MOX–Report No. 39/2010

Finite Element Approximation of Elliptic Problems with Dirac Measure Terms in Weighted Spaces. Applications to 1D-3D Coupled Problems

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Finite Element Approximation of Elliptic Problems with Dirac Measure Terms in Weighted Spaces. Applications to 1D-3D Coupled Problems.

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Abstract

In this work we study the stability and the convergence rates of the finite element approximation of elliptic problems involving Dirac measures, using weighted Sobolev spaces and weighted discrete norms. Our approach handles both the cases where the measure is simply a right hand side or it represents an additional term, i.e. solution-dependent, in the formulation of the problem. The main motivation of this study is to provide a methodological tool to treat elliptic problems in fractured domains, where the coupling terms are seen as Dirac measures concentrated on the fractures.

We first establish a decomposition lemma, which is our fundamental tool for the analysis of the considered problems in the non-standard setting of weighted spaces. Then, we consider the stability of the Galerkin approximation with finite elements in weighted norms, with uniform and graded meshes. We introduce a discrete decomposition lemma that extends the continuous one and allows to derive discrete inf-sup conditions in weighted norms. Then, we focus on the convergence of the finite element method. Due to the lack of regularity, the convergence rates are suboptimal for uniform meshes; we show that using graded meshes optimal rates are recovered. Our theoretical results are supported by several numerical experiments. Finally, we show how our theoretical results apply to certain coupled problems involving fluid flow in porous three-dimensional media with one-dimensional fractures, that are found in the analysis of microvascular flows. *Keywords:* elliptic problems, measure, Dirac measure, weighted spaces, finite element method, graded mesh, error estimates, reduced models, multiscale models, microcirculation.

1 Introduction

Reduced models of fluid flows or mass transport in heterogeneous media are often used to save computational resources when the system to be simulated is too complex. An example is provided by Darcy's flow in fractured domains: usually, reduced models treat thin planar fractures as surfaces embedded in the porous domain, providing suitable interface conditions (see for instance Angot et al. [2009], Alboin et al. [2002], Martin et al. [2005], Frih et al. [2008] and Lesinigo et al. [2010]).

However, if the dimensional gap between the space dimension $N = 3$ of the considered porous domain Ω and the manifold representing the fracture is higher, for instance when the fractures are thin tubes or vessels, the situation is more complicated, since the solution may be strongly singular on the fracture. A typical

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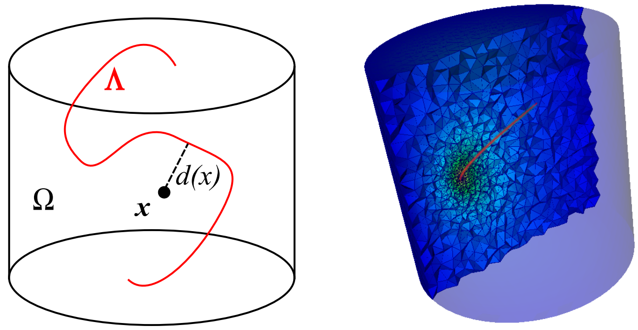

Figure 1: On the left: the three-dimensional domain $\Omega \subset \mathbb{R}^3$, the embedded line Λ, and the function $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \Lambda)$. On the right: an example of graded mesh, refined near Λ.

example would be the following Darcy's equation in a three-dimensional domain Ω enclosing a one-dimensional fracture Λ (see fig. 1),

$$\nabla \cdot (-\mathsf{K}\nabla u) = f \quad \text{in } \Omega, \tag{1a}$$

$$\langle f, v\rangle = \int_\Lambda q(s)v(s)\,\mathrm{d}s \quad \forall v \in C(\Omega), \tag{1b}$$

where K is the positive hydraulic permeability of the medium and $q \in L^2(\Lambda)$ is the linear mass flux (i.e. flux per unit length) from the fracture into the porous domain. Equation (1), equipped with suitable boundary conditions, is in fact an elliptic problem whose datum f is a Dirac measure. The finite element approximation of such problems was previously studied by Babuška [1971], Scott [1973], Casas [1985], using "weak" norms (i.e. L^2 or H^s for s small). Only recently, the analysis of the FEM approximation of such problems with graded meshes has been considered [Apel et al., 2009]. To our knowledge, the techniques based on weighted norms and a suitable augmented formulation presented in this work are new.

Such techniques allow the treatment of even more complex situations. A relevant example is when f depends on u itself (this is, f is a *measure term*), for instance due to an averaged Starling filtration law

$$q(s) = 2\pi R L_p(\hat{u}(s) - \bar{u}(s)), \tag{2}$$

where L_p is a permeability coefficient, R is the actual radius of the fracture, $\hat{u}(s)$ is the given pressure inside the fracture, and $\bar{u}(s)$ is the average pressure in the porous medium on a circle normal to Λ, at distance R from the considered point s on Γ. Finally, if the fluid pressure $\hat{u}$ in the fracture is not known *a priori*, but is rather to be computed using a suitable fracture flow model, the problem to be solved will be a coupled 1D-3D problem.

Although mathematically non-standard, this kind of coupled problems is of great interest in many applications, for instance in the computational analysis of tissue perfusion or drug delivery by a network of blood vessels, which has been addressed by several authors to study the physiology of cancerous tissues and related drug administration strategies (we will present an example of such computations in section 5). Drug delivery to the stenosed arterial wall by a thin implanted device (stent) has been investigated in D'Angelo and Zunino [2009] using similar 1D-3D models. A mathematical foundation for such kind of problems was provided in a previous work by D'Angelo and Quarteroni [2008], where functional tools based on weighted Sobolev spaces were introduced to prove the well-posedness of the model.

In this paper, we reconsider the problem from a new perspective, thanks to augmented formulations that allow us to establish the inf-sup conditions needed for both the continuous problem and the discrete Galerkin approximation. We than focus on the finite element approximation, studying the convergence of the method, and deriving convergence rates on uniform and graded meshes (see fig 2, on the right), thanks to suitable interpolation operators. Finally, we apply our results to the study of 1D-3D coupled problems with applications to blood flow in biological tissues.

2 Definition of the model problem

We consider a three-dimensional domain $\Omega \subset \mathbb{R}^3$ and a one-dimensional subdomain $\Lambda \subset \Omega$. The latter is actually a line, that we will assume to be smooth enough.

For $\alpha \in (-1,1)$, we denote by $L^2_\alpha(\Omega)$ the Hilbert space of measurable functions u such that

$$\int_\Omega u(\mathbf{x})^2 d^{2\alpha}(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty,$$

where d is the distance from Λ, $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x},\Lambda)$, equipped with the scalar product

$$(u,v)_{L^2_\alpha} = \int_\Omega u(\mathbf{x})v(\mathbf{x})d^{2\alpha}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

We will denote by bold symbols vector spaces, for instance $\boldsymbol{L}^2_\alpha(\Omega) = [L^2(\Omega)]^3$, and by $\langle\cdot,\cdot\rangle_E$ the $L^2(E)$ scalar product on the generic domain E. We will often consider $\langle\cdot,\cdot\rangle_E$ as a duality pairing, for instance between $L^2_\alpha(E)$ and $L^2_{-\alpha}(E)$. Note that $\|u\|_{L^2_\alpha} = \|d^\alpha u\|_{L^2}$, $\langle u,v\rangle_\Omega = \langle d^\alpha u, d^{-\alpha} v\rangle_\Omega \le \|u\|_{L^2_\alpha}\|v\|_{L^2_{-\alpha}}$ and that the mapping $u \mapsto d^{2\alpha}u$ is an isometry from $L^2_\alpha(\Omega)$ to $L^2_{-\alpha}(\Omega)$.

Let $m \in \mathbb{N}$; we define the weighted Sobolev spaces

$$H^m_\alpha(\Omega) = \left\{D^{\boldsymbol{\beta}}u \in L^2_\alpha(\Omega), |\boldsymbol{\beta}| \le m\right\},$$

where $\boldsymbol{\beta}$ is a multi-index and $D^{\boldsymbol{\beta}}$ denotes the corresponding distributional partial derivative. $H^m_\alpha(\Omega)$ is equipped with the following seminorm and norm,

$$|u|_{H^m_\alpha} = \left(\sum_{|\boldsymbol{\beta}|=m}\|D^{\boldsymbol{\beta}}u\|^2_{L^2_\alpha}\right)^{\frac{1}{2}}, \qquad \|u\|_{H^m_\alpha} = \left(\sum_{k=0}^m |u|^2_{H^k_\alpha}\right)^{\frac{1}{2}}.$$

We shall also use Kondrat′ev-type weighted spaces $V^m_\alpha(\Omega)$, $m \in \mathbb{N}$, with

$$|u|_{V^m_\alpha} = |u|_{H^m_\alpha}, \quad \|u\|_{V^m_\alpha} = \left(\sum_{k=0}^m |u|^2_{V^k_{\alpha-m+k}}\right)^{\frac{1}{2}}. \tag{3}$$

For instance, $\|u\|^2_{V^1_\alpha} = \|\nabla u\|^2_{L^2_\alpha} + \|u\|^2_{L^2_{\alpha-1}}$. Similar spaces have been extensively used to treat boundary value problems in domain with corners, see Kozlov et al. [1997]. Let $m, s \in \mathbb{N}$, $s \ge 0$, $\alpha \in (0,1)$. Note that the embedding $V^{m+s}_{\alpha+s} \hookrightarrow V^m_\alpha$ is continuous, since as a direct consequence of (3) we have

$$\|u\|_{V^m_\alpha} \le \|u\|_{V^{m+s}_{\alpha+s}}. \tag{4}$$

The embedding $H^1_\alpha \subset L^2_{\alpha-1}$ is also continuous: indeed it can be shown [Babuška and Rosenzweig, 1972, th. 2.2], [Kufner, 1985] that

$$\|u\|_{L^2_{\alpha-1}} \le C_\alpha \|u\|_{H^1_\alpha}, \tag{5}$$

with $C_\alpha = \mathcal{O}(\alpha^{-1})$ for $\alpha \to 0$. As a result, the H^1_α and V^1_α norm are equivalent, but not uniformly w.r.t. α. Finally, we recall [Kufner, 1985] the following Poincaré inequality:

$$\|u\|_{L^2_\alpha} \leq C_P \|\nabla u\|_{L^2_\alpha} \quad \forall u \in H^1_\alpha(\Omega) : u_{|\partial\Omega} = 0. \tag{6}$$

2.1 The model problem

The Darcy's problem (1) is an elliptic problems with measure data f. In particular, f is not a bounded functional on $H^1(\Omega)$, since there is no bounded trace operator $\gamma_\Lambda : H^1(\Omega) \to L^2(\Lambda)$, Λ being a $(N-2)$-dimensional manifold. In fact, the solution u has a logarithmic singularity on Λ, with $|\nabla u| = \mathcal{O}(d^{-1}(\mathbf{x}))$ for $\mathbf{x}$ close to Λ, which is not L^2, so that $u \notin H^1(\Omega)$. However [D'Angelo and Quarteroni, 2008, th. 4.2], for any $\alpha > 0$ there exists a unique linear continuous map $\gamma_\Lambda : H^1_{-\alpha}(\Omega) \to L^2(\Lambda)$ such that $\gamma_\Lambda \phi = \phi_{|\Lambda}$ for any smooth function $\phi \in C^\infty(\Omega)$. On the other hand, a $\mathcal{O}(d^{-1}(\mathbf{x}))$ singularity is L^2_α for any $\alpha > 0$. As also pointed out in D'Angelo and Quarteroni [2008], these observations suggest a variational formulation in $H^1_\alpha \times H^1_{-\alpha}$. More precisely, let us consider homogeneous Dirichlet boundary conditions, and define

$$W_\alpha = \{u \in H^1_\alpha(\Omega) : u_{|\partial\Omega} = 0\}, \quad \text{normed by} \quad \|u\|_{W_\alpha} := \|\nabla u\|_{L^2_\alpha}.$$

Thanks to (6), the latter is indeed a norm equivalent to $\|\cdot\|_{H^1_\alpha}$ (uniformly w.r.t. α) and $\|\cdot\|_{V^1_\alpha}$. Problem (1) admits the following variational formulation: *find $u \in W_\alpha$ such that*

$$\langle \mathsf{K}\nabla u, \nabla v\rangle_\Omega = \langle f, v\rangle = \langle q, \gamma_\Lambda v\rangle_\Lambda \quad \forall v \in W_{-\alpha}. \tag{7}$$

This simple example is the basic motivation for the analysis of a model elliptic problem with a variational formulation in the previously weighted spaces.

