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# A discontinuous Galerkin Reduced Basis Element method for elliptic problems

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## Abstract

We propose and analyse a new discontinuous reduced basis element method for the approximation of parametrized elliptic PDEs in partitioned domains. The method is built upon an offline stage (parameter independent) and an online (parameter dependent) one. In the offline stage we build a non-conforming (discontinuous) global reduced space as a direct sum of local basis functions built independently on each subdomain. In the online stage, for a given value of the parameter, the approximate solution is obtained by ensuring the weak continuity of the fluxes and of the solution itself thanks to a discontinuous Galerkin approach. The new method extends and generalizes the methods introduced in [16, 18]. We prove stability and convergence properties of the proposed method, as well as conditioning properties of the associated algebraic online system. We also propose a two-level preconditioner for the online problem which exploits the pre-existing decomposition of the domain and is based upon the introduction of a global coarse finite element space. Numerical tests are performed to validate our theoretical results.

## Introduction

The Reduced Basis (RB) method, see *e.g.* [26, 25], for elliptic Parametrized Partial Differential Equations (PPDEs) has been successfully developed to approximate the solution of problems like:

$$\text{find } u(\boldsymbol{\mu}) \in V \text{ such that } A(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = F(v; \boldsymbol{\mu}) \quad \forall v \in V, \quad (1)$$

where  $V$  is a suitable Hilbert space,  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_P)$  is a  $P$ -tuple of parameters which belongs to a subspace  $\mathcal{D}$  of  $\mathbb{R}^P$ ,  $A$  is a continuous coercive bilinear form defined on  $V \times V$  and  $F$  is a linear continuous functional on  $V$ .

When the domain  $\Omega(\boldsymbol{\mu})$  of the PPDE is decomposed into several subdomains, a convenient numerical approach is provided by the so-called Reduced Basis Element (RBE) method, presented in [22, 20, 21, 23], in which local (*i.e.*, defined on each subdomain) reduced bases are built by restriction of global solutions, while the global continuity of the RB solution is guaranteed either by the introduction of suitable Lagrange multipliers, as in [18], or by adopting a discontinuous Galerkin (DG) approach, as in [11].

Several improvements of the RBE idea have been recently proposed. One instance is the so-called static condensation Reduced Basis Element method [14, 15, 12], where a RB approximation of the Schur complement is proposed and rigorous *a posteriori* error estimators are derived. Another approach is represented by the so-called Reduced Basis Hybrid Method (RBHM) [16, 18], where a global coarse solution, responsible for ensuring interface continuity of normal fluxes, is overlaid to the subspace of local reduced basis computed off-line at subdomain levels. The continuity of the global reduced solution is enforced using Lagrange multipliers. A further instance is provided by the Reduced Basis - Domain Decomposition - Finite Element (RDF) method [16, 17], in which the continuity of the elements of the reduced space on the whole domain  $\Omega(\boldsymbol{\mu})$  is guaranteed by the introduction of additional degrees of freedom on the interfaces, corresponding to the fine-grid Finite Element (FE) Lagrangian basis functions associated with the nodes on each interface. Ideas related to the RBE approach can also be applied to the RB approximation of multiscale phenomena, as done in [1, 19].

In this work we propose a discontinuous Galerkin Reduced Basis Element (DGRBE) method which represents in fact a generalization and an improvement of both RDF and RBHM. As a matter of fact, as in the RDF method, the DGRBE approximation is based upon a set of local basis functions that feature non-homogeneous Neumann conditions, however it does not require the introduction of additional degrees of freedom on the interfaces. Moreover, a possible preconditioner for the reduced problem makes use of a coarse space correction on the local basis inspired from the RBHM. We point out that this correction was essential to ensure interface stress continuities in RBHM, whereas in DGRBE method it only serves the purpose of improving the spectral properties of the preconditioner to solve the associated online linear system.

Furtherly, the underlying DG approach allows for the use of independent elementwise representation of the numerical solution, without necessitating Lagrange multipliers to ensure the continuity across the internal interfaces, as was the case for RBHM. The DGRBE method is then well suited for global meshes which are non-conforming on the subdomain interfaces. We point out that the local bases are constructed by solving local problems with suitably chosen boundary conditions. No approximate solution of the global problem (1) is therefore required. This makes the DGRBE method particularly well suited for problems defined on “modular” domains, namely composed by an arbitrary number of subdomains that are geometrical transformation of few parameter-independent reference subdomains, *cf.* [14, 18]. We show by numerical experiments that, as regards the accuracy, the DGRBE approximation of (1) on a partitioned domain is comparable to a fine-grid FE one, but is based on a significantly lower dimensional approximation space. After introducing the DGRBE method, we carry out its analysis in the case of elliptic problems. More precisely, we prove: the well-posedness of the method, its stability and some convergence estimates.

An outline of the paper is as follows. In Section 1 the main features of the DGRBE method are introduced, while in Section 2 the theoretical analysis is carried out. In Sec-

tion 3 a two level preconditioner is presented and it is meant to make the preconditioned online system weakly scalable. Finally, in Section 4 some numerical tests are shown. In the appendices some implementation details are reported.

## 1 The DGRBE method

We assume that a parameter dependent open subset  $\Omega(\boldsymbol{\mu}) \subset \mathbb{R}^2$  is given, where  $\boldsymbol{\mu}$  is a parameter belonging to the space  $\mathcal{D} \subseteq \mathbb{R}^P$ ,  $P \geq 1$ . Given an integer  $N_S > 1$ , we assume that the domain is composed of a finite number of non-overlapping subdomains,

$$\bar{\Omega}(\boldsymbol{\mu}) = \bigcup_{i=1}^{N_S} \bar{\Omega}_i(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

where each  $\Omega_i(\boldsymbol{\mu})$  is an open bounded subset of  $\mathbb{R}^2$ . The model problem we are considering is the following:

$$\begin{aligned} -\nu(\boldsymbol{\mu})\Delta u(\boldsymbol{\mu}) + \sigma(\boldsymbol{\mu})u(\boldsymbol{\mu}) &= f(\boldsymbol{\mu}) \quad \text{in } \Omega(\boldsymbol{\mu}), \\ u(\boldsymbol{\mu}) &= 0 \quad \text{on } \partial\Omega(\boldsymbol{\mu}), \end{aligned} \tag{2}$$

where  $f \in L^2(\Omega(\boldsymbol{\mu}))$  is a given source term and  $\nu(\boldsymbol{\mu})$ ,  $\sigma(\boldsymbol{\mu})$  are  $\boldsymbol{\mu}$ -dependent constant coefficients. We point out that our results can be extended to the case of subdomain-wise constant coefficients or, under suitable regularity assumptions, to the case of space-dependent functions. We also remark that the parameter dependence can be both *physical* and *geometrical*, that is both the coefficients and the domain can depend on the parameters. To follow a reduced basis approach, we define a reference domain  $\Omega = \Omega(\bar{\boldsymbol{\mu}})$ , for a suitably chosen  $\bar{\boldsymbol{\mu}} \in \mathcal{D}$ . Correspondingly, we define the reference subdomains  $\Omega_i = \Omega_i(\bar{\boldsymbol{\mu}})$ , for  $i = 1 \dots N_S$ . Let  $T_i^\boldsymbol{\mu}: \Omega_i \rightarrow \Omega_i(\boldsymbol{\mu})$ , be the local geometrical transformation mapping the reference subdomains into the ‘‘physical’’ ones. By patching together these local transformations, we can define a global transformation  $T^\boldsymbol{\mu}$  which maps the reference domain  $\Omega$  onto  $\Omega(\boldsymbol{\mu})$ . We assume that the global map  $T^\boldsymbol{\mu}$  is continuous and bijective. Setting  $V = H_0^1(\Omega)$ , we define

$$A(w, v; \boldsymbol{\mu}) = \int_{\Omega(\boldsymbol{\mu})} \nu(\boldsymbol{\mu}) \nabla(w \circ T^\boldsymbol{\mu}) \cdot \nabla(v \circ T^\boldsymbol{\mu}) \, dx + \int_{\Omega(\boldsymbol{\mu})} \sigma(\boldsymbol{\mu}) (w \circ T^\boldsymbol{\mu}) (v \circ T^\boldsymbol{\mu}) \, dx,$$

for all  $w, v \in V$ . For the sake of notation, in what follows we let the composition with  $T^\boldsymbol{\mu}$  to be understood.

The reduced-order method we are going to introduce features two main components:

- a local reduced basis for each subdomain;
- a DG-type interface conditions at subdomain boundaries.

In what follows we explain the role played by these components during the two stages of the DGRBE method: the *offline stage* and the *online stage*. Details about the implementation aspects can be found in Appendix A.

### 1.1 Offline stage

The offline stage of the DGRBE method is inspired to the offline stage of the RDF method introduced in [16] and follows ideas which can also be found in [14].

For each  $i = 1, \dots, N_S$ , we define a (parameter independent) conforming quasi-uniform triangulation  $\mathcal{T}_{h,i}$  on  $\Omega_i$ . Setting  $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ , we define the local spaces

$$\begin{aligned} V_i &= \{v_i \in L^2(\Omega) \mid v_i|_{\Omega_i} \in H^1(\Omega_i), v_i = 0 \text{ on } \partial\Omega_i \setminus \Gamma_i, v_i = 0 \text{ on } \Omega \setminus \Omega_i\}, \\ V_{h,i} &= \{v_{h,i} \in V_i \mid v_{h,i}|_K \in \mathbb{P}^1(K) \forall K \in \mathcal{T}_{h,i}\}. \end{aligned}$$

The idea behind the construction of the local bases is the following: on each subdomain we build a local reduced basis such that, for each value of the parameters, it allows a good approximation of the solution of the two following problems:

$$\begin{aligned} -\nu(\boldsymbol{\mu})\Delta \hat{u}_i(\boldsymbol{\mu}) + \sigma(\boldsymbol{\mu})\hat{u}_i(\boldsymbol{\mu}) &= f(\boldsymbol{\mu}), & \text{in } \Omega_i(\boldsymbol{\mu}) \\ \frac{\partial \hat{u}_i(\boldsymbol{\mu})}{\partial n_i} &= 0, & \text{on } \Gamma_i(\boldsymbol{\mu}) \\ \hat{u}_i(\boldsymbol{\mu}) &= 0, & \text{otherwise,} \end{aligned} \quad (3)$$

and

$$\begin{aligned} -\nu(\boldsymbol{\mu})\Delta w_i(\boldsymbol{\mu}, \beta) + \sigma(\boldsymbol{\mu})w_i(\boldsymbol{\mu}, \beta) &= 0, & \text{in } \Omega_i(\boldsymbol{\mu}) \\ \frac{\partial w_i(\boldsymbol{\mu}, \beta)}{\partial n_i} &= g(\beta), & \text{on } \Gamma_i(\boldsymbol{\mu}) \\ w_i(\boldsymbol{\mu}, \beta) &= 0, & \text{otherwise,} \end{aligned} \quad (4)$$

where  $g(\beta)$  is a Neumann datum which depends on an artificial parameter  $\beta \in \mathbb{R}$ . Recalling that  $u(\boldsymbol{\mu})$  is the solution of (2), we now observe that if  $\sum_j g(\beta_j)$  is an approximation of  $\frac{\partial u(\boldsymbol{\mu})}{\partial n}|_{\Gamma_i}$ , for a given set  $\{\beta_1, \dots, \beta_{\bar{n}}\} \subset \mathbb{R}$ , then by linearity  $u_i(\boldsymbol{\mu}) = \hat{u}_i(\boldsymbol{\mu}) + \sum_j w_i(\boldsymbol{\mu}, \beta_j)$  is an approximation of  $u(\boldsymbol{\mu})|_{\Omega_i(\boldsymbol{\mu})}$ . Thus, building a space able to approximate the solutions of problems (3) and (4) on each subspace allows to approximate also the solution of the initial problem (2). In the following we made the above idea more clear. We first introduce the local forms

