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Agosti, A.; Formaggia, L.;Scotti, A.

MOX, Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox@mate.polimi.it http://mox.polimi.it

Analysis of a model for precipitation and dissolution coupled with a Darcy flux

ABRAMO AGOSTI♯ and LUCA FORMAGGIA♯ and ANNA SCOTTI♯

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♯ MOX– Modellistica e Calcolo Scientifico
Dipartimento di Matematica “F. Brioschi”
Politecnico di Milano
via Bonardi 9, 20133 Milano, Italy
abramo.agosti@polimi.it
luca.formaggia@mail.polimi.it
anna.scotti@polimi.it

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Abstract

In this paper we deal with the numerical analysis of an upscaled model of a reactive flow in a porous medium, which describes the transport of solutes undergoing precipitation and dissolution, leading to the formation/degradation of crystals inside the porous matrix. The model is defined at the Darcy scale, and it is coupled to a Darcy flow characterized by a permeability field that changes in space and time according to the precipitated crystal concentration. The model involves a non-linear multi-valued reaction term, which is treated exactly by solving an inclusion problem for the solutes and the crystals dynamics. We consider a weak formulation for the coupled system of equations expressed in a dual mixed form for the Darcy field and in a primal form for the solutes and the precipitate, and show its well posedness without resorting to regularization of the reaction term. Convergence to the weak solution is proved for its finite element approximation. We perform numerical experiments to study the behavior of the system and to assess the effectiveness of the proposed discretization strategy. In particular we show that a method that captures the discontinuity yields sharper dissolution fronts with respect to methods that regularize the discontinuous term.

1 Introduction

The study of reactive flow in porous media is of particular relevance in several applications ranging from geochemical reactions in sedimentary basins [3, 4], kerogen degradation and expulsion in oil reservoirs [22], groundwater contamination and remediation processes[20], biomedical applications such as drug release from drug eluting devices [18, 12, 2].

These processes can be characterized by the presence of phenomena such as adsorption that, at the macroscale, can be effectively modeled by discontinuous reaction terms. This is, for instance, the case of oil generation from kerogen in the source rock where retention processes occur and result in discontinuous reaction rates. The process of crystal precipitation/dissolution, which is the subject of this work, is usually modeled with a discontinuous dissolution rate to account for the fact that dissolution can only take place if the crystal concentration is above a given value. From the mathematical point of view, this type of discontinuous equations exhibit a discontinuity that depends on the solution itself. Therefore, to prove the existence of a solution, and to determine its behavior at the discontinuity, we have to resort to Filippov theory [11]. Indeed, when the trajectory of the solution reaches the discontinuity, according to the properties of the discontinuous right hand side it can cross the surface or slide onto it until a suitable exit condition is met. This class of problem may be interpreted as differential inclusions: a recent review on their numerical treatment, in the context of ODEs, is found in [9]. Indeed there are three basic ways of treating discontinuous differential problems. The first is to rely on a regularization approach, where the discontinuity is eliminated by using a suitable smoothing operator, which typically depends on a single parameter. The main limitation of this approach is that the solution depends on the value of the parameter and a compromise must be set between accuracy and stiffness of the regularized problem. A second technique consists in simply ignoring the discontinuity and relying on adaptive stepping. The error indicator associated to several time advancing scheme will detect the discontinuity and refine the step in its vicinity. The drawback here is that the refinement may be extremely fine, with the resulting computational cost, and may be difficult to maintain high order accuracy. Moreover, both techniques may fail, or give unsatisfactory results, in the case of sliding motion, i.e. when the solution after reaching the discontinuity surface, slides onto it [7]. A third approach is based on detecting when the solution reaches the discontinuity and select its behavior according to Filippov theory [11]. These method, often called event driven methods, may guarantee optimal convergence at reasonable computational cost, and allow for the resolution of sliding motions.

In this work we will consider the simplified, yet realistic model proposed in [10] whose numerical solution has been carried out and analyzed in [17, 16]. The model is defined at the Darcy scale and describes the dissolution and precipitation process leading to the formation/degradation of crystals inside the porous matrix. We will consider here the coupling between dissolution/precipitation and a Darcy flow whose permeability is affected by the crystal concentration. The model involves a non-linear discontinuous reaction term and is cast as a differential inclusion that accounts for the discontinuity in the reaction term that describes the fact that the dissolution process starts only when the concentration of the reactants reaches a certain critical value.

Differently from the quoted references, where a regularization approach was followed, we will solve here the inclusion problem given by the set-valued reaction term, and we will follow a numerical treatment of the discontinuity based on the techniques proposed in [7].

In this work we focus on the analysis of the coupled problem, showing for the first time its well posedness, and the convergence of a fully discrete finite element approximation to the continuous weak solution. For the sake of brevity, we have omitted the details of the numerical techniques based on event-driven methods [7] that we have employed, which are the subject of a forthcoming work. We will show some numerical results, which underline the fact that by using a method that captures the discontinuity accurately we get sharper dissolution fronts than regularization methods.

