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Abstract

We propose a mathematical model describing the interactions among students during work group activities aimed at solving a math problem. The model, which hinges upon the pioneering work of Hegselmann and Krause, is able to incorporate: 1) the feelings of each student towards the classmates (building upon the "I can"-"You can" framework); 2) the influence of the correct solution to model the students' preparation; 3) the presence of the teacher, who may or may not intervene to drive students towards the correct solution of the problem. Several numerical experiments are presented to assess the capability of the model in reproducing typical realistic scenarios.

1 Introduction

Interactionist research in Mathematics Education sees learning as becoming participant in a mathematical activity. Hence, the mathematical activity is sensitive to the context and allows the growth of mutual understanding and coordination between the individual and the rest of the community. Accordingly, each activity has its roots in our cultural heritage and can be shaped and re-shaped by the group of practitioners. It is within this framework that thinking is conceptualized as a case of communication, since interactionist research postulates the inherently social origin of all human activities [23]. Sfard points out that "communication may be defined as a person's attempt to make an interlocutor act, think or feel according to her intentions" [23, p. 13]. Following this view, thinking is thus subordinated to and informed by the demand of making communication effective.

Group work activities in classroom have gained more and more attention in the recent years, since during such activities the students act, interact and communicate much more than in usual frontal lessons settings [23, 22]. It is well acknowledged that, when students interact in a group, the interaction has not a purely cognitive nature, but it is shaped by affective factors such as emotions, values, wills, attitudes and so on (see, e.g., [9] and [7]).

In recent years, many researches have provided evidence that student's attitudes towards mathematics, the classmates and oneself are crucial for the learning process, in particular the feelings "I sense", "I like" and "I can" [7, 2]. On these premises, the "I can"-"You can" framework has been proposed [1] and it is on this paradigm that we will build our student opinion dynamics model in this paper.

The study of human behaviours and interactions is a field of investigation that has gained the attention of researcher in mathematics, physics and computer science (see, e.g., [4] and the references therein). One of the most recent mathematical area which deal with this topic is the so called opinion dynamics. With this term researchers describe a wide class of models for several different social phenomena, such as collective decision making, emergence of ideas, influence of social network on people's opinion and behaviour [16]. Opinion dynamics dates back to French, who defines it as "a way by which many complex phenomena about groups can be deduced from a few simple postulates about interpersonal relations" [8, p. 1]. Since then, many other models have been proposed, but the milestone is the bounded confidence model introduced by Hegselmann and Krause in 2002 [10], where the evolution of the state of a agent (i.e., the opinion) depends on interactions among the agents taking place in a bounded domain of confidence.

More recently, the huge variety of applications of such a class of models, which range from engineering to the life and social sciences, has prompted the researchers to new theoretical and experimental achievements (see, e.g, [5, 6, 14] and [17], respectively). Furthermore, very recently a flourishing research activity has focused on the problem of controlling or influencing the opinion dynamics so to reach, for instance, the consenus among agents [13, 25, 15, 11].

However, despite the massive research activity of the last years, to the best our knowledge, there are no mathematical models addressing small group-dynamics in educational context, such as students who are asked to solve a mathematical task working in a group. The present paper aims at filling this gap.

In the construction of our model, in addition to the interaction among students, we take into account also the role of the teacher. This in agreement with Radford (2013) who assigns a central role to the teacher, since she is the only one who knows where the activity should go [18]. The teacher can intervene or not during a small group activity, and her intervention (if any) can have either a mathematical/cognitive or an affective/social nature (see also [24]). For example, a teacher may notice that a student is remaining silent for a long time, hence she can take the pen from the student who is leading the activity and give it to the silent student. This intervention has a social nature. Or, the teacher can comment on a part of a solution the students are discussing. As Huberman (1993) points out, in fact, a teacher in her classroom is exposed to a continuous readjustment process, and the success of her teaching depends on how quickly and how accurately she is able to read the situation [12]. In other words, a teacher needs to make many in-the-moment decisions (see also [21]), on the basis of her ability to read a situation in a specific moment. In our extensive and intensive observation of group interactions [3], in fact, we have observed many secondary school teachers intervene during their students' groupwork activities: what has struck us is the fact that sometimes the teacher's intervention had been held by the students, while other times it had been ignored.

Our interpretation, that will be encoded in the mathematical model proposed in this work, is that the acceptance or the refusal of a teacher's comment (as well as the acceptance/refusal of a mate's intervention in the group activity) depends both on the perceived competence that each student of the group has about herself in that particular moment, and the competence that she recognises to her mates/the teacher in that particular moment.

The outline of this paper is as follows. In Section 2 we introduce the "I can"-"You can" framework in order to single out the variables that are used to describe the students' feelings; then in Section 3 we present the mathematical model describing the student opinion dynamics. In Section 4 we explore the capability of the model in reproducing realistic scenarios, while in Section 5 we draw some conclusions and we discuss possible future extensions of the present work.

2 Codifying the students' feelings

During any group work activities, students' feelings play a crucial role. This is also true in mathematical activities (i.e problem solving). In particular, "I can" and "You can" feelings drive and shape both social interactions and mathematical understanding. For example, "I can't" may induce a student to remain silent for a long time, but one of his classmates' invitation (driven by a "You can") may prompt her to intervene. This has a social nature. Or, the same "I can't" may induce the student to comment on a part of the mathematical solution she is not grasping, seeking for clarification. This has a mathematical nature. The "I can" dimension is the perceived competence of oneself, it identifies how (if so) a student feels capable with a particular mathematical task, while the "You can" dimension represents the perceived competence towards his classmate, hence the "You can" identifies how much a student is willing to trust and to listen to each one of his classmates.

