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Mimetic finite difference approximation of quasilinear elliptic problems *

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Abstract

In this work we approximate the solution of a quasilinear elliptic problem of monotone type by using the Mimetic Finite Difference (MFD) method. Under a suitable approximation assumption, we prove that the MFD approximate solution converges, with optimal rate, to the exact solution in a mesh-dependent energy norm. The resulting nonlinear discrete problem is then solved iteratively via linearization by applying the Kačanov method. The convergence of the Kačanov algorithm in the discrete mimetic framework is also proved. Several numerical experiments confirm the theoretical analysis.

Keywords: mimetic finite difference method, quasilinear elliptic problems, Kačanov method.

Mathematics Subject Classification: 65N12, 65N15.

1 Introduction

In this paper we address the discretization of the following quasilinear elliptic problem:

$$-\operatorname{div}(\kappa(|\nabla u|^2)\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (1)$$

where f is a given function, and Ω is an open, bounded set of \mathbb{R}^2 . The approximation of problem (1) is a challenging standalone problem, see, e.g., [23, 25, 27, 40] and [31, 32], respectively, in the context of conforming and discontinuous finite element methods. At the same time, efficient discretizations of problem (1) represent a necessary intermediate step towards the solution of non-newtonian flow problems, see, e.g., [4–8, 16, 34, 39] and [24] again in the conforming and discontinuous finite element context, respectively. It is important to remark that non-newtonian fluids are ubiquitous in industrial applications. For example, for polymeric extrusion, robust and reliable numerical algorithms are mandatory in order to predict and, possibly, to optimize the production process, c.f. [3, 38], for example.

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The aim of this paper is to show that the flexibility of the Mimetic Finite Difference (MFD) method (see, e.g., [18–21] for a detailed introduction) can be successfully exploited to discretize quasilinear elliptic problems, and that the Kačanov algorithm can be successfully employed to solve the discrete nonlinear system resulting from MFD discretizations; we refer to [42] for more details on the Kačanov method and to [29] for an application in the context of adaptive finite element methods. The MFD method can be naturally employed on very general decompositions made of (possibly non convex) polyhedrals which do not have to fulfill matching conditions. Thanks to such a flexibility, the MFD method has been rapidly applied to a wide range of problems [9, 13–15, 18, 36, 37], and to Maxwell’s equations and magnetostatic fields problems [33, 35], Stokes equations [11, 12], eigenvalue problems [22], problems governed by variational inequalities [1] and optimal control problems [2]. Very recently, a very promising evolution of the MFD scheme, namely the Virtual Element Method (VEM), has been introduced in [10]. However, the application to nonlinear elliptic problems has not been taken into account yet. The aim of this work is to approximate the solution of a quasilinear elliptic problem of monotone type by using the MFD method. We show that the MFD solution converges to the exact analytical solution in a suitable discrete energy norm. The convergence holds under a suitable approximation assumption which is verified through numerical experiments. The resulting nonlinear problem is then solved iteratively via linearization by applying the Kačanov method, and the convergence of the Kačanov algorithm is proved in the discrete mimetic framework.

The rest of the paper is organized as follows. In Section 2 we introduce the weak form of (1) and state its well-posedness. The MFD discrete formulation and its well-posedness is discussed in Section 3, and the *a priori* error estimates are derived in Section 4. In Section 5, we introduce the Kačanov method in the mimetic context, and prove that the algorithm converges to the mimetic discrete solution. Finally, in Section 6 we present some numerical results to confirm our theoretical analysis.

2 A quasilinear elliptic problem

Throughout the paper we will use standard notation for Sobolev spaces, norms and seminorms (see, e.g., [28]). Moreover, the symbol \lesssim denotes an upper bound that holds up to a positive constant independent of the discretization parameters.

We first introduce the following semilinear form

$$b(u; v, w) := \int_{\Omega} \kappa(|\nabla u|^2) \nabla v \cdot \nabla w \, dx \quad \forall u, v, w \in H_0^1(\Omega), \quad (2)$$

where $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nonlinear function whose properties will be stated below (cf. Assumption 2.1). We are interested in solving the following quasilinear elliptic problem: find $u \in H_0^1(\Omega)$ such that

$$b(u; u, v) = F(v) \quad \forall v \in H_0^1(\Omega), \quad (3)$$

where

$$F(v) := \int_{\Omega} f v \, dx,$$

for a given $f \in L^2(\Omega)$. We remark that relevant physical applications, as the modeling of non-Newtonian fluids governed by the Carreau law:

$$\kappa(t) := \eta_{\infty} + (\eta_0 - \eta_{\infty})(1 + \lambda t)^{\frac{p-2}{2}}, \quad t \geq 0, \quad (4)$$

with $\eta_0 \geq \eta_{\infty} \geq 0$, $\lambda > 0$ and $p \in (1, \infty)$, fit in this framework.

Throughout the paper, the nonlinear function $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is assumed to satisfy the following.

Assumption 2.1 (Assumptions on the nonlinearity). *The nonlinear function $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the following*

- i) $\kappa(\cdot)$ is continuous over $[0, +\infty)$;
- ii) there exist two positive constants k_*, k^* such that:

$$k_*(t-s) \leq \kappa(t^2)t - \kappa(s^2)s \leq k^*(t-s) \quad \forall t > s \geq 0. \quad (5)$$

We observe that, by choosing $s = 0$ into (5) we get

$$k_* \leq \kappa(t^2) \leq k^* \quad \forall t > 0. \quad (6)$$

Remark 2.1. *The Carreau law (4) satisfies Assumption (2.1) provided that $p \in (1, 2)$ and $\eta_\infty > 0$. To prove the upper bound in (5), it is sufficient to observe that the function*

$$\varphi(t) := \frac{t}{(1 + \lambda t^2)^\nu} \quad \nu \in (0, 1)$$

is Lipschitz continuous with constant equal to one. Indeed,

$$\varphi'(t) = \frac{1}{(1 + \lambda t^2)^\nu} \left[1 - \frac{2\nu t^2}{1 + t^2} \right]$$

is positive and bounded from above (the maximum is equal to one) and it tends to zero as t goes to infinity. Therefore, setting $\nu := \frac{2-p}{2}$, we have

$$\eta(t^2)t - \eta(s^2)s = \eta_\infty(t-s) + (\eta_0 - \eta_\infty) \left[\frac{t}{(1 + \lambda t^2)^\nu} - \frac{s}{(1 + \lambda s^2)^\nu} \right] \leq \eta_0(t-s).$$

The lower bound in (5) follows straightforwardly

$$\eta(t^2)t - \eta(s^2)s = \eta_\infty(t-s) + (\eta_0 - \eta_\infty) \left[\frac{t}{(1 + \lambda t^2)^\nu} - \frac{s}{(1 + \lambda s^2)^\nu} \right] \geq \eta_\infty(t-s),$$

since the term in the square brackets is positive.

Remark 2.2. *The nonlinear function $\kappa(\cdot)$ could also be a function of the space position, i.e., $\kappa : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$. In such cases, Assumption 2.1 has to be modified as follows:*

- i) $\kappa(\cdot, \cdot)$ is continuous over $\bar{\Omega} \times [0, +\infty)$;
- ii) there exist two positive constants k_*, k^* such that:

$$k_*(t-s) \leq \kappa(x, t^2)t - \kappa(x, s^2)s \leq k^*(t-s) \quad \forall t > s \geq 0 \quad \forall x \in \Omega.$$

Since we are interested in non-Newtonian fluids governed by the Carreau law (4) we will not address in this paper this case specifically.