The model problem that we will focus on reads: *given $f \in W'_{-\alpha}$, find $u \in W_\alpha$ such that*

$$\langle \nabla u, \nabla v\rangle_\Omega = \langle f, v\rangle \quad \forall v \in W_{-\alpha}. \tag{8}$$

The unique solvability of problem (8) requires to show that the bilinear form $\langle \nabla u, \nabla v\rangle_\Omega$ satisfies the Brezzi-Nečas-Babuška (BNB) theorem (Ern and Guermond [2004], th. 2.6; see also Nečas [1962], Babuška [1971], Quarteroni and Valli [1997]) i.e. the usual inf-sup conditions on $W_\alpha \times W_{-\alpha}$. That has been shown in D'Angelo and Quarteroni [2008] for α sufficiently small. The proof was constructive: given $u \in W_\alpha$, we choose $v = d^{2\alpha}u + 2\alpha\Psi$, where $\Psi \in W_{-\alpha}$ is a corrective term. In fact, we have

$$\begin{aligned} \langle \nabla u, \nabla v\rangle_\Omega &= \langle \nabla u, d^{2\alpha}\nabla u\rangle_\Omega + 2\alpha\langle \nabla u, d^{2\alpha-1}u\nabla d + \nabla\Psi\rangle_\Omega \\ &\geq (1 - C\alpha)\|\nabla u\|^2_{L^2_\alpha} \gtrsim \|u\|^2_{W_\alpha} \quad \text{for } \alpha < \frac{1}{C}, \end{aligned} \tag{9}$$

provided that $2\|d^{\alpha-1}u\nabla d + \nabla\Psi\|_{L^2_{-\alpha}} \leq C\|\nabla u\|_{L^2_\alpha}$, with C independent of α. The corrective term Ψ is necessary. In fact, $\|d^{\alpha-1}u\nabla d\|_{L^2_{-\alpha}} = \|u\|_{L^2_{\alpha-1}} \leq C_\alpha\|u\|_{H^1_\alpha}$ thanks to (5); but since C_α is not uniformly upper-bounded for small α, we are not allowed to proceed to the second line in (9) if $\Psi = 0$. One drawback of this approach, is that function Ψ is constructed resorting to a rather technical Fourier expansion, that cannot be replicated in the discrete setting of the Galerkin approximation.

The first point addressed in this paper is a much simpler alternative approach, based on an augmented formulation, that admits a simple extension to the discrete setting. As a major consequence, we will obtain not only the existence and uniqueness (for any $\alpha \in [0, 1)$) of problem (8); we will also show the stability of its Galerkin approximation using the finite element method.

The main idea is the following. The corrective term $2\alpha\Psi$ is needed in (9) since we cannot choose $v \in W_{-\alpha}$ s.t. $\nabla v = d^{2\alpha}\nabla u$. The previous approach was based on keeping that correction "small". However, what we really need is to decompose $\boldsymbol{q} = d^{2\alpha}\nabla u \in \boldsymbol{L}^2_{-\alpha}(\Omega)$ as $\boldsymbol{q} = \nabla v + \boldsymbol{\sigma}$, with $v \in W_{-\alpha}$, and $\boldsymbol{\sigma} \in \boldsymbol{L}^2_{-\alpha}(\Omega)$ *orthogonal* to ∇W_α, i.e. $\langle \boldsymbol{\sigma}, \nabla w\rangle_\Omega = 0\ \forall w \in W_\alpha$. That is what is stated in the following fundamental lemma.

Lemma 2.1 (Decomposition of $\boldsymbol{L}^2_s$)**.** *Let $s \in (-1,1)$. For each $\boldsymbol{q} \in \boldsymbol{L}^2_s(\Omega)$, there exists a unique couple $(\boldsymbol{\sigma}, u) \in \boldsymbol{L}^2_s(\Omega) \times W_s$ such that*

$$\boldsymbol{q} = \nabla u + \boldsymbol{\sigma}, \quad \langle \boldsymbol{\sigma}, \nabla v\rangle_\Omega = 0\ \forall v \in W_{-s},$$

$$\|\nabla u\|_{L^2_s} \leq 2\|\boldsymbol{q}\|_{L^2_s}, \quad \|\boldsymbol{\sigma}\|_{L^2_s} \leq \|\boldsymbol{q}\|_{L^2_s}.$$

In other words, the following is a direct sum:

$$\boldsymbol{L}^2_s(\Omega) = (\nabla W_s) \bigoplus (\nabla W_{-s})^\perp.$$

Proof. Let $M_s = \boldsymbol{L}^2_s(\Omega)$. The problem can be recast as a generalized saddle point problem. In fact, $(\boldsymbol{\sigma}, u) \in M_s \times W_s$ are such that

$$\begin{cases} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(u, \boldsymbol{\tau}) = F(\boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in M_{-s}, \\ b(v, \boldsymbol{\sigma}) = 0 & \forall v \in W_{-s}, \end{cases} \tag{10}$$

where $a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \langle \boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_\Omega$, $b(u, \boldsymbol{\tau}) = \langle \nabla u, \boldsymbol{\tau}\rangle_\Omega$, $F(\boldsymbol{\tau}) = \langle \boldsymbol{q}, \boldsymbol{\tau}\rangle_\Omega$. Note that a is a bilinear form on $M_s \times M_{-s}$, and b is considered as a bilinear form on $W_s \times M_{-s}$ and on $W_{-s} \times M_s$ in equations (10). F is a linear functional on M_{-s}, and $\|F\|_{M'_{-s}} = \|\boldsymbol{q}\|_{M_s}$.

First, we have $a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \leq \|\boldsymbol{\sigma}\|_{M_s}\|\boldsymbol{\tau}\|_{M_{-s}}$, $b(u, \boldsymbol{\tau}) \leq \|\nabla u\|_{M_s}\|\boldsymbol{\tau}\|_{M_{-s}}$ and $b(v, \boldsymbol{\sigma}) \leq \|\nabla v\|_{M_{-s}}\|\boldsymbol{\sigma}\|_{M_s}$, so that all the forms are bounded with continuity constant equal to 1. For any $\boldsymbol{\sigma} \in M_s$, the function $\boldsymbol{\tau} = d^{2s}\boldsymbol{\sigma}$ is such that $\|\boldsymbol{\tau}\|_{M_{-s}} = \|\boldsymbol{\sigma}\|_{M_s}$, $a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \|\boldsymbol{\sigma}\|^2_{M_s}$. Those identities still hold if we swap $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ and change the sign of s. Hence, we have

$$\sup_{\boldsymbol{\tau}\neq \boldsymbol{0}} \frac{a(\boldsymbol{\sigma}, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{M_{-s}}} \geq \alpha_1\|\boldsymbol{\sigma}\|_{M_s}, \quad \sup_{\boldsymbol{\sigma}\neq \boldsymbol{0}} \frac{a(\boldsymbol{\sigma}, \boldsymbol{\tau})}{\|\boldsymbol{\sigma}\|_{M_s}} \geq \alpha_2\|\boldsymbol{\tau}\|_{M_{-s}},$$

with $\alpha_1 = \alpha_2 = 1$. Moreover, for any $u \in W_s$, choosing $\boldsymbol{\tau} = d^{2s}\nabla u$ yields $b(u, \boldsymbol{\tau}) = \|\nabla u\|^2_{M_s} = \|u\|^2_{W_s}$, $\|\boldsymbol{\tau}\|_{M_{-s}} = \|\nabla u\|_{M_s} = \|u\|_{W_s}$. Again, the same identities are obtained by replacing u by v, $\boldsymbol{\tau}$ by $\boldsymbol{\sigma}$ and changing the sign of s, so that we have

$$\sup_{\boldsymbol{\tau}\neq \boldsymbol{0}} \frac{b(u, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{M_{-s}}} \geq \beta_1\|u\|_{W_s}, \quad \sup_{\boldsymbol{\sigma}\neq \boldsymbol{0}} \frac{b(v, \boldsymbol{\sigma})}{\|\boldsymbol{\sigma}\|_{M_s}} \geq \beta_2\|v\|_{W_{-s}},$$

with $\beta_1 = \beta_2 = 1$. All the hypotheses of the generalized BNB theorem (Bernardi et al. [1988]; see also Ern and Guermond [2004], Nicolaides [1982]) are thus satisfied. As a consequence, there exists a unique couple $(\boldsymbol{\sigma}, v) \in M_s \times W_s$ satisfying (10), and we have

$$\|\boldsymbol{\sigma}\|_{M_s} \leq \alpha_1^{-1}\|\boldsymbol{q}\|_{M_s} = \|\boldsymbol{q}\|_{M_s}, \quad \|u\|_{W_s} \leq \frac{1 + \alpha_1^{-1}}{\beta_1}\|\boldsymbol{q}\|_{M_s} = 2\|\boldsymbol{q}\|_{M_s}.$$

□

As a first consequence, we can improve the existence result of D'Angelo and Quarteroni [2008] as follows.

Corollary 2.1. *For any $\alpha \in (-1,1)$, the model problem (8) is well-posed, and we have*

$$\|u\|_{W_\alpha} \leq 3\|f\|_{W'_{-\alpha}}.$$

In particular, the Laplace operator $-\Delta$ is an isomorphism from W_α to $W'_{-\alpha}$.

Proof. Let $v \in W_{-\alpha}$, and $\boldsymbol{q} = d^{-2\alpha}\nabla v \in \boldsymbol{L}^2_\alpha(\Omega)$. Thanks to lemma 2.1 with $s = \alpha$, there exist $\boldsymbol{\sigma} \in \boldsymbol{L}^2_\alpha(\Omega)$ and $u \in W_\alpha$ such that $\boldsymbol{q} = \nabla u + \boldsymbol{\sigma}$, $\|u\|_{W_\alpha} \leq 2\|\boldsymbol{q}\|_{L^2_\alpha} = 2\|v\|_{W_{-\alpha}}$, and $\boldsymbol{\sigma} \perp \nabla W_{-\alpha}$. As a consequence,

$$\langle \nabla u, \nabla v\rangle_\Omega = \langle \boldsymbol{q}, \nabla v\rangle_\Omega - \langle \boldsymbol{\sigma}, \nabla v\rangle_\Omega = \langle \boldsymbol{q}, \nabla v\rangle_\Omega = \|v\|^2_{W_{-\alpha}}.$$

This implies $\sup_{u\in W_\alpha} \frac{\langle \nabla u, \nabla v\rangle_\Omega}{\|u\|_{W_\alpha}} \geq c_1\|v\|_{W_{-\alpha}}$, with $c_1 = \frac{1}{2}$. The same estimates still hold if we swap u and v and change the sign of α, leading to $\sup_{v\in W_{-\alpha}} \frac{\langle \nabla u, \nabla v\rangle_\Omega}{\|v\|_{W_{-\alpha}}} \geq c_2\|u\|_{W_\alpha}$, $c_2 = \frac{1}{2}$. The thesis follows thanks to the BNB theorem, noting that $\|u\|_{W_\alpha} \leq \left(1 + c_2^{-1}\right)\|f\|_{W'_{-\alpha}}$. □

Remark 2.1. *Corollary 2.1 applies immediately to (7); as a consequence, problem (1) is well-posed in W_α, for any $\alpha \in (0,1)$, and if $\mathsf{K} > 0$ is constant we have $\|u\|_{W_\alpha} \leq 3C_\Lambda(\alpha)\mathsf{K}^{-1}\|q\|_{L^2(\Lambda)}$, where $C_\Lambda(\alpha) = \sup_{v\neq 0} \frac{\|\gamma_\Lambda v\|_{L^2(\Lambda)}}{\|v\|_{W_{-\alpha}}}$ is the norm of the continuous trace operator γ_Λ.*

3 Numerical approximation of the model problem

Assume that the domain Ω is a polyhedron, and introduce a regular family of triangulations $\{\mathcal{T}_h\}$ of Ω. As usual, each triangulation $\mathcal{T}_h$ is a collection of tetrahedral elements K so that $\Omega = \bigcup_{K\in\mathcal{T}_h} K$. The index h stands for the maximum element size; the diameter of each element K will be denoted by h_K. We assume that $\mathcal{T}_h$ is shape-regular, i.e. $\rho_K \leq h_K \lesssim \rho_K$ $\forall K \in \mathcal{T}_h$ uniformly w.r.t. K, ρ_K being the radius of the largest ball contained in K. The notation $x \lesssim y$ will be used if $x \leq Cy$ for a generic constant $C > 0$, indipendent of h. Moreover, we will make the following assumption on the line Λ:

$$|\Lambda \cap K| \lesssim h_K \quad \forall K \in \mathcal{T}_h. \tag{11}$$

Note that (11) will always hold for smooth Λ provided that h is small enough.

Then, we introduce the family $\{W^k_h\}$ of $\mathbb{P}_1$ finite element subspaces of degree $k \geq 1$,

$$W^k_h = \{u_h \in C(\Omega): \ u_{h|K} \in \mathbb{P}_k(K)\ \forall K \in \mathcal{T}_h,\ u_{h|\partial\Omega} = 0\}. \tag{12}$$

Note that $W^k_h \subset W_s$ for all $s \in (-1,1)$. The Galerkin approximation of the model problem (8) reads: *find $u_h \in W^k_h$ such that*

$$\langle \nabla u_h, \nabla v_h\rangle_\Omega = \langle f, v_h\rangle \quad \forall v \in W^k_h. \tag{13}$$

In the following sections, we shall study the stability of the discrete problem (13) in weighted spaces W_α, and estimate the approximation error $\|u - u_h\|_{W_\alpha}$ in weighted Sobolev norms.