The simplest "I can"-"You can" framework considers the affective dimensions as *dichotomous* variables: a student can be characterized by two variables, which assume only two values, $+1$ and -1, and the situation can be represented on a coordinate system called the *diagram "I can"-"You can"* as shown in Figure 1. Given this dichotomous nature of the affective dimensions, four different profiles, that will be referred to as "traits", can be identified:

Trait I *Cooperative* student is characterized by positive "I can" and positive "You can"; this student is willing to share knowledge and mediate his opinion listening to his classmate.

Trait II *Obstinate* student has positive "I can" but negative "You can", namely he has "You can't"; this student is not willing to listen to others whilst he wants his classmates to listen to him.

Trait III *Isolated* student has "I can't" and "You can't", he does not take part to the group activity and discussion.

Trait IV *Follower* student is willing to listen to others but he does not trust himself, since positive "You can" and negative low level of "I can" characterize this profile.

As a starting scenario, we considered a dichotomous "I can"-"You can" framework. However, it is more realistic to consider its continuous extension in order to identify many nuances of students' prototypes. More precisely, we assume that the dimensions "I can" and "You can" continuously range within the interval $[0, 1]$, where the value 0 corresponds to the lack of the corresponding dimension (i.e., extreme "I can't" and extreme "You can't").

For future reference, we introduce a matrix K collecting the traits of the students. More specifically, let us consider N students and, for $i, j = 1, \ldots, N$, let k_{ij} denote the level of

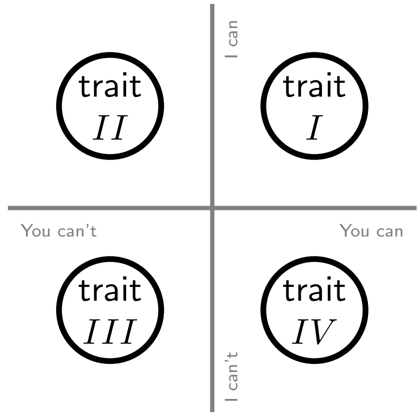


Figure 1: Schema of student traits on diagram "I can"- "You can".

competence of student j perceived by student i, that is the "You can" of the student i towards the student j, while k_{ii} indicates the level of "I can" of student i. Then the matrix

$$K \in \mathbb{R}^{N \times N}, \qquad K = [k_{ij}]_{i,j=1}^{N} \tag{1}$$

will be referred to as the *trait matrix*. For the sake of clarity, we present now a simple example.

Example 2.1. *Let us consider $N = 3$ students. The following matrix*

$$K = \begin{bmatrix} 0.9 & 0.2 & 0.15 \\ 0.18 & 0.97 & 0.89 \\ 0.95 & 0.08 & 0.21 \end{bmatrix}$$

encodes the following traits of the students:

- *student 1 has high "I can" ($k_{1,1} = 0.9$) and low "You can" towards both his classmates ($k_{1,2} = 0.2, k_{1,3} = 0.15$); hence he can be classified as obstinate.*
- *student 2 has mixed traits: he is quite cooperative with student 3 ($k_{2,3} = 0.89$) whilst he is almost obstinate with student 1 ($k_{2,1} = 0.18$). Moreover, since his level of "I can" is high ($k_{2,2} = 0.97$) he wants his classmates to listen to him*
- *student 3 has low "I can" ($k_{3,3} = 0.21$) and he has mixed traits like student 2, but he is willing to listen to student 1 following him in silence, since his "You can" towards him is high ($k_{3,1} = 0.95$). In addition, he is isolated with respect to student 2.*

We can further notice that the 3×3 matrix can reduce to the dichotomous case in a quite straightforward way, rounding each element to the closest integer:

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

3 The student opinion dynamics model

In this section, building upon the "I can"-"You can" framework discussed in Section 2, we describe a multi-agent model, based on the concept of bounded confidence introduced in [10], which describes the opinion dynamics of a group of students working together during a classroom activity. The general form of the model is rather standard and reads as follows. Let N denote the number of involved students whose opinion is represented by $x_i \in [0,1]$, $i = 1, \ldots, N$, then we assume that for each student i his opinion evolves according to the following:

$$\dot{x}_i(t) = \sum_{j=1}^{N} w_{ij}(t)(x_j(t) - x_i(t)), \qquad x_i(0) = x_i^0, \tag{2}$$

where the initial opinion $x_i^0 \in [0,1]$ and $w_{ij}(t)$ are suitable (time dependent) weights measuring how much the student i takes into account the opinion of the student j. The corresponding matrix

$$W(t) = [w_{ij}(t)]_{i,j=1}^{N}$$

is named *confidence matrix*.