Next, we address the well-posedness of problem (3). To this aim, we introduce the nonlinear operator

$$\mathcal{B} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega) \quad \langle \mathcal{B}u, v \rangle := b(u; u, v) \quad \forall v \in H_0^1(\Omega). \quad (7)$$

The existence and uniqueness of such an operator follows from the Riesz's Theorem. Hence, problem (3) can be equivalently stated as follows: find $u \in H_0^1(\Omega)$ such that

$$\mathcal{B}u = F, \quad (8)$$

where $F \in H^{-1}(\Omega)$ is given. The next result is a key point towards the proof of the well-posedness of problem (8).

Lemma 2.1. *The operator \mathcal{B} defined as in (7) is Lipschitz continuous and strongly monotone with respect to the $H^1(\Omega)$ -norm, i.e.,*

$$\|\mathcal{B}u - \mathcal{B}v\|_{H^{-1}(\Omega)} \lesssim \|u - v\|_{H^1(\Omega)} \quad \forall u, v \in H_0^1(\Omega), \quad (9)$$

$$\langle \mathcal{B}u - \mathcal{B}v, u - v \rangle \gtrsim \|u - v\|_{H^1(\Omega)}^2 \quad \forall u, v \in H_0^1(\Omega). \quad (10)$$

Proof. We first show inequality (9). From the Cauchy-Schwarz inequality and (5) of Assumption 2.1 we get

$$\begin{aligned} |\langle \mathcal{B}u - \mathcal{B}v, u - v \rangle| &\leq \int_{\Omega} |\kappa(|\nabla u|^2)\nabla u - \kappa(|\nabla v|^2)\nabla v| |\nabla(u - v)| \, dx \\ &\leq k^* \int_{\Omega} |\nabla(u - v)|^2 \, dx \lesssim \|u - v\|_{H^1(\Omega)}^2. \end{aligned}$$

By dividing by $\|u - v\|_{H^1(\Omega)}$ we obtain (9). Inequality (10) can be proved in a similar way by using (5) and the Poincaré inequality. \square

Finally, we can state the well-posedness of problem (8).

Proposition 2.1. *Let \mathcal{B} be the operator defined as in (7) and let $F \in H^{-1}(\Omega)$ be given. Under Assumption 2.1, equation (8) admits a unique solution $u \in H_0^1(\Omega)$.*

Proof. By using the Zarantonello's Theorem (see, e.g., [42, Theorem 25.B]) and Lemma 2.1 we can conclude that equation (8) admits a unique solution $u \in H_0^1(\Omega)$. \square

3 The mimetic finite difference discretization

In this section we derive a mimetic discretization of problem (3) by introducing suitable mesh assumptions and the discrete space endowed with a proper scalar product. We refer [1, 18, 21] for more details on the MFD method for elliptic problems.

3.1 Mesh assumptions, discrete space and norms

Let Ω_h be a non-overlapping partition of Ω into, possibly non-convex polygonal elements E with granularity $h := \sup_{E \in \Omega_h} h_E$, being h_E the diameter of $E \in \Omega_h$. We denote by \mathcal{N}_h° and \mathcal{N}_h^∂ the sets of interior and boundary mesh vertexes, respectively, and set $\mathcal{N}_h = \mathcal{N}_h^\circ \cup \mathcal{N}_h^\partial$. Proceeding as in [21] we also assume the following.

Assumption 3.1 (Mesh regularity assumptions). *There exist an integer number N , independent of h , and a compatible sub-decomposition \mathcal{T}_h of every Ω_h into shape-regular triangles in such a way that every element E can be decomposed in at most N triangles.*

We point out that Assumption 3.1 only requires the existence of a *compatible* sub-mesh that does not have to be constructed in practice. Moreover, Assumption 3.1 guarantees that the following mesh regularity properties are satisfied (cf. [21]).

- i)* There exists $N_e > 0$ such that every element E has at most N_e edges;
- ii)* There exists $\tau > 0$ such that every element E is star-shaped with respect to every point of a ball centered at a point $C_E \in E$ and with radius τh_E ;
- iii)* There exists $\gamma > 0$ such that for every element E and for every edge \mathbf{e} of E , it holds $|\mathbf{e}| \geq \gamma h_E$, where $|\mathbf{e}|$ is the length of \mathbf{e} ;
- iv)* For every $E \in \Omega_h$ and for every edge \mathbf{e} of E , the following *trace inequality* holds

$$\|\psi\|_{L^2(\mathbf{e})}^2 \lesssim h_E^{-1} \|\psi\|_{L^2(E)}^2 + h_E |\psi|_{H^1(E)}^2 \quad \forall \psi \in H^1(E). \quad (11)$$

Next, we introduce the discrete space by choosing as degrees-of-freedom the nodal values in each vertex $\mathbf{v} \in \mathcal{N}_h$. More precisely, every discrete function v_h is a vector of real components $v_h := \{v^\mathbf{v}\}_{\mathbf{v} \in \mathcal{N}_h}$, one per mesh vertex. The space of unknowns V_h is then defined taking into account the homogenous Dirichlet boundary conditions, i.e.,

$$V_h := \{v_h := \{v^\mathbf{v}\}_{\mathbf{v} \in \mathcal{N}_h} : v_h^\mathbf{v} = 0 \ \forall \mathbf{v} \in \mathcal{N}_h^\partial\}.$$

Clearly, the number of unknowns is equal to the number of interior mesh vertexes. The space V_h is endowed with the following discrete norm that mimics the usual $H_0^1(\Omega)$ -norm:

$$\|v_h\|_{1,h}^2 := \sum_{E \in \Omega_h} \|v_h\|_{1,h,E}^2 = \sum_{E \in \Omega_h} |E| \sum_{\substack{\mathbf{e} \in \mathcal{E}_h \\ \mathbf{e} \subset \partial E}} \left[\frac{1}{|\mathbf{e}|} (v^{\mathbf{v}_2} - v^{\mathbf{v}_1}) \right]^2, \quad (12)$$

where \mathbf{v}_1 and \mathbf{v}_2 are the two vertexes of $\mathbf{e} \in \mathcal{E}_h$, and $|E|$ is the area of the element $E \in \Omega_h$.

Finally, we introduce the standard nodal interpolation operator satisfying classical approximation estimates [17]. For every function $v \in \mathcal{C}^0(\bar{\Omega}) \cap H_0^1(\Omega)$, the interpolation $v_{\mathbf{I}} \in V_h$ is given by

$$v_{\mathbf{I}}^\mathbf{v} := v(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{N}_h. \quad (13)$$

A local interpolation operator from $\mathcal{C}^0(\bar{E}) \cap H^1(E)$ into V_h^E can be defined analogously, where we denote by V_h^E the restriction of V_h to the $E \in \Omega_h$, i.e., $V_h^E = V_h|_E$.

3.2 The discrete formulation of the problem

The mimetic approximation of problem (3) is essentially based on the following observation. Let us consider the local semilinear form given by the restriction of the form (2) on each element E of the partition, i.e.,

$$b^E(u; v, w) := \int_E \kappa(|\nabla u|^2) \nabla v \cdot \nabla w \, dx \quad \forall u, v, w \in H^1(E). \quad (14)$$

Next, denoting by $\mathbb{P}^1(E)$ the space of linear polynomials on $E \in \Omega_h$, we can observe that it holds

$$b^E(\varphi; v, w) = \kappa(|\nabla \varphi|^2) \int_E \nabla v \cdot \nabla w \, dx \quad \forall \varphi \in \mathbb{P}^1(E) \quad \forall v, w \in H^1(E).$$

In view of the above relation, the idea is to construct a suitable discrete approximation of the nonlinear term $\kappa(|\nabla \varphi|^2)$ and of the integral one.