3.1 Uniform and graded meshes

For each element $K \in \mathcal{T}_h$ let us define the following quantities,

$$r_K := \mathrm{dist}(K, \Lambda), \quad \bar{r}_K := \max_{\mathbf{x}\in K} \mathrm{dist}(\mathbf{x}, \Lambda), \quad h_K := \mathrm{diam}(K). \tag{14}$$

Let $\mu \in (0,1]$ be a mesh grading parameter. We will assume that the local element size h_K scales as $r_K^{1-\mu}$ and that $h_K \simeq h^{1/\mu}$ if K is close to Λ. Precisely, let $\delta > 0$ be a fixed safety coefficient (say, $\delta = \frac{1}{2}$); we assume that there exist two positive constants c, C, such that

$$c h r_K^{1-\mu} \le h_K \le C h r_K^{1-\mu} \quad \text{if } r_K > \delta h_K, \tag{15a}$$

$$c h^{1/\mu} \le h_K \le C h^{1/\mu} \quad \text{otherwise.} \tag{15b}$$

Such refinements are usually introduced for the approximation of elliptic problems in polygonal domains to handle corner singularity, see Apel et al. [1996]. In this work we will take advantage of mesh grading to study the convergence rates of the finite element approximation in weighted norms. Note that for $\mu = 1$, $h \lesssim h_K \lesssim h$ $\forall K \in \mathcal{T}_h$, and the mesh is quasi-uniform. As $\mu \to 0$, the elements cluster in a small neighborhood of Λ. It can be shown (Eriksson [1985]) that the total number of elements is independent of μ (in other words, grading corresponds to a sort of redistribution of the elements to capture the singularity on Λ).

Note that r_K is the distance of K from a line Λ, which is non-conformal with respect to the mesh, and embedded into it. We will isolate the line inside a suitable collection of elements (that we will refer to as $\mathcal{T}_h^{\text{in}}$) and carry out a separate analysis for elements $K \in \mathcal{T}_h^{\text{in}}$ and elements belonging to the rest of the mesh. Specifically, we will split the mesh according to (15a,b), introducing the following partition:

$$\mathcal{T}_h^{\text{in}} = \{K \in \mathcal{T}_h : r_K \le \delta h_K\}, \quad \mathcal{T}_h^{\text{out}} = \mathcal{T}_h \backslash \mathcal{T}_h^{\text{in}}.$$

Referring to fig. 2 (left), $\mathcal{T}_h^{\text{in}}$ consists of the shaded elements. With little abuse of notation, the symbols $\mathcal{T}_h^{\text{in,out}}$ will denote also the regions $\bigcup_{K \in \mathcal{T}_h^{\text{in,out}}} K$.

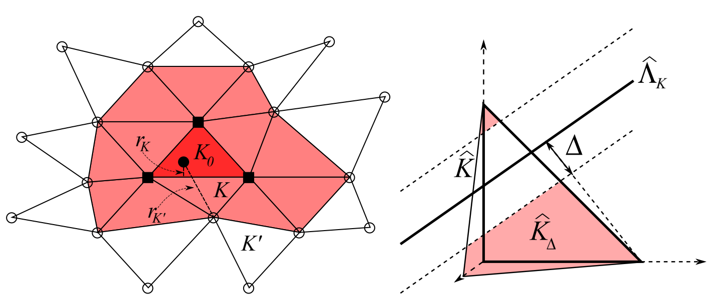

Figure 2: On the left: sectional view of the 3D mesh onto a plane normal to the line Λ; the filled circle indicates the intersection between the normal plane and Λ. The shaded elements belong to $\mathcal{T}_h^{\text{in}}$. Marked are $K_0 \in \mathcal{T}_h$ such that $r_{K_0} = 0$, $K \in \mathcal{T}_h^{\text{in}}$ (i.e. $r_K \le \delta h_K$), $K' \in \mathcal{T}_h^{\text{out}}$. Empty circles denote nodes $\mathbf{x}_i$ in $\overline{\mathcal{T}_h^{\text{out}}}$ where the interpolant $I_h u$ introduced in sec. 3.3 is equal to $u(\mathbf{x}_i)$; filled squares denote the rest of nodes where $I_h u = 0$. On the right: the reference element $\hat{K}$ and the image $\hat{\Lambda}_K = T_K^{-1}\Lambda$ of Λ under T_K^{-1}, where $K \in \mathcal{T}_h^{\text{in}}$ is an element crossed by Λ. The distance Δ is a fraction of $\Delta_K = \max_{\hat{\mathbf{x}} \in \hat{K}} \text{dist}(\hat{\mathbf{x}}, \hat{\Lambda}_K) \lesssim 1$.

We will make use of the following auxiliary inequalities.

Lemma 3.1. *The mesh splitting $\mathcal{T}_h = \mathcal{T}_h^{\text{in}} \cup \mathcal{T}_h^{\text{out}}$ satisfies the following properties,*

$$r_K \lesssim h_K, \; h_K \lesssim \bar{r}_K \lesssim h_K \quad \forall K \in \mathcal{T}_h^{\text{in}}, \tag{16a}$$

$$\bar{r}_K \lesssim r_K \quad \forall K \in \mathcal{T}_h^{\text{out}}. \tag{16b}$$

In other words, the minimal distance of elements $K \in \mathcal{T}_h^{\mathrm{in}}$ from Λ is controlled by their size, and for any $K \in \mathcal{T}_h$ the minimal and maximal distance are equivalent, uniformly with respect to K.

Proof. By definition, $r_K \leq \delta h_K \; \forall K \in \mathcal{T}_h^{\mathrm{in}}$ and, for any $K \in \mathcal{T}_h$, we clearly have $\bar{r}_K \gtrsim \frac{1}{2} h_K$. Moreover, if $K \in \mathcal{T}_h^{\mathrm{in}}$, $\bar{r}_K \leq r_K + h_K \leq (\delta+1) h_K$ so that (16a) follows. Conversely, if $K \in \mathcal{T}_h^{\mathrm{out}}$ we have $\bar{r}_K \leq r_K + h_K \leq r_K + \frac{1}{\delta} r_K = \frac{1+\delta}{\delta} r_K$, yielding (16b). □

3.2 Stability of the finite element approximation

We introduce the following discrete norm:

$$\|u_h\|_{h,\alpha}^2 := \sum_{K \in \mathcal{T}_h} (\bar{r}_K)^{2\alpha} \|u_h\|_{L^2(K)}^2. \tag{17}$$

Lemma 3.2. *Let $|\alpha| < t$, $t \in [0,1)$. We have the following norm equivalence, where the constants of the inequalities only depend on t:*

$$\|u_h\|_{h,\alpha} \lesssim \|u_h\|_{L^2_\alpha(\Omega)} \lesssim \|u_h\|_{h,\alpha} \quad \forall u_h \in W_h^k.$$

Proof. For the sake of simplicity, we consider only the case $\alpha \geq 0$. Let $K \in \mathcal{T}_h$, $\mathbf{x} \in K$; we have $d(\mathbf{x})^{2\alpha} \leq (\bar{r}_K)^{2\alpha}$, so that $\|u_h\|_{L^2_\alpha(\Omega)} \leq \|u_h\|_{h,\alpha}$. To obtain the inverse estimate, let us distinguish two cases. If $K \in \mathcal{T}_h^{\mathrm{out}}$, we use Lemma 3.1, eq. (16b), and we have $(\bar{r}_K)^{2\alpha} \|u_h\|_{L^2(K)}^2 \lesssim r_K^{2\alpha} \|u_h\|_{L^2(K)}^2 \leq \|d^\alpha u_h\|_{L^2(K)}^2$. Now let us show that a similar estimate holds true if $K \in \mathcal{T}_h^{\mathrm{in}}$. In that case, let $\hat{K}$ be the reference element, and let $T_K : \hat{K} \to K$ be the affine transformation mapping $\hat{K}$ onto the actual element K. Let $\hat{u}_h = u_h \circ T_K$, let $\hat{\Lambda}_K = T_K^{-1} \Lambda$ such that Λ is the image of $\hat{\Lambda}_K$ under T_K, and let $\hat{d}(\hat{\mathbf{x}}) = \mathrm{dist}(\hat{\mathbf{x}}, \hat{\Lambda}_K)$. Thanks to shape regularity, the eigenvalues of the jacobian matrix of T_K are uniformly upper and lower bounded by h_K. Hence, distances are transformed according to $d(T_K \hat{\mathbf{x}}) \gtrsim h_K \hat{d}(\hat{\mathbf{x}})$. As a result,

$$\|d^\alpha u_h\|_{L^2(K)}^2 = \int_K d^{2\alpha} u_h^2 = \frac{|K|}{|\hat{K}|} \int_{\hat{K}} [d(T_K \hat{\mathbf{x}})]^{2\alpha} \hat{u}_h^2 \gtrsim h_K^{2\alpha} \frac{|K|}{|\hat{K}|} \int_{\hat{K}} \hat{d}^{2\alpha} \hat{u}_h^2.$$

Let us introduce the subset $\hat{K}_\Delta = \{\hat{\mathbf{x}} \in \hat{K} : \mathrm{dist}(\hat{\mathbf{x}}, \hat{\Lambda}_K) > \Delta\}$, where $\Delta > 0$ is a parameter (see fig. 2); we have

$$\Delta^{2\alpha} \|\hat{u}_h\|_{L^2(\hat{K}_\Delta)}^2 \leq \int_{\hat{K}} \hat{d}^{2\alpha} \hat{u}_h^2.$$

Note that, at least for Δ small, $\inf_K |\hat{K}_\Delta|$ cannot degenerate with respect to $|\hat{K}| \gtrsim 1$. Indeed, thanks to (11), we can estimate $|\hat{K}| - |\hat{K}_\Delta| \lesssim \pi \Delta^2$, irrespective of the position of $\hat{\Lambda}_K$; hence, choosing Δ sufficiently small, we have $|\hat{K}_\Delta| \geq c'_\Delta |\hat{K}|$ where the constant $c'_\Delta = 1 - \mathcal{O}(\Delta^2)$ depends on Δ but not on $K \in \mathcal{T}_h^{\mathrm{in}}$. Similarly, it is easy to see that $\|\hat{u}_h\|_{L^2(\hat{K}_\Delta)} \geq c_\Delta \|\hat{u}_h\|_{L^2(\hat{K})}^2 \; \forall \hat{u}_h \in \mathbb{P}_k(\hat{K})$ (it suffices to see that the estimates hold for the local basis functions on $\hat{K}$), where again the constant c_Δ depends only on Δ and not on the shape of $\hat{K}_\Delta$. Hence, using $\alpha \leq t$ and (16a), we conclude

$$\begin{aligned}\|d^\alpha u_h\|_{L^2(K)}^2 &\gtrsim c_\Delta \Delta^{2\alpha} h_K^{2\alpha} \frac{|K|}{|\hat{K}|} \|\hat{u}_h\|_{L^2(\hat{K})}^2 \geq c_\Delta \Delta^{2t} h_K^{2\alpha} \frac{|K|}{|\hat{K}|} \|\hat{u}_h\|_{L^2(\hat{K})}^2 \\ &\gtrsim c_\Delta \Delta^{2t} (\bar{r}_K)^{2\alpha} \|u_h\|_{L^2(K)}^2.\end{aligned}$$

□

Let us establish the discrete analogous of Lemma 2.1. To this end, let $M_h^{k-1} = \{\boldsymbol{q}_h \in L^2(\Omega) : \boldsymbol{q}_h \in \mathbb{P}^{k-1}(K),\ \forall K \in \mathcal{T}_h\}$ be the space of vector, discontinuous $\mathbb{P}_{k-1}$ finite element functions.

Lemma 3.3 (Decomposition of M_h^{k-1})**.** *Let $s \in (-1,1)$. For each $\boldsymbol{q}_h \in M_h^{k-1}$, there exists a unique couple $(\boldsymbol{\sigma}_h, u_h) \in M_h^{k-1} \times W_h^k$ such that*

$$\boldsymbol{q}_h = \nabla u_h + \boldsymbol{\sigma}_h, \quad \langle \boldsymbol{\sigma}_h, \nabla v_h \rangle_\Omega = 0\ \forall v \in W_h^k,$$

$$\|\nabla u_h\|_{h,s} \leq 2\|\boldsymbol{q}_h\|_{h,s}, \quad \|\boldsymbol{\sigma}_h\|_{h,s} \leq \|\boldsymbol{q}_h\|_{h,s}.$$

In other words, we have the decomposition $M_h^{k-1} = (\nabla W_h^k) \bigoplus (\nabla W_h^k)^\perp$.