$$A_i(w_i, v_i; \boldsymbol{\mu}) = \int_{\Omega_i(\boldsymbol{\mu})} \nu(\boldsymbol{\mu}) \nabla w_i \cdot \nabla v_i \, dx + \int_{\Omega_i(\boldsymbol{\mu})} \sigma(\boldsymbol{\mu}) w_i v_i \, dx, \quad F_i(v_i; \boldsymbol{\mu}) = \int_{\Omega_i(\boldsymbol{\mu})} f(\boldsymbol{\mu}) v_i \, dx,$$

for all  $w_i, v_i \in V_i$ . We then define

$$(w_i, v_i)_{V_i} = A_i(w_i, v_i; \bar{\boldsymbol{\mu}}), \quad \|v_i\|_{V_i} = (v_i, v_i)_{V_i}^{1/2} \quad \forall w_i, v_i \in V_i,$$

where  $\bar{\boldsymbol{\mu}}$  is the parameter value chosen to define the reference domain. Next, we define the extended parameter space  $\tilde{\mathcal{D}} = \mathcal{D} \times \{0, \dots, n_{BC,i}\}$  and we denote with  $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \beta)$  the generic element of  $\tilde{\mathcal{D}}$ . Denoting with  $V_{h,i}^{\Gamma_i}$  the space of the traces on  $\Gamma_i$  of the elements of  $V_{h,i}$ , we introduce a  $\beta$ -dependent functional  $\mathcal{I}_i^\beta$  belonging to the dual space of  $V_{h,i}^{\Gamma_i}$ , for all  $\beta \in \{0, \dots, n_{BC,i}\}$ . We are now able to define the local problems which we use to build the local basis. Given a parameter value  $\tilde{\boldsymbol{\mu}} \in \tilde{\mathcal{D}}$ , find  $\tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}) \in V_{h,i}$  such that

$$A_i(\tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}), v_{h,i}; \tilde{\boldsymbol{\mu}}) = \tilde{F}_i(v_{h,i}; \tilde{\boldsymbol{\mu}}) + \langle \mathcal{I}_i^\beta, v_{h,i}|_{\Gamma_i} \rangle \quad \forall v_{h,i} \in V_{h,i}, \quad (5)$$

We assume that, for each choice of  $w_{h,i}$  and  $v_{h,i}$  in  $V_{h,i}$ , it holds

$$\begin{aligned} \tilde{F}_i(v_{h,i}; \tilde{\boldsymbol{\mu}}) &= F_i(v_{h,i}; \boldsymbol{\mu}) \quad \text{and} \quad \langle \mathcal{I}_i^\beta, v_{h,i}|_{\Gamma_i} \rangle = 0 \quad \forall \tilde{\boldsymbol{\mu}} \in \mathcal{D} \times \{0\}, \quad \text{i.e., when } \beta = 0, \\ \tilde{F}_i(v_{h,i}; \tilde{\boldsymbol{\mu}}) &= 0 \quad \forall \tilde{\boldsymbol{\mu}} \in \mathcal{D} \times \{1, \dots, n_{BC,i}\} \end{aligned}$$

We observe that we are considering the FE approximation of problems (3) and (4) when  $\beta = 0$  and  $\beta \neq 0$ , respectively. The linear functional  $\mathcal{I}_i^\beta$  serves the purpose of (weakly)

imposing the non-homogeneous Neumann boundary conditions. We propose two possible choices of  $\mathcal{I}_i^\beta$ : the former is based on the approximation of the weak normal derivative of the fine FE solution, the latter on the approximation of the normal derivative of the continuous global solution along the internal interfaces.

The well-posedness of the local problem (5) is guaranteed by the following lemma, which can be proven using a standard energy argument (cf. [24]).

**Lemma 1.1.** *Let  $D_i^\mu$  be the Jacobian matrix of  $T^\mu$  and let  $J_i^\mu$  be its determinant. For every  $\mu$  in  $\mathcal{D}$ , we assume that  $\nu(\mu) > 0$ ,  $\sigma(\mu) > 0$  and  $J_i^\mu > 0$  on  $\Omega_i$ ,  $i = 1, \dots, N_S$ , and define*

$$\begin{aligned}\alpha_i(\mu) &= \min \left\{ \frac{\nu(\mu)}{\nu(\bar{\mu})}, \frac{\sigma(\mu)}{\sigma(\bar{\mu})} \right\} \min_{x \in \Omega_i} [\min\{\lambda_{\min}((D_i^\mu)^{-1}(D_i^\mu)^{-\top}), 1\} J_i^\mu], \\ K_i(\mu) &= \max \left\{ \frac{\nu(\mu)}{\nu(\bar{\mu})}, \frac{\sigma(\mu)}{\sigma(\bar{\mu})} \right\} \max_{x \in \Omega_i} [\max\{\lambda_{\max}((D_i^\mu)^{-1}(D_i^\mu)^{-\top}), 1\} J_i^\mu].\end{aligned}$$

Then, for  $i = 1, \dots, N_S$ , it holds:

- i)  $0 < \alpha_i(\mu) < K_i(\mu)$ ;
- ii) for each  $v_i, w_i$  in  $V_i$  and for each  $\mu$  in  $\mathcal{D}$ ,

$$\alpha_i(\mu) \|v_i\|_{V_i}^2 \leq A_i(v, v; \mu), \quad |a_i(w, v; \mu)| \leq K_i(\mu) \|w_i\|_{V_i} \|v_i\|_{V_i}.$$

By applying the Greedy algorithm [25, 26] to problem (5) we obtain a local RB space

$$V_i^{RB} = \text{span}\{\tilde{u}_{h,i}(\tilde{\mu}_i^1), \dots, \tilde{u}_{h,i}(\tilde{\mu}_i^{N_i})\},$$

where the parameters  $\tilde{\mu}_i^k$ ,  $k = 1, \dots, N_i$ , are chosen by the Greedy algorithm. What we obtain is that the solution  $u_i^{RB}(\tilde{\mu}) \in V_i^{RB}$  of

$$A_i(\tilde{u}_i^{RB}(\tilde{\mu}), v_i^{RB}; \tilde{\mu}) = \tilde{F}_i(v_i^{RB}; \tilde{\mu}) + \langle \mathcal{I}_i^\beta, v_i^{RB} |_{\Gamma_i} \rangle \quad \forall v_i^{RB} \in V_i^{RB},$$

satisfies

$$\|\tilde{u}_{h,i}(\tilde{\mu}) - \tilde{u}_i^{RB}(\tilde{\mu})\|_{\mu} \leq \varepsilon^*, \quad (6)$$

for a given (small) tolerance. The global DGRBE space is now defined as

$$V^{RB} = \bigoplus_{i=1}^{N_S} V_i^{RB}. \quad (7)$$

A basis of the space  $V^{RB}$  is  $\mathcal{B}_{RB} = \bigcup_{i=1}^{N_S} \mathcal{B}_{RB,i}$ .

**Remark 1.2.** The set of parameters on which a single local problem depend can be smaller than the global set of parameters associated with problem (2). For instance if the domain  $\Omega(\mu)$  depends on the parameter, it can happen that the geometry of a single subdomain  $\Omega_i(\mu)$  depends only on some components of  $\mu$ , thus the  $i$ -th local problem depends on  $\mu_i = (\mu_{i_1}, \dots, \mu_{i_{P_i}})$ , where  $\{i_1, \dots, i_{P_i}\} \subseteq \{1, \dots, P\}$ . This fact can be very favourable in terms of the offline computational cost, because the local Greedy algorithm could be performed on a parameter space which has a smaller dimension than the global one.

We discuss two possible definitions of  $\mathcal{I}_i^\beta$ , used to construct the local bases.

**Method A: approximation of the weak normal derivative of a fine-grid discrete solution** First of all, we observe that the weak normal derivative of the fine-grid solution is actually a functional  $\mathcal{F}_{h,i}^\mu \in (V_{h,i}^{\Gamma_i})'$ , which corresponds to an element  $w_{h,i}^{\Gamma_i}(\tilde{\mu}) \in V_{h,i}^{\Gamma_i}$  by the Riesz representation theorem. We then consider a basis  $\mathcal{B}^{\Gamma_i} = \{\phi_{h,1}, \dots, \phi_{N_{\Gamma_i}}\}$  of  $V_{h,i}^{\Gamma_i}$ , denote by  $\omega_m(\mu)$  the expansion coefficients of  $w_{h,i}^{\Gamma_i}(\mu)$  with respect to  $\mathcal{B}^{\Gamma_i}$ , and set  $n_{BC} = N_{\Gamma_i}$ . Then,

$$\langle \mathcal{F}_{h,i}^\mu, v_{h,i}|_{\Gamma_i} \rangle = \sum_{m=1}^{n_{BC}} \omega_m(\mu) (\phi_{h,m}, v_{h,i}|_{\Gamma_i})_{\Gamma_i} \quad \forall v_{h,i} \in V_{h,i}. \quad (8)$$

Inspired to (8), we define the functional  $\mathcal{I}_i^\beta$  such that

$$\langle \mathcal{I}_i^\beta, v_{h,i}|_{\Gamma_i} \rangle = \sum_{m=1}^{n_{BC}} \mathbb{1}_{\{m\}}(\beta) (\phi_{h,m}, v_{h,i}|_{\Gamma_i})_{\Gamma_i} \quad \forall v_{h,i} \in V_{h,i},$$

where  $\mathbb{1}_{\{m\}}(\beta) = 1$  if and only if  $\beta = m$ , otherwise it is null. In this way, the local Greedy procedure can take into account the Neumann data associated with each interface basis  $\phi_{h,i}$ ,  $i = 1, \dots, N_{\Gamma_i}$ .

**Method B: Legendre approximation of the normal derivative of the continuous solution** We assume now that the interface  $\Gamma_i$  of  $\Omega_i$  is regular (or, at least, it is a finite union of regular components). The idea is now to approximate the normal derivative with a properly chosen  $L^2$  orthogonal basis  $\{\phi_m\}_{m=0}^\infty$ . We write

$$\frac{\partial u(\mu)}{\partial \mathbf{n}} = \sum_{m=1}^{\infty} \omega_m(\mu) \phi_m, \quad (9)$$

and consider its approximation obtained by truncating the series. We denote the truncated sum with  $\tilde{w}^{\Gamma_i}(\mu)$  and choose the Legendre polynomials as basis functions. We now define the functional  $\mathcal{I}_i^\beta$  in the following way:

$$\langle \mathcal{I}_i^\beta, v_{h,i}|_{\Gamma_i} \rangle = \sum_{m=1}^{n_{BC}} \mathbb{1}_{\{m\}}(\beta) (\phi_m, v_{h,i}|_{\Gamma_i})_{\Gamma_i} \quad \forall v_{h,i} \in V_{h,i}.$$

We chose  $n_{BC}$  in order to achieve a good local approximation, as we will discuss later.