The construction of the confidence matrix is crucial as it encodes the behaviour of the student i towards the student j in terms of listening, sharing, talking, and mediating. It is clear that such behaviour might depend on the student attitudes, therefore the trait matrix K introduced in the previous section will enter into the construction of the matrix W. This will be addressed in the next section

Remark 3.1 (On the meaning of the opinion x_i)**.** *In this work the opinion x_i is the solution that student i has in mind for the proposed task. For the sake of simplicity, it is represented by a scalar value in the range $[0,1] \subset \mathbb{R}$. More involved characterizations of the student opinion (for instance, a vector value $\mathbf{x}_i \in \mathbb{R}^d$ describing the different steps of a solution strategy) could be also considered to account for more complex situations. However, this goes well beyond the scope of the work.*

3.1 Construction of the confidence matrix

In this section we will build $W(t)$ as a time dependent matrix with elements depending on the opinions $x_i(t)$, $i = 1, \ldots, N$. The construction will be done in several steps.

First, we introduce the rough weight $w_{ij}^*(t)$ defined as the product of the *attitude score* a_{ij} and the *perceived distance score* $\varphi_{ij}(t)$:

$$w_{ij}^*(t) = a_{ij} \cdot \varphi_{ij}(t). \tag{3}$$

The element $w_{ij}(t)$ is then built as normalization of the rough weights $w_{ij}^*(t)$, i.e.

$$w_{ij}(t) = \frac{w_{ij}^*(t)}{\sum_{j=1}^{N} w_{ij}^*(t)}, \ \forall i = 1, \ldots, N. \tag{4}$$

The score a_{ij} encodes the attitude of the student i towards the student j and it is defined as

$$a_{ij} = k_{ij} + 0.1, \tag{5}$$

where $k_{i,j}$ is an element of the trait matrix K. Hence, the more the student i feels that student j is reliable, the more the opinion of the latter will influence the opinion of the former. The presence of the shift term 0.1 introduces a small level of interaction even when student i has a high "I can't" towards student j ($k_{ij} = 0$).

The perceived distance score $\varphi_{ij}(t)$ modulates the *opinion distance* $d_{ij}(t) = x_j(t) - x_i(t)$ according to the reciprocal attitudes of the students i and j. Thus, we propose to define the perceived distance score as:

$$\varphi_{ij}(t) = \beta_{ij}(\psi_j(|d_{ij}(t)|)), \tag{6}$$

where β_{ij} is the confidence function

$$\beta_{ij}(z) = \begin{cases} 1 - \dfrac{1}{\varepsilon_{ij}}|z|, & |z| \leq \varepsilon_{ij} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

and ψ_j is the perceived distance function

$$\psi_j(z) = z \cdot 2^{-k_{jj}}. \tag{8}$$

The perceived distance function ψ_j has been designed so that a student with an high value k_{jj} of "I can" is able to get his classmates to listen. Roughly speaking, a student with high level of "I can" reduces the opinion distance between himself and his classmates.

The confidence function β_{ij} is a simple variation of the confidence interval introduced by Hegenselman and Krause, and it allows to reproduce the typical dynamics where the student i is only willing to listen to students with "close" opinions. This is obtained by introducing suitable student dependent thresholds ε_{ij}. The novelty in our approach is that the threshold ε_{ij} is highly influenced by the trait of student i:

$$\varepsilon_{ij} = [\varepsilon_2 + (k_{ii} - 1)(\varepsilon_2 - \varepsilon_3)](1 - k_{ij}) + [\varepsilon_1 + (k_{ii} - 1)(\varepsilon_1 - \varepsilon_4)]k_{ij}, \tag{9}$$

where ε_l, with $l = 1, 2, 3, 4$, are thresholds parameters associated to the four traits discussed in the previous section.

Remark 3.2 (On the choice of ε_l)**.** *The parameters ε_l have been chosen so that the solution of* (2) *qualitatively reproduces realistic student work group activities. High threshold values* ($\varepsilon_4 = 0.7$ *and* $\varepsilon_1 = 0.5$) *have been employed for students with high "You can" (trait I and IV) since they are open to listen to students who have relatively different opinions from them. However, the cooperative student (trait I), who has a high "I can", is willing to listen less than the follower student (trait IV).*

Low threshold values ($\varepsilon_2 = 0$ *and* $\varepsilon_3 = 0.05$) *are used for the other two types of student, since the obstinate student (trait II) does not listen to other opinion, whilst the isolated student (trait III) might change his opinion only if he listens to an opinion very close to his own.*

Example 3.1. *Let us compute the confidence matrix for a simple case where only two students interact. The first student is cooperative (trait I) while the second one is a follower (trait IV). Their initial opinions are* $x_1^0 = 1$ *and* $x_2^0 = 0$*, respectively, and the correspondent attitude matrix* K *is given by:*

$$K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the attitude score (5)*, the perceived distance score* (6) *and the threshold values introduced in Remark 3.2, the confidence matrix for this simple case is given by*

$$W(0) = \begin{bmatrix} 1 & 0 \\ \frac{22}{29} & \frac{7}{29} \end{bmatrix}. \tag{10}$$

Note that matrix $W(0)$ *is not symmetric, due to the different traits of the students. The resulting dynamical system* (2) *is numerically solved using an adaptive Runge-Kutta method (*ode23 *solver of Matlab) and the evolution of the opinions is displayed in Figure 2. Initially, student 2 (red line) moves towards student 1 (blue line), while the latter does not change his opinion until student 2 is enough close to him. Since then, the weights* $w_{1,2}(t)$ *becomes not null and also student 1 starts to mediate his opinion moving slowly towards student 2. Finally, they reach a consensus, namely an agreement about the solution (right or not) to the task.*