Proof. As for Lemma 2.1, the considered problem can be recast as a generalized saddle point problem. In fact, $(\boldsymbol{\sigma}_h, u_h) \in M_h^{k-1} \times W_h^k$ are such that

$$\begin{cases} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(u_h, \boldsymbol{\tau}_h) = F(\boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in M_h^{k-1}, \\ b(v_h, \boldsymbol{\sigma}_h) = 0 & \forall v_h \in W_h^k, \end{cases}$$

where a, b and F are as in the proof of Lemma 2.1, eq. (10). Let us consider the inf-sup inequalities needed in the generalized BNB theorem using the discrete norms. We have $a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \leq \|\boldsymbol{\sigma}_h\|_{h,s}\|\boldsymbol{\tau}_h\|_{h,-s}$, $b(u_h, \boldsymbol{\tau}_h) \leq \|\nabla u_h\|_{h,s}\|\boldsymbol{\tau}_h\|_{h,-s}$ and $b(v_h, \boldsymbol{\sigma}_h) \leq \|\nabla v_h\|_{h,-s}\|\boldsymbol{\sigma}_h\|_{h,s}$, so that all the forms are bounded in the respective discrete weighted norms, with continuity constant equal to 1. For any $\boldsymbol{\sigma}_h \in M_h^{k-1}$, the function $\boldsymbol{\tau}_h \in M_h^{k-1}$ defined by $\boldsymbol{\tau}_{h|K} = (\bar{r}_K)^{2s}\boldsymbol{\sigma}_{h|K}$ $\forall K \in \mathcal{T}_h$ is such that $\|\boldsymbol{\tau}_h\|_{h,-s} = \|\boldsymbol{\sigma}\|_{h,s}$, $a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = \|\boldsymbol{\sigma}_h\|^2_{h,s}$. Those identities still if we swap $\boldsymbol{\sigma}_h$ and $\boldsymbol{\tau}_h$ and change the sign of s. Hence, we have

$$\sup_{\boldsymbol{\tau}_h \in M_h^{k-1}} \frac{a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{h,-s}} \geq \alpha_1 \|\boldsymbol{\sigma}_h\|_{h,s}, \qquad \sup_{\boldsymbol{\sigma}_h \in M_h^{k-1}} \frac{a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\sigma}_h\|_{h,s}} \geq \alpha_2 \|\boldsymbol{\tau}_h\|_{h,-s},$$

with $\alpha_1 = \alpha_2 = 1$. Moreover, for any $u_h \in W_h^k$, choosing $\boldsymbol{\tau}_{h|K} = (\bar{r}_K)^{2s}\nabla u_{h|K}$ $\forall K \in \mathcal{T}_h$ yields $b(u_h, \boldsymbol{\tau}_h) = \|\nabla u_h\|^2_{h,s}$, $\|\boldsymbol{\tau}_h\|_{h,-s} = \|\nabla u_h\|_{h,s}$. Again, the same identities are obtained by replacing u_h by v_h, $\boldsymbol{\tau}_h$ by $\boldsymbol{\sigma}_h$ and changing the sign of s, so that we have

$$\sup_{\boldsymbol{\tau}_h \in M_h^{k-1}} \frac{b(u_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{h,s}} \geq \beta_1 \|\nabla u_h\|_{h,s}, \qquad \sup_{\boldsymbol{\sigma}_h \in M_h^{k-1}} \frac{b(v_h, \boldsymbol{\sigma}_h)}{\|\boldsymbol{\sigma}_h\|_{h,s}} \geq \beta_2 \|\nabla v_h\|_{h,-s},$$

with $\beta_1 = \beta_2 = 1$. All the hypotheses of the generalized BNB theorem are thus satisfied, and the proof is concluded. □

We are now in a position to establish the stability of the Galerkin approximation (13).

Theorem 3.1. *Let $0 < \alpha \leq t < 1$. We have*

$$\sup_{v_h \in W_h^k} \frac{\langle \nabla u_h, \nabla v_h \rangle_\Omega}{\|\nabla v_h\|_{L^2_{-\alpha}}} \gtrsim \|\nabla u_h\|_{L^2_\alpha}, \qquad \sup_{u_h \in W_h^k} \frac{\langle \nabla u_h, \nabla v_h \rangle_\Omega}{\|\nabla u_h\|_{L^2_\alpha}} \gtrsim \|\nabla v_h\|_{L^2_{-\alpha}}, \tag{18}$$

where the constants in the above inequalities depend t but not on α. The Galerkin approximation (13) is thus stable and we have the optimal error estimate

$$\|u - u_h\|_{W_\alpha} \leq C(t) \inf_{v_h \in W_h^k} \|u - v_h\|_{W_\alpha}. \tag{19}$$

Proof. Let $u_h \in W_h^k$; define $\boldsymbol{q}_h \in M_h^{k-1}$ by $\boldsymbol{q}_{h|K} = (\bar{r}_K)^{2\alpha}\nabla u_{h|K}$ $\forall K \in \mathcal{T}_h$. Thanks to Lemma 3.3 with $s = -\alpha$, there exists a couple $(\boldsymbol{\tau}_h, v_h) \in M_h^{k-1} \times W_h^k$ s.t. $\|\nabla v_h\|_{h,-\alpha} \leq 2\|\boldsymbol{q}_h\|_{h,-\alpha} = 2\|\nabla u_h\|_{h,\alpha}$, and

$$\langle \nabla u_h, \nabla v_h\rangle_\Omega = \langle \nabla u_h, \boldsymbol{q}_h\rangle_\Omega - \langle \nabla u_h, \boldsymbol{\tau}_h\rangle_\Omega = \langle \nabla u_h, \boldsymbol{q}_h\rangle_\Omega = \|\nabla u_h\|^2_{h,\alpha}.$$

Hence, the first inf-sup condition in (18) follows, thanks to the norm equivalence established by Lemma 3.2. The second condition is obtained similarly. The error estimate (19) then follows thanks to Galerkin orthogonality and (18) in the standard way, see Ern and Guermond [2004]. □

The last step for the numerical analysis of our model problem (8) concerns the convergence rates of the finite element approximation, addressed in sec. 3.3. To this end, we will use the Kondrat′ev-type weighted spaces $V^{l+1}_{l+\alpha}(\Omega)$, $l \geq 1$, and we will derive suitable error estimates on uniform and graded meshes assuming that the exact solution $u \in V^{l+1}_{l+\alpha}(\Omega)$. Note that the logarithmic singularity $u(\mathbf{x}) = -\ln d(\mathbf{x})$ (our "reference" fundamental solution) is in $V^{l+1}_{l+\alpha}$ for any $\alpha > 0$, $l \geq 0$.

3.3 Convergence rates of the finite element approximations

In this section α will be always considered to be *non negative*, i.e. $\alpha \in [0, 1)$. Let l be a positive integer, $1 \leq l \leq k$; we will introduce a suitable interpolation operator $I_h : V^{l+1}_{l+\alpha} \to W_h^k$ that will be employed to study the convergence rates of our finite element scheme. Let us recall that functions in $V^{l+1}_{l+\alpha}$ are non-smooth on Λ while being locally H^{l+1} on elements whose closure does not intersect Λ. Following this observation we require that I_h satisfies the following approximation properties: for any $u \in V^{l+1}_{l+\alpha}$ and for $0 \leq m \leq l$,

$$|u - I_h u|_{H^m(K)} \leq C^{I_h} h_K^{l+1-m} |u|_{H^{l+1}(K)}, \qquad \text{if } K \in \mathcal{T}_h^{\text{out}}, \tag{20}$$

$$|I_h u|_{V^m_{m-1+\alpha}(K)} \leq C^{I_h}_\alpha \|u\|_{V^{l+1}_{l+\alpha}(\mathcal{T}_h^{\text{in}})}, \qquad \text{if } K \in \mathcal{T}_h^{\text{in}}, \tag{21}$$

where C^{I_h}, $C^{I_h}_\alpha$ are positive constants independent of h. An interpolator I_h satisfying (20) and (21) can be constructed as follows. We will consider a Lagrange interpolant, i.e. $I_h u = \sum_{i=1}^{N_h} I_{h,i}(u)\phi_i$, where $\{\phi_i,\ i = 1, \ldots, N_h\}$ are the piecewise $\mathbb{P}_k$ Lagrange basis functions, N_h is the number of nodes (degrees of freedom), and $I_{h,i}(u) \in \mathbb{R}$ is the i-th nodal value. We shall denote $\mathbf{x}_i$, $i = 1, \ldots, N_h$, the corresponding nodes.

First, on all $K \in \mathcal{T}_h^{\text{out}}$ we define I_h as the standard Lagrange interpolant Π_h^k of degree k, which is well-defined since $H^{l+1}(K) \hookrightarrow C(K)$:

$$I_h u_{|K} = \Pi_h^k u_{|K} \quad \forall K \in \mathcal{T}_h^{\text{out}}. \tag{22}$$

Equation (22) is equivalent to $I_{h,i}(u) = u(\mathbf{x}_i)$ on all nodes $\mathbf{x}_i \in \overline{\mathcal{T}_h^{\text{out}}}$. Since W_h^k consists of continuous piecewise polynomials functions, this implies $I_{h,i}(u) = u(\mathbf{x}_i)$ for all nodes $\mathbf{x}_i \in \partial\mathcal{T}_h^{\text{in}}$ on the boundary of $\mathcal{T}_h^{\text{in}}$. Next, we define $I_{h,i}(u) = 0$ at all nodes $\mathbf{x}_i$ in the *interior* of $\mathcal{T}_h^{\text{in}}$,

$$I_{h,i}(u) = 0 \quad \text{if } \mathbf{x}_i \notin \overline{\mathcal{T}_h^{\text{out}}}, \tag{23}$$

which completes the definition of I_h (see fig. 2). Let us show that this is sufficient for (21) to hold.

Lemma 3.4. *The interpolant $I_h : V^{l+1}_{l+\alpha} \to W_h^k$ defined by (22-23) satisfies (20) and (21). The constant $C^{I_h}_\alpha$ depends on α; precisely, $C^{I_h}_\alpha = \mathcal{O}(1/\sqrt{\alpha})$ for $\alpha \to 0$.*

Proof. The interpolant I_h clearly satisfies (20); let us show that (21) holds as well.

Let $K \in \mathcal{T}_h^{\text{in}}$ such that K shares a node with at least one element $K' \in \mathcal{T}_h^{\text{out}}$. It suffices to consider this case, since otherwise we have $I_h u_{|K} = 0$, so that (21) trivially holds.

Let $D_i = \{i = 0, \dots, N_h : \mathbf{x}_i \in \overline{K} \cap \overline{K'}, K' \in \mathcal{T}_h^{\text{out}}\}$ be the set of indexes i related to the nodes of K shared with elements of $\mathcal{T}_h^{\text{out}}$. Thanks to (23), we have

$$|I_h u|_{V^m_{m-1+\alpha}(K)} \leq \sum_{i \in D_i} |I_{h,i}(u)| |\phi_i|_{V^m_{m-1+\alpha}(K)}. \tag{24}$$

For all $i \in D_i$, $\mathbf{x}_i \in \overline{\mathcal{T}_h^{\text{out}}}$. So let K' be any element of $\mathcal{T}_h^{\text{out}}$ sharing the node $\mathbf{x}_i$ with K. Using the standard affine mapping $T_{K'} : \hat{K} \to K'$ from the reference element $\hat{K}$, defining $\hat{u}(\hat{\mathbf{x}}) = u(T_{K'}(\hat{\mathbf{x}}))$, thanks to the Sobolev embedding theorem in dimension $N = 3$ on the reference element and mapping back to K' we have

$$\begin{aligned} |I_{h,i}(u)| &\leq \|u\|_{L^\infty(K')} = \|\hat{u}\|_{L^\infty(\hat{K})} \lesssim \|\hat{u}\|_{H^{l+1}(\hat{K})} = \left(\sum_{i=0}^{l+1} |\hat{u}|^2_{H^i(\hat{K})} \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{i=0}^{l+1} h_{K'}^{2i-N} |u|^2_{H^i(K')} \right)^{\frac{1}{2}}. \end{aligned} \tag{25}$$

Note that, since K' shares a node with $K \in \mathcal{T}_h^{\text{in}}$, it is sufficiently close to Λ for the following estimates to be true:

$$h^{\frac{1}{\mu}} \lesssim r_{K'} \leq \bar{r}_{K'} \lesssim h^{\frac{1}{\mu}}. \tag{26}$$

In fact, $r_{K'} \geq \bar{r}_K \gtrsim h^{\frac{1}{\mu}}$ thanks to (16a); moreover, owing to (16a) and (15a,b), we have $\bar{r}_{K'} \leq \bar{r}_K + h_{K'} \lesssim h^{\frac{1}{\mu}} + Chr_{K'}^{1-\mu} = h^{\frac{1}{\mu}}[1 + C(h^{\frac{1}{\mu}}/r_{K'})^{1-\mu}] \lesssim h^{\frac{1}{\mu}}$, where we used $r_{K'} \gtrsim h^{\frac{1}{\mu}}$ in the last inequality. Now, observing that

$$\|u\|^2_{L^2(K')} \leq (\bar{r}_{K'})^{-2(\alpha-1)} \|u\|_{L^2_{\alpha-1}(K')}, \quad |u|^2_{H^i(K')} \leq r_{K'}^{-2(i-1+\alpha)} |u|^2_{V^i_{i-1+\alpha}(K')} \ \forall i \geq 1,$$

eq. (25) leads to

$$|I_{h,i}(u)| \lesssim h_{K'}^{-N/2} \left((\bar{r}_{K'})^{-2(\alpha-1)} \|u\|^2_{L^2_{\alpha-1}(K')} + \sum_{i=1}^{l+1} h_{K'}^{2i} r_{K'}^{-2(i-1+\alpha)} |u|^2_{V^i_{i-1+\alpha}(K')} \right)^{\frac{1}{2}}$$

and, owing to (26),

$$|I_{h,i}(u)| \lesssim h^{-N/2} \left(h^{-2(\alpha-1)/\mu} \|u\|^2_{L^2_{\alpha-1}(K')} + \sum_{i=1}^{l+1} h^{-2(\alpha-1)/\mu} |u|^2_{V^i_{i-1+\alpha}(K')} \right)^{\frac{1}{2}}. \tag{27}$$