## 1.2 Online stage

The elements of  $V^{RB}$  are obviously discontinuous functions across subdomain interfaces. To compensate for that, we introduce the following DG-type bilinear form

$$A_{DG}(w^{RB}, v^{RB}; \mu) = \sum_{i=1}^{N_S} A_i(w_i^{RB}, v_i^{RB}; \mu) + d_{DG}(w^{RB}, v^{RB}; \mu) + c_{DG}(w^{RB}, v^{RB}; \mu), \quad (10)$$

where

$$\begin{aligned} d_{DG}(w^{RB}, v^{RB}; \mu) &= - \int_{\Gamma(\mu)} \nu(\mu) \{\nabla w^{RB}\} \cdot \llbracket v^{RB} \rrbracket ds - \int_{\Gamma(\mu)} \nu(\mu) \{\nabla v^{RB}\} \cdot \llbracket w^{RB} \rrbracket ds \\ c_{DG}(w^{RB}, v^{RB}; \mu) &= \frac{\nu(\mu)\gamma}{h} \int_{\Gamma(\mu)} \llbracket w^{RB} \rrbracket \cdot \llbracket v^{RB} \rrbracket ds, \end{aligned} \quad (11)$$



for a suitable constant  $\gamma > 0$ . Here  $\Gamma(\boldsymbol{\mu})$  is the union of all internal interfaces, and we used the standard notation for jump and average operators, see [6], that on  $\bar{\Gamma}_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$  becomes:

$$\begin{aligned} \{\nabla v^{RB}\}_{|\Gamma_{ij}(\boldsymbol{\mu})} &= \frac{1}{2} ((\nabla v_i^{RB})_{|\Gamma_{ij}(\boldsymbol{\mu})} + (\nabla v_j^{RB})_{|\Gamma_{ij}(\boldsymbol{\mu})}), \\ \llbracket v^{RB} \rrbracket_{|\Gamma_{ij}(\boldsymbol{\mu})} &= (v_i^{RB} \cdot \mathbf{n}_i)_{|\Gamma_{ij}(\boldsymbol{\mu})} + (v_j^{RB} \cdot \mathbf{n}_j)_{|\Gamma_{ij}(\boldsymbol{\mu})}, \end{aligned}$$

where  $\mathbf{n}_k$  is the normal unit vector pointing outwards  $\Omega_k(\boldsymbol{\mu})$ ,  $k = i, j$ .

For any given value  $\boldsymbol{\mu} \in \mathcal{D}$ , the corresponding global reduced approximation takes the following form

$$\text{find } u^{RB}(\boldsymbol{\mu}) \in V^{RB} \text{ such that } A_{DG}(u^{RB}(\boldsymbol{\mu}), v^{RB}; \boldsymbol{\mu}) = F(v^{RB}; \boldsymbol{\mu}) \quad \forall v^{RB} \in V^{RB}. \quad (12)$$

where  $F(v^{RB}; \boldsymbol{\mu}) = \sum_{i=1}^{N_S} F_i(v_i^{RB}; \boldsymbol{\mu})$ .

## 2 Convergence analysis

In this section we present the convergence analysis of the DGRBE method, focusing in particular on the approximation properties of the online problem (12). We assume that the weak solution  $u(\boldsymbol{\mu})$  of the continuous problem (2) has  $H^2(\Omega)$  regularity. We then set  $V = H_0^1(\Omega) \cap H^2(\Omega)$  and  $V(RB) = V^{RB} + V$ . In the following, we will denote with the symbol  $\lesssim$  all the inequalities valid up to a multiplicative constant, which can depend also on the parameter  $\boldsymbol{\mu}$ . We define the norms

$$\begin{aligned} \|v\|_{DG, \boldsymbol{\mu}}^2 &= \sum_{i=1}^{N_S} A_i(v, v; \boldsymbol{\mu}) + c_{DG}(v, v; \boldsymbol{\mu}) \quad \forall v \in V(RB) \quad \forall \boldsymbol{\mu} \in \mathcal{D}, \\ \|v\|_{DG, \boldsymbol{\mu}}^2 &= \|v\|_{DG, \boldsymbol{\mu}}^2 + h^2 \sum_{i=1}^{N_S} \sum_{K \in \mathcal{T}_{h,i}} |v|_{H^2(K)}^2 \quad \forall v \in V(RB) \quad \forall \boldsymbol{\mu} \in \mathcal{D}, \end{aligned}$$

and observe that  $\|v^{RB}\|_{DG, \boldsymbol{\mu}} = \|v^{RB}\|_{DG, \boldsymbol{\mu}}$  for any  $v^{RB} \in V^{RB}$ , as each element of  $V^{RB}$  is piecewise linear. We define also the reference parameter independent norms  $\|\cdot\|_{DG} = \|\cdot\|_{DG, \bar{\boldsymbol{\mu}}}$  and  $\|\cdot\|_{DG} = \|\cdot\|_{DG, \bar{\boldsymbol{\mu}}}$ , which are equivalent to the parameter dependent norms. We observe that, as we are using piecewise linear polynomials,  $\|v^{RB}\|_{DG} = \|v^{RB}\|_{DG}$  for all  $v^{RB} \in V^{RB}$ .

We assume that, for  $i = 1, \dots, N_S$ , the local basis  $\{\zeta_1^i, \dots, \zeta_{N_i}^i\}$  of  $V_i^{RB}$  satisfies the following orthogonality conditions

$$(\zeta_j^i, \zeta_k^i)_{V_i} = \delta_{jk} \quad j, k = 1, \dots, N_i, \quad i = 1, \dots, N_S, \quad (13)$$

as guaranteed by the Greedy algorithm [25, 26]. Exploiting (7), we observe that we can uniquely express every element of  $V^{RB}$  as  $v^{RB} = \sum_{i=1}^{N_S} v_i^{RB}$ , and that

$$\|v^{RB}\|_{DG}^2 = \sum_{i=1}^{N_S} (v_i^{RB}, v_i^{RB})_{V_i} + c_{DG}(v^{RB}, v^{RB}; \bar{\boldsymbol{\mu}}). \quad (14)$$

We prove that the form  $A_{DG}$  is coercive and continuous with respect to the DG-type reference norm (14). This guarantees that the DGRBE online problem (12) is well posed. We assume that the mesh is quasi uniform. We recall the following trace inequality

$$\left\| \frac{\partial v}{\partial n} \right\|_{L^2(e)}^2 \lesssim h_e^{-1} |v|_{H^1(K)}^2 + h_e |v|_{H^2(K)}^2 \quad \forall v \in H^2(K), \quad (15)$$

see [5, 6], where  $K$  is an arbitrary element of the fine triangulation,  $e$  is an edge of  $K$  and  $h_e$  is the length of  $e$ .

**Proposition 2.1** (Stability of the DGRBE method). *The following estimates hold.*

1. For all  $\boldsymbol{\mu} \in \mathcal{D}$ , there exists  $\alpha_{DG}(\boldsymbol{\mu}) > 0$  such that for all  $v^{RB} \in V^{RB}$

$$A_{DG}(v^{RB}, v^{RB}; \boldsymbol{\mu}) \geq \alpha_{DG}(\boldsymbol{\mu}) \|v^{RB}\|_{DG}^2,$$

provided the stability parameter  $\gamma$  in (10) has been chosen large enough.

2. For all  $\boldsymbol{\mu} \in \mathcal{D}$ , there exists  $K_{DG}(\boldsymbol{\mu}) > 0$  such that for every  $w, v \in V(RB)$

$$A_{DG}(w, v; \boldsymbol{\mu}) \leq K_{DG}(\boldsymbol{\mu}) \|w\|_{DG} \|v\|_{DG}.$$

*Proof.* As regards the coercivity, given  $v^{RB} \in V^{RB}$  and  $\boldsymbol{\mu} \in \mathcal{D}$ , we need to find a suitable upper bound for the term  $d_{DG}(v^{RB}, v^{RB}; \boldsymbol{\mu})$ . Using Young's inequality we obtain

$$\begin{aligned} \nu(\boldsymbol{\mu}) \int_{\Gamma(\boldsymbol{\mu})} \{\nabla v^{RB}\} \cdot \llbracket v^{RB} \rrbracket &\leq \nu(\boldsymbol{\mu}) \|\{\nabla v^{RB}\} \cdot \mathbf{n}_1\|_{L^2(\Gamma(\boldsymbol{\mu}))} \|\llbracket v^{RB} \rrbracket \cdot \mathbf{n}_1\|_{L^2(\Gamma(\boldsymbol{\mu}))} \\ &\leq \nu(\boldsymbol{\mu}) \left( \frac{h}{2\varepsilon(\boldsymbol{\mu})} \|\{\nabla v^{RB}\} \cdot \mathbf{n}_1\|_{L^2(\Gamma(\boldsymbol{\mu}))}^2 + \frac{\varepsilon(\boldsymbol{\mu})}{2h} \|\llbracket v^{RB} \rrbracket \cdot \mathbf{n}_1\|_{L^2(\Gamma(\boldsymbol{\mu}))}^2 \right), \end{aligned}$$

with  $\varepsilon(\boldsymbol{\mu}) > 0$ . Then, using (15), together with the quasi-uniformity of the mesh coercivity follows from Lemma 1.1 provided  $\gamma$  is sufficiently large. As for the continuity, the volume terms and  $c_{DG}$  can be easily bounded, by the Cauchy-Schwarz inequality. Let us then consider again the derivative consistency term. Given  $w, v \in V(RB)$ , inequality (15) and the quasi-uniformity of the mesh lead to

$$\begin{aligned} \nu(\boldsymbol{\mu}) \int_{\Gamma(\boldsymbol{\mu})} \{\nabla w\} \cdot \llbracket v \rrbracket &\leq \nu(\boldsymbol{\mu}) \|\{\nabla w\} \cdot \mathbf{n}_1\|_{L^2(\Gamma(\boldsymbol{\mu}))} \|\llbracket v \rrbracket \cdot \mathbf{n}_1\|_{L^2(\Gamma(\boldsymbol{\mu}))} \\ &\lesssim h^{1/2} \left[ \sum_{i=1}^{N_S} \sum_{K \in \mathcal{T}_{h,i}} (h^{-1} |w|_{H^1(K)}^2 + h |w|_{H^2(K)}^2)^{1/2} \right] \frac{1}{h^{1/2}} \|\llbracket v \rrbracket \cdot \mathbf{n}_1\|_{L^2(\Gamma(\boldsymbol{\mu}))} \lesssim \|w\|_{DG} \|v\|_{DG}. \end{aligned}$$

and the thesis follows.  $\square$

We now prove global error estimates. To ease the notation, we will omit the parameter dependence of the domain. Given a parameter value  $\boldsymbol{\mu} \in \mathcal{D}$ , the idea is to build  $z^{RB}(\boldsymbol{\mu}) \in V^{RB}$  for which it is possible to estimate  $\|u(\boldsymbol{\mu}) - z^{RB}(\boldsymbol{\mu})\|_{DG}$ . To show that we can bound the approximation error with  $\|u(\boldsymbol{\mu}) - z^{RB}(\boldsymbol{\mu})\|_{DG}$ , we observe that

$$\|u(\boldsymbol{\mu}) - u^{RB}(\boldsymbol{\mu})\|_{DG} \leq \|u(\boldsymbol{\mu}) - z^{RB}(\boldsymbol{\mu})\|_{DG} + \|z^{RB}(\boldsymbol{\mu}) - u^{RB}(\boldsymbol{\mu})\|_{DG},$$

and that, as  $A_{DG}(u(\boldsymbol{\mu}) - u^{RB}(\boldsymbol{\mu}), z^{RB}(\boldsymbol{\mu}) - u^{RB}(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0$  by strong consistency,

$$\begin{aligned} \|z^{RB}(\boldsymbol{\mu}) - u^{RB}(\boldsymbol{\mu})\|_{DG}^2 &\leq \frac{1}{\alpha_{DG}(\boldsymbol{\mu})} A_{DG}(z^{RB}(\boldsymbol{\mu}) - u(\boldsymbol{\mu}), z^{RB}(\boldsymbol{\mu}) - u^{RB}(\boldsymbol{\mu}); \boldsymbol{\mu}) \\ &\leq \frac{K_{DG}(\boldsymbol{\mu})}{\alpha_{DG}(\boldsymbol{\mu})} \|z^{RB}(\boldsymbol{\mu}) - u(\boldsymbol{\mu})\|_{DG} \|z^{RB}(\boldsymbol{\mu}) - u^{RB}(\boldsymbol{\mu})\|_{DG}. \end{aligned}$$

We follow the approach used in [6], which first requires a local best approximation result. Usually, the local best approximation is given employing a piecewise polynomial interpolant. As we do not dispose of such an interpolant, we will have to build an element of our local

spaces (best fit) which reasonably provides a good approximation of the continuous solution. We finally observe that, for each  $\boldsymbol{\mu} \in \mathcal{D}$ , the reduced space built by applying the Greedy algorithm to problem (5) contains the element

$$z_i^{RB}(\boldsymbol{\mu}) = \tilde{u}_i^{RB}(\boldsymbol{\mu}, 0) + w_i^{RB}(\boldsymbol{\mu}), \quad \text{where} \quad w_i^{RB}(\boldsymbol{\mu}) = \sum_{m=1}^{n_{BC}} \omega_m(\boldsymbol{\mu}) \tilde{u}_i^{RB}(\boldsymbol{\mu}, m),$$

and the weights  $\omega_m(\boldsymbol{\mu})$  are defined in (8) and (9) for Method A and Method B, respectively. The following lemmas hold.