To this aim, we introduce a symmetric bilinear form $a_h^E(\cdot, \cdot) : V_h^E \times V_h^E \rightarrow \mathbb{R}$ that locally mimics

$$a_h^E(v_h, w_h) \sim \int_E \nabla \tilde{v}_h \cdot \nabla \tilde{w}_h \, dx. \quad (15)$$

In this context, \tilde{v}_h, \tilde{w}_h can be interpreted as regular functions living on E which ‘‘extend the data’’ v_h, w_h inside the element and the symbol \sim stands for approximation. As shown in [21], the local forms $a_h^E(\cdot, \cdot)$ can be constructed in such a way that the following properties are satisfied.

Proposition 3.1. *For every $E \in \Omega_h$, it holds*

$$\|v_h\|_{1,h,E}^2 \lesssim a_h^E(v_h, v_h), \quad a_h^E(u_h, v_h) \lesssim \|u_h\|_{1,h,E} \|v_h\|_{1,h,E} \quad \forall u_h, v_h \in V_h^E.$$

Moreover, for every $E \in \Omega_h$ and every linear function $\varphi \in \mathbb{P}^1(E)$

$$a_h^E(v_h, \varphi_I) = \sum_{\substack{\mathbf{e} \in \mathcal{E}_h \\ \mathbf{e} \subset \partial E}} (\nabla \varphi \cdot \mathbf{n}_E^{\mathbf{e}}) \frac{|\mathbf{e}|}{2} (v_h^{\mathbf{v}_1} + v_h^{\mathbf{v}_2}) \quad \forall v_h \in V_h^E, \quad (16)$$

where \mathbf{v}_1 and \mathbf{v}_2 are the two vertexes of $\mathbf{e} \in \mathcal{E}_h$, and $\mathbf{n}_E^{\mathbf{e}}$ is the unit normal vector to $\mathbf{e} \in \mathcal{E}_h$ that points outside E .

Identity (16) is usually referred to as *local consistency* property and means that the discrete bilinear forms $a_h^E(\cdot, \cdot)$ respect integration by parts whenever tested with linear functions. As pointed out in [1], the local form $a_h^E(\cdot, \cdot)$ can be written in an algebraic manner by introducing a suitable symmetric and positive definite local matrix $\mathbb{A}_h^E \in \mathbb{R}^{k_E \times k_E}$ such that

$$a_h^E(u_h, v_h) = u_h^T \mathbb{A}_h^E v_h \quad \forall u_h, v_h \in V_h^E,$$

where k_E denotes the number of vertexes of $E \in \Omega_h$.

Next, we discuss the mimetic discretization of the nonlinear term within each element. To this aim, for every $E \in \Omega_h$, we introduce the operator

$$\mathcal{G}_h^E : V_h^E \longrightarrow \mathbb{R}^+ \quad \mathcal{G}_h^E(u_h) := \frac{a_h^E(u_h, u_h)}{|E|} = \frac{u_h^T \mathbb{A}_h^E u_h}{|E|}. \quad (17)$$

Bearing in mind (15), the above operator turns out to be a good candidate to approximate $|\nabla u_h|^2$ within each element, as the following heuristics shows:

$$\frac{\int_E |\nabla u|^2}{|E|} \sim \mathcal{G}_h^E(u_I) \quad \forall u \in H^1(E)$$

where $u_I \in V_h^E$ is defined according to (13).

In view of the above discussion, we define the mimetic discretization of the local semi-linear form (14) as follows:

$$b_h^E(u_h; v_h, w_h) := \kappa(\mathcal{G}_h^E(u_h)) a_h^E(v_h, w_h) \quad \forall u_h, v_h, w_h \in V_h^E,$$

and set

$$b_h(u_h; v_h, w_h) := \sum_{E \in \Omega_h} b_h^E(u_h; v_h, w_h) \quad \forall u_h, v_h, w_h \in V_h. \quad (18)$$

Then, the mimetic discretization of problem (3) reads: find $u_h \in V_h$, such that

$$b_h(u_h; u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h. \quad (19)$$

The right-hand-side of problem (19) is defined as

$$F_h(v_h) := \sum_{E \in \Omega_h} \bar{f}_E \sum_{i=1}^{k_E} v^{\mathbf{v}_i} \omega_E^i,$$

where $\mathbf{v}_1, \dots, \mathbf{v}_{k_E}$ are the vertexes of E , $\omega_E^1, \dots, \omega_E^{k_E}$ are positive weights such that $\sum_{i=1}^{k_E} \omega_E^i = |E|$, and

$$\bar{f}_E := \frac{1}{|E|} \int_E f \, dx.$$

3.2.1 Well-posedness of the discrete problem

This section is devoted to prove the well-posedness of the discrete problem (19). To this aim we start stating some preliminary results.

Let us introduce the function $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by

$$\beta(s) := \frac{1}{2} \int_0^{s^2} \kappa(t) dt.$$

By observing that $\beta'(s) = \kappa(s^2)s$, for all $s \in \mathbb{R}^+$, and using Assumption 2.1, we get

$$k_*(t-s) \leq \beta'(t) - \beta'(s) \leq k^*(t-s) \quad \forall t > s \geq 0.$$

Next, let us recall the following result. We refer to [42, Lemma 25.26] for the proof.

Lemma 3.1. *Let $\gamma(x) := \beta(|x|)$ for $x \in \mathbb{R}^d$. Under Assumption 2.1, it holds*

$$\langle \gamma'(x) - \gamma'(y), x - y \rangle \geq k_* \|x - y\|^2 \quad \forall x, y \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product of \mathbb{R}^d and $\|\cdot\|$ the associated norm.

The following straightforward generalization of Lemma 3.1 will be useful in the forthcoming analysis.

Corollary 3.1. *Let $\mathbb{M} \in \mathbb{R}^{d \times d}$ be a symmetric and positive definite matrix, and let $\langle \cdot, \cdot \rangle_{\mathbb{M}}$ and $\|\cdot\|_{\mathbb{M}}$ the induced scalar product and norm, respectively, i.e.,*

$$\langle x, y \rangle_{\mathbb{M}} := x^T \mathbb{M} y, \quad \|x\|_{\mathbb{M}}^2 := x^T \mathbb{M} x \quad \forall x, y \in \mathbb{R}^d.$$

Under Assumption 2.1, the function $\gamma_{\mathbb{M}}(x) := \beta(\|x\|_{\mathbb{M}})$ satisfies

$$\langle \gamma'_{\mathbb{M}}(x) - \gamma'_{\mathbb{M}}(y), x - y \rangle_{\mathbb{M}} \geq k_* \|x - y\|_{\mathbb{M}}^2 \quad \forall x, y \in \mathbb{R}^d.$$

Proceeding as before, we introduce the discrete operator

$$\mathcal{B}_h : V_h \rightarrow \mathbb{R}, \quad \langle \mathcal{B}_h u_h, v_h \rangle := b_h(u_h; u_h, v_h) \quad \forall v_h \in V_h, \quad (20)$$

and rewrite problem (19) as: find $u_h \in V_h$ such that

$$\mathcal{B}_h u_h = F_h, \quad (21)$$

where F_h is given.

