_D(|\mathbf{v}_h^\mathcal{D}|)\mathbf{v}_h^\mathcal{D})\cdot\mathbf{w}_h^\mathcal{D}\, dx\right| \begin{aligned}[t] &\leq \frac{\overline{\nu}_D}{k_{\max}}\|\mathbf{u}_h^\mathcal{D} - \mathbf{v}_h^\mathcal{D}\|\|\mathbf{w}_h^\mathcal{D}\| \\ &\lesssim \|\mathbf{u}_h^\mathcal{D} - \mathbf{v}_h^\mathcal{D}\|_{\mathbf{X}_h^D}\|\mathbf{w}_h^\mathcal{D}\|_{\mathbf{X}_h^D}, \end{aligned}$$

$$\left|\sum_{F\in\mathscr{F}_\mathcal{D}^i}\int_F \frac{\sigma_D}{h_K}[\![\mathbf{u}_h^\mathcal{D} - \mathbf{v}_h^\mathcal{D}]\!]_\mathbf{n}[\![\mathbf{w}_h^\mathcal{D}]\!]_\mathbf{n}\, ds\right| \leq \|\mathbf{u}_h^\mathcal{D} - \mathbf{v}_h^\mathcal{D}\|_{\mathbf{X}_h^D}\|\mathbf{w}_h^\mathcal{D}\|_{\mathbf{X}_h^D},$$

$$\left|\sum_{F\in\Gamma}\int_F \frac{\sigma_\Gamma}{h_K}[\![\mathbf{u}_h - \mathbf{v}_h]\!]_\mathbf{n}[\![\mathbf{w}_h]\!]_\mathbf{n}\, ds\right| \leq \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{X}_h}\|\mathbf{w}_h\|_{\mathbf{X}_h},$$

where the hidden constants depend on $\overline{\nu_S}, \overline{\nu_D}, k_{\max}$, and the inverse inequality constant. Putting together all this estimates we obtain (4.8). □

Now, we prove the monotonicity.

Proof. Let $\mathbf{u}_h, \mathbf{v}_h \in X_h$, by the discrete Korn's inequality, we can write:

$$\begin{aligned} &\sum_{K\in\mathscr{K}_\mathcal{S}}\int_K (g_S(|\mathbf{D}(\mathbf{u}_h^\mathcal{S})|)\mathbf{D}(\mathbf{u}_h^\mathcal{S}) - g_S(|\mathbf{D}(\mathbf{v}_h^\mathcal{S})|)\mathbf{D}(\mathbf{v}_h^\mathcal{S})) : \mathbf{D}(\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S})dx \\ &\qquad \gtrsim \|\mathbf{D}(\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S})\|_{L^2(\Omega_S)}^2 \gtrsim \|\nabla_h(\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S})\|_{L^2(\Omega_S)}^2 - \sum_{F\in\mathscr{F}_\mathcal{S}}\frac{1}{h_K}\|[\![\mathbf{u}_h^\mathcal{S} - \mathbf{v}_h^\mathcal{S}]\!]\|_{L^2(F)}^2. \end{aligned}$$

Now we apply the Lipschitz continuity of $g_S(\cdot)$ and Young's inequality, (2.3), to obtain:

$$
\begin{aligned}
-\sum_{F\in\mathscr{F}_\mathcal{S}}\int_F &\{\!\{(g_S(|\mathbf{D}(\mathbf{u}_h^\mathcal{S})|)\mathbf{D}(\mathbf{u}_h^\mathcal{S}) - g_S(|\mathbf{D}(\mathbf{v}_h^\mathcal{S})|)\mathbf{D}(\mathbf{v}_h^\mathcal{S}))\mathbf{n}\}\!\}[\![\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S}]\!]ds \\
&\gtrsim -\sum_{F\in\mathscr{F}_\mathcal{S}}\|\{\!\{\mathbf{D}(\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S})\}\!\}\|_{L^2(F)}\|[\![\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S}]\!]\|_{L^2(F)} \\
&\gtrsim -\frac{1}{4\eta}\|\nabla_h(\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S})\|^2_{L^2(\Omega_S)} - \sum_{F\in\mathscr{F}_\mathcal{S}}\frac{\eta}{h_K}\|[\![\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S}]\!]\|^2_{L^2(F)}.
\end{aligned}
$$