**Lemma 2.2** (Method A). *If the local reduced basis are built using Method A, then  $z_i^{RB}(\boldsymbol{\mu})$  approximates  $u_{h,i}(\boldsymbol{\mu}) \in V_{h,i}$ , i.e., the restriction to  $\Omega_i$  of the global fine-grid solution  $u_h(\boldsymbol{\mu})$  such that*

$$A_{DG}(u_h(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}) = F(v_h; \boldsymbol{\mu}) \quad \forall v_h \in V_h, \quad (16)$$

where  $V_h = \bigoplus_{i=1}^{N_S} V_{h,i}$ . Moreover it holds that

$$\|u_{h,i}(\boldsymbol{\mu}) - z_i^{RB}(\boldsymbol{\mu})\|_{\boldsymbol{\mu},i} \leq \Phi_{\boldsymbol{\mu}}^A \varepsilon^*, \quad (17)$$

$$\|u_{h,i}(\boldsymbol{\mu}) - z_i^{RB}(\boldsymbol{\mu})\|_{L^2(\Omega_i)} \lesssim \Phi_{\boldsymbol{\mu}}^A \varepsilon^*, \quad (18)$$

with  $\Phi_{\boldsymbol{\mu}}^A = 1 + N_{\Gamma_i} \max_m \omega_m(\boldsymbol{\mu})$ . Here  $N_{\Gamma_i}$  is the number of fine-grid interface basis.

*Proof.* We first observe that  $u_{h,i}(\boldsymbol{\mu})$  is the solution of

$$A_i(u_{h,i}(\boldsymbol{\mu}), v_{h,i}; \boldsymbol{\mu}) = F_i(v_{h,i}; \boldsymbol{\mu}) + \langle \mathcal{F}_{h,i}^{\boldsymbol{\mu}}, v_{h,i} \rangle \quad \forall v_{h,i} \in V_{h,i},$$

where  $\mathcal{F}_{h,i}^{\boldsymbol{\mu}} \in (V_{h,i}^{\Gamma_i})'$  can be decomposed as in (8). We then note that  $u_{h,i}(\boldsymbol{\mu})$  can be written as  $u_{h,i}(\boldsymbol{\mu}) = \tilde{u}_{h,i}(\boldsymbol{\mu}, 0) + w_{h,i}(\boldsymbol{\mu})$  where  $\tilde{u}_{h,i}(\boldsymbol{\mu}, 0)$  solves problem (5) with  $\beta = 0$  and  $w_{h,i}(\boldsymbol{\mu}) = \sum_{m=1}^{N_{\Gamma_i}} \omega_m(\boldsymbol{\mu}) \tilde{u}_{h,i}(\boldsymbol{\mu}, m)$ . Using (6) and observing that

$$\|w_{h,i}(\boldsymbol{\mu}) - w_{RB,i}(\boldsymbol{\mu})\|_{\boldsymbol{\mu},i} \leq \sum_{m=1}^{N_{\Gamma_i}} \omega_m(\boldsymbol{\mu}) \|\psi_{RB,i}^m(\boldsymbol{\mu}) - \tilde{u}_i^{RB}(\boldsymbol{\mu}, m)\|_{\boldsymbol{\mu},i} \leq \varepsilon^* N_{\Gamma_i} \max_i \omega_i(\boldsymbol{\mu}), \quad (19)$$

estimate (17) follows from triangular inequality. As for (18), we observe that the norm  $\|\cdot\|_{\boldsymbol{\mu},i}$  is associated with a diffusion reaction operator and we conclude.  $\square$

**Lemma 2.3** (Method B). *If the local reduced basis are built using Method B, then  $z_i^{RB}(\boldsymbol{\mu})$  is an approximation of  $u(\boldsymbol{\mu})|_{\Omega_i}$ , where  $u(\boldsymbol{\mu})$  is the weak solution of (16) in  $H_0^1(\Omega)$ . Setting*

$$\mathcal{E}_i(\boldsymbol{\mu}) = \left\| \frac{\partial u(\boldsymbol{\mu})}{\partial \mathbf{n}} - \tilde{w}^{\Gamma_i}(\boldsymbol{\mu}) \right\|_{L^2(\Gamma_i)},$$

where  $\tilde{w}^{\Gamma_i}(\boldsymbol{\mu})$  is the truncated sum (9), it holds that

$$\|u(\boldsymbol{\mu})|_{\Omega_i} - z_i^{RB}(\boldsymbol{\mu})\|_{\boldsymbol{\mu},i} \lesssim h \left( \|f\|_{L^2(\Omega_i)} + \left\| \frac{\partial u(\boldsymbol{\mu})}{\partial \mathbf{n}} \right\|_{L^2(\Gamma_i)} \right) + \mathcal{E}_i(\boldsymbol{\mu}) + \Phi_{\boldsymbol{\mu}}^B \varepsilon^*, \quad (20)$$

$$\|u(\boldsymbol{\mu})|_{\Omega_i} - z_i^{RB}(\boldsymbol{\mu})\|_{L^2(\Omega_i)} \lesssim h^2 \left( \|f\|_{L^2(\Omega_i)} + \left\| \frac{\partial u(\boldsymbol{\mu})}{\partial \mathbf{n}} \right\|_{L^2(\Gamma_i)} \right) + \mathcal{E}_i(\boldsymbol{\mu}) + \Phi_{\boldsymbol{\mu}}^B \varepsilon^*. \quad (21)$$

with  $\Phi_{\boldsymbol{\mu}}^A = 1 + M_{BC} \max_m \omega_m(\boldsymbol{\mu})$ . Here  $M_{BC}$  is the number of Legendre polynomials considered on each interface.

*Proof.* We first split the restriction of the exact solution to  $\Omega_i$  as  $u(\boldsymbol{\mu})|_{\Omega_i} = \hat{u}_i(\boldsymbol{\mu}) + w_i(\boldsymbol{\mu})$ , where  $\hat{u}_i(\boldsymbol{\mu})$  and  $w_i(\boldsymbol{\mu})$  are such that

$$A_i(\hat{u}_i(\boldsymbol{\mu}), v_i; \boldsymbol{\mu}) = F_i(v_i; \boldsymbol{\mu}), \quad A_i(w_i(\boldsymbol{\mu}), v_i; \boldsymbol{\mu}) = \left( \nu(\boldsymbol{\mu}) \frac{\partial u(\boldsymbol{\mu})}{\partial \mathbf{n}}, v_i|_{\Gamma_i} \right)_{\Gamma_i},$$

for all  $v_i \in V_i$ . We then define  $\tilde{w}_i(\boldsymbol{\mu})$  such that

$$A_i(\tilde{w}_i(\boldsymbol{\mu}), v_i; \boldsymbol{\mu}) = (\tilde{w}^{\Gamma_i}(\boldsymbol{\mu}), v_i|_{\Gamma_i})_{\Gamma_i} \quad \forall v_i \in V_i.$$

where  $\tilde{w}^{\Gamma_i}(\boldsymbol{\mu})$  is the polynomial expansion (9) truncated after the first  $M_{BC}$  terms. Moreover, it holds that  $\tilde{w}_i(\boldsymbol{\mu}) = \sum_{m=1}^{M_{BC}} \omega_m(\boldsymbol{\mu}) \psi_{m,i}(\boldsymbol{\mu})$ , where  $\psi_{m,i}(\boldsymbol{\mu}) \in V_i$ ,  $m = 1, \dots, M$ , are the harmonic extensions of the basis of Neumann boundary data, *i.e.*,

$$A_i(\psi_{m,i}(\boldsymbol{\mu}), v_i; \boldsymbol{\mu}) = (\phi_m(\boldsymbol{\mu}), v_i|_{\Gamma_i})_{\Gamma_i} \quad \forall v_i \in V_i.$$

Note that  $\tilde{u}_{h,i}(\boldsymbol{\mu}, m)$  is the FE approximation of  $\psi_{m,i}(\boldsymbol{\mu})$ , then the FE approximation of  $\tilde{w}_i(\boldsymbol{\mu})$  is  $\tilde{w}_{h,i}(\boldsymbol{\mu}) = \sum_{m=1}^{M_{BC}} \omega_m(\boldsymbol{\mu}) \tilde{u}_{h,i}(\boldsymbol{\mu}, m)$ . By triangular inequality,

$$\begin{aligned} \|u(\boldsymbol{\mu})|_{\Omega_i} - z_i^{RB}(\boldsymbol{\mu})\|_{\bar{\boldsymbol{\mu}},i} &\leq \|\hat{u}_i(\boldsymbol{\mu}) - \tilde{u}_{h,i}(\boldsymbol{\mu}, 0)\|_{\bar{\boldsymbol{\mu}},i} + \|\tilde{u}_{h,i}(\boldsymbol{\mu}, 0) - \tilde{u}_i^{RB}(\boldsymbol{\mu}, 0)\|_{\bar{\boldsymbol{\mu}},i} \\ &\quad + \|w_i(\boldsymbol{\mu}) - \tilde{w}_i(\boldsymbol{\mu})\|_{\bar{\boldsymbol{\mu}},i} + \|\tilde{w}_i(\boldsymbol{\mu}) - \tilde{w}_{h,i}(\boldsymbol{\mu})\|_{\bar{\boldsymbol{\mu}},i} \\ &\quad + \|\tilde{w}_{h,i}(\boldsymbol{\mu}) - w_{RB,i}(\boldsymbol{\mu})\|_{\bar{\boldsymbol{\mu}},i}. \end{aligned}$$

Recalling the standard error estimates of the FE method (*cf.* [24]), exploiting (6) an reasoning as in (19) inequality (20) follows. As regards (21), the proof is similar.  $\square$

In order to prove a global approximation estimate, we use an argument similar to that used in [6], Section 4.3. First of all, we need to recall the following trace inequalities:

$$\|v\|_{L^2(e)}^2 \lesssim h_e^{-1} \|v\|_{L^2(K)}^2 + h_e |v|_{H^1(K)}^2 \quad \forall v \in H^1(K) \quad \forall K \in \mathcal{T}_h,$$

where  $e$  is an edge of  $K$ ,

$$\|v_i\|_{L^2(\Gamma_i)} \lesssim h^{-1} \|v_i\|_{L^2(\Omega_i)}^2 + h |v_i|_{H^1(\Omega_i)}^2 \quad \forall v_i \in H^1(\Omega_i), \quad i = 1, \dots, N_S. \quad (22)$$

We define our global best fit as:  $z^{RB}(\boldsymbol{\mu}) = \sum_{i=1}^{N_S} z_i^{RB}(\boldsymbol{\mu})$ . The following result holds.