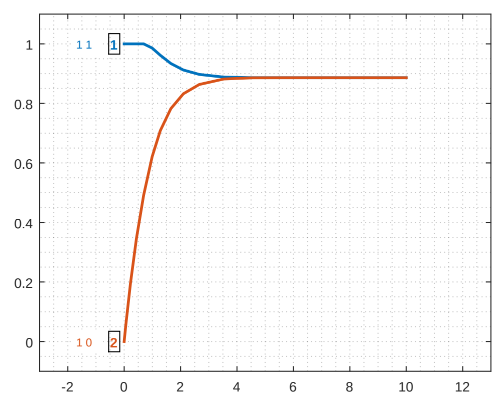


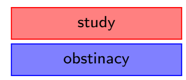
Figure 2: Example of 2-student opinion dynamics. Student 1 (blue line) is a cooperative while student 2 (red line) is a follower.

3.2 The role of the correct solution

It is worth noting that at each time the model (2) gives rise to opinions that belong to the convex hull of the initial opinions. Thus, the opinions at the final time $t = T$ will range within the extreme opinions at the initial time $t = 0$. This is in general acceptable if one has in mind, for example, a group of experts judging a project (in this case the "true opinion" does not exist!). However, such a restriction on the range of the final opinions is not acceptable in our context. Indeed, during a work group activity the students are typically challenged to find *the correct answer* to a given exercise and, unfortunately, the correct solution to the task may not belong to the convex hull of the initial opinions. However, it may happen in practice that the interactions between students contribute to make a completely "new" opinion (i.e. not belonging to the convex hull of the initial opinions) appear. For example, this is the case when two students with different, and complementary, competences work together.

In view of the above discussion, we add an extra "agent" who plays the role of the *correct solution*. More precisely, the correct solution acts as an agent who never changes his opinion (i.e., the the correct answer to the task), $\dot{x}_0(t) = 0$ and $w_{00}(t) = 1$. In addition, it may strongly attract a certain class of students, namely those who have studied and are confident with the problem. However, a student who has a high level of obstinacy will hardly change his opinion; for instance the attitude to have misconceptions and consequently making mathematical mistakes may not prompt the student to get the correct solution. In order to introduce the level of study and obstinacy, the trait matrix K is extended by adding an extra row k_{0i} which represents the level of study of the student i and an extra column k_{i0} which represents the level of "obstinacy" of student i. Accordingly, we define the *extended trait matrix* $\mathcal{K}$ which considers the student traits and the correct solution, as follows

$$\mathcal{K} = \left[\begin{array}{c|ccc} 0 & k_{0,1} & \cdots & k_{0,N} \\ \hline k_{1,0} & & & \\ \vdots & & \text{student traits} & \\ k_{N,0} & & & \end{array}\right] \qquad (11)$$



The weight w_{i0} encoding the interaction between the student i and the correct solution is defined as

$$w_{i0} = c_1 \left(k_{0i}(1 - k_{i0}) + c_0 k_{0i} k_{i0}\right) \qquad (12)$$

with $c_0 > 0$ and $0 < c_1 \leq 1$. A convenient choice made in our simulations is $c_0 = \frac{1}{10}$, $c_1 = \frac{1}{2}$. The heuristics behind equation (12) is that a student who has studied and is not obstinate is attracted by the correct solution, while a student who has studied but he is obstinate, is not directly attracted by the correct solution.

In summary, the opinion dynamics of student i is given by

$$\dot{x}_i(t) = \sum_{j=0}^{N} w_{ij}(t)(x_j(t) - x_i(t)), \quad i = 1, \ldots, N \qquad (13)$$

where $x_0(t) = x_0$ for any t. Finally, the rough weights w^*_{ij} with $j = 1, \ldots, N$ are normalized to obtain the final weights w_{ij} such that

$$\sum_{j=1}^{N} w_{ij} = 1 - w_{i0},$$

while the weight w_{i0}, defined by Equation (12), is not rescaled.

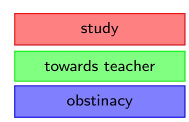
3.3 The teacher as a leader

A crucial aspect in describing the student opinion dynamics is the presence of the teacher, who plays an important role as she knows both the correct solution and the procedure that students should follow to get the correct answer. Therefore, we introduce the teacher as a new agent with opinion x_{N+1} who influences the opinions of the students (but it is not influenced by them!). More precisely, x_{N+1} evolves according to an a-priori defined strategy, e.g., staying silent until at certain moment when she starts interacting by giving hints. Adding the contribution of the teacher, the student opinion dynamics model becomes

$$\begin{cases} x_0(t) = x_0 \\ \dot{x}_i(t) = \displaystyle\sum_{j=0}^{N+1} w_{ij}(t)(x_j(t) - x_i(t)), \quad i = 1, \ldots, N \\ \dot{x}_{N+1}(t) = u(t) \end{cases} \tag{14}$$

where $u(t) : \mathbb{R}_0^+ \to \mathbb{R}$ represents the evolution of the opinion of the teacher. Roughly speaking, $u(t)$ represents the strategy implemented by the teacher. The weight $w_{i,N+1}$, which encodes the attitude of the student i towards the teacher, is computed according to the general procedure described above; this means that the teacher can be listened or not by a student according to the value of $k_{i,N+1}$ of the trait matrix K. The traits are now collected in the further extended trait matrix $\mathcal{K}$ which now assumes the following form:

$$\mathcal{K} = \left[\begin{array}{c|ccc|c} 0 & k_{0,1} & \ldots & k_{0,N} & 0 \\ \hline k_{1,0} & & & & k_{1,N+1} \\ \vdots & & \text{student traits} & & \vdots \\ k_{N,0} & & & & k_{N,N+1} \\ \hline 0 & 0 & \ldots & 0 & 1 \end{array}\right] \tag{15}$$



4 Numerical results

This section reports two sets of numerical results dealing with the opinion dynamics of a group of 3 students. Each set includes the dynamics with and without the influence of the correct solution, and, eventually, with the interventions of the teacher. These dynamics are different due to the different traits of the students involved.