Lemma 3.2. *The operator \mathcal{B}_h defined as in (20) is Lipschitz continuous and strongly monotone with respect to the discrete norm (12), i.e.,*

$$|\langle \mathcal{B}_h u_h - \mathcal{B}_h v_h, u_h - v_h \rangle| \lesssim \|u_h - v_h\|_{1,h}^2 \quad \forall u_h, v_h \in V_h \quad (22)$$

$$\langle \mathcal{B}_h u_h - \mathcal{B}_h v_h, u_h - v_h \rangle \gtrsim \|u_h - v_h\|_{1,h}^2 \quad \forall u_h, v_h \in V_h. \quad (23)$$

Proof. Inequality (23) follows by applying Corollary 3.1 to the following function

$$\tilde{\gamma}(u_h) := |E| \beta \left(\frac{\|u_h\|_{\mathbb{A}_h^E}}{|E|^{1/2}} \right) = |E| \beta(\sqrt{\mathcal{G}_h^E u_h}),$$

after having observed that it holds

$$\langle \gamma'(u_h), v_h \rangle = \kappa(\mathcal{G}_h^E u_h) a_h^E(u_h, v_h) = b_h^E(u_h; u_h, v_h) \quad \forall u_h, v_h \in V_h.$$

Next, we first prove (22). A simple algebraic manipulation yields

$$\begin{aligned}
|\langle \mathcal{B}_h u_h - \mathcal{B}_h v_h, u_h - v_h \rangle| &= |b_h(u_h; u_h, u_h - v_h) - b_h(v_h; v_h, u_h - v_h)| \\
&\leq \sum_{E \in \Omega_h} |b_h^E(u_h; u_h, u_h - v_h) - b_h^E(v_h; v_h, u_h - v_h)| \\
&= \sum_{E \in \Omega_h} [\kappa(\mathcal{G}_h^E u_h) a_h^E(u_h - v_h, u_h - v_h) + \delta_E], \tag{24}
\end{aligned}$$

where we have set

$$\delta_E := [\kappa(\mathcal{G}_h^E u_h) - \kappa(\mathcal{G}_h^E v_h)] a_h^E(v_h, u_h - v_h) \quad \forall E \in \Omega_h.$$

We preliminary observe that Proposition 3.1 together with (5) of Assumption 2.1 yields

$$\begin{aligned}
|\kappa(\mathcal{G}_h^E u_h) \|u_h\|_{1,h,E} - \kappa(\mathcal{G}_h^E v_h) \|v_h\|_{1,h,E}| &\lesssim |E|^{1/2} \left| \kappa(\mathcal{G}_h^E u_h) \frac{\|u_h\|_{\mathbb{A}_h^E}}{|E|^{1/2}} - \kappa(\mathcal{G}_h^E v_h) \frac{\|v_h\|_{\mathbb{A}_h^E}}{|E|^{1/2}} \right| \\
&\lesssim |E|^{1/2} \left| \frac{\|u_h\|_{\mathbb{A}_h^E}}{|E|^{1/2}} - \frac{\|v_h\|_{\mathbb{A}_h^E}}{|E|^{1/2}} \right| \\
&\lesssim \|u_h - v_h\|_{\mathbb{A}_h^E}.
\end{aligned}$$

Then, we can estimate $|\delta_E|$ as follows:

$$\begin{aligned}
|\delta_E| &\leq |\kappa(\mathcal{G}_h^E u_h) - \kappa(\mathcal{G}_h^E v_h)| \|v_h\|_{1,h,E} \|u_h - v_h\|_{1,h,E} \\
&\leq \kappa(\mathcal{G}_h^E u_h) \|v_h\|_{1,h,E} - \|u_h\|_{1,h,E} \|u_h - v_h\|_{1,h,E} \\
&\quad + |\kappa(\mathcal{G}_h^E u_h) \|u_h\|_{1,h,E} - \kappa(\mathcal{G}_h^E v_h) \|v_h\|_{1,h,E}| \|u_h - v_h\|_{1,h,E} \\
&\lesssim \|u_h - v_h\|_{1,h,E}^2 + \|v_h\|_{1,h,E} - \|u_h\|_{1,h,E} \|u_h - v_h\|_{1,h,E} \\
&\lesssim \|u_h - v_h\|_{1,h,E}^2,
\end{aligned}$$

where we have used Assumption 2.1 and the following observation:

$$\| \|u_h\|_{1,h,E} - \|v_h\|_{1,h,E} \| \leq \|u_h - v_h\|_{1,h,E} \quad \forall u_h, v_h \in V_h.$$

Then, by substituting the estimate of $|\delta_E|$ into (24) we get

$$|\langle \mathcal{B}_h u_h - \mathcal{B}_h v_h, u_h - v_h \rangle| \lesssim \|u_h - v_h\|_{1,h}^2,$$

and the proof is complete. \square

Finally, the following well-posedness result holds.

Proposition 3.2. *Let \mathcal{B}_h be the operator defined as in (20). Under Assumption 2.1, equation (21) admits a unique solution $u_h \in V_h$.*

Proof. The thesis follows by the Zarantonello's Theorem and the Lipschitz continuity and strongly monotonicity of the operator \mathcal{B}_h proved in Lemma 3.2. \square

4 Error analysis

In this section we derive the *a-priori* error estimates in the mesh-dependent norm (12) to prove the convergence of the MFD approximate solution to the exact solution of problem (3).

Let us preliminary prove the following result that will be useful in the subsequent analysis.

Lemma 4.1. For every $u_h \in V_h$ it holds

$$|b_h(u_h; v_h, w_h)| \lesssim \|v_h\|_{1,h} \|w_h\|_{1,h} \quad \forall w_h, v_h \in V_h. \quad (25)$$

and

$$b_h(u_h; w_h - v_h, w_h - v_h) \gtrsim \|w_h - v_h\|_{1,h}^2 \quad \forall w_h, v_h \in V_h. \quad (26)$$

Proof. We first show (25). By observing that $b_h(\cdot; \cdot, \cdot)$ is symmetric and linear in its second and third arguments and by using (6) and Proposition 3.1, we get

$$\begin{aligned} |b_h(u_h; v_h, w_h)| &\leq \sum_{E \in \Omega_h} |\kappa(\mathcal{G}_h^E(u_h)) a_h^E(v_h, w_h)| \\ &\lesssim \sum_{E \in \Omega_h} \|v_h\|_{1,h,E} \|w_h\|_{1,h,E} \lesssim \|v_h\|_{1,h} \|w_h\|_{1,h}. \end{aligned}$$

Similarly, we have

$$b_h(u_h; v_h, w_h) \gtrsim \sum_{E \in \Omega_h} \|v_h\|_{1,h,E} \|w_h\|_{1,h,E} \gtrsim \|v_h\|_{1,h} \|w_h\|_{1,h},$$

which is (26), and the proof is complete. \square

Next, we introduce the following lifting operator.

Lemma 4.2. For every $E \in \Omega_h$, there exists a local lifting operator

$$\mathcal{R}_h^E : V_h^E \longrightarrow H^1(E) \cap \mathcal{C}^0(\bar{E}),$$

that satisfies the following properties:

- (L1) it vanishes on the boundary of Ω_h ;
- (L2) $\mathcal{R}_h^E v_h|_{\mathbf{e}}$ is a linear function $\forall \mathbf{e} \in \mathcal{E}_h$, $\mathbf{e} \subseteq \partial E$ and $\forall v_h \in V_h^E$;
- (L3) $|\mathcal{R}_h^E v_h|_{H^1(E)}^2 \lesssim \|v_h\|_{1,h,E}^2 \quad \forall v_h \in V_h^E$;
- (L4) $\|\mathcal{R}_h^E v_h - v_h^\vee\|_{L^2(E)} \lesssim h_E \|v_h\|_{1,h,E} \quad \forall v_h \in V_h^E$.