**Theorem 2.4.** *Let  $u(\boldsymbol{\mu})$  be the exact weak solution of problem (2), and let  $u_A^{RB}(\boldsymbol{\mu})$  and  $u_B^{RB}(\boldsymbol{\mu})$  be the approximated solution obtained with Method A and Method B, respectively. Then,*

$$\|u(\boldsymbol{\mu}) - u_A^{RB}(\boldsymbol{\mu})\|_{DG} \lesssim h \|f\|_{L^2(\Omega_i)} + \sqrt{N_S} \left(1 + \frac{1}{h}\right) \Phi_{\boldsymbol{\mu}}^A \varepsilon^*, \quad (23)$$

$$\|u(\boldsymbol{\mu}) - u_B^{RB}(\boldsymbol{\mu})\|_{DG} \lesssim h \|f\|_{L^2(\Omega)} + \left(1 + \frac{1}{h}\right) \sum_{i=1}^{N_S} \mathcal{E}_i(\boldsymbol{\mu}) + \sqrt{N_S} \left(1 + \frac{1}{h}\right) \Phi_{\boldsymbol{\mu}}^B \varepsilon^*, \quad (24)$$

where  $\Phi_{\boldsymbol{\mu}}^A$ ,  $\Phi_{\boldsymbol{\mu}}^B$  and  $\mathcal{E}_i(\boldsymbol{\mu})$  are defined as in Lemma 2.2 and in 2.3. All the hidden constants depend on the domain  $\Omega$  and the parameter  $\boldsymbol{\mu}$ , but are independent of  $h$  and  $\varepsilon^*$ .

*Proof.* Let us start with estimate (23). First of all we observe that the solution  $u_h(\boldsymbol{\mu})$  of (16) is such that

$$\|u(\boldsymbol{\mu}) - u_h(\boldsymbol{\mu})\|_{DG} \lesssim h \|f\|_{L^2(\Omega)},$$

see, for instance, [27]. We then observe that as  $V_h$  and thus  $V^{RB}$  are piecewise polynomials space, in order to prove the thesis it is sufficient to estimate the error  $\|u_h(\boldsymbol{\mu}) - z^{RB}(\boldsymbol{\mu})\|_{DG}$ . Moreover, we observe that we need just to estimate the jump terms, as the others are already bounded by (17). For every  $i = 1, \dots, N_S$ , denoting with  $n(i)$  the number of neighbours of  $\Omega_i$  and applying inequality (22) together with Lemma 2.2, we obtain

$$\begin{aligned} & \frac{\gamma}{h} \sum_{\{i,j:\bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset\}} \|[u_h(\boldsymbol{\mu}) - z^{RB}(\boldsymbol{\mu})]\|_{L^2(\Gamma_{ij})}^2 \\ & \lesssim \frac{1}{h} \sum_{\{i,j:\bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset\}} \|u_h(\boldsymbol{\mu}) - z_i^{RB}(\boldsymbol{\mu})\|_{L^2(\Gamma_{ij})}^2 + \|u_h(\boldsymbol{\mu}) - z_j^{RB}(\boldsymbol{\mu})\|_{L^2(\Gamma_{ij})}^2 \\ & \lesssim \sum_{i=1}^{N_S} n(i) \left( \|u_h(\boldsymbol{\mu}) - z_i^{RB}(\boldsymbol{\mu})\|_{\boldsymbol{\mu},i}^2 + h^{-2} \|u_h(\boldsymbol{\mu}) - z_i^{RB}(\boldsymbol{\mu})\|_{L^2(\Omega_i)}^2 \right) \\ & \lesssim \left( \sqrt{N_S} \left( 1 + \frac{1}{h} \right) \Phi_{\boldsymbol{\mu}}^A \varepsilon^* \right)^2 \end{aligned} \tag{25}$$

where the hidden constant depends also on  $M = \max_i n(i)$ , and the thesis follows.

As for (24), it is sufficient to find an upper bound for the jump term of  $\|\cdot\|_{DG}$ , because the other terms can be controlled by the local estimate (20) and by observing that

$$\sum_{i=1}^{N_S} \sum_{K \in \mathcal{T}_{h,i}} |u(\boldsymbol{\mu}) - z^{RB}(\boldsymbol{\mu})|_{\mathbb{H}^2(K)}^2 = |u(\boldsymbol{\mu})|_{\mathbb{H}^2(\Omega)}^2 \lesssim \|f(\boldsymbol{\mu})\|_{L^2(\Omega)}^2.$$

Let  $n(i)$  be the number of neighbours of  $\Omega_i$ , for  $i = 1, \dots, N_S$ . Thanks to inequality (22), reasoning as in (25) and recalling Lemma 2.3, we have

$$\begin{aligned} & \frac{\gamma}{h} \sum_{\{i,j:\bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset\}} \|[u(\boldsymbol{\mu}) - z^{RB}(\boldsymbol{\mu})]\|_{L^2(\Gamma_{ij})}^2 \\ & \lesssim \sum_{i=1}^{N_S} n(i) \left( \|u(\boldsymbol{\mu})|_{\Omega_i} - z_i^{RB}(\boldsymbol{\mu})\|_{\boldsymbol{\mu},i}^2 + h^{-2} \|u(\boldsymbol{\mu})|_{\Omega_i} - z_i^{RB}(\boldsymbol{\mu})\|_{L^2(\Omega_i)}^2 \right) \\ & \lesssim \left( h \|f\|_{L^2(\Omega)} + \left( 1 + \frac{1}{h} \right) \sum_{i=1}^{N_S} \mathcal{E}_i(\boldsymbol{\mu}) + \sqrt{N_S} \left( 1 + \frac{1}{h} \right) \Phi_{\boldsymbol{\mu}}^B \varepsilon^* \right)^2, \end{aligned}$$

where we have also used that

$$\sum_{i=1}^{N_S} \left\| \frac{\partial u(\boldsymbol{\mu})}{\partial n} \right\|_{L^2(\Omega_i)} \lesssim \sum_{i=1}^{N_S} \|\nabla u(\boldsymbol{\mu})\|_{L^2(\Omega_i)} \lesssim \|\nabla u(\boldsymbol{\mu})\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)},$$

and the thesis follows.  $\square$

**Remark 2.5.** Note that for both methods the contribution to the error due to the local RB increases as the square root of the number of subdomains.

**Remark 2.6.** As for Method B, we observe that the quality of the global approximation relies strongly on how well the normal derivative of  $u(\boldsymbol{\mu})$  on the interfaces can be approximated by a polynomial expansion, on each internal interface. In our numerical tests, shown in Section 4, we chose  $M_{BC} = h^{-1/2}$ . This choice can be motivated by some theoretical results about the Legendre polynomials approximation of  $L^2$  functions, provided that we assume high regularity on the solution  $u(\boldsymbol{\mu})$ . We note that, if  $u(\boldsymbol{\mu}) \in H^6(\Omega)$  then  $\frac{\partial u(\boldsymbol{\mu})}{\partial \mathbf{n}}$  has  $H^4(\Omega)$  regularity on each regular component of the internal interface. Thus, the following approximation result holds:

$$\left\| \frac{\partial u(\boldsymbol{\mu})}{\partial \mathbf{n}} - \tilde{w}^{\Gamma_i}(\boldsymbol{\mu}) \right\|_{L^2(\Gamma_i)} \lesssim M^{-4} \left| \frac{\partial u(\boldsymbol{\mu})}{\partial \mathbf{n}} \right|_{H^4(\Gamma_i)},$$

cf. [10]. In this way, the error due to the approximation of the normal derivative on the interfaces scales as  $h$ . A possible alternative to the requirement of a high regularity of the solution is the introduction of a suitable *a posteriori* error estimator which allow to automatically tune the number of polynomial bases at the interface. The latter approach is currently under investigation.

## 2.1 Spectral bounds

We prove now some spectral bounds on the condition number of the matrix associated with the online problem (12) through the basis  $\mathcal{B}_{RB}$ .

We observe that for  $i = 1, \dots, N_S$ , every element  $v_i^{RB} \in V_i^{RB}$  can be expressed as  $v_i^{RB} = \sum_{j=1}^{N_i} v_j^i \zeta_j^i$ . We set  $\mathbf{v}_i = (v_1^i, \dots, v_{N_i}^i)$ . Then, each  $v_i^{RB} \in V_i^{RB}$  is associated with a vector  $\mathbf{v}^{RB} = (\mathbf{v}_1, \dots, \mathbf{v}_{N_S})$ . In the following, denote with  $|\cdot|$  the Euclidean norm.

**Lemma 2.7.** *It holds:*

$$\|v_i^{RB}\|_{V_i}^2 = |\mathbf{v}_i|^2 \quad \forall v_i^{RB} \in V_i^{RB}, i = 1, \dots, N_S.$$

*Proof.* Thanks to the orthogonality assumption (13), for  $i = 1, \dots, N_S$ , it holds

$$\|v_i^{RB}\|_{V_i}^2 = (v_i^{RB}, v_i^{RB})_{V_i} = \left( \sum_{j=1}^{N_i} v_j^i \zeta_j^i, \sum_{k=1}^{N_i} v_k^i \zeta_k^i \right)_{V_i} = \sum_{j,k=1}^{N_i} v_j^i v_k^i (\zeta_j^i, \zeta_k^i)_{V_i} = \mathbf{v}_i^T \mathbf{v}_i.$$

□

**Lemma 2.8.** *It holds*

$$c_{DG}(v^{RB}, v^{RB}; \bar{\boldsymbol{\mu}}) \lesssim \frac{\nu(\bar{\boldsymbol{\mu}})\gamma}{h} |\mathbf{v}^{RB}|^2 \quad \forall v^{RB} \in V^{RB},$$

where  $\gamma$  is the penalization coefficient defined in (11) and the hidden constant depends only on the reference domain  $\Omega$ .

*Proof.* Let us consider the interface  $\Gamma_{ij}$  between  $\Omega_i$  and  $\Omega_j$ . Recalling that:

$$\|v_k\|_{L^2(\Gamma_{ij})} \leq C_{\Gamma_{ij}} \|v_k\|_{V_k} \quad \forall v_k \in V_k, k = i, j, \quad (26)$$

cf. [9], we observe that, using the Schwarz inequality and (26)

$$\begin{aligned} \left| \int_{\Gamma_{ij}} \llbracket v^{RB} \rrbracket^2 \right| &\leq C_{\Gamma_{ij}}^2 \left( \|v_i^{RB}\|_{V_i}^2 + \|v_j^{RB}\|_{V_j}^2 + 2 \|v_i^{RB}\|_{V_i} \|v_j^{RB}\|_{V_j} \right) \\ &\lesssim C_{\Gamma_{ij}}^2 \left( \|v_i^{RB}\|_{V_i}^2 + \|v_j^{RB}\|_{V_j}^2 \right). \end{aligned}$$

Using Lemma 2.7 we finally have

$$\left| \int_{\Gamma_{ij}} \llbracket v^{RB} \rrbracket^2 \right| \lesssim C_{\Gamma_{ij}}^2 \left( |\mathbf{v}_i|^2 + |\mathbf{v}_j|^2 \right). \quad (27)$$

Summing (27) over  $i$  and  $j$ , we get

$$\left| \int_{\Gamma} \llbracket v^{RB} \rrbracket^2 \right| \lesssim C_{\Gamma_{ij}}^2 |\mathbf{v}_{RB}|^2,$$

where the hidden constant depends on the maximum number of neighbouring subdomains a given subdomain can have. Recalling (11), the thesis follows.  $\square$

From Lemma 2.7 and Lemma 2.8 we can obtain the following result.