Test case 1: cooperative students at work. Figure 3(a) shows an example of 3 cooperative students, who share and mediate their initial (guessed) opinion reaching a consensus, that is they reach an agreement about the task solution even if it is not the correct one. Since students have not studied (the correct solution has no effect) and they are cooperative, they are willing to listen to each other and mediate their opinions, but they move away from the correct solution. Figure 3(b) reports the dynamics in which the correct solution plays an active role. In particular student 3 has studied ($k_{0,3} = 1$) but he typically makes mathematical mistakes ($k_{3,0} = 0.5$), while the other two students have not studied ($k_{0,1} = k_{0,2} = 0$) but they have good procedural skills ($k_{1,0} = k_{2,0} = 0.1$). The dynamics reported in Figure 3(b) can be interpreted as follows: student 3, who has studied, has an initial opinion close to the correct solution, then he moves towards the classmates explaining them the topic of the task and how to solve it. Finally, followed by the other two students, he goes towards the solution slowly because of his difficulty in making computations.

In Figure 4 we explore the influence of the teacher. In the dynamics reported in Figure 4(a) the teacher, from the beginning, takes part to the discussion hinting the correct solution. Note that all students listen to the teacher ($k_{i,4} = 1$, $i = 1, 2, 3$) and her opinion is constant, namely $u(t) = 0$ in Equation (14).

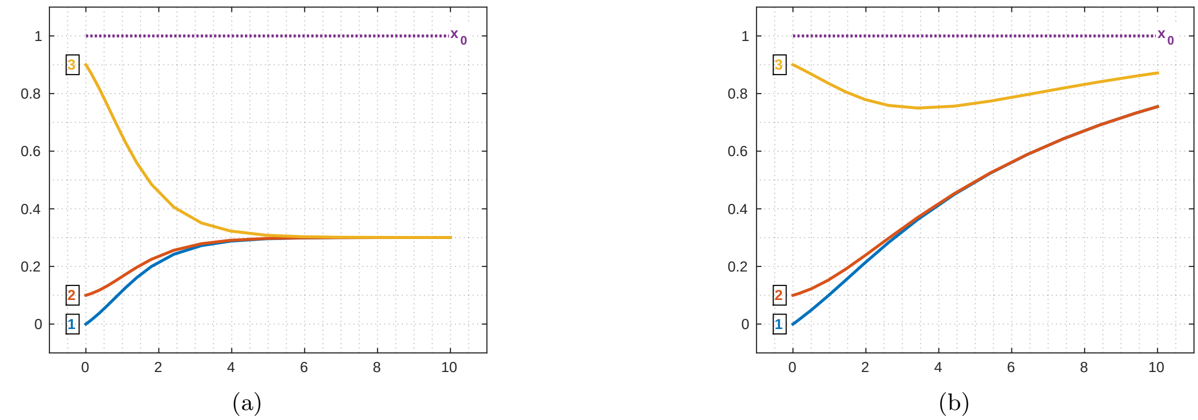


Figure 3: Test case 1: 3-Student opinion dynamics and the influence of correct solution. The solid lines represent the evolution of the opinions associated to each student. The dotted line, labelled by x_0, identifies the correct solution.

In principle, the teacher acts to improve the work group discussion allowing students to give the correct solution, however it is not easy for her to identify an effective strategy to adopt. In Figures 4(b)-4(d) we report the dynamics associated to different strategies ($u(t) \neq 0$) corresponding to different choices for the function $u(t)$, namely

$$u_b(t) = \frac{2}{25}(t-5), \tag{16}$$

$$u_c(t) = \begin{cases} -1, & t \in [0;1] \\ \frac{1}{9}, & t \in (1;10] \end{cases}, \tag{17}$$

$$u_d(t) = \begin{cases} -\frac{1}{2}, & t \in [0;1/3] \\ 0, & t \in (1/3;7] \\ \frac{1}{20}, & t \in (7;10] \end{cases} \tag{18}$$

where the subscripts correspond to the sub-figure in Figure 4.

Employing the strategy u_b the teacher ranges on the time interval $[0, 10]$ all the possible opinions, starting from the correct one ($x = 1$), reaching $x = 0$ at $t = 5$ and going back to the correct solution. The aim of the strategy is to pick the students up and drive them to the correct solution. However, such a strategy is not completely effective because of the velocity of the evolution. Indeed the teacher rushes to achieve her goal and the students do not manage to take fully advantage of her hints. Adopting the strategy u_c the teacher wants to reach the "worst" students, namely student 1 and 2 who are far from the correct solution, and drive them towards the correct solution, i.e. x_0. If compared with strategy u_b, this strategy allows to drive students 1 and 2 closer to the correct solution. However student 3 is driven far from correct solution, with respect to his initial opinion. Finally, employing the strategy u_d the teacher moves closer to student 3 and somehow endorses him forming a "pole", which attracts the other two students. When all the opinions are clustered ($t = 7$), the teacher moves towards x_0 bringing the students closer to the correct solution.