Additionally, using the monotonicity of $g_D(\cdot)$, (cf. (2.7)) we can obtain:

$$
\sum_{K\in\mathscr{K}_\mathcal{D}}\int_K (g_D(|\mathbf{u}_h^\mathcal{D}|)\mathbf{u}_h^\mathcal{D} - g_D(|\mathbf{v}_h^\mathcal{D}|)\mathbf{v}_h^\mathcal{D})\cdot(\mathbf{u}_h^\mathcal{D}-\mathbf{v}_h^\mathcal{D})dx \gtrsim \|\mathbf{u}_h^\mathcal{D}-\mathbf{v}_h^\mathcal{D}\|^2_{L^2(\Omega_D)}.
$$

We can conclude that, for large enough σ_S, it holds

$$
\begin{aligned}
(a_h(\mathbf{u}_h,\mathbf{u}_h-\mathbf{v}_h) - a_h(\mathbf{v}_h,\mathbf{u}_h-\mathbf{v}_h)) &\gtrsim \left(1-\frac{1}{4\eta}\right)\|\nabla_h(\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S})\|^2_{L^2(\Omega_S)} + \sum_{F\in\mathscr{F}_\mathcal{S}}\left(\sigma_S - \frac{1-\eta}{h_K}\right)\|[\![\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S}]\!]\|^2_{L^2(F)} \\
&+\sum_{j=1}^{d-1}\sum_{F\in\Gamma}\rho^{-1}\|(\mathbf{u}_h^\mathcal{S}-\mathbf{v}_h^\mathcal{S})\cdot\mathbf{t}_{\Gamma,j}\|^2_{L^2(F)} + \|\mathbf{u}_h^\mathcal{D}-\mathbf{v}_h^\mathcal{D}\|^2_{L^2(\Omega_D)} \\
&+\sum_{F\in\mathscr{F}_\mathcal{D}^i}\sigma_D\|[\![\mathbf{u}_h^\mathcal{D}-\mathbf{v}_h^\mathcal{D}]\!]\|^2_{L^2(F)} + \sum_{F\in\Gamma}\sigma_\Gamma\|[\![(\mathbf{u}_h-\mathbf{v}_h)]\!]_\mathbf{n}\|^2_{L^2(F)} \\
&\gtrsim \|\mathbf{u}_h-\mathbf{v}_h\|^2_{\mathbf{X}_h},
\end{aligned}
$$

and the proof is complete. □

C Appendix

In this section, we prove a bound for the L^2 norm of the divergence of $\mathbf{u}_h^\mathcal{D}$. We start by considering the second equation of (3.3) and test it against $\tilde{q}_h = (0, \nabla\cdot\mathbf{u}_h^\mathcal{D})$, to obtain

$$
\begin{aligned}
-b_h(\mathbf{u}_h,\tilde{q}_h)+s_h(p_h,\tilde{q}_h) &= \sum_{K\in\mathscr{K}_\mathcal{D}}\int_K \nabla\cdot\mathbf{u}_h^\mathcal{D}\nabla\cdot\mathbf{u}_h^\mathcal{D}dx - \sum_{F\in\mathscr{F}_\mathcal{D}}\int_F\{\!\{\nabla\cdot\mathbf{u}_h^\mathcal{D}\}\!\}[\![\mathbf{u}_h^\mathcal{D}]\!]_\mathbf{n}ds \\
&- \sum_{F\in\Gamma_h}\int_K \nabla\cdot\mathbf{u}_h^\mathcal{D}[\![\mathbf{u}_h]\!]_\mathbf{n}ds + \sum_{F\in\mathscr{F}_\mathcal{D}^i}\int_F \xi_D[\![p_h^\mathcal{D}]\!][\![\nabla\cdot\mathbf{u}_h^\mathcal{D}]\!]ds + \sum_{F\in\Gamma}\int_F\xi_\Gamma[\![p_h]\!]\nabla\cdot\mathbf{u}_h^\mathcal{D}ds = 0.
\end{aligned}
$$