Now let us consider $|\phi_i|_{V^m_{m-1+\alpha}(K)}$: letting $\hat{\phi}_i(\hat{\mathbf{x}}) = \phi_i(T_K(\hat{\mathbf{x}}))$, for $m \geq 1$ we have (see for instance Apel [2004], Lemma 2)

$$\begin{aligned} |\phi_i|_{V^m_{m-1+\alpha}(K)} &\leq (\bar{r}_K)^{m-1+\alpha} |\phi_i|_{H^m(K)} \\ &\lesssim (\bar{r}_K)^{m-1+\alpha} h_K^{-m+N/2} |\hat{\phi}_i|_{H^m(\hat{K})} \lesssim h_K^{\alpha-1+N/2}, \end{aligned} \tag{28}$$

where in the last estimate we used eq. (16a) and $|\hat{\phi}_i|_{H^m(\hat{K})} \lesssim 1$. For $m = 0$ we have[1]:

$$|\phi_i|_{V^0_{\alpha-1}(K)} = \|\phi_i\|_{L^2_{\alpha-1}(K)} \leq \|d^{\alpha-1}\|_{L^2(K)} \lesssim \alpha^{-\frac{1}{2}} h_K^{\alpha-1+N/2}. \tag{29}$$

[1]Note that in $N = 3$ dimensions and using cylindrical coordinates:

$$\|d^{\alpha-1}\|^2_{L^2(K)} \leq 2\pi \int_{\Lambda \cap K} \int_0^{h_K} r^{2(\alpha-1)} \cdot r \, \mathrm{d}r \, \mathrm{d}s \leq \frac{\pi}{\alpha} h_K^{2\alpha+1} = \frac{\pi}{\alpha} h_K^{2(\alpha-1+N/2)}.$$

Substituting (28), (29) and (27) in (24), observing that $h_K^{\alpha-1+N/2} \lesssim h^{(N/2+\alpha-1)/\mu}$ by (15b), we get

$$|I_h u|_{V^m_{m-1+\alpha}(K)} \lesssim \left(\frac{1}{\alpha} \|u\|^2_{L^2_{\alpha-1}(K')} + \sum_{i=1}^{l+1} |u|^2_{V^i_{i-1+\alpha}(K')} \right)^{\frac{1}{2}} \leq C^{I_h}_\alpha \|u\|_{V^{l+1}_{l+\alpha}(\mathcal{T}_h^{\text{in}})},$$

with $C^{I_h}_\alpha = \mathcal{O}(1/\sqrt{\alpha})$, which concludes the proof. □

We are now in a position to establish our principal interpolation estimate.

Theorem 3.2. *Let $\alpha \in (0,1)$, $0 \leq m \leq l$, $l \leq k$, and let $\epsilon \in (0, \alpha)$; then we have*

$$|u - I_h u|_{V^m_{m-1+\alpha}(\Omega)} \lesssim C^{I_h}_\alpha h^{l+1-m} \|u\|_{V^{l+1}_{l+\epsilon}(\Omega)} \quad \forall u \in V^{l+1}_{l+\epsilon}, \tag{30}$$

provided that the mesh grading parameter satisfies

$$\mu \leq \frac{\alpha - \epsilon}{l + 1 - m}. \tag{31}$$

Otherwise, we have the following suboptimal interpolation estimate,

$$|u - I_h u|_{V^m_{m-1+\alpha}(\Omega)} \lesssim C^{I_h}_\alpha h^{\frac{\alpha-\epsilon}{\mu}} \|u\|_{V^{l+1}_{l+\epsilon}(\Omega)}. \tag{32}$$

Proof. Let us derive the local interpolation estimate on each element $K \in \mathcal{T}_h$. We consider first the case in which $K \in \mathcal{T}_h^{\text{out}}$. In this case, owing to (20) and (15b) we have

$$\begin{aligned} |u - I_h u|_{V^m_{m-1+\alpha}(K)} &\leq (\bar{r}_K)^\alpha |u - I_h u|_{H^m(K)} \leq (\bar{r}_K)^\alpha C^{I_h} h_K^{l+1-m} |u|_{H^{l+1}(K)} \\ &\leq (\bar{r}_K)^\alpha r_K^{-(l+1-m)-\epsilon} C^{I_h} h_K^{l+1-m} |u|_{V^{l+1}_{l+1-m+\epsilon}(K)} \\ &\lesssim (\bar{r}_K)^\alpha r_K^{-\epsilon-\mu(l+1-m)} h^{l+1-m} |u|_{V^{l+1}_{l+1-m+\epsilon}(K)}. \end{aligned} \tag{33}$$

Thanks to Lemma 3.1, we have $\bar{r}_K \lesssim r_K$, so that

$$\begin{aligned} |u - I_h u|_{V^m_\alpha(K)} &\lesssim r_K^{\alpha-\epsilon-\mu(l+1-m)} h^{l+1-m} |u|_{V^{l+1}_{l+1-m+\epsilon}(K)} \\ &\lesssim h^{l+1-m} |u|_{V^{l+1}_{l+1-m+\epsilon}(K)} \end{aligned} \tag{34}$$

provided that $\alpha - \epsilon - \mu(l+1-m) \geq 0$, that is precisely eq. (31).

Now assume that $K \in \mathcal{T}_h^{\text{in}}$. In this case, we simply write

$$\begin{aligned} |u - I_h u|_{V^m_{m-1+\alpha}(K)} &\lesssim |u|_{V^m_{m-1+\alpha}(K)} + |I_h u|_{V^m_{m-1+\alpha}(K)} \lesssim C^{I_h}_\alpha \|u\|_{V^{l+1}_{l+\alpha}(K)} \\ &\lesssim C^{I_h}_\alpha (\bar{r}_K)^{\alpha-\epsilon} \|u\|_{V^{l+1}_{l+\epsilon}(K)} \lesssim C^{I_h}_\alpha h^{(\alpha-\epsilon)/\mu} \|u\|_{V^{l+1}_{l+\epsilon}(K)} \\ &\lesssim C^{I_h}_\alpha h^{l+1-m} \|u\|_{V^{l+1}_{l+\epsilon}(K)}, \end{aligned} \tag{35}$$

where we used the the continuity of the embedding $V^{l+1}_{l+\alpha}(K) \hookrightarrow V^m_{m-1+\alpha}(K)$ for $m \leq l$ (eq. (4)), the continuity of the intepolant $I_h : V^{l+1}_{l+\alpha}(K) \to V^m_{m-1+\alpha}(K)$ (eq. (21)), the estimates (16a,15b), and finally (31).

If (31) is not satisfied, (34) and (35) give the suboptimal estimate (32). □

As a consequence of theorem 3.2 for $m = 0, 1$, $l = k$, we have the following estimates of the interpolation errors using $\mathbb{P}_k$ finite elements on (uniform or) graded meshes.

Corollary 3.1. *If $u \in V^{k+1}_{k+\epsilon}(\Omega)$ for some $\epsilon \in (0,\alpha)$, then we have*

$$\text{if } \frac{\alpha-\epsilon}{k} < \mu \le 1: \quad \|u - I_h u\|_{V^1_\alpha} \lesssim C^{I_h}_\alpha h^{\frac{\alpha-\epsilon}{\mu}} \|u\|_{V^{k+1}_{k+\epsilon}};$$
$$\text{if } 0 < \mu \le \frac{\alpha-\epsilon}{k}: \quad \|u - I_h u\|_{V^1_\alpha} \lesssim C^{I_h}_\alpha h^k \|u\|_{V^{k+1}_{k+\epsilon}}.$$

In particular, optimal convergence rates in V^1_α are obtained for $\mu = \frac{\alpha-\epsilon}{k}$.

Proof. Immediate consequences of Theorem 3.2. □

We conclude our analysis with the following result, establishing the convergence rates of the finite element approximation of the model problem (8) on uniform and graded meshes. It is obtained by choosing $v_h = I_h u$ in (19) and using corollary 3.1.

Corollary 3.2. *Let $u \in W_\alpha$ be the unique solution of the model problem (8), and let $u_h \in W^k_h$ be the finite element approximation defined by (13). If $u \in V^{k+1}_{k+\epsilon}(\Omega)$ for some $\epsilon \in (0,\alpha)$, we have the following error bounds:*

$$\begin{aligned} &\text{if } \frac{\alpha-\epsilon}{k} < \mu \le 1: \quad \|u - u_h\|_{V^1_\alpha} \lesssim C^{I_h}_\alpha h^{\frac{\alpha-\epsilon}{\mu}} \|u\|_{V^{k+1}_{k+\epsilon}}, \\ &\text{if } 0 < \mu \le \frac{\alpha-\epsilon}{k}: \quad \|u - u_h\|_{V^1_\alpha} \lesssim C^{I_h}_\alpha h^k \|u\|_{V^{k+1}_{k+\epsilon}}. \end{aligned} \tag{36}$$

3.4 Validation and numerical results

In order to verify the theoretical estimates presented in this section, we performed several convergence tests on a simple problem.

Let $\Omega = (0,1)^3 = \widetilde{\Omega} \times (0,1)$, where $\widetilde{\Omega} = (0,1)^2$ is the unit square in $\mathbb{R}^2$. Let $\Lambda = \widetilde{\mathbf{x}}_0 \times (0,1)$, where $\widetilde{\mathbf{x}}_0 \in \widetilde{\Omega}$ (in our computations we considered $\widetilde{\mathbf{x}}_0 = (0,3,0.3)$, see fig. 3). For any $q \in L^2(\Lambda)$, we denote by $q\delta_\Lambda$ the linear functional defined by $\langle q\delta_\Lambda, v\rangle = \int_\Lambda q(s)v(s)$. This is a measure, but, as already pointed out, it is also bounded on $H^1_{-\alpha}(\Omega)$, $\alpha > 0$.

We consider the following problem:

$$\begin{cases} -\Delta u + 2\pi RD(u_0 - \bar{u})\delta_\Lambda = 0 & \text{in } \Omega, \\ u = u_e & \text{on } \partial\Omega, \end{cases} \tag{37}$$

where u_0 is a positive constant, $\bar{u}(\mathbf{x})$ is the average value of u computed on circles centered on $\mathbf{x} \in \Lambda$ with radius R, i.e. $\bar{u}(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi} u(\mathbf{x} + R\mathbf{e}_\theta)\,\mathrm{d}\theta$, being $\mathbf{e}_\theta = (\cos\theta, \sin\theta, 0)^T$. The Dirichlet data u_e are provided by an exact solution, whose value at any point $\mathbf{x} = (\widetilde{\mathbf{x}}, z)$, $\mathbf{x} \in \widetilde{\Omega}$, is given by

$$u_e(\mathbf{x}) = -u_0 \frac{RD}{1 - RD\ln R} \ln r, \tag{38}$$

where $r = |\widetilde{\mathbf{x}} - \widetilde{\mathbf{x}}_0|$. Note that (38) is independent of the z coordinate: problem (37) is indeed invariant w.r.t. traslations along z and hence we can solve the corresponding 2D problem in which Ω is replaced by $\widetilde{\Omega}$. In this case, $\bar{u}$ is a scalar value given by $\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(\widetilde{\mathbf{x}}_0 + R\widetilde{\mathbf{e}}_\theta)\,\mathrm{d}\theta$, $\widetilde{\mathbf{e}}_\theta = (\cos\theta, \sin\theta)$.

Problem (37) is a special instance of elliptic problem with measure data that depend on the solution itself; it corresponds to (1) with $\mathsf{K} = 1$, $q = q(u) = 2\pi RD(u_0 - \bar{u})$. However, at least for $\beta = 2\pi RD$ small, the variational formulation is inf-sup stable w.r.t. weighted norms, in both the continuous the discrete settings (see sec. 4). Moreover, u_e is of class $V^{k+1}_{k+\epsilon}$ for any $\epsilon > 0$ and positive integer k. Hence, corollary 3.2 applies and we expect convergence rates $p = k$ in V^1_α.

We computed the standard finite element solution with uniform and graded meshes, and reported the errors in different weighted norms. Specifically, in Table 1 we have the case of uniform mesh ($\mu = 1$), with polynomial degree $k = 1, 2$. We immediately recognize the suboptimal rates predicted by Corollary 3.2. Indeed, we observe (for $\alpha = 0.5$) convergence of order $p = \alpha$ in the (equivalent) V^1_α and H^1_α norms, $p = \alpha/2$ in the (equivalent) $V^1_{\alpha/2}$ and $H^1_{\alpha/2}$ norms. We also observe convergence of order $p = 1 + \alpha$ in the L^2_α norms and $p = 1 \pm \alpha/2$ in the $L^2_{\pm\alpha/2}$ norms, as predicted by the Aubin-Nitsche theorem in weighted norm (that holds true thanks to our interpolation error estimates under the assumption of weighted elliptic regularity $\Delta^{-1} : L^2_\alpha \to V^2_{1+\alpha}$).

In Table 2 we have the case of graded meshes, with polynomial degree $k = 1, 2$, and consider always $\alpha = 0.5$. From our interpolation estimates, we expect optimal convergence rates provided that $\mu \leq \frac{\alpha-\epsilon}{k} < \frac{\alpha}{k}$. In the case $k = 1$ of Table 2(a), we have $\mu = \alpha/2 < \alpha/k$; we observe optimal rates of order $p \simeq 1$ in the (equivalent) V^1_α and H^1_α norms and $p \simeq 2$ in the L^2_α norm. If we turn to $k = 2$ in Table 2(b) with $\mu = \alpha/2.2 < \alpha/k$ we still observe optimal rates of order $p \simeq 2$ in the (equivalent) H^1_α and V^1_α norms, $p \simeq 3$ in the L^2_α norm.