As it is clear from the above discussion, it is not easy to define an a priori optimal strategy that ensures a desired dynamics. Thus it becomes crucial to identify a systematic ways to design optimal interventions. In the next case we explore a way to intervene, and postpone to Section 5 a discussion on possible different approaches to deal with this topic.

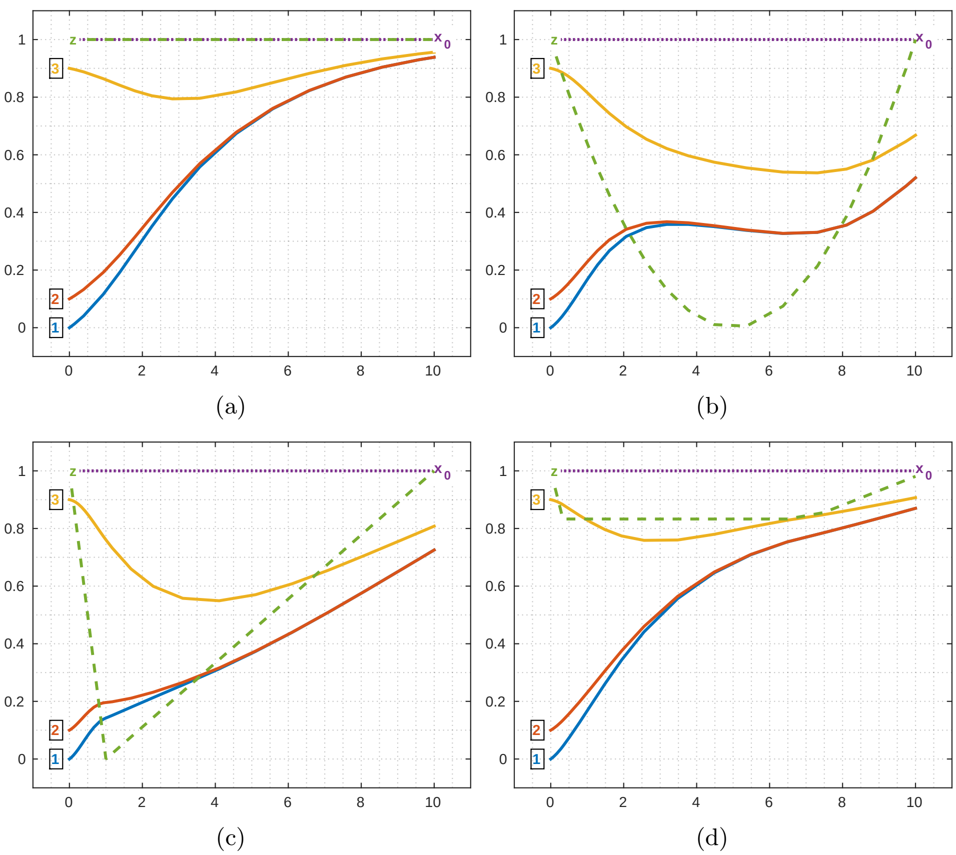


Figure 4: Test case 1: 3-Student opinion dynamics with different strategies for the teacher intervention. The dashed green line represents the evolution of the teacher (z) opinion.

Test case 2: Obstinate controls the dynamics. In this set of simulations an heterogeneous group of student is considered. The trait matrix is reported in (19), student 1 is obstinate, student 2 is a follower and student 3 has a mixed trait between follower and cooperative. The resulting dynamics is showed in Figure 5: student 1, who does not change his opinion (blue line), monopolizes the dynamics attracting the other students since these ones have low level of "I can" while trusting other classmates; student 2, who trusts more student 1 ($k_{2,1} = 1$) than student 3 ($k_{2,3} = 0.5$), moves quickly towards student 1's opinion and slowly towards student 3's one. Finally, student 3 cooperates with the other two classmates reaching them rather quickly. Once students 2 and 3 are close to each other and to student 1, they follow this latter reaching a consensus. It is worth noting that student 3 was initially close to the correct solution, but, because of his attitude, he ends very far from it. Finally, we remark that the correct solution has no effect since none of them has studied.

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0.5 \\ 0.5 & 1 & 0.5 \end{bmatrix}. \tag{19}$$