The paper is organized as follows: in Section 2 we outline the mathematical model for dissolution-precipitation which is taken from [17], and for the coupling with a Darcy velocity field. We first give the main result of uniqueness and regularity of the solution of the coupled problem. The existence of solutions is proved in subsection 2.5, via a Faedo-Galerkin approach. We construct a finite element approximation and we prove that the limit solution exists and coincide with that of the original differential problem. The final Section is devoted to the illustration of a numerical result that shows the effectiveness of the model.

2 A simplified model of dissolution and precipitation

We introduce a model that describes the flow in a porous medium with ions dissolved in water that move under the action of transport and diffusion and precipitate in a crystal form [10]. We model the problem at the Darcy scale: the medium is a continuum, and the pores and solid particles are homogenized on a reference volume element. We are interested in studying the interaction of transport and diffusion, with the chemical reactions that determine the process of ion (anion and cation) precipitation and dissolution. These processes transform the dissolved ions into immobile solid species, with the consequent formation/dissolution of crystals.

2.1 Nomenclature

With $\Omega \subset \mathbb{R}^2$ we indicate the domain of the problem, occupied by the porous medium. The domain is open and polygonal, with boundary $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$. To obtain uniqueness and regularity results we may need to impose further restrictions later on. With $T > 0$ we denote a given final time. We further define

$$\Omega^T = (0, T] \times \Omega, \quad \Gamma^T_{D,N} = (0, T] \times \Gamma$$

and we indicate with $L^p(\Omega)$, $H^p(\Omega)$ and $L^p((0, T); V)$ the usual Sobolev spaces and spaces with values in Sobolev spaces[1], for a $p \in (0, \infty]$. While, $H_{div} = \{\mathbf{v} \in [L^2(\Omega)]^2,\ \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$. With $||\cdot||$ we indicate the $L^2(\Omega)$ norm.

For a function $u : \Omega^T \to \mathbb{R} :\ u = u(t, \mathbf{x})$ we set $u(t) : \Omega \to \mathbb{R} :\ u(t)(\mathbf{x}) = u(t, \mathbf{x})$, and analogously for vector functions.

Furthermore, $C, D, E, \ldots$ denote throughout generic positive constants independent of the unknown variables or the discretization parameters, the value of which might change from line to line.

2.2 The model

We use the simplified model considered in [17, 16], here briefly recalled for readers' convenience. While reactions usually take place between various cations and anions, this simplified model considers only one mobile species, whose mass concentration is denoted by u. The mass concentration of the (immobile) precipitate is denoted by v.

From the mass conservation principle and the chemical balance dynamics the following adimensionalized problem for the evolution of the reactant u is derived[17, 16]:

$$\begin{cases} \dfrac{\partial}{\partial t}(u+v) - \operatorname{div}\big(\nabla \mathrm{u} - \mathbf{q}\mathrm{u}\big) = 0 \quad \text{in} \quad \Omega^{\mathrm{T}}, \\ u = g \quad \text{in} \quad \Gamma_{\mathrm{D}}^{\mathrm{T}}, \\ \nabla u \cdot n = h \quad \text{in} \quad \Gamma_{\mathrm{N}}^{\mathrm{T}}, \\ u = u_0 \quad \text{in} \quad \Omega \quad \text{for} \quad \mathrm{t} = 0, \end{cases} \tag{1}$$

where g and h are given data, and $\mathbf{q}$ is the velocity vector field of the Darcy flow, describing the water flow. The full coupling whit the Darcy problem will be described later. The diffusion coefficient has been taken equal to 1 for simplicity. System 1 is coupled with the adimensionalized equation for the precipitate that reads:

$$\begin{cases} \dfrac{\partial}{\partial t} v = r(u) - H(v) \quad \text{in} \quad \Omega^{\mathrm{T}} \\ v = v_0 \quad \text{in} \quad \Omega \quad \text{for} \quad \mathrm{t} = 0, \end{cases} \tag{2}$$

where $r(u)$ and $H(v)$ are the production and dissolution rates, respectively, so that the rate of change in the precipitate concentration is the net result of the process of precipitation and dissolution. It is assumed that $r : \mathbb{R} \to [0, \infty)$ be locally Lipschitz continuous function with the following properties

$$\begin{cases} r(0) = 0 \quad \text{for } u \le u_* \quad \text{with } 0 \le u_* < 1 \\ r \quad \text{monotonically increasing for} \quad u \ge u_* \\ r(u^*) = 1 \quad \text{for } u_* < u^*. \end{cases}$$

Here, $0 \le u_* < u^* \le 1$ are two limiting values. The former sets the minimal concentration for the reaction to occur. The latter limits the maximal (adimensional) reaction rate to one.