**Theorem 2.9.** *The minimum and maximum eigenvalues of  $\mathbf{A}_{RB}$  satisfy*

$$\lambda_{\min}(\mathbf{A}_{RB}) \geq \alpha_{DG}(\boldsymbol{\mu}), \quad \lambda_{\max}(\mathbf{A}_{RB}) \lesssim K_{DG}(\boldsymbol{\mu}) \left( 1 + \frac{\nu(\bar{\boldsymbol{\mu}})\gamma}{h} \right),$$

for every  $\boldsymbol{\mu} \in \mathcal{D}$ , where  $\gamma$  is defined in (11) and  $\Lambda_c$  is defined as in Lemma 2.8. The condition number of  $\mathbf{A}_{RB}$  can therefore be bounded by

$$\kappa(\mathbf{A}_{RB}) \lesssim \frac{K_{DG}(\boldsymbol{\mu})}{\alpha_{DG}(\boldsymbol{\mu})} \left( 1 + \frac{\nu(\bar{\boldsymbol{\mu}})\gamma}{h} \right).$$

### 3 Preconditioning the online system

In this section we propose a possible two-level preconditioner to efficiently solve the online system. More precisely, we aim to find a preconditioner for the parameter independent bilinear form

$$B(w_h, v_h) = \sum_{i=1}^{N_S} (w_{h,i}, v_{h,i})_{V_i} + c_{DG}(v_h, w_h; \bar{\boldsymbol{\mu}}),$$

which is the scalar product associated with the norm  $\|\cdot\|_{DG}$  (see (14)) and is spectrally equivalent to the form  $A_{DG}(\cdot, \cdot; \boldsymbol{\mu})$  (see Proposition 2.1). In the following,  $\mathbf{B}$  will be the matrix associated with  $B$  through the basis  $\mathcal{B}_{RB}$  and  $\mathbf{P}$  the preconditioner of the online algebraic system.

The key ingredient of our two-level preconditioner is the construction of a coarse solver. For  $i = 1, \dots, N_S$  we define a parameter independent coarse triangulation  $\mathcal{T}_{H,i}$ , and assume that each fine triangulation  $\mathcal{T}_{h,i}$  introduced in Section 1.1 is a refinement of the corresponding  $\mathcal{T}_{H,i}$ . On each subdomain we define the local reduced space  $V_i^{RB}$  as

$$V_i^{RB} = V_{H,i} \oplus V_{N_i,i}, \quad (28)$$

where  $V_{H,i}$  is the piecewise discontinuous linear FE space associated with  $\mathcal{T}_{H,i}$ , while  $V_{N_i,i}$  is the space spanned by local reduced basis computed in  $V_{h,i}$ . We denote with  $\mathcal{B}_{H,i}$  the standard FE basis of  $V_{H,i}$  and with  $\mathcal{B}_{N_i,i}$  the basis of  $V_{N_i,i}$ . A basis for  $V_i^{RB}$  is then  $\mathcal{B}_{RB,i} = \mathcal{B}_{H,i} \cup \mathcal{B}_{N_i,i}$ . We show in Appendix B how to build a basis which satisfies the direct sum assumption in (28). The offline-online decomposition of the method enriched with a coarse space is very similar to that shown in Section 1. The main difference is that now the global reduced space is defined as

$$V^{RB} = \bigoplus_{i=1}^{N_S} V_i^{RB} = \bigoplus_{i=1}^{N_S} V_{H,i} \oplus \bigoplus_{i=1}^{N_S} V_{N_i,i}. \quad (29)$$

We note that  $V_H = \bigoplus_{i=1}^{N_S} V_{H,i}$  is a non-conforming global coarse approximation space.

**Remark 3.1.** As regards the stability of the online problem associated with the reduced space (29) and the approximation properties, the results proven in Section 2 still hold.

### 3.1 Two level Schwarz preconditioner

In this section we introduce a two level Schwarz preconditioner based on the reduced space enriched with a coarse FE space, following the approach used in [2, 3, 4]. We recall that the global space is the DGRBE space  $V^{RB}$  defined in (29), we then define the global coarse space  $W_0 = \bigoplus_{i=1}^{N_S} V_{H,i}$ , and the local spaces  $W_i = V_i^{RB}$ , for  $i = 1, \dots, N_S$  (see (28) for the definition). We observe that  $W_0 \subseteq \bigoplus_{i=1}^{N_S} W_i = V^{RB}$ .

Let  $R_i^T: W_i \rightarrow V^{RB}$  be the inclusion  $W_i \rightarrow V^{RB}$ . Now, it is possible to define the local operators  $B_i$ ,  $i = 0, \dots, N_S$  such that

$$B_i(w_i, v_i) = B(R_i^T w_i, R_i^T v_i) \quad \forall w_i, v_i \in W_i.$$

We then introduce some projection-like operators  $P_i = R_i^T \tilde{P}_i: V^{RB} \rightarrow R_i^T W_i$ , for  $i = 0, \dots, N_S$ , where  $\tilde{P}_i: V^{RB} \rightarrow W_i$  is such that:

$$B_i(\tilde{P}_i w, v_i) = B(w, R_i^T v_i) \quad \forall v_i \in W_i.$$

The two level additive Schwarz preconditioner is then defined by  $P_{ad} = \sum_{i=0}^{N_S} P_i$ . Employing the matrix notation we have  $\mathbf{P}_{ad} = \mathbf{P}^{-1} \mathbf{B}$  with  $\mathbf{P}^{-1} = \sum_{i=0}^{N_S} \mathbf{R}_i^T \mathbf{B}_i^{-1} \mathbf{R}_i$ , being  $\mathbf{R}_i^T$  and  $\mathbf{B}_i$  the matrix representation of  $R_i^T$  and  $B_i(\cdot, \cdot)$ , respectively.

We next provide an estimate for the condition number of the preconditioned matrix  $\mathbf{P}^{-1} \mathbf{B}$ . The arguments used are similar to [2]. Given  $w \in V^{RB}$ , we define  $w_0 \in W_0$  such that:

$$w_0 = \sum_{i=1}^{N_S} w_0^i, \quad w_0^i = \Pi_{L^2}^{V_{H,i}} w|_{\Omega_i}, \quad i = 1, \dots, N_S, \quad (30)$$

where  $\Pi_{L^2}^{V_{H,i}}$  is the  $L^2$  projection onto  $V_{H,i}$ . It holds that:

$$\|w - w_0^i\|_{L^2(\Omega_i)} \lesssim H |w|_{H^1(\Omega_i)}, \quad |w_0^i| \lesssim |w|_{H^1(\Omega_i)}, \quad (31)$$

for  $i = 1, \dots, N_S$ , cf. [8]. We now report some preliminary lemmas whose proofs are based upon standard arguments (cf. [2]).



**Lemma 3.2.** For any  $w, v \in V^{RB}$ , we consider their unique decompositions as  $w = \sum_{i=1}^{N_S} R_i^T w_i$ ,  $v = \sum_{i=1}^{N_S} R_i^T v_i$ , with  $w_i, v_i \in W_i$  for  $i = 1, \dots, N_S$ . It holds that:

$$B(w, v) = \sum_{i=1}^{N_S} B_i(w_i, w_i) + I(w, v), \quad \text{where} \quad I(w, v) = \frac{\nu(\bar{\mu})}{h} \sum_{\substack{i,j=1 \\ i < j}}^{N_S} \int_{\Gamma_{ij}} u_i \mathbf{n}_i \cdot v_j \mathbf{n}_j + u_j \mathbf{n}_j \cdot v_i \mathbf{n}_i \, ds.$$

Moreover,

$$|I(w, w)| \lesssim \gamma \frac{1}{Hh} \|w\|_{L^2(\Omega)}^2 + \gamma \frac{H}{h} \sum_{i=1}^{N_S} |w|_{\mathbb{H}^1(\Omega_i)}^2,$$

where the hidden constant is independent of the mesh sizes  $h$  and  $H$  and of the penalty parameter  $\gamma$ .

**Lemma 3.3.** For any  $w \in V^{RB}$ , let  $w_0$  be the piecewise  $L^2$  projection defined in (30). Then the following estimates hold:

$$B_0(w_0, w_0) \lesssim \left(2 + \gamma \frac{H}{h}\right) B(w, w), \quad B(w - \tilde{w}_0, w - \tilde{w}_0) \lesssim \left(1 + \gamma \frac{H}{h}\right) B(w, w)$$

where the hidden constant is independent of  $h$ ,  $H$  and  $\gamma$ .

**Lemma 3.4** (Stable decomposition). For any  $w \in V^{RB}$ , let  $w_0$  be the element of  $W_0$  defined by (30) and let  $w_i \in W_i$ ,  $i = 1, \dots, N_S$ , be the uniquely determined elements such that  $w - R_0^T w_0 = \sum_{i=1}^{N_S} R_i^T w_i$ . Then:

$$\sum_{i=0}^{N_S} B_i(w_i, w_i) \leq \gamma C_0^2 B(w, w), \quad \text{with} \quad C_0^2 = O\left(\frac{H}{h}\right).$$

*Proof.* We denote by  $\tilde{w}_0 = R_0^T w_0$ . We have that

$$\sum_{i=0}^{N_S} B_i(w_i, w_i) = B(w - \tilde{w}_0, w - \tilde{w}_0) + B_0(w_0, w_0) - I(w - \tilde{w}_0, w - \tilde{w}_0).$$

We observe that, thanks to Lemma 3.2, and to (31), we have

$$\begin{aligned} |I(w - \tilde{w}_0, w - \tilde{w}_0)| &\lesssim \gamma \frac{1}{Hh} \|w - \tilde{w}_0\|_{L^2(\Omega)}^2 + \gamma \frac{H}{h} \sum_{i=1}^{N_S} |w - \tilde{w}_0|_{\mathbb{H}^1(\Omega_i)}^2 \\ &\lesssim \gamma \frac{H}{h} \sum_{i=1}^{N_S} |w|_{\mathbb{H}^1(\Omega_i)}^2 \lesssim \gamma \frac{H}{h} B(w, w). \end{aligned}$$

Exploiting Lemma 3.3 we can conclude.  $\square$

We can finally prove the following proposition about the condition number of the preconditioned matrix  $\mathbf{P}^{-1}\mathbf{B}$ .

**Proposition 3.5.** The following estimate holds

$$\kappa(\mathbf{P}^{-1}\mathbf{B}) \leq \gamma C_0^2 (2 + M) \lesssim \gamma (2 + M) \frac{H}{h},$$

where  $M$  is the maximum neighbours of each subdomain.

*Proof.* The proof is similar to [2], Theorem 5.1, and follows the general theory of Schwarz methods, see [28].  $\square$

	$l = 1$		$l = 2$		$l = 3$		$l = 4$		$l = 5$	
$N_{\Gamma_i}$	15		29		57		113		225	
$M_{BC}$	4		6		7		10		13	
$N_h$	324		1328		5376		21632		86784	
Method	A	B	A	B	A	B	A	B	A	B
$N_{RB}$	35	17	57	20.5	97.5	21	168	27.5	293	33.5
$N_{DGRBE}$	70	34	114	41	195	42	336	55	586	67
$N_h/N_{DGRBE}$	4.6	9.5	11.6	32.4	27.5	128	64.4	393	148	1295

Table 1: Example 1. Dimensions of the space involved in the computations.