We observe that it is not necessary to build explicitly the lifting operator, but it is sufficient to show that it exists. A detailed way to construct it and more details about its properties can be found in [21]. We can then extend this definition, by introducing the *global lifting operator* given by

$$\begin{aligned} \mathcal{R}_h : V_h &\longrightarrow H^1(\Omega_h) \cap \mathcal{C}^0(\bar{\Omega}_h), \\ \mathcal{R}_h(v_h)|_E &:= \mathcal{R}_h^E(v_h|_E) \quad \forall v_h \in V_h, E \in \Omega_h. \end{aligned}$$

Finally, the following result states the convergence of the MFD approximation towards the exact solution of problem (3).

Theorem 4.1. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the continuous problem (3) and let u_h be the solution of the discrete problem (19). Moreover, let u_I be the interpolation of the exact solution u defined according to (13). Finally, assume that there exists $\alpha > 0$ so that

$$\|\kappa(|\nabla v|^2) - \kappa(\mathcal{G}_h^E(v_I))\|_\infty \lesssim h^\alpha \quad \forall v \in H^1(\Omega). \quad (27)$$

Then, it holds

$$\|u_I - u_h\|_{1,h} \lesssim h^{\min(1,\alpha)}. \quad (28)$$

Proof. Throughout the proof we set $e_h := u_h - u_{\mathbb{I}}$. For every $E \in \Omega_h$, let u_E^1 be the $L^2(E)$ -projection of u onto the space of linear polynomials on E . We then define the piecewise discontinuous linear function u^1 as $u^1|_E := u_E^1$ for all $E \in \Omega_h$. With a little abuse of notation, in the following we indicate with $(u^1)_{\mathbb{I}}$ a collection of nodal values such that for all elements E the restriction of $(u^1)_{\mathbb{I}}$ to $E \in \Omega_h$ is given by the local interpolation $(u_E^1)_{\mathbb{I}}$. Note that both the $\|\cdot\|_{1,h}$ norm and the bilinear form $a_h(\cdot, \cdot)$ can be immediately extended to $(u^1)_{\mathbb{I}}$, since both operators are a sum of local terms. By using (26), the discrete problem (19) and Proposition 3.1 we get

$$\begin{aligned} \|e_h\|_{1,h}^2 &\lesssim b_h(u_h; u_h, e_h) - b_h(u_{\mathbb{I}}; u_{\mathbb{I}}, e_h) = F_h(e_h) - b_h(u_{\mathbb{I}}; u_{\mathbb{I}}, e_h) \\ &= F_h(e_h) - b_h(u_{\mathbb{I}}; u_{\mathbb{I}} - (u^1)_{\mathbb{I}}, e_h) - b_h(u_{\mathbb{I}}; (u^1)_{\mathbb{I}}, e_h) \\ &\lesssim F_h(e_h) + \|u_{\mathbb{I}} - (u^1)_{\mathbb{I}}\|_{1,h} \|e_h\|_{1,h} + \sum_{E \in \Omega_h} \delta_E, \end{aligned} \quad (29)$$

where $\delta_E := -\kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) a_h^E((u^1)_{\mathbb{I}}, e_h)$. We next estimate the local term δ_E . We first observe that, thanks to (L2)

$$\frac{|e|}{2} (e_h^{v_1} - e_h^{v_2}) = \int_{\mathbf{e}} \mathcal{R}_h^E e_h \, dx \quad \forall E \in \Omega_h, \mathbf{e} \in \mathcal{E}_h^E.$$

Then, by integrating twice by parts and using the fact that $\mathcal{R}_h e_h$ vanishes on the boundary of Ω_h , it follows

$$\begin{aligned} \delta_E &= -\kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \sum_{\substack{\mathbf{e} \in \mathcal{E}_h \\ \mathbf{e} \subseteq \partial E}} \frac{\partial u^1}{\partial \mathbf{n}_{\mathbf{e}}} \int_{\mathbf{e}} \mathcal{R}_h^E e_h \, dx \\ &= -\kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \int_E \nabla \mathcal{R}_h^E e_h \cdot \nabla u^1 \, dx \\ &= \kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \int_E \nabla \mathcal{R}_h^E e_h \cdot \nabla (u - u^1) \, dx - \kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \int_E \nabla \mathcal{R}_h^E e_h \cdot \nabla u \, dx \\ &= \kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \int_E \nabla \mathcal{R}_h^E e_h \cdot \nabla (u - u^1) \, dx \\ &\quad + \int_E \{\kappa(|\nabla u|^2) - \kappa(\mathcal{G}_h^E(u_{\mathbb{I}}))\} \nabla \mathcal{R}_h^E e_h \cdot \nabla u \, dx \\ &\quad + \int_E \operatorname{div} (\kappa(|\nabla u|^2) \nabla u) \mathcal{R}_h^E e_h \, dx, \end{aligned} \quad (30)$$

Finally, by using problem (3) and employing (30) into (29) we have

$$\begin{aligned} \|e_h\|_{1,h}^2 &\lesssim |F_h(e_h) - (f, R_h^E e_h)| + \|u_{\mathbb{I}} - (u^1)_{\mathbb{I}}\|_{1,h} \|e_h\|_{1,h} \\ &\quad + \left| \sum_{E \in \Omega_h} \kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \int_E \nabla \mathcal{R}_h^E e_h \cdot \nabla (u - u^1) \, dx \right| \\ &\quad + \left| \sum_{E \in \Omega_h} \int_E \{\kappa(|\nabla u|^2) - \kappa(\mathcal{G}_h^E(u_{\mathbb{I}}))\} \nabla \mathcal{R}_h^E e_h \cdot \nabla u \, dx \right| \\ &:= \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Let us estimate separately the terms I – IV. By using (L4) and proceeding as in the estimate of the *First Piece* in [18], there holds

$$\text{I} = |F_h(e_h) - (f, R_h^E e_h)| \lesssim h \|f\|_{L^2(\Omega)} \|e_h\|_{1,h} \lesssim h \|e_h\|_{1,h}.$$

Applying the trace inequality (11) and employing a standard interpolation error estimate yield to

$$\text{II} \lesssim \sum_{E \in \Omega_h} \left[\|\nabla(u - u^1)\|_{L^2(E)}^2 + h_E^2 |u|_{H^2(E)}^2 \right] \|e_h\|_{1,h} \lesssim h^2 |u|_{H^2(\Omega)} \|e_h\|_{1,h}.$$

The term term III can be estimated using the Cauchy-Schwarz inequality, assumption (L3) and again standard interpolation error estimates

$$\text{III} \lesssim \|\nabla R_h^E e_h\|_{L^2(\Omega_h)} \|\nabla(u - u^1)\|_{L^2(\Omega_h)} \lesssim h \|e_h\|_{1,h} |u|_{H^2} \lesssim h \|e_h\|_{1,h}.$$

Finally, by using Assumption (27) and (L3), we get

$$\begin{aligned} \text{IV} &\leq \sum_{E \in \Omega_h} \|\kappa(|\nabla u|^2) - \kappa(\mathcal{G}_h^E(u_T))\|_{\infty} \int_E |\nabla \mathcal{R}_h^E e_h \cdot \nabla u \, dx| \\ &\lesssim h^\alpha \|\nabla R_h^E e_h\|_{L^2(\Omega_h)} \|\nabla(u)\|_{L^2(\Omega_h)} \lesssim h^\alpha \|e_h\|_{1,h} |u|_{H^2}. \end{aligned}$$

Combining the estimates of I – IV with (4) yields the result. \square

Remark 4.1 (On Assumption (27)). *Theorem 4.1 relies on assumption (27), whose theoretical justification is still object of an on-going research. However, the numerical investigation performed in Section 6 suggests the validity of (27) with $\alpha = 1$, so that the error estimate (28) reduces to $\|u_T - u_h\|_{1,h} \lesssim h$.*

5 The Kačanov method

In this section we present the Kačanov method (see, e.g., [42]) to solve the nonlinear problem (19). For $k \geq 0$, the idealized algorithm without any stopping criterion (that will be discussed later) reads as follows: given $u_h^{(0)} \in V_h$, find $u_h^{(k+1)} \in V_h$ such that

$$b_h(u_h^{(k)}; u_h^{(k+1)}, v_h) = F_h(v_h) \quad \forall v_h \in V_h. \quad (31)$$

The convergence of the sequence $\{u_h^{(k)}\}_{k \geq 0}$ to the “exact” discrete solution u_h of problem (19) is stated in Theorem 5.1, below. The proof takes the steps from the general ideas of [42, Theorem 25.B] and requires some preliminary results that will be collected in the following.