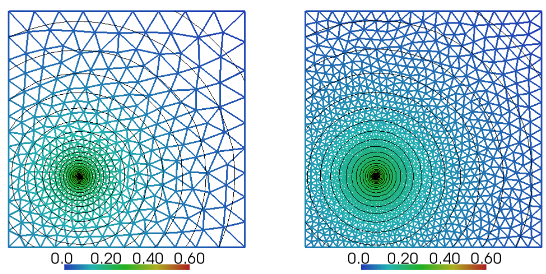

Figure 3: A pair of graded meshes ($\mu = 0.25$) with isolines of the solution u_h on the square $\widetilde{\Omega} = (0, 1)^2$.

(a)

h	L^2 $(\times 10^{-3})$	L^2_α $(\times 10^{-3})$	H^1_α $(\times 10^{-1})$	V^1_α $(\times 10^{-1})$	$L^2_{\alpha/2}$ $(\times 10^{-3})$	$L^2_{-\alpha/2}$ $(\times 10^{-2})$	$H^1_{\alpha/2}$ $(\times 10^{-1})$	$V^1_{\alpha/2}$ $(\times 10^{-1})$
1	0.83086	0.20577	0.11643	0.11671	0.38670	0.20196	0.24041	0.24122
1/2	0.41872	0.07527	0.08375	0.08385	0.16459	0.12118	0.20311	0.20347
1/4	0.20619	0.02654	0.05985	0.05989	0.06831	0.07097	0.17123	0.17137
1/8	0.10542	0.01006	0.04246	0.04247	0.02951	0.04316	0.14397	0.14404
1/16	0.05222	0.00330	0.03011	0.03011	0.01212	0.02550	0.12112	0.12115
p	0.99	1.49	0.49	0.49	1.25	0.75	0.25	0.25

(b)

h	L^2 $(\times 10^{-3})$	L^2_α $(\times 10^{-3})$	H^1_α $(\times 10^{-2})$	V^1_α $(\times 10^{-2})$	$L^2_{\alpha/2}$ $(\times 10^{-3})$	$L^2_{-\alpha/2}$ $(\times 10^{-3})$	$H^1_{\alpha/2}$ $(\times 10^{-1})$	$V^1_{\alpha/2}$ $(\times 10^{-1})$
1	0.22584	0.05573	0.64311	0.64348	0.10989	0.48447	0.13725	0.13733
1/2	0.11200	0.01942	0.45511	0.45524	0.04579	0.28501	0.11548	0.11551
1/4	0.05617	0.00697	0.32176	0.32181	0.01934	0.17012	0.09709	0.09711
1/8	0.02800	0.00243	0.22757	0.22758	0.00810	0.10076	0.08165	0.08166
1/16	0.01399	0.00086	0.16092	0.16093	0.00340	0.05987	0.06866	0.06867
p	1.00	1.5	0.5	0.5	1.25	0.75	0.25	0.25

Table 1: Convergence rates of the error $\|u - u_h\|$ on uniform meshes ($\mu = 1$), for polynomial degrees $k = 1$ (a) and $k = 2$ (b), in different norms, for $\alpha = 0.5$

(a)

h	L^2_α ($\times 10^{-2}$)	H^1_α ($\times 10^{-1}$)	V^1_α ($\times 10^{-1}$)
1/1	0.17521	0.27274	0.27394
1/2	0.03950	0.13880	0.13892
1/4	0.00978	0.07055	0.07056
1/8	0.00237	0.03470	0.03470
1/16	0.00121	0.01721	0.01720
p	2.07	0.99	0.99

(b)

h	L^2_α ($\times 10^{-4}$)	H^1_α ($\times 10^{-2}$)	V^1_α ($\times 10^{-2}$)
1/1	1.36870	1.07253	1.07276
1/2	0.18435	0.17712	0.17713
1/4	0.02417	0.04048	0.04048
1/8	0.00229	0.01008	0.01008
1/16	0.00032	0.00263	0.00263
p	3.01	2.17	2.17

Table 2: Convergence rates of the error $\|u-u_h\|$ on graded meshes in different norms, for $\alpha = 0.5$. Grading parameter values are (a) $\mu = \alpha/2 = 0.25$, for polynomial degrees $k = 1$ and (b) $\mu = \alpha/2.2 = 0.2273$ for polynomial degrees $k = 2$

4 Coupled 1D-3D problems

Consider problem (1), with the Starling filtration law (2) for the flux $q(s)$ on Λ. Note that (2) contains an "averaging" operator $\Pi : H^1_\alpha(\Omega) \to L^2(\Lambda)$, representing the mean value of u around each point on Λ, namely $\Pi u = \bar{u}$. This is the average of u on circles of radius R and normal to Λ, i.e.

$$\Pi u(s) = \bar{u}(s) := \frac{1}{2\pi}\int_0^{2\pi} u(\mathbf{x}(s,R,\theta))\,\mathrm{d}\theta. \tag{39}$$

In (39) we make use of local cylindrical coordinates (s, r, θ) around Λ, defined by the mapping

$$\mathbf{x}(s,r,\theta) = \mathbf{x}_\Lambda(s) + \mathbf{n}(s) r\cos\theta + \mathbf{b}(s) r \sin\theta,$$

where s is the arc length, $\mathbf{x}_\Lambda : [s_1, s_2] = \hat{\Lambda} \to \Lambda$ is the canonical parametrization of Λ, $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are respectively the normal and binormal versor on Λ (see D'Angelo and Quarteroni [2008], section 2). If $\beta = 2\pi R L_p$, we have $q(s) = \beta(\hat{u} - \Pi u) \in L^2(\Lambda)$. Note that β plays the role of a hydraulic conductance per unit length, so that $q(s)$ represents the linear density of flow rate from Λ to Ω.

Very often, the internal fracture pressure $\hat{u}(s)$ is not known a priori, but rather computed through a fracture flow model. In the simple case of axial Darcy's flow we have

$$-\frac{\mathrm{d}}{\mathrm{d}s}\hat{\mathsf{K}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{u} + q(s) = \quad \text{in } \Lambda, \tag{40}$$

equipped with suitable boundary conditions (BCs). Let us consider the following BCs for the 3D and 1D problems:

$$-\mathsf{K}\frac{\partial u}{\partial n} = 0 \;\text{ on } \partial\Omega, \qquad -\hat{\mathsf{K}}\frac{\mathrm{d}\hat{u}}{\mathrm{d}s}(s_1) = q_1, \;\hat{u}(s_2) = 0. \tag{41}$$

Introducing the spaces $W_\alpha = H^1_\alpha(\Omega)$, $\hat{W} = \{\hat{u} \in H^1(\Lambda) : \hat{u}(s_2) = 0\}$, the weak formulation of the coupled problem (1), (2), (40), (41) reads: *find* $\mathbf{u} = (u, \hat{u}) \in W_\alpha \times \hat{W}$ *such that*

$$A(\mathbf{u};\mathbf{v}) := \langle \mathsf{K}\nabla u, \nabla v\rangle_\Omega + \langle \hat{\mathsf{K}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{u}, \frac{\mathrm{d}}{\mathrm{d}s}\nabla\hat{v}\rangle_\Lambda + \langle \beta(\Pi u - \hat{u}), \gamma_\Lambda v - \hat{v}\rangle_\Lambda$$
$$= q_1\hat{v}(s_1) \quad \forall \mathbf{v} = (v, \hat{v}) \in W_{-\alpha} \times \hat{W}, \tag{42}$$

where $\gamma_\Lambda : W_{-\alpha} \to L^2(\Lambda)$ is the continuous trace operator on Λ. We point out that in this section $W_\alpha = H^1_\alpha(\Omega)$ does not include any homogeneous Dirichlet boundary, so that the Poincaré inequality does not hold and $\|\nabla u\|_{L^2_\alpha(\Omega)}$ is not a norm on W_α.

Problem (42) is challenging since it contains a measure term, which is solution-dependent, that is also the coupling term between the two flow models. In the sequel, we will see how an augmented formulation similar to the one used in lemmas 2.1 and 3.3 allows us to study the well-posedness and FEM approximation of such coupled problems.

For the finite element approximation, we will need to introduce a one-dimensional mesh $\hat{\mathcal{T}}_h$ by partitioning $\hat{\Lambda}$ into one-dimensional elements $K \in \hat{\mathcal{T}}_h$; again, the index h stands for the maximum element size.

Each 1D element is mapped via the parametrization $\mathbf{x}_\Lambda$ to a "curved" element in Λ; of course there is no need of referring to such curved elements, since the reference 1D domain is actually the interval $\hat{\Lambda} = [s_1, s_2]$.

In analogy with the three-dimensional finite element space W_h^k, we define the one-dimensional finite element space by considering continuous function that are piecewise linear on $\hat{\Lambda}$:

$$\hat{W}_h^k = \{\hat{u}_h \in C(s_1, s_2) : \ \hat{u}_{h|K} \in \mathbb{P}_1(K) \ \forall K \in \hat{\mathcal{T}}_h\}. \tag{43}$$

Note that $\mathcal{T}_h$ and $\hat{\mathcal{T}}_h$ are non-matching. The fact that Ω and Λ are meshed independently is of course an attractive feature of this approach.

The analysis and numerical approximation of problem (42) is, again, non-standard due to the non-symmetric trial and test spaces $W_{\pm\alpha}$. This feature makes it difficult to verify the inf-sup conditions characterizing the unique solvability of the problem.

Our fundamental tool, as in lemma 2.1 and in its discrete counterpart (lemma 3.3), is to circumvent the difficulties related to the non-symmetric spaces and/or norms by reformulating our variational problem as a saddle point problem. In particular, we will exploit this technique to get suitable stability estimates for the continuous and discrete problem in the correct, physically meaningful weighted norms.

To this end, we will assume that K, $\hat{\mathsf{K}}$ and β are strictly positive. We then define $\boldsymbol{M}_1 = \boldsymbol{L}^2_\alpha \times L^2(\Lambda)^3$, $\boldsymbol{M}_2 = \boldsymbol{L}^2_{-\alpha} \times L^2(\Lambda)^3$, $\boldsymbol{W}_1 = W_\alpha \times \hat{W}$, $\boldsymbol{W}_2 = W_{-\alpha} \times \hat{W}$, and equip these spaces with the following norms:

$$\begin{aligned}
\|\mathbf{s}\|^2_{\boldsymbol{M}_1} &= \|\mathsf{K}^{\frac{1}{2}}\boldsymbol{\sigma}\|^2_{L^2_\alpha(\Omega)} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\hat{\sigma}\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\lambda - \hat{\lambda})\|^2_{L^2(\Lambda)},\\
\|\mathbf{t}\|^2_{\boldsymbol{M}_2} &= \|\mathsf{K}^{\frac{1}{2}}\boldsymbol{\tau}\|^2_{L^2_{-\alpha}(\Omega)} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\hat{\tau}\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\mu - \hat{\mu})\|^2_{L^2(\Lambda)},\\
\|\mathbf{u}\|^2_{\boldsymbol{W}_1} &= \|\mathsf{K}^{\frac{1}{2}}\nabla u\|^2_{L^2_\alpha(\Omega)} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{u}\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\Pi u - \hat{u})\|^2_{L^2(\Lambda)},\\
\|\mathbf{v}\|^2_{\boldsymbol{W}_2} &= \|\mathsf{K}^{\frac{1}{2}}\nabla v\|^2_{L^2_{-\alpha}(\Omega)} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{v}\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\gamma_\Lambda v - \hat{v})\|^2_{L^2(\Lambda)},
\end{aligned}$$

where $\mathbf{s} = (\boldsymbol{\sigma}, \hat{\sigma}, \lambda, \hat{\lambda})$, $\mathbf{t} = (\boldsymbol{\tau}, \hat{\tau}, \mu, \hat{\mu})$, $\mathbf{u} = (u, \hat{u})$, $\mathbf{v} = (v, \hat{v})$. Note that $\|\cdot\|_{\boldsymbol{W}_{1,2}}$ are indeed norms on $\boldsymbol{W}_{1,2}$, thanks to the Poincaré inequality in $\hat{W}$ (so that $\|\hat{\mathsf{K}}^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{u}\|_{L^2(\Lambda)}$ is a norm on $\hat{W}$) and owing to the boundedness and invariance on the constants of Π and γ_Λ.

Let us introduce the following bilinear forms:

$$\begin{aligned}
a(\mathbf{s}, \mathbf{t}) &= \langle \mathsf{K}\boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_\Omega + \langle \hat{\mathsf{K}}\hat{\sigma}, \hat{\tau}\rangle_\Lambda + \langle \beta(\lambda - \hat{\lambda}), \mu - \hat{\mu}\rangle_\Lambda;\\
b_1(\mathbf{u}, \mathbf{t}) &= \langle \mathsf{K}\nabla u, \boldsymbol{\tau}\rangle_\Omega + \langle \hat{\mathsf{K}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{u}, \hat{\tau}\rangle_\Lambda + \langle \beta(\Pi u - \hat{u}), \mu - \hat{\mu}\rangle_\Lambda;\\
b_2(\mathbf{v}, \mathbf{s}) &= \langle \mathsf{K}\boldsymbol{\sigma}, \nabla v\rangle_\Omega + \langle \hat{\mathsf{K}}\hat{\sigma}, \frac{\mathrm{d}}{\mathrm{d}s}\hat{v}\rangle_\Lambda + \langle \beta(\lambda - \hat{\lambda}), \gamma_\Lambda v - \hat{v}\rangle_\Lambda.
\end{aligned}$$

Note how these forms were constructed: they are obtained from the expression (42) of A by replacing the "fluxes" ∇u, ∇v, $\mathrm{d}\hat{u}/\mathrm{d}s$, $\mathrm{d}\hat{v}/\mathrm{d}s$, Πu, $\gamma_\Lambda v$, $\hat{u}$ and $\hat{v}$ respectively by the new variables $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, $\hat{\sigma}$, $\hat{\tau}$, λ, μ, $\hat{\lambda}$ and $\hat{\mu}$.