## 4 Numerical results

In this section we show some numerical tests in order to validate the theoretical results presented in the previous sections. We recall that we denote with “Method A” and “Method B” the strategies for the construction of the local basis introduced in Section 1.1

### 4.1 Example 1. Comparison of the two enrichment strategies

We make a comparison between the two proposed strategies for the construction of the local bases. We tested both strategies on a diffusion reaction problem defined on  $\Omega = \Omega_1 \cup \Omega_2$  with  $\Omega_1 = (0, 1) \times (0, 1)$  and  $\Omega_2 = (1, 2) \times (0, 1)$ . We then consider a parameter  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_4)$  belonging to  $\mathcal{D} = [0.1, 10] \times [0, 1]^3 \subset \mathbb{R}^4$  and we set  $\nu(\boldsymbol{\mu}) = \mu_1$ ,  $\sigma \equiv 1$ . The right-hand side function  $f(\boldsymbol{\mu})$  is chosen such that the exact solution is:

$$u(\boldsymbol{\mu}) = \frac{\mu_2}{\mu_1} \sin(\pi x) \sin(\pi y) + \frac{\mu_3}{\mu_1} \frac{1}{2} \sin(2\pi x) \sin(2\pi y) + \frac{\mu_4}{\mu_1} \frac{1}{3} \sin(3\pi x) \sin(3\pi y).$$

We consider a sequence of uniform refinements  $\mathcal{T}_{h_l}$ ,  $l = 2, \dots, 5$ , of a given initial grid  $\mathcal{T}_{h_1}$ , such that  $h_l = \frac{h_{l-1}}{2}$  for  $l = 2, \dots, 5$ .

In Table 1 we report the dimensions of the spaces involved in the computations. We have denoted with  $N_h$  the dimension of the fine FE space upon which the reduced bases are built, with  $N_{RB}$  average number of local basis on each subdomain and with  $N_{DGRBE}$  the dimension of the whole DGRBE space. From the results, it is evident that Method A produces bigger DGRBE spaces and thus has a greater computational cost than Method B. In Figure 1 we show the relative approximation error in energy norm of the enriched DGRBE method and we compare it with the fine-grid FE solution relative approximation error. The errors are computed with respect to  $u(\boldsymbol{\mu})$ , and represent an average on a sample of 24 parameter values. We see that the DGRBE recovers the convergence order of the FE solution. Indeed, we observe that the curves relative to the DGRBE approximation and the fine FE one are almost overlapped.

### 4.2 Example 2. Several subdomain case

We tested Method B on a domain composed by many subdomains. We define a global domain of the form  $\Omega = (0, S) \times (0, S)$  with  $S \in \mathbb{N}$ , partitioned into  $N_S = S^2$  subdomains. We built a Legendre basis on each non-Dirichlet side of the square-shaped subdomains. In Figure 2 we plot the relative DGRBE approximation error as a function of the fine mesh

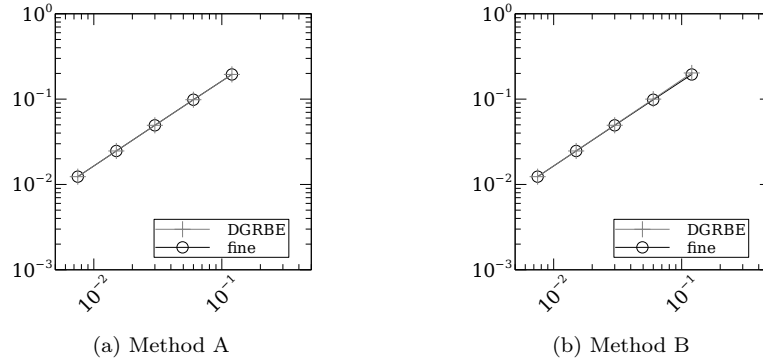


Figure 1: Example 1. Relative approximation error of the two different enrichment methods versus  $h$ .

size, and we compare it with the error given by the fine and the coarse FE approximations. The method does not show a worsening of the performances as the number of subdomain increases.

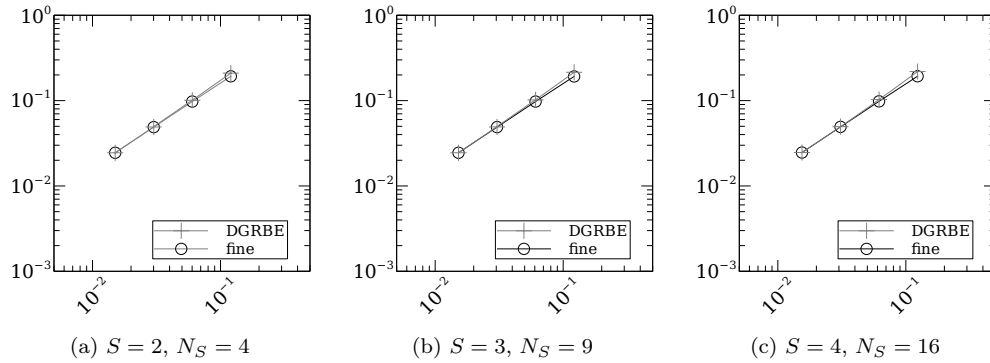


Figure 2: Example 2. Comparison of the DGRBE relative approximation error and the FE coarse one, as functions of the fine mesh size  $h$ .

### 4.3 Example 3. Two-level preconditioner

We tested the two level preconditioner of Section 3 on the problem introduced in the Example 2, see Section 4.2. We chose to use  $N_S = 16, 64$  and we applied Method B of Section 1.1 to build the local basis. As for the meshes, we define a coarse mesh  $\mathcal{T}_H$  and a sequence of uniform refinements  $\mathcal{T}_{h_l}$ ,  $l = 1, \dots, 6$ , assuming that the restrictions of these meshes to each subdomain are conforming triangulations. We then set

$$\mathcal{T}'_H = \mathcal{T}_{h_1}, \quad \mathcal{T}'_{h_l} = \mathcal{T}_{h_{l+1}}, \quad l = 1, \dots, 5, \quad \text{and} \quad \mathcal{T}''_H = \mathcal{T}_{h_2}, \quad \mathcal{T}'_{h_l} = \mathcal{T}_{h_{l+2}}, \quad l = 1, \dots, 4.$$

In Table 2 and in Table 3 we show the condition number of the non-preconditioned online system and the preconditioned one, for  $N_S = 16$  and  $N_S = 64$ , respectively. In brackets we

	No coarse	$H$	$H/2$	$H/4$
$h$	9.68 e2 (44)	1.94 e2 (51)	/	/
$h/2$	1.91 e3 (98)	4.04 e2 (100)	5.17 e2 (109)	/
$h/4$	3.78 e3 (131)	8.20 e2 (135)	1.06 e3 (152)	1.66 e3 (163)
$h/8$	7.50 e3 (176)	1.65 e3 (195)	2.15 e3 (184)	3.40 e3 (223)
$h/16$	1.49 e4 (218)	3.31 e3 (252)	4.32 e3 (264)	6.87 e3 (271)

(a) Non preconditioned

	$H$	$H/2$	$H/4$
$h$	11.4 (21)	/	/
$h/2$	25.6 (32)	11.6 (25)	/
$h/4$	52.7 (41)	23.7 (33)	9.87 (22)
$h/8$	104 (52)	54.1 (43)	21.5 (29)
$h/16$	205 (63)	109 (53)	44.0 (37)

(b) Preconditioned

Table 2: Example 3,  $N_S = 16$ . Condition number of the online system (fine mesh size on the rows, coarse mesh size on the columns) and iteration counts (between parentheses).

report the numbers of conjugate gradient iterations needed to solve the online system (12) (with a tolerance of  $10^{-9}$ ). As regards the non-preconditioned case, the values reported are referred to the online matrices associated with the space (7) (column “no coarse”) and with the spaces (29) for different values of the coarse mesh size (columns “ $H$ ”, “ $H/2$ ” and “ $H/4$ ”). The condition number of the non-preconditioned matrix has been evaluated by explicitly computing the extremal eigenvalues. Differently, the condition number of the preconditioned system has been computed as in [2], by exploiting the connections between the Lanczos technique and the Preconditioned Conjugate Gradient (PCG) method, as presented in detail in [13]. We observe that the condition number of the preconditioned system scales as expected by Proposition 3.5. We observe also the condition number of the preconditioned system and thus the number of PCG iterations are independent of the number of subdomains. In our tests we used an initial coarse mesh size  $H \approx 0.5$ .

## 5 Conclusions

We have introduced the DGRBE method and carried out its convergence analysis. In particular we have proven the well-posedness of the method and we have shown that the DGRBE approximation error scales as the size of the fine mesh on which the local bases are built, provided we use properly defined local problems to build the local bases and set a sufficiently small tolerance in the local Greedy procedures. We have presented two possible methods to generate the local bases. We have finally proposed a possible preconditioner for the online problem, which exploits the pre-existing decomposition of the domain, but requires a slight modification in the Greedy procedure used to build the local bases. The modified Greedy algorithm is needed to ensure the linear independence between the coarse space and the local bases, which is crucial to guarantee the well-posedness of the method. Through numerical experiments, we have compared the DGRBE approximation with the FE one based on the fine meshes on which the local basis are built. We have shown that the accuracy of both

	No coarse	$H$	$H/2$	$H/4$
$h$	1.73 e3 (127)	6.04 e2 (87)	/	/
$h/2$	2.73 e3 (230)	1.24 e3 (157)	1.67 e3 (181)	/
$h/4$	5.43 e3 (304)	2.52 e3 (207)	3.42 e3 (215)	5.41 e3 (222)
$h/8$	1.08 e4 (408)	5.08 e3 (300)	6.93 e3 (311)	1.11 e4 (335)
$h/16$	2.17 e4 (539)	1.02 e4 (384)	1.39 e4 (331)	2.23 e4 (341)

(a) Non preconditioned

	$H$	$H/2$	$H/4$
$h$	11.4 (21)	/	/
$h/2$	25.6 (33)	11.6 (26)	/
$h/4$	53.0 (43)	23.8 (33)	9.92 (23)
$h/8$	105 (54)	54.2 (46)	21.6 (31)
$h/16$	205 (63)	109 (53)	44.0 (37)

(b) Preconditioned

Table 3: Example 3,  $N_S = 64$ . Condition number of the online system (fine mesh size on the rows, coarse mesh size on the columns) and iteration counts (between parentheses).

methods is similar, but the former is based on a lower dimensional approximation space. This property holds for any given fine triangulation. We also tested the performances of the proposed preconditioner. Further developments of this work are the study of *a posteriori* error estimators which provide upper bounds for the approximation error of the reduced solution with respect to the fine-grid one. Moreover, the strategy proposed can be extended to more general problems, *e.g.* the Stokes problem, in which some stabilization techniques may be needed to guarantee the stability of the method.

## Appendix A Implementation issues

We now sum up the main ingredients of the DGRBE method, from the implementation point of view.

As regards the *offline stage* of the DGRBE method, we point out that it can be split into two sub-steps: *i)* the *local offline stage*, where the local reduced bases and the local discrete operators are built; *ii)* the *global offline stage*, where the global DGRBE space and the interface discrete operators are constructed.