We introduce the functional $E_h : V_h \rightarrow \mathbb{R}$ defined as

$$E_h(v_h) := \frac{1}{2} \sum_{E \in \Omega_h} |E| \beta \left(\sqrt{\mathcal{G}_h^E(v_h)} \right) \quad (32)$$

that represents the energy associated to system (19). It is easy to see that

$$\langle E_h'(u_h), v_h \rangle = b_h(u_h; u_h, v_h) \quad \forall u_h, v_h \in V_h. \quad (33)$$

Finally, we state two auxiliary lemmas that will be crucial to prove the convergence of the Kačanov algorithm.

Lemma 5.1. *Let $E_h(\cdot)$ be defined according to (32) and let assume that $\kappa(\cdot)$ is monotone decreasing, i.e., $\kappa(t_1) \geq \kappa(t_2)$ for all $0 \leq t_1 \leq t_2$. Then,*

$$E_h(u_h) - E_h(v_h) \leq \frac{1}{2} [b_h(u_h; u_h, u_h) - b_h(u_h; v_h, v_h)], \quad (34)$$

$$\langle E_h'(u_h) - E_h'(v_h), u_h - v_h \rangle \gtrsim \|u_h - v_h\|_{1,h}^2 \quad (35)$$

for all $u_h, v_h \in V_h$.

Proof. Inequality (34) can be proved by using that $k(\cdot)$ is monotone decreasing and (17), (18) and (32)

$$\begin{aligned} E_h(u_h) - E_h(v_h) &= \frac{1}{2} \sum_{E \in \Omega_h} |E| \left[\int_0^{\mathcal{G}_h^E(u_h)} \kappa(s) \, ds + \int_0^{\mathcal{G}_h^E(v_h)} \kappa(s) \, ds \right] \\ &\leq \frac{1}{2} \sum_{E \in \Omega_h} |E| \kappa(\mathcal{G}_h^E(v_h)) \left[\frac{a_h^E(u_h, u_h)}{|E|} - \frac{a_h^E(v_h, v_h)}{|E|} \right] \\ &\leq \frac{1}{2} [b_h(u_h; u_h, u_h) - b_h(u_h; v_h, v_h)]. \end{aligned}$$

Inequality (35) easily follows from (33) and the strongly monotonicity of the operator \mathcal{B}_h shown in Lemma 3.2. \square

We also have the following result which is a straightforward consequence of Lemma 4.1.

Lemma 5.2. *For every $k \geq 0$, given $u_h^{(k)} \in V_h$, the variational problem (31) admits a unique solution $u_h^{(k+1)} \in V_h$.*

We are now ready to prove the convergence of the Kačanov method (31).

Theorem 5.1. *Let $\{u_h^{(k)}\}_{k \geq 0}$ be the sequence built by the Kačanov method, then $u_h^{(k)} \rightarrow u_h$ in V_h , as $k \rightarrow +\infty$.*

Proof. We preliminary observe that, by defining $\mathcal{L}_h(u_h) := E_h(u_h) - F_h(u_h)$, the following minimization problem

$$\min_{v_h \in V_h} \mathcal{L}_h(v_h)$$

admits a unique solution thanks to the well-posedness of problem (19) together with (33). Next, let us define

$$g(u_h) := \frac{1}{2} \left(b_h(u_h^{(k)}; u_h, u_h) - b_h(u_h^{(k)}; u_h^{(k)}, u_h^{(k)}) \right) + E_h(u_h^{(k)}) - F_h(u_h).$$

Thanks to (34), we get

$$\begin{aligned} E_h(u_h^{(k+1)}) - F_h(u_h^{(k+1)}) &\leq \frac{1}{2} \left(b_h(u_h^{(k)}; u_h^{(k+1)}, u_h^{(k+1)}) - b_h(u_h^{(k)}; u_h^{(k)}, u_h^{(k)}) \right) \\ &\quad + E_h(u_h^{(k)}) - F_h(u_h^{(k+1)}), \end{aligned}$$

that is

$$\mathcal{L}_h(u_h^{(k+1)}) \leq g(u_h^{(k+1)}).$$

Next, by using that $u_h^{(k+1)} \in V_h$ is the minimum of the functional $g(\cdot)$, we get

$$\mathcal{L}_h(u_h^{(k+1)}) \leq g(u_h^{(k+1)}) \leq g(u_h^{(k)}) = \mathcal{L}_h(u_h^{(k)}). \quad (36)$$

So, we can conclude that the sequence $\{\mathcal{L}_h(u_h^{(k)})\}_{k \geq 0}$ is monotone decreasing and is limited from below, since the operator $\mathcal{L}_h(\cdot)$ admits a minimum. Consequently,

$$\lim_{k \rightarrow \infty} \left\{ \mathcal{L}_h(u_h^{(k+1)}) - \mathcal{L}_h(u_h^{(k)}) \right\} = 0.$$

Finally, by using (36), the linearity of the form $b_h(\cdot; \cdot, \cdot)$ in its second and third arguments and property (26) we get

$$\begin{aligned} \mathcal{L}(u_h^{(k)}) - \mathcal{L}(u_h^{(k+1)}) &\geq \mathcal{L}(u_h^{(k)}) - g(u_h^{(k+1)}) \\ &= F_h(u_h^{(k+1)} - u_h^{(k)}) - \frac{1}{2}(b_h(u_h^{(k)}; u_h^{(k+1)}, u_h^{(k)}) - b_h(u_h^{(k)}; u_h^{(k)}, u_h^{(k)})) \\ &= \frac{1}{2}b_h(u_h^{(k)}; u_h^{(k+1)} - u_h^{(k)}, u_h^{(k+1)} - u_h^{(k)}) \gtrsim \|u_h^{(k+1)} - u_h^{(k)}\|_{1,h}^2. \end{aligned}$$

So we can conclude that

$$\|u_h^{(k+1)} - u_h^{(k)}\|_{1,h} \rightarrow 0, \quad (37)$$

as k tends to $+\infty$. Next, we prove that the sequence $\{u_h^{(k)}\}_{k \geq 0}$ converges to the solution u_h , as k tends to $+\infty$. By using (35), (33) and that u_h is the solution of problem (19) we have

$$\begin{aligned} \|u_h^{(k)} - u_h\|_{1,h}^2 &\lesssim \langle E_h'(u_h^{(k)}) - E_h'(u_h), u_h^{(k)} - u_h \rangle \\ &\leq \langle E_h'(u_h^{(k)}), u_h^{(k)} - u_h \rangle + F_h(u_h - u_h^{(k)}) \\ &= b_h(u_h^{(k)}; u_h^{(k)}, u_h^{(k)} - u_h) + F_h(u_h - u_h^{(k)}). \end{aligned} \quad (38)$$