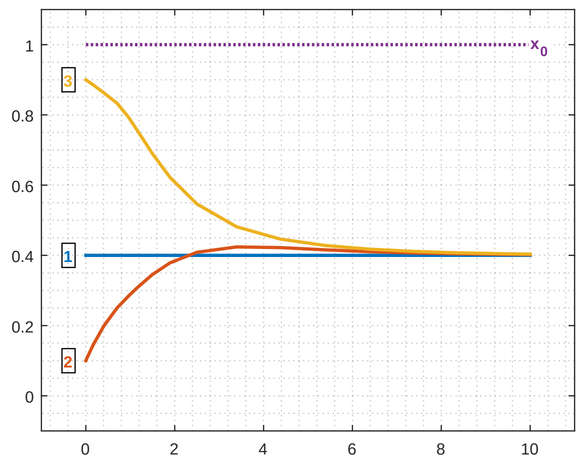
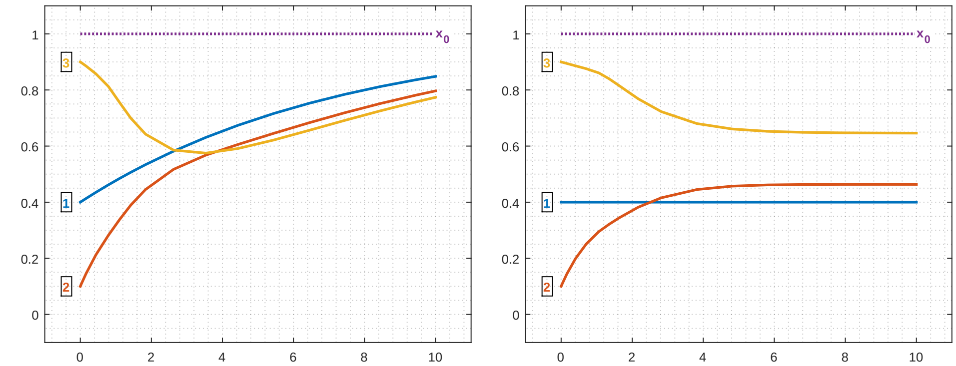


Figure 5: Test case 2: 3-Student opinion dynamics (of heterogenous group of students). Solid lines represent the student opinion while the dotted one represents the correct solution, which has no effect on students in this case. Student 1 is obstinate, while students 2 and 3 are almost cooperative.

On the contrary, when the mathematical attitudes are considered in the dynamics, different behaviours and results occur, as shown in Figure 6. We assume that student 1 has studied a bit ($k_{0,1} = 0.5$) and his level of obstinacy towards the correct solution is $k_{1,0} = 0.5$, while the other two students have not studied at all ($k_{0,2} = k_{0,3} = 0$) but they have good skill of calculus ($k_{2,0} = k_{3,0} = 0$). There follows that student 1 drives his classmates towards the correct solution, even if they do not reach it because of lack of time, see Figure 6(a). Figure 6(b) shows the dynamics in case the only student who has studied is student 3. In this case, where $k_{0,3} = 1, k_{3,0} = 0.5$ and the other traits relative to mathematical attitudes are null, the knowledge and the mathematical skills of student 3 are confused by the eloquence of student 1, who somehow controls the dynamics, whilst student 2 is between two poles represented by the obstinate student 1 and the shy student 3. Thus the discussion ends with 3 very different opinions, that is there is no clue what could be the correct answer to the given task.

When the dynamics is getting more complex and the answer to the task could be not achieved by the students during the work group activity, the role of the teacher becomes more and more crucial. Figure 7 reports different teachers' strategies to drive the student to the correct solution in a priori fixed amount of time. The extended trait matrix used in these simulations is

$$\mathcal{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mathbf{0.5} \\ 0 & 1 & 0 & 0.5 & \mathbf{1} \\ 0 & 0.5 & 1 & 0.5 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{20}$$



(a) (b)

Figure 6: Test case 2: 3-Student opinion dynamics (of heterogenous group of students). Student 1 is cooperative while students 2 and 3 are almost cooperative. Two different settings: (a) Student 1 has studied; (b) Student 3 has studied.

We recall that the last column of $\mathcal{K}$ is related to the teacher influence. In this example, student 2 and 3 are willing to listen to teacher ($k_{2,4} = k_{3,4} = 1$) whilst student 1, due to his attitude, has some resistance to listen to her.

In the corresponding dynamics reported in Figure 7 the teacher interacts with the students during the whole discussion using the following different strategies

$$u_a(t) = 0 \tag{21}$$

$$u_b(t) = \frac{2}{50}(t-5) \tag{22}$$

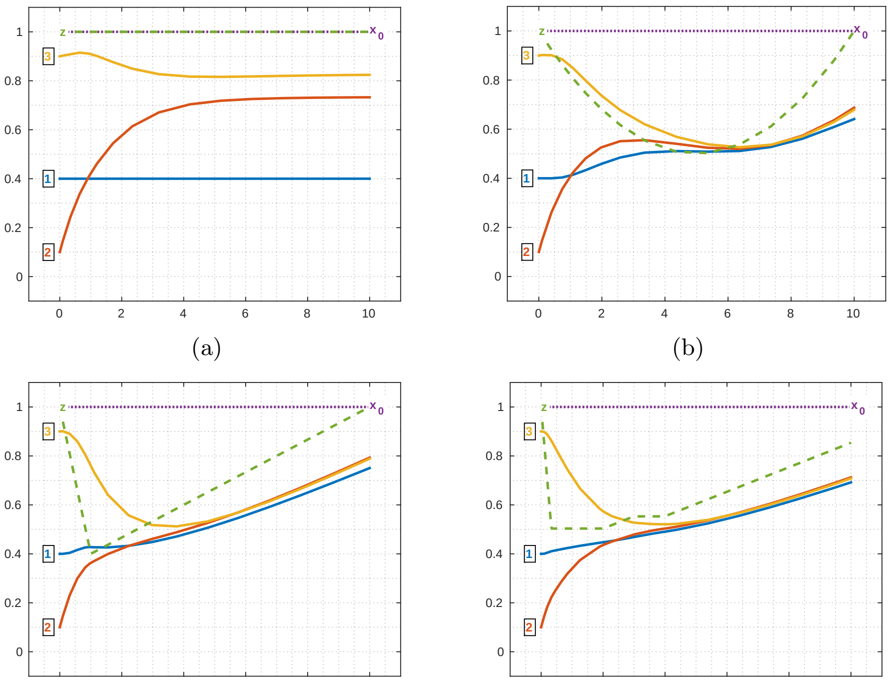
$$u_c(t) = \begin{cases} -\frac{3}{5}, & t \in [0;1] \\ \frac{1}{15}, & t \in (1;10] \end{cases} \tag{23}$$