We consider now the *local offline stage* on the subdomain  $\Omega_i$ . With respect to a given FE basis  $\mathcal{B}_{h,i} = \{\varphi_1^i, \dots, \varphi_{N_{h,i}}^i\}$  of  $V_{h,i}$ , given  $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \beta) \in \tilde{\mathcal{D}}$  the matrix form of the reference problem (5) reads

$$\mathbf{A}_{h_i}(\boldsymbol{\mu})\mathbf{u}_{h_i}(\boldsymbol{\mu}) = \mathbf{F}_{h_i}(\boldsymbol{\mu}) + \mathbf{I}_{h_i}(\beta).$$

As usual in the RB context, we assume that the local operators depends “affinely” on the parameter [26, 25], *i.e.*,

$$\mathbf{A}_{h_i}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_{A_i}} \Theta_{A_i}^q(\boldsymbol{\mu}) \mathbf{A}_{h,i}^q, \quad \mathbf{F}_{h_i}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_{F_i}} \Theta_{F_i}^q(\boldsymbol{\mu}) \mathbf{F}_{h,i}^q, \quad \mathbf{I}_{h_i}(\beta) = \sum_{q=1}^{Q_{I_i}} \Theta_{I_i}^q(\beta) \mathbf{I}_{h,i}^q, \quad (32)$$

where  $\mathbf{A}_{h,i}^q$ ,  $\mathbf{F}_{h,i}^q$  and  $\mathbf{I}_{h,i}^q$  are parameter independent arrays, while  $\Theta_{A_i}^q$ ,  $\Theta_{F_i}^q$  and  $\Theta_{\mathcal{I}_i}^q$  are real valued functions of the parameter  $\boldsymbol{\mu} \in \mathcal{D}$  and  $\beta \in \mathcal{D}_{BC}$ . In order to recover such ‘‘affine’’ decomposition of the operators, suitable empirical interpolation techniques may be needed [7]. We denote with  $\mathcal{B}_{RB,i} = \{\zeta_1^i, \dots, \zeta_{N_i}^i\} \subseteq V_{h,i}$  the basis of the space  $V_i^{RB}$  produced by the Greedy algorithm, *i.e.*, the local reduced basis. We denote with  $\mathbf{Z}_{RB,i}$  the matrices whose columns are the coefficients of the expansion of the elements of  $\mathcal{B}_{RB,i}$  with respect to the fine basis  $\mathcal{B}_{h,i}$ . In the local offline stage the following matrices and vectors are then built and stored:

$$\begin{aligned} \mathbf{A}_{RB,i,i}^q &= \mathbf{Z}_{RB,i}^T \mathbf{A}_{h,i}^q \mathbf{Z}_{RB,i} & \forall q = 1, \dots, Q_{A_i}, \\ \mathbf{F}_{RB,i}^q &= \mathbf{Z}_{RB,i}^T \mathbf{F}_{h,i}^q & \forall q = 1, \dots, Q_{F_i}, \\ \mathbf{I}_{RB,i}^q &= \mathbf{Z}_{RB,i}^T \mathbf{I}_{h,i}^q & \forall q = 1, \dots, Q_{\mathcal{I}_i}. \end{aligned}$$

As for the *global offline stage*, we start by considering the matrices associated to the interface terms through the bases  $\mathcal{B}_{h,i}$  and  $\mathcal{B}_{h,j}$ . Setting  $\bar{\Gamma}_{ij}(\boldsymbol{\mu}) = \bar{\partial\Omega}_i(\boldsymbol{\mu}) \cap \bar{\partial\Omega}_j(\boldsymbol{\mu})$ , as regards the jump term we have

$$(\mathbf{C}_{r,r'}(\boldsymbol{\mu}))_{pq} = \frac{\nu(\boldsymbol{\mu}) \gamma}{h} \int_{\Gamma_{ij}(\boldsymbol{\mu})} \varphi_q^{r'} \varphi_p^r ds \quad \forall p = 1, \dots, N_{h,r'} \quad \forall q = 1, \dots, N_{h,r} \quad \forall r, r' \in \{i, j\}.$$

As for the derivative consistency term, we define

$$(\mathbf{D}_{r,r'}(\boldsymbol{\mu}))_{pq} = \int_{\Gamma_{ij}(\boldsymbol{\mu})} \frac{1}{2} \nu(\boldsymbol{\mu}) (\nabla \varphi_q^{r'} \cdot \mathbf{n}_{r'}) \varphi_p^r ds \quad \forall p = 1, \dots, N_{h,r'} \quad \forall q = 1, \dots, N_{h,r} \quad \forall r, r' \in \{i, j\},$$

where  $\mathbf{n}_r$  is the normal unit vector of  $\Gamma_{ij}(\boldsymbol{\mu})$ , exiting from  $\Omega_r$ . We assume that also these interface matrices admit an affine decomposition as in (32), *i.e.*,

$$\mathbf{C}_{r,r'}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_{C_{r,r'}}} \Theta_{C_{r,r'}}^q(\boldsymbol{\mu}) \mathbf{C}_{r,r'}^q, \quad \mathbf{D}_{r,r'}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_{D_{r,r'}}} \Theta_{D_{r,r'}}^q(\boldsymbol{\mu}) \mathbf{D}_{r,r'}^q,$$

for suitably chosen parameter independent matrices  $\mathbf{C}_{r,r'}^q$ ,  $\mathbf{D}_{r,r'}^q$  and real valued functions  $\Theta_{C_{r,r'}}^q$ ,  $\Theta_{D_{r,r'}}^q$  of  $\boldsymbol{\mu}$ .

In the global offline stage we build and store the following matrices:

$$\begin{aligned} \mathbf{C}_{RB,r,r'}^q &= S(r, r') \mathbf{Z}_{RB,r}^T \mathbf{C}_{r,r'}^q \mathbf{Z}_{RB,r'} & \forall q = 1, \dots, Q_{C_{r,r'}} \quad \forall r, r' \in \{i, j\}, \\ \mathbf{D}_{RB,r,r'}^q &= S(r, r') \mathbf{Z}_{RB,r}^T \mathbf{D}_{r,r'}^q \mathbf{Z}_{RB,r'} & \forall q = 1, \dots, Q_{D_{r,r'}} \quad \forall r, r' \in \{i, j\}, \end{aligned}$$

where  $S(r, r') = 1$  if  $r = r'$ ,  $S(r, r') = -1$  otherwise, and  $Q_{C_{r,r'}}$ ,  $Q_{D_{r,r'}}$  are the numbers of affine terms of  $\mathbf{C}_{r,r'}$  and  $\mathbf{D}_{r,r'}$ , respectively. These matrices have to be built for each interface.

During the *online stage*, the matrices built and stored during the offline stage have to be properly assembled exploiting the affine decomposition property, for a given value of the parameter  $\tilde{\boldsymbol{\mu}} \in \mathcal{D}$ . First of all the arrays  $\mathbf{A}_{RB,i,j}(\boldsymbol{\mu})$ ,  $\mathbf{C}_{RB,i,j}(\boldsymbol{\mu})$ ,  $\mathbf{D}_{RB,i,j}(\boldsymbol{\mu})$  and  $\mathbf{F}_{RB,i}(\boldsymbol{\mu})$  have to be built, exploiting the affine decomposition (which follows from the fine-grid operators affine decomposition (32) and (33)), by summing the previously stored quantities. In order to simplify the exposition, if  $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ , we assume that  $\mathbf{C}_{RB,i,j}(\boldsymbol{\mu})$  and  $\mathbf{D}_{RB,i,j}(\boldsymbol{\mu})$  are null matrices of dimension  $N_i \times N_j$ . Moreover, if  $i \neq j$ , we assume that  $\mathbf{A}_{RB,i,j}(\boldsymbol{\mu})$  is the null matrix of dimension  $N_i \times N_j$ . For the sake of notation, we now omit the parameter

dependence of matrices and vectors. We define:

$$\mathbf{A}_{\text{On}} = (\mathbf{A}_{RB,i,j})_{i,j=1,\dots,N_S}, \quad \mathbf{C}_{\text{On}} = (\mathbf{C}_{RB,i,j})_{i,j=1,\dots,N_S}, \quad \mathbf{D}_{\text{On}} = (\mathbf{D}_{RB,i,j})_{i,j=1,\dots,N_S}. \quad (33)$$

The matrix associated to the online problem (12) is then:  $\mathbf{A}_{RB} = \mathbf{A}_{\text{On}} + \mathbf{C}_{\text{On}} + \mathbf{D}_{\text{On}} + \mathbf{D}_{\text{On}}^T$ . We finally define  $\mathbf{u}_{RB} = (\mathbf{u}_{N_i})_{i=1,\dots,N_S}$ , and  $\mathbf{F}_{RB} = (\mathbf{F}_{N_i})_{i=1,\dots,N_S}$ . The algebraic system associated with the online problem (12) is then

$$\mathbf{A}_{RB}\mathbf{u}_{RB} = \mathbf{F}_{RB}.$$

## Appendix B Construction of the local basis for the two-level preconditioner

To make sure that the local reduced spaces  $V_{N_i,i}$  and the coarse ones  $V_{H,i}$  are in direct sum as assumed in (29) the standard implementation of the RB Greedy algorithm [26] will not serve the purpose. We propose a possible way to build the space  $V_{N_i,i}$ , which is described in detail in the pseudo-algorithm below (Algorithm B.1). It is a proper modification of the algorithm shown in [26, 25]. We consider local *a posteriori* error estimators  $\Delta_{H_i,N_i}^k: \tilde{\mathcal{D}} \rightarrow R$ ,  $i = 1, \dots, N_S$ , based on the dual norm of the residual, similarly to the *a posteriori* estimator of the standard RB method (*cf.* [26, 25]). The estimator  $\Delta_{H_i,i}^k$  satisfies

$$\|\tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}) - u_{H,i}^k(\tilde{\boldsymbol{\mu}})\|_{\tilde{\boldsymbol{\mu}},i} \leq \Delta_{H_i,i}^k(\tilde{\boldsymbol{\mu}}) \quad \forall \tilde{\boldsymbol{\mu}} \in \mathcal{D},$$

where  $\tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}})$  solves (5) and  $u_{H,i}^k(\tilde{\boldsymbol{\mu}}) \in V_{H,i}^k$  is such that

$$A_i(u_{H,i}^k(\tilde{\boldsymbol{\mu}}), v_{H,i}^k(\tilde{\boldsymbol{\mu}})) = F_i(v_{H,i}^k(\tilde{\boldsymbol{\mu}})) + \langle \mathcal{I}_i^\beta, v_{H,i}^k(\tilde{\boldsymbol{\mu}})|_{\Gamma_i} \rangle \quad \forall v_{H,i}^k \in V_{H,i}^k, \quad (34)$$

with  $V_{H,i}^k = V_{H,i} \oplus \text{span}\{\tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}_i^1), \dots, \tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}_i^k)\}$ . Note that  $V_{H,i}^0 = V_{H,i}$ . It holds that

$$\Delta_{H_i,i}^k(\tilde{\boldsymbol{\mu}}) = 0 \iff \tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}) \in V_{H,i}^k, \quad (35)$$

*cf.* [26]. Thanks to (35), if the parameter value  $\tilde{\boldsymbol{\mu}}_i^{k+1}$  picked up at the  $k$ -th step of the Greedy algorithm is such that  $\Delta_{H_i,i}^k(\tilde{\boldsymbol{\mu}}_i^{k+1}) > 0$ , then  $\tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}_i^{k+1})$  is independent of  $V_{H,i}^k$  and, in particular of  $V_{H,i}$ .

At each step of our Greedy algorithm, we ensure also some orthogonality properties on the local basis. This is useful to control the conditioning of the linear system associated with problem (34), to be solved many times during the Greedy procedure.

**Algorithm B.1** (Greedy algorithm for the  $i$ -th local problem).

```

 $k = 0; X_0 = \{0\}; \varepsilon^0 = \varepsilon^* + 1;$ 
while  $\varepsilon^k > \varepsilon^*$  do
   $\tilde{\boldsymbol{\mu}}_i^{k+1} = \text{argmax}_{\tilde{\boldsymbol{\mu}} \in \mathcal{D}} \Delta_{H_i,i}^k(\tilde{\boldsymbol{\mu}});$ 
   $\varepsilon^{k+1} = \Delta_{H_i,i}^k(\tilde{\boldsymbol{\mu}}_i^{k+1});$ 
  computation of  $\tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}_i^{k+1});$ 
   $\zeta_{k+1}^i = \text{orthonormalization of } \tilde{u}_{h,i}(\tilde{\boldsymbol{\mu}}_i^{k+1}) \text{ w.r.t. } V_{H,i}^k \text{ and } (\cdot, \cdot)_{V_i};$ 
   $X_{k+1} = X_k \oplus \text{span}\{\zeta_{k+1}^i\};$ 
   $k = k + 1;$ 
end while
 $N_i = k; V_{N_i,i} = X_{N_i}.$ 

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