Then, by using twice (31) we get

$$\begin{aligned} b_h(u_h^{(k)}; u_h^{(k)}, u_h^{(k)} - u_h) + F_h(u_h - u_h^{(k)}) &= \\ &= b_h(u_h^{(k)}; u_h^{(k)} - u_h^{(k+1)}, u_h^{(k)} - u_h) \pm b_h(u_h^{(k)}; u_h^{(k+1)}, u_h^{(k+1)} - u_h^{(k)}) \\ &= b_h(u_h^{(k)}; u_h^{(k)} - u_h^{(k+1)}, u_h^{(k)} - u_h + u_h^{(k+1)}) + F_h(u_h^{(k+1)} - u_h^{(k)}) \end{aligned}$$

Finally, by inserting the above estimate into (38), using (25) and that $|F_h(v_h)| \leq \|f\|_{L^2(\Omega)} \|v_h\|_{1,h}$ we get

$$\|u_h^{(k)} - u_h\|_{1,h}^2 \lesssim \|u_h^{(k+1)} - u_h^{(k)}\|_{1,h} \left(\delta \|u_h^{(k)} - u_h + u_h^{(k+1)}\|_{1,h} + \|f\|_{L^2(\Omega)} \right). \quad (39)$$

The proof is complete by observing that the quantity $\|u_h^{(k)} - u_h + u_h^{(k+1)}\|_{1,h}$ is bounded from above since each term is bounded thanks to the well-posedness of problem (19) and of the corresponding linearized ones (see Proposition 5.2). Therefore, thanks to (37) we get $\|u_h^{(k)} - u_h\| \rightarrow 0$ as k tends to ∞ . \square

The following result contains a computable estimate for the error $\|u_h - u_h^{(k)}\|_{1,h}$, which can be employed as a reliable stopping criterion for the Kačanov method (see also Section 6 for further details).

Proposition 5.1. *Assume the function $\kappa(\cdot)$ satisfies Assumption 2.1 and also $\kappa'(t) \leq 0$ and $2\kappa'(t)t + \kappa(t) \gtrsim 1$. Then*

$$\|u_h - u_h^{(k)}\|_{1,h}^2 \lesssim E_h(u_h^{(k)}) - F_h(u_h^{(k)}) + E_h^*(p_h^*)$$

where $p_h^*|_E := \kappa(\mathcal{G}_h^E(u_h^{(k-1)}))(\mathbb{A}_h^E)^{1/2} u_h^{(k)}|_E$, and the conjugate functional $E_h^*(\cdot)$ is defined as

$$E_h^*(p_h^*) := \sum_{E \in \Omega_h} |E| \beta^* \left(\sqrt{\mathcal{G}_h^E(p_h^*)} \right).$$

Here, $\beta^*(s^*) := \kappa(\bar{s}^2)\bar{s}^2 - \beta(\bar{s})$ for every $s^* \in \mathbb{R}$ being $\bar{s} = \bar{s}(s^*)$ the unique solution to the algebraic equation

$$\beta'(\bar{s}) = s^*.$$

Proof. Setting, as before, $\mathcal{L}_h(\cdot) := E_h(\cdot) - F_h(\cdot)$, we first observe that it holds

$$\begin{aligned} \mathcal{D}_h(u_h^{(k)}) &:= \mathcal{L}_h(u_h^{(k)}) - \mathcal{L}_h(u_h) - \langle \mathcal{L}'_h(u_h), u_h^{(k)} - u_h \rangle \\ &= \int_0^1 (1-t) \langle \mathcal{L}''_h(u_h + t(u_h^{(k)} - u_h))(u_h^{(k)} - u_h), u_h^{(k)} - u_h \rangle dt, \end{aligned} \quad (40)$$

where the second order Fréchet derivative satisfies

$$\begin{aligned} \langle \mathcal{L}''_h(u_h)v_h, v_h \rangle &= \sum_E 2\kappa' \left(\frac{\|u_h\|_{\mathbb{A}_h^E}^2}{|E|} \right) \frac{(a_h^E(u_h, v_h))^2}{|E|} + \kappa \left(\frac{\|u_h\|_{\mathbb{A}_h^E}^2}{|E|} \right) a_h^E(v_h, v_h) \\ &\gtrsim \sum_E \left(2\kappa' \left(\frac{\|u_h\|_{\mathbb{A}_h^E}^2}{|E|} \right) \frac{\|u_h\|_{1,h,E}^2}{|E|} + \kappa \left(\frac{\|u_h\|_{\mathbb{A}_h^E}^2}{|E|} \right) \right) \|v_h\|_{1,h,E}^2 \\ &\gtrsim \|v_h\|_{1,h}^2, \end{aligned} \quad (41)$$

where we used Proposition 3.1 together with the fact that $\kappa'(z) \leq 0$ and $2\kappa'(t)t + \kappa(t) \gtrsim 1$. Observing that u_h is the unique solution to (19) yields $\langle \mathcal{L}'_h(u_h), v_h \rangle = 0$ for every $v_h \in V_h$. Hence, combining (40) and (41) we get

$$\mathcal{L}_h(u_h^{(k)}) - \mathcal{L}_h(u_h) = \mathcal{D}_h(u_h^{(k)}) \gtrsim \|u_h^{(k)} - u_h\|_{1,h}^2. \quad (42)$$

Using the standard theory of duality in optimization (see, e.g., [26, 41] for a general introduction, and [30] for an application to a posteriori error estimates), we obtain

$$\mathcal{L}_h(u_h^{(k)}) - \mathcal{L}_h(u_h) \leq \mathcal{L}_h(u_h^{(k)}) + E_h^*(p_h^*), \quad (43)$$

where it is crucial to observe that p_h^* has been chosen to solve the linear elliptic problem $p_h^T \mathbb{A}_h v_h = F(v_h)$ for every $v_h \in V_h$ (cf. [41, Theorem 51.B] and [42, Theorem 25.K]). Indeed, a closer look reveals that p_h^* is solution of the k -th linearized problem (31). Finally, combining (42) and (43) yields the thesis. \square

Remark 5.1. *It is easy to see that the Carreau law defined in (4) satisfies the assumptions on the function $\kappa(\cdot)$ of Proposition 5.1.*

We conclude this section by observing that Theorem 4.1 and Theorem 5.1 can be combined to obtain an estimate of the error $\|u_I - u_h^{(k)}\|_{1,h}$.

Corollary 5.1. *Let $u_I \in V_h$ be the interpolation of the exact solution of problem (3) as defined in (13). Then, for every $h > 0$ there exists $k \in \mathbb{N}$, $k = k(h)$, such that*

$$\|u_I - u_h^{(k)}\|_{1,h} \lesssim O(h) + \|u_h - u_h^{(k)}\|_{1,h},$$

where $u_h^{(k)} \in V_h$ is the k -th iterate obtained with the Kačanov method (31).

6 Numerical experiments

This section is devoted to investigate the performance of the MFD method for the approximate solution of problem (3) and to validate the convergence analysis stated in Theorem 5.1.