$$u_d(t) = \begin{cases} -\frac{1}{5}, & t \in [0;1/3] \\ 0, & t \in (1/3;2] \\ \frac{1}{20}, & t \in (2;3] \\ 0, & t \in (3;4] \\ \frac{1}{20} & t \in (4;10] \end{cases} \tag{24}$$

where the subscripts of the control $u(t)$ refer to the Figure 7(a)-7(d), respectively.

In Figure 7(a) the teacher's opinion is constant during the activity and she hints the correct solution which is got only by student 3, the closest one, who, in turn, mediates between the hint of the teacher and the opinion of the obstinate student. Furthermore, student 3 is able to attract student 2 with the help of teacher since this student trusts both of them. In Figure 7(b), the teacher ranges all the possible opinions catching the student 1 as well; however, from time $t = 5$ she comes back to the correct solution too quickly loosing the students, namely the complexity of her suggestions increases too much to be fully caught by them.

The last two dynamics shown in Figure 7 are quite similar since the idea, already discussed above, which underlies is the same. In details, with strategy u_c (Figure 7(c)) the teacher



(a) (b) (c) (d)

Figure 7: Tet case 2: 3-Student opinion dynamics (of heterogenous group of students). The presence of the dashed green line means teacher intervines and it represents the evolution of the teacher (z) opinion.

reaches the obstinate student and wants him to follow her; however, since the evolution of her opinion is too fast, the students stop following her very early. Similarly, the strategy u_d (see Figure 7(d)) allows the teacher to keep the students closer to her but not enough to drive them towards the solution. Nevertheless, in all dynamics, she drives student 1 and 2 closer to the correct solution even though student 3 moves far away from x_0.

We remark that the teacher has another possibility: instead of an intervention with a mathematical nature, her intervention can have a social nature. For instance she can silence students or prompt the silent student to talk. The last dynamics presented in Figure 8, reports an example of this kind of strategy. Firstly the teacher observes the discussion studying its dynamics, at a certain point ($t = 2$) she intervenes on two fronts: transforming the obstinate student 1 in a follower ($k_{1,1} = 0, k_{1,4} = 1$) and suggesting the correct solution to all students ($k_{i,4} = 1, u(t) = 0$). Since she is confident that student 1 has understood the task and the resolution strategy, she stops the intervention at time $t = 6$ and leaves the dynamics to evolve naturally. Such a strategy allows the teacher to focus on the obstinate student (student 1), who may acts as a leader, leaving him to drive his classmates towards the solution. Of course, up to a certain extent, the hints of the teacher are taken into account by student 2 and 3 as well. At the end of the dynamics, all students are very close to the correct solution even though they do not completely reach the correct solution x_0.

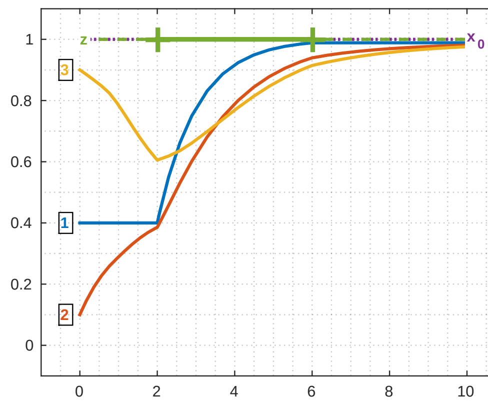


Figure 8: Set 2: 3-Student opinion dynamics of heterogenous group. Mix intevention strategy. The green solid line represents the teacher hints, while triangle green markers represent the instant where teacher changes the trait of student 1 (blue line).

5 Conclusion and future work

In this paper we introduced and discussed a mathematical model suited for describing the student opinion dynamics during work-group activity. Although several numerical experiments show the capability of the model to reproduce realistic scenarios, there are still open issues that need to be addressed in future works. In the following, we highlight the most important ones.

- The meaning and the precise mathematical characterization of the opinion x require a deeper investigation, see Remark 3.1.
- The construction of the confidence matrix W needs a surplus of understanding. In particular, a more precise characterization of the interaction between the teacher and the students would be necessary. To this aim we are planning to implement a set of video-recorded experiments involving students and teachers interacting during "prototypal" situations.
- A mathematical understanding of the "optimal" intervention of the teacher to drive the students towards the correct solution represents a challenging development of the present work. This task requires setting up a suitable optimal control problem. For related works, see e.g. [25, 11].
- As an ultimate goal, we would like to study the possible role that improved versions of the present model can play in supporting teachers who want to handle student-centred activities in classroom. In particular, a simulator based on the present model can help teachers to improve their abilities to manage in-the-moment decision making [20, 19] .

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