The next result establishes the well-posedness of the continuous coupled problem (42), as well as the stability and convergence properties of the finite element approximation for R (or β) small.

Theorem 4.1. *Let $0 < \alpha \le t < 1$. The coupled problem (42) admits a unique solution $(u, \hat{u}) \in W_\alpha \times \hat{W}$.*

Let $\zeta = \frac{R^{2\alpha}\beta_{\max}}{2\alpha \mathsf{K}_{\min}}$, where $\beta_{\max} = \sup \beta$ and $\mathsf{K}_{\min} = \inf \mathsf{K} > 0$. Then, for ζ small and for any integer $k \ge 1$, there exists a unique discrete solution $(u, \hat{u}) \in W_h^k \times \hat{W}_h^k$ such that

$$A(u - u_h, \hat{u} - \hat{u}_h; v_h, \hat{v}_h) = 0 \quad \forall (v_h, \hat{v}_h) \in W_h^k \times \hat{W}_h^k.$$

Finally, if $(u, \hat{u}) \in V_{k+\epsilon}^{k+1}(\Omega) \times H^{k+1}(\Lambda)$, $\epsilon \in (0, \alpha)$ we have the following error estimates,

$$\|(u - u_h, \hat{u} - \hat{u}_h)\|_{\boldsymbol{W}_1} \le C(\alpha; \mathsf{K}, \beta) h^p \|u\|_{V_{k+\epsilon}^{k+1}(\Omega)}, + \hat{C}(\hat{\mathsf{K}}, \beta) h^k \|u\|_{H^k(\Lambda)}, \tag{44}$$

where $p = k$ if $\mu \le \frac{\alpha - \epsilon}{k}$, $p = \frac{\alpha-\epsilon}{\mu}$ otherwise.

Proof. As in lemma 2.1, we consider a suitable reformulation as a generalized saddle point problem. We seek $(\mathbf{s}, \mathbf{u}) \in \boldsymbol{M}_1 \times \boldsymbol{W}_1$, such that

$$\begin{cases} a(\mathbf{s}, \mathbf{t}) + b_1(\mathbf{u}, \mathbf{t}) = F(\mathbf{t}) & \forall \mathbf{t} \in \boldsymbol{M}_2, \\ b_2(\mathbf{v}, \mathbf{s}) = 0 & \forall \mathbf{v} \in \boldsymbol{W}_2, \end{cases} \tag{45}$$

where F is a continuous functional on $\boldsymbol{M}_2$. The bilinear forms a, b_1 and b_2 are respectively bounded on $\boldsymbol{M}_1 \times \boldsymbol{M}_2$, $\boldsymbol{W}_1 \times \boldsymbol{M}_2$ and $\boldsymbol{W}_2 \times \boldsymbol{M}_1$. For any $(\boldsymbol{\sigma}, \hat{\sigma}, \lambda, \hat{\lambda}) = \mathbf{s} \in \boldsymbol{M}_1$, we have that $\mathbf{t} = (d^{2\alpha}\boldsymbol{\sigma}, \hat{\sigma}, \lambda, \hat{\lambda})$ satisfies $a(\mathbf{s}, \mathbf{t}) = \|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}_1}$, $\|\mathbf{t}\|_{\boldsymbol{M}_2} = \|\boldsymbol{\sigma}\|_{\boldsymbol{M}_1}$. The latter estimates still hold if we swap $\mathbf{s}$ and $\mathbf{t}$, $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, and we change the sign of α. Similarly, for any $(u, \hat{u}) = \mathbf{u} \in \boldsymbol{W}_1$, choosing $\mathbf{t} = (d^{2\alpha}\nabla u, \frac{\mathrm{d}}{\mathrm{d}s}\hat{u}, \Pi u, \hat{u})$ yields $b_1(\mathbf{u}, \mathbf{t}) = \|\mathbf{u}\|^2_{\boldsymbol{M}_1}$, $\|\mathbf{t}\|_{\boldsymbol{M}_2} = \|\mathbf{u}\|_{\boldsymbol{W}_1}$. Finally, for any $(v, \hat{v}) = \mathbf{v} \in \boldsymbol{W}_2$, choosing $\mathbf{s} = (d^{-2\alpha}\nabla v, \frac{\mathrm{d}}{\mathrm{d}s}\hat{v}, \gamma_\Lambda v, \hat{v})$ yields $b_2(\mathbf{v}, \mathbf{s}) = \|\mathbf{v}\|^2_{\boldsymbol{M}_2}$, $\|\mathbf{s}\|_{\boldsymbol{M}_1} = \|\mathbf{v}\|_{\boldsymbol{W}_2}$.

Thus we have

$$\begin{aligned} &\sup_{\mathbf{t}\neq 0} \frac{a(\mathbf{s}, \mathbf{t})}{\|\mathbf{t}\|_{\boldsymbol{M}_2}} \ge \|\mathbf{s}\|_{\boldsymbol{M}_1}, \quad \sup_{\mathbf{s}\neq 0} \frac{a(\mathbf{s}, \mathbf{t})}{\|\mathbf{s}\|_{\boldsymbol{M}_1}} \ge \|\mathbf{t}\|_{\boldsymbol{M}_2}, \\ &\sup_{\mathbf{t}\neq 0} \frac{b_1(\mathbf{u}, \mathbf{t})}{\|\mathbf{t}\|_{\boldsymbol{M}_2}} \ge \|\mathbf{u}\|_{\boldsymbol{W}_1}, \quad \sup_{\mathbf{s}\neq 0} \frac{b_2(\mathbf{v}, \mathbf{s})}{\|\mathbf{s}\|_{\boldsymbol{M}_1}} \ge \|\mathbf{v}\|_{\boldsymbol{W}_2}. \end{aligned}$$

All the hypotheses of the generalized BNB theorem are thus satisfied; problem (45) is well-posed and we have the estimate $\|(\mathbf{s}, \mathbf{u})\|_{\boldsymbol{M}_1 \times \boldsymbol{W}_1} \le 2\|F\|_{\boldsymbol{W}_2'}$.

We claim that, as a consequence, also the original problem (42) is well-posed, and, for $F(\mathbf{t}) = F(\boldsymbol{\tau}, \hat{\tau}, \mu, \hat{\mu}) = q_1\hat{\mu}(s_1)$, the solution $(\mathbf{s}, \mathbf{u})$ of (45) also provides the solution $\mathbf{u}$ of (42). To see this, for any $(v, \hat{v}) = \mathbf{v} \in \mathbf{W}_2$, choose $\mathbf{t}_v = (\nabla v, \frac{\mathrm{d}}{\mathrm{d}s}\hat{v}, \gamma_\Lambda v, \hat{v}) \in \boldsymbol{M}_2$ in such a way that $\|\mathbf{t}_v\|_{\boldsymbol{M}_2} = \|\mathbf{v}\|_{\boldsymbol{W}_2}$ and $a(\mathbf{s}, \mathbf{t}_v) = b_2(\mathbf{v}, \mathbf{s}) = 0$ for all $\mathbf{s} \in \boldsymbol{M}_1$. This leads to

$$b_1(\mathbf{u}, \mathbf{t}_v) = A(\mathbf{u}, \mathbf{v}) = F(\mathbf{t}_v) \quad \forall \mathbf{v} \in \mathbf{W}_2,$$

i.e. $\mathbf{u}$ satisfies (42). Moreover, $\|\mathbf{u}\|_{\boldsymbol{W}_1} \le 2\|F\|_{\boldsymbol{W}_2'} = 2\hat{C}|q_1|$ where $\hat{C}$ is the norm of the trace operator $\hat{v} \in \hat{W} \mapsto \hat{v}(s_1)$.

Now, the same techniques of lemma 3.3 and theorem 3.1 can be employed in the discrete setting. However, in this case we have to treat the following issue: even if u_h and v_h are both discrete functions in W_h^k, the discrete variables $\lambda_h = \Pi u_h$ and $\mu_h = \gamma_\Lambda v_h$ do not belong to the same discrete subspace of $L^2(\Lambda)$. Hence, we first consider a "symmetrized" problem, and then go back to the original formulation (42). First, let us introduce the discrete spaces

$$\begin{aligned} M_h^{k-1} &= \{\boldsymbol{q}_h \in L^2(\Omega) : \boldsymbol{q}_h \in \mathbb{P}^{k-1}(K),\ \forall K \in \mathcal{T}_h\}, \\ \hat{M}_h^{k-1} &= \{\hat{q}_h \in L^2(\Lambda) : \hat{q}_h \in \mathbb{P}^{k-1}(K),\ \forall K \in \hat{\mathcal{T}}_h\}, \\ \widetilde{M}_h^k &= \{\widetilde{u}_h \in L^2(\Lambda) : \widetilde{u}_h = \Pi u_h, u_h \in W_h^k\}, \end{aligned}$$

$$\boldsymbol{M}_{1,h} = \boldsymbol{M}_{1,h} = M_h^{k-1} \times \hat{M}_h^{k-1} \times \widetilde{M}_h^k \times \hat{W}_h^k, \quad \boldsymbol{W}_{1,h} = \boldsymbol{W}_{2,h} = W_h^k \times \hat{W}_h^k.$$

The discrete spaces are then equipped with the following discrete norms, where in the sequel we set $\mathbf{s}_h = (\boldsymbol{\sigma}_h, \hat{\sigma}_h, \widetilde{\lambda}_h, \hat{\lambda}_h) \in \boldsymbol{M}_{1,h}$, $\mathbf{t}_h = (\boldsymbol{\tau}_h, \hat{\tau}_h, \widetilde{\mu}_h, \hat{\mu}_h) \in \boldsymbol{M}_{2,h}$, $\mathbf{u}_h = (u_h, \hat{u}_h) \in \boldsymbol{W}_{1,h}$ and $\mathbf{v}_h = (v_h, \hat{v}_h) \in \boldsymbol{W}_{2,h}$:

$$\begin{aligned}
\|\mathbf{s}_h\|^2_{\boldsymbol{M}_{1,h}} &= \|\mathsf{K}^{\frac{1}{2}}\boldsymbol{\sigma}_h\|^2_{h,\alpha} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\hat{\sigma}_h\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\widetilde{\lambda}_h - \hat{\lambda}_h)\|^2_{L^2(\Lambda)},\\
\|\mathbf{t}_h\|^2_{\boldsymbol{M}_{2,h}} &= \|\mathsf{K}^{\frac{1}{2}}\boldsymbol{\tau}_h\|^2_{h,-\alpha} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\hat{\tau}_h\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\widetilde{\mu}_h - \hat{\mu}_h)\|^2_{L^2(\Lambda)},\\
\|\mathbf{u}_h\|^2_{\boldsymbol{W}_{1,h}} &= \|\mathsf{K}^{\frac{1}{2}}\nabla u_h\|^2_{h,\alpha} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{u}_h\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\Pi u_h - \hat{u}_h)\|^2_{L^2(\Lambda)},\\
\|\mathbf{v}_h\|^2_{\boldsymbol{W}_{2,h}} &= \|\mathsf{K}^{\frac{1}{2}}\nabla v_h\|^2_{h,-\alpha} + \|\hat{\mathsf{K}}^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{v}_h\|^2_{L^2(\Lambda)} + \|\beta^{\frac{1}{2}}(\Pi v_h - \hat{v}_h)\|^2_{L^2(\Lambda)}.
\end{aligned}$$

The symmetrized formulation reads: *find* $\mathbf{u}_h = (u_h, \hat{u}_h) \in \boldsymbol{W}_{1,h}$ *such that*

$$\begin{aligned}
A_s(\mathbf{u}_h; \mathbf{v}_h) := \langle \mathsf{K}\nabla u_h, \nabla v_h\rangle_\Omega + \langle \hat{\mathsf{K}}\frac{\mathrm{d}}{\mathrm{d}s}\hat{u}_h, \frac{\mathrm{d}}{\mathrm{d}s}\nabla\hat{v}_h\rangle_\Lambda + \langle \beta(\Pi u_h - \hat{u}_h), \Pi v_h - \hat{v}_h\rangle_\Lambda \\
= q_1\hat{v}_h(s_1) \quad \forall \mathbf{v}_h = (v_h, \hat{v}_h) \in \boldsymbol{W}_{2,h}. \quad (46)
\end{aligned}$$