Throughout the section, the function $\kappa(\cdot)$ is chosen equal to the Carreau law defined in (4), with $\eta_0 = 3$, $\eta_\infty = 1$ and $p = 1.7$. We consider four different sequences of decompositions of the domain $\Omega = (0, 1)^2$. Figure 1 shows an example of all the considered computational grids, that we denote by *hexagons type1*, *hexagons type2*, *quadrilaterals*

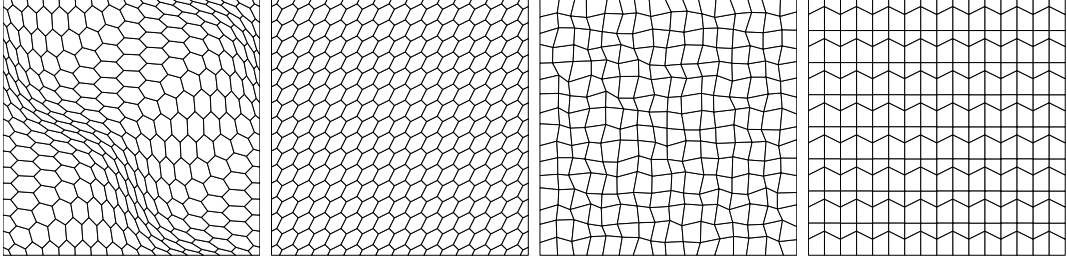


Figure 1: Examples of the considered decompositions of $\Omega = (0,1)^2$. From left to right: *hexagons type1*, *hexagons type2*, *quadrilaterals* and *trapezes*.

and *trapezes*, respectively.

To solve problem (19) we have employed the *feasible* Kačanov method supplemented with a suitable stopping criterion, as shown in Algorithm 6.1. The tolerance in the stopping criterion has been set equal to 10^{-8} . We remark that, the reliability of the stopping criterion is justified in view of inequality (39). Alternatively, one can resort to the computable estimate presented in Proposition 5.1.

Algorithm 6.1: *Feasible* Kačanov method

- 1 Given the initial guess $u_h^{(0)}$, set **tol1**, $k = -1$, $u_h^{(-1)} = u_h^{(0)}$;
 - 2 **while** $\|u_h^{(k+1)} - u_h^{(k)}\|_{1,h} \geq \mathbf{tol1}$ **do**
 - 3 $k + 1 \leftarrow k$;
 - 4 **SOLVE** $b_h(u_h^{(k)}; u_h^{(k+1)}, v_h) = F_h(v_h) \quad \forall v_h \in V_h$;
 - 5 **end**
 - 6 **SET** $u_h := u_h^{(k+1)}$;
-

6.1 Example 1

In the first example we choose $u(x, y) = \log(1 + x + y) + x + y$ as the exact solution and set the data and the non-homogeneous boundary conditions accordingly. The relative errors in the discrete norm (12) computed on all the sequences of the considered grids are reported in Figure 2(a) (loglog scale). We clearly observe a linear convergence as predicted by our *a-priori* error estimates stated in Theorem 5.1.

6.2 Example 2

The second example is taken from [32]. Here, the source term f is selected so that $u(x, y) = x(1-x)y(1-y)(1-2y) \exp(-20((2x-1)^2))$ is the analytical solution of problem (19). The numerical results are reported in Figure 2(b), where the computed relative errors in the discrete norm defined in (12) are shown (loglog scale). Also in this case, we observe that the error goes to zero linearly as the mesh is refined.

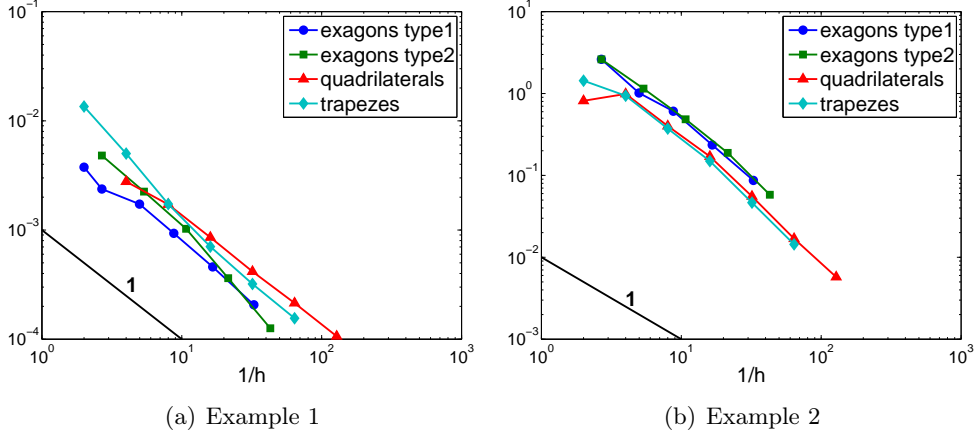


Figure 2: Computed relative errors $\|u_{\mathbb{I}} - u_h\|_{1,h} / \|u_{\mathbb{I}}\|_{1,h}$ versus $1/h$ (loglog scale). Example 1 (left) and Example 2 (right), .

6.3 On Assumption (27).

Finally, we check numerically the validity of hypothesis (27) in Theorem 4.1. On the sequence of considered grids, we have computed the quantity

$$\left\| \kappa \left(\Pi^0 |\nabla u|^2 \right) - \kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \right\|_{\infty,h},$$

where Π^0 denotes the projection onto the space of piecewise constant functions defined on Ω_h , $u_{\mathbb{I}}$ is the interpolation of the exact solutions of Example 6.1 and Example 6.2, respectively, and

$$\|v_h\|_{\infty,h} := \sup_{v \in \mathcal{N}_h} |v_h^*| \quad \forall v_h \in V_h.$$

The computed results are reported in Figure 3 (loglog scale). From the numerical results

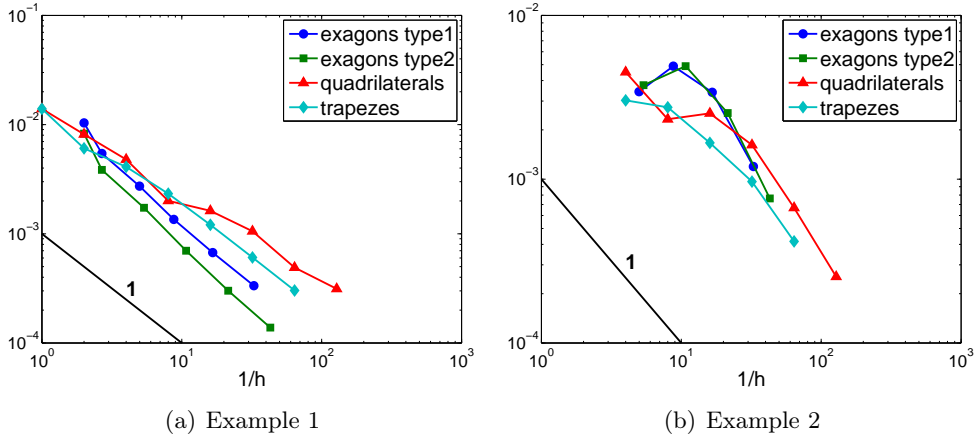


Figure 3: $\left\| \kappa \left(\Pi^0 |\nabla u|^2 \right) - \kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \right\|_{\infty,h}$ versus $1/h$ (loglog scale). Here u is the exact solution of Example 1 (left) and Example 2 (right).

it seems that

$$\left\| \kappa \left(\Pi^0 |\nabla u|^2 \right) - \kappa(\mathcal{G}_h^E(u_{\mathbb{I}})) \right\|_{\infty,h} \lesssim h.$$

This, together with standard interpolation error estimates suggest the validity of (27) with $\alpha = 1$.

7 Conclusions

We have proposed a mimetic finite difference discretization of a quasilinear elliptic problem. Under a suitable approximation assumption, we have shown optimal *a-priori* error estimates in a mesh-dependent energy norm. A Kačanov iterative scheme has been proposed and analyzed to solve the resulting nonlinear discrete problem. All the theoretical results have been confirmed by numerical experiments. Future research will be devoted to extend the presented results to (free-boundary) quasi-newtonian flow problems governed by the non-linear Stokes equations by taking advantage of the Uzawa iterative algorithm, where at each step a quasilinear elliptic problem similar to (1) has to be solved.

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