From this we derive:

$$
\begin{aligned}
\|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(\Omega_D)} &= \sum_{F\in\mathscr{F}_{\mathcal{D}}}\int_F \{\!\{\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\}\!\}[\![\mathbf{u}_h^{\mathcal{D}}]\!]_{\mathbf{n}} ds + \sum_{F\in\Gamma_h}\int_F \nabla\cdot\mathbf{u}_h^{\mathcal{D}}[\![\mathbf{u}_h]\!]_{\mathbf{n}} ds - \sum_{F\in\mathscr{F}_{\mathcal{D}}}\int_F \xi_D[\![p_h^{\mathcal{D}}]\!][\![\nabla\cdot\mathbf{u}_h^{\mathcal{D}}]\!] ds \\
&\quad - \sum_{F\in\Gamma}\int_F \xi_\Gamma[\![p_h]\!][\![\nabla\cdot\mathbf{u}_h^{\mathcal{D}}]\!] ds \\
&\lesssim \sum_{F\in\mathscr{F}_{\mathcal{D}}} \|\{\!\{\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\}\!\}\|_{L^2(F)}\|[\![\mathbf{u}_h^{\mathcal{D}}]\!]_{\mathbf{n}}\|_{L^2(F)} + \sum_{K\in\Gamma_h} \|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|_{L^2(F)}\|[\![\mathbf{u}_h]\!]_{\mathbf{n}}\|_{L^2(F)} \\
&\quad + \sum_{F\in\mathscr{F}^i_{\mathcal{D}}} \xi_D\|[\![p_h^{\mathcal{D}}]\!]\|_{L^2(F)}\|[\![\nabla\cdot\mathbf{u}_h^{\mathcal{D}}]\!]\|_{L^2(F)} + \sum_{F\in\Gamma}\xi_\Gamma\|[\![p_h]\!]\|_{L^2(F)}\|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|_{L^2(F)} \\
&\lesssim \sum_{K\in\mathscr{K}_{\mathcal{D}}}\frac{1}{\eta}\|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(K)} + \sum_{F\in\mathscr{F}_{\mathcal{D}}}\sigma_D\|[\![\mathbf{u}_h^{\mathcal{D}}]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{K:\partial K\cap\Gamma\neq\emptyset}\frac{1}{\eta}\|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(K)} \\
&\quad + \sum_{F\in\Gamma_h}\sigma_\Gamma\|[\![\mathbf{u}_h]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{F\in\mathscr{F}^i_{\mathcal{D}}}\eta\xi_D\|[\![p_h^{\mathcal{D}}]\!]\|^2_{L^2(F)} + \sum_{K\in\mathscr{K}_{\mathcal{D}}}\frac{1}{2}\|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(K)} \\
&\quad + \sum_{F\in\Gamma}\eta\xi_\Gamma\|[\![p_h]\!]\|^2_{L^2(F)} + \sum_{K:\partial K\cap\Gamma\neq\emptyset}\frac{1}{2}\|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(K)}.
\end{aligned}
$$

Finally, we obtain the following estimate.

$$
\begin{aligned}
\|\nabla\cdot\mathbf{u}_h^{\mathcal{D}}\|^2_{L^2(\Omega_D)} \lesssim& \sum_{F\in\mathscr{F}_{\mathcal{D}}}\sigma_D\|[\![\mathbf{u}_h^{\mathcal{D}}]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{F\in\Gamma}\sigma_\Gamma\|[\![\mathbf{u}_h]\!]_{\mathbf{n}}\|^2_{L^2(F)} + \sum_{F\in\mathscr{F}^i_{\mathcal{D}}}\xi_D\|[\![p_h^{\mathcal{D}}]\!]\|^2_{L^2(F)} \\
&+ \sum_{F\in\Gamma}\xi_\Gamma\|[\![p_h^{\mathcal{D}}]\!]\|^2_{L^2(F)}.
\end{aligned}
$$

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