Problem (42) can be recast in an augmented formulation, as in the continuous case, to treat the non-symmetric trial and test norms. In particular, it is immediately verified that the generalized saddle point problem of finding $(\mathbf{s}_h, \mathbf{u}_h) \in \boldsymbol{M}_{1,h} \times \boldsymbol{W}_{1,h}$ such that

$$\begin{cases} a(\mathbf{s}_h, \mathbf{t}_h) + b_1(\mathbf{u}_h, \mathbf{t}_h) = F(\mathbf{t}_h) & \forall \mathbf{t}_h \in \boldsymbol{M}_{2,h}, \\ b_1(\mathbf{v}_h, \mathbf{s}_h) = 0 & \forall \mathbf{v}_h \in \boldsymbol{W}_{2,h}, \end{cases} \quad (47)$$

passes the discrete inf-sup conditions related to the discrete norms. That is obtained proceeding as in lemma 3.3. In fact, for any $(\boldsymbol{\sigma}_h, \hat{\sigma}_h, \widetilde{\lambda}_h, \hat{\lambda}_h) = \mathbf{s}_h \in \boldsymbol{M}_{1,h}$, we have that $(\boldsymbol{\tau}_h, \hat{\tau}_h, \widetilde{\mu}_h, \hat{\mu}_h) = \mathbf{t}_h \in \boldsymbol{M}_{2,h}$ defined by $\boldsymbol{\tau}_{h|K} = \bar{r}_K^{2\alpha}\boldsymbol{\sigma}_{|K}$ $\forall K \in \mathcal{T}_h$, $\hat{\tau}_h = \hat{\sigma}_h$, $\widetilde{\mu}_h = \widetilde{\lambda}_h$ and $\hat{\mu}_h = \hat{\lambda}_h$ satisfies $a(\mathbf{s}_h, \mathbf{t}_h) = \|\boldsymbol{\sigma}_h\|^2_{\boldsymbol{M}_{1,h}}$, $\|\mathbf{t}\|_{\boldsymbol{M}_{2,h}} = \|\boldsymbol{\sigma}\|_{\boldsymbol{M}_{1,h}}$. Those identities still hold if we swap $\mathbf{s}_h$ and $\mathbf{t}_h$, $\boldsymbol{M}_{1,h}$ and $\boldsymbol{M}_{2,h}$, and change the sign of α. Moreover, for any $\mathbf{u}_h = (u_h, \hat{u}_h) \in \mathbf{W}_{1,h}$, choosing $\mathbf{t}_h$ with $\boldsymbol{\tau}_{h|K} = \bar{r}_K^{2\alpha}\nabla u_{h|K}$ $\forall K \in \mathcal{T}_h$, $\hat{\tau}_h = \frac{\mathrm{d}}{\mathrm{d}s}\hat{u}_h$, $\widetilde{\mu}_h = \Pi u_h$ and $\hat{\mu}_h = \hat{u}_h$ yields $b_1(\mathbf{u}_h, \mathbf{t}_h) = \|\mathbf{u}_h\|^2_{\boldsymbol{W}_{1,h}}$, $\|\mathbf{t}_h\|_{\boldsymbol{M}_{2,h}} = \|\mathbf{u}_h\|_{\boldsymbol{W}_{1,h}}$. Again, the same identities are obtained by replacing $\mathbf{u}_h$ by $\mathbf{v}_h$, $\mathbf{t}_h$ by $\mathbf{s}_h$ and changing the sign of α.

Hence, (47) is well-posed, the solution $(\mathbf{s}_h, \mathbf{u}_h)$ is such that $\mathbf{u}_h$ solves also (46), and $\|\mathbf{u}_h\|_{\boldsymbol{W}_{1,h}} \leq 2\|F\|_{\boldsymbol{W}'_{2,h}}$. As a consequence, we have the discrete stability estimate

$$\|\mathbf{u}_h\|_{\boldsymbol{W}_{1,h}} \leq 2\|F\|_{\boldsymbol{W}'_{2,h}} = 2\sup_{\mathbf{v}_h \neq 0} \frac{A_s(\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\boldsymbol{W}_{2,h}}}. \quad (48)$$

The final step is to show that, if R is small, also the original formulation (42) is stable. To this end, for any $v \in W_{-\alpha}$, we observe that for all cylindrical coordinates s and θ we have

$$\gamma_\Lambda v(s) = v(s, R, \theta) - \int_0^R \partial_r v(s, r, \theta)\,\mathrm{d}r = \Pi v(s) - \frac{1}{2\pi}\int_0^{2\pi}\int_0^R \partial_r v(s, r, \theta)\,\mathrm{d}r\,\mathrm{d}\theta,$$

so that, thanks to the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|(\Pi - \gamma_\Lambda)v\|^2_{L^2(\Lambda)} &\leq \int_\Lambda \left(\frac{1}{2\pi}\int_0^{2\pi}\int_0^R \partial_r v(s, r, \theta)\,\mathrm{d}r\,\mathrm{d}\theta\right)^2 \mathrm{d}s \\
&\leq \frac{1}{4\pi^2}\frac{R^{2\alpha}}{2\alpha}\int_\Lambda\int_0^{2\pi}\int_0^R [\partial_r v(s, r, \theta)]^2 r^{1-2\alpha}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}s \leq \frac{R^{2\alpha}}{8\alpha\pi^2}\|\nabla v\|^2_{L^2_{-\alpha}(\Omega)}. \quad (49)
\end{aligned}$$

Let $\mathbf{u}_h \in \boldsymbol{W}_{1,h}$ be given. Observing that

$$A(\mathbf{u}_h, \mathbf{v}_h) = A_s(\mathbf{u}_h, \mathbf{v}_h) + \langle \beta(\Pi u_h - \hat{u}_h), (\gamma_\Lambda - \Pi) v_h \rangle_\Lambda$$

and owing to (48), there exists $\mathbf{v}_h \in \boldsymbol{W}_{2,h}$ such that $\|\mathbf{v}_h\|_{\boldsymbol{W}_{2,h}} = \|\mathbf{u}_h\|_{\boldsymbol{W}_{1,h}}$ and

$$\begin{aligned} A(\mathbf{u}_h, \mathbf{v}_h) &\geq A_s(\mathbf{u}_h, \mathbf{v}_h) - \frac{\epsilon}{2} \|\beta^{\frac{1}{2}} (\Pi u_h - \hat{u}_h)\|^2_{L^2(\Lambda)} - \frac{1}{2\epsilon} \|\beta^{\frac{1}{2}} (\Pi - \gamma_\Lambda) v_h\|^2_{L^2(\Lambda)} \\ &\geq \frac{1-\epsilon}{2} \|\mathbf{u}_h\|^2_{\boldsymbol{W}_{1,h}} - \frac{1}{2\epsilon} \|\beta^{\frac{1}{2}} (\Pi - \gamma_\Lambda) v_h\|^2_{L^2(\Lambda)}. \end{aligned}$$

Choosing for instance $\epsilon = 1/2$, thanks to (49) and to lemma 3.2, we conclude

$$A(\mathbf{u}_h, \mathbf{v}_h) \geq \frac{1}{4} \|\mathbf{u}_h\|^2_{\boldsymbol{W}_{1,h}} - \frac{R^{2\alpha}}{8\alpha\pi^2} \frac{\beta_{\max}}{\mathsf{K}_{\min}} \|\mathsf{K}^{\frac{1}{2}} \nabla v_h\|^2_{L^2_{-\alpha}} \geq \frac{1}{4} \left[1 - C_t \frac{R^{2\alpha}}{2\alpha} \frac{\beta_{\max}}{\mathsf{K}_{\min}} \right] \|\mathbf{u}_h\|^2_{\boldsymbol{W}_{1,h}},$$

where C_t is a positive constant depending on t only. Similarly, let $\mathbf{v}_h \in \boldsymbol{W}_{2,h}$ be given. Then, there exists $\mathbf{u}_h \in \boldsymbol{W}_{1,h}$ such that $\|\mathbf{u}_h\|_{\boldsymbol{W}_{1,h}} = \|\mathbf{v}_h\|_{\boldsymbol{W}_{2,h}}$ and

$$\begin{aligned} A(\mathbf{u}_h, \mathbf{v}_h) &\geq \frac{1}{2} \|\mathbf{v}_h\|^2_{\boldsymbol{W}_{2,h}} - \frac{\epsilon}{2} \|\mathbf{u}_h\|^2_{\boldsymbol{W}_{2,h}} - \frac{R^{2\alpha}}{16\epsilon\alpha\pi^2} \frac{\beta_{\max}}{\mathsf{K}_{\min}} \|\mathsf{K}^{\frac{1}{2}} \nabla v_h\|^2_{L^2_{-\alpha}} \\ &\geq \frac{1}{4} \left[1 - C'_t \frac{R^{2\alpha}}{2\alpha} \frac{\beta_{\max}}{\mathsf{K}_{\min}} \right] \|\mathbf{v}_h\|^2_{\boldsymbol{W}_{2,h}} \quad (\text{with } \epsilon = 1/2), \end{aligned}$$

where $C'_t > 0$ only depends on t. The stability and solvability of the finite element approximation of the coupled problem for $\zeta = \frac{R^{2\alpha}}{2\alpha} \frac{\beta_{\max}}{\mathsf{K}_{\min}}$ small follows. The error estimates (44) are immediately obtained thanks to Corollary 3.1 and standard interpolation error estimates. □

5 Applications to microvascular flows

Microcirculation is a relevant instance of the coupled 1D-3D problems considered in section 4. It concerns blood flow through a network of small vessels (arterioles / capillaries / venules) surrounded by a tissue called *interstitial matrix*. Blood flows from arterioles to venules; however, part of the fluid (*plasma*) crosses the vessel walls entering the interstitial tissue (*transmural flow*), where it percolates before being drained by the lymphatic system.

Models of microcirculation and interstitial flow are exhaustively discussed for instance in Baxter and Jain [1989], Pries and Secomb [2008] (see also Lee and Skalak [1990] for a review). For a single vessel Λ of radius R, the blood pressure $\hat{u}$ and flow rate $\hat{v}$ satisfy the Poiseuille law

$$\hat{v}(s) = -\frac{\pi R^4}{8\mu} \frac{\mathrm{d}\hat{u}(s)}{\mathrm{d}s}, \qquad \frac{\mathrm{d}\hat{v}(s)}{\mathrm{d}s} = q(s), \tag{50}$$

where μ is the (effective) blood viscosity and $q(s)$ is the transmural flux (exiting the vessel). Note that this is precisely eq. (40) with $\hat{\mathsf{K}} = \frac{\pi R^4}{8\mu}$.

Let the vessel Λ be embedded in a region Ω representing the interstitial tissue, and let u be interstitial pressure. The latter satisfies the Darcy's law (1), where (if the osmotic pressure is neglected) the transmural flux q is related to the interstitial plasma pressure u and to blood vascular pressure $\hat{u}$ by means of the Starling's law (2), where L_p is the permeability of the vessel walls. In fig. 4 we show the numerical simulation of microvascular flow inside a network of 12 vessels embedded in a "brick" of interstitial tissue by solving problem (42).

The tissue brick measures $35 \times 20 \times 15\mu$m. We assume that the interstitial hydraulic conductivity K is $2.0 \cdot 10^{-7}$ cm^2 mmHg^{-1} s^{-1}. We consider small capillaries with $R = 5\mu$m and effective blood viscosity $\mu = 3 \cdot 10^{-5}$mmHg s, yielding $\hat{\mathsf{K}} = 8.18 \cdot 10^{-10}$cm^4 mmHg^{-1} s^{-1}. We set $L_p = 2.8\ 10^{-6}$cm mmHg^{-1} s^{-1}. These values, with K and L_p higher than physiological levels, are typical of tumor tissues. Blood flow and plasma filtration are clearly seen in fig. 4, with plasma being exchanged between capillaries. The 1D microvascular mesh and the 3D tissue mesh are completely independent.

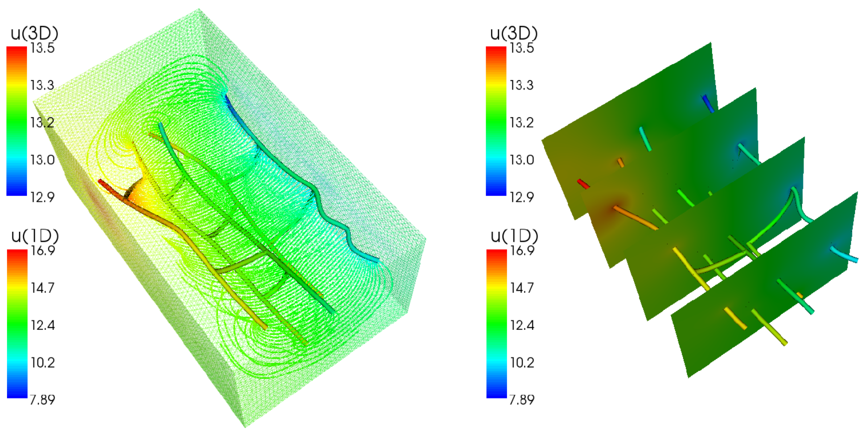

Figure 4: Microvascular 1D network embedded in a 3D interstitial tissue. Pressure distributions (1D and 3D on slices) and plasma fluid paths.

Acknowledgments

This work has been supported by the ERC Advanced Grant N.227058 *Mathematical Modelling and Simulation of the Cardiovascular System (MATHCARD)*. The author thanks Ana Maria Soane for a fruitful discussion and Marco D'Angelo for the 3D microvascular network model.

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