The dissolution rate is described by the Heaviside distribution

$$H(v) = \begin{cases} 0 \quad \text{for} \quad v < 0, \\ [0,1] \quad \text{for} \quad v = 0, \\ 1 \quad \text{for} \quad v > 0, \end{cases}$$

which is a set-valued function. Thus, the equal symbol in equations (1) and (2) should in fact be replaced by an inclusion symbol. Following [16, 17], we set

$$H(v) = \min\{1, r(u)\} \quad \text{if } v = 0. \tag{3}$$

Note that, if $r(u) = u^n,\ n \in \mathbb{N}^+$, (in [17] $r(u) = u^2$, as given by the mass balance law), then $u_* = 0$, i.e. $r(u) = \mathrm{Proj}_{\mathbb{R}^+}(\mathrm{u^n})$, and $u^\star = 1$. If $u = u^\star$ for all $\mathbf{x}$ at a given time t^*, then the system is in equilibrium for $t > t^*$, *i.e.* no precipitation or dissolution occur, since the precipitation rate is balanced by the dissolution rate regardless of the value of v. Analogously, if $v = 0$ for all $\mathbf{x}$ at a given time t^*, then the system is in equilibrium for $t > t^*$. This model describes the fact that there is a threshold value for the concentration of the reactant above which dissolution starts.

2.3 Analysis of the differential inclusion problem

Contrary to what has been done in [17] we do not regularize the dissolution term. Since it is a jump discontinuous function, the solution may not be everywhere differentiable in $(0, T)$. We need then to exploit some results on finite dimensional Ordinary Differential Equations with Discontinuous Right Hand Side (ODE with DRH) and adopt special techniques for the numerical solution of the problem.

Let us introduce a family of finite dimensional subspaces V_h of $H^1(\Omega)$ such that $H^1(\Omega)$ is an Hilbertian sum of the V_h. Let us denote by $P_h : L^2(\Omega) \to V_h$ the projection operator, and by $u_h = P_h u$, $v_h = P_h v$. Problem (2) becomes: at any time t find u_h and v_h such that:

$$\begin{cases} \dfrac{\partial}{\partial t} v_h \in r(u_h) - H(v_h) & \text{in} \quad \Omega^{\mathrm{T}}, \\ v_h = P_h[v(0)] & \text{in} \quad \Omega \quad \text{for} \quad \mathrm{t} = 0, \end{cases} \tag{4}$$

where we have supposed that $v(0) \in L^2(\Omega)$.

Lemma 2.1 *System (4) has a unique solution.*

Proof. The proof is based on the fact that (4) represents a family of autonomous finite dimensional inclusion problems on the parameter h. For any $\mathbf{x} \in \Omega$ this problem is isomorphic to $\dot{\mathbf{z}} \in F(\mathbf{z}(t)),\quad t \in [0, T],\ z(0) = z_0$ with $z : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $n \in \mathbb{N}$. An existence and uniqueness result for this inclusion problem is expressed in the following theorem,

Theorem 2.1 *Let the set valued map $F = F(\mathbf{z})$ satisfy the following conditions:*

1. *the sets $F(\mathbf{z})$ are closed and convex,*
2. *the map F is Upper Semi-Continuous (USC), i.e. the closure of the set $\{F(\mathbf{z}) \mid ||\mathbf{z}-\mathbf{z}_0|| < \delta\}$, $\delta > 0$, is compact $\forall \mathbf{z}_0 \in \mathbb{R}^n$,*
3. *$F(\mathbf{z})$ satisfies a growth condition: $\forall \mathbf{y} \in F(\mathbf{z})\ \exists k > 0$ and $a \mid ||\mathbf{y}|| \leq k||\mathbf{z}|| + a$, for all $\mathbf{z} \in \mathbb{R}^n$.*

Then there is an absolutely continuous solution to the differential inclusion, for every $\mathbf{z}_0 \in \mathbb{R}^n$. If moreover the map $F(t, \mathbf{z})$ is One-Sided Lipschitz Continuous (OSLC), i.e. if there exists a constant L such that for every $\mathbf{z}_1$, $\mathbf{z}_2$, and for every $\mathbf{y}_1 \in F(\mathbf{z}_1)$ and $\mathbf{y}_2 \in F(\mathbf{z}_2)$:

$$(\mathbf{z_1} - \mathbf{z_2}, \mathbf{y_1} - \mathbf{y_2}) \leq L||\mathbf{z_1} - \mathbf{z_2}||^2,$$

then the solution is unique. Analogous results exist at the continuous level, given by the Hille-Yosida theory for maximal monotone nonlinear operators.

The proof may be found in [11]. The set-valued map $H(v_k)$ has the following properties:

1. it is convex, compact and maximal monotone (indeed it can be characterized as the sub-differential of a convex function);
2. it satisfies the growth condition, with $k = a = 1$ (in particular it satisfies a boundedness condition with $k = 0$);
3. it is USC and the term $-H(v_k)$ is OSLC with $L = 0$ (due to the monotonicity property of $H(v_k)$).

Therefore, according to Theorem 2.1, problem (4) admits a unique absolutely continuous solution. □

To integrate system (4) at discrete level, we have to select an element of the set $H(0)$ when $v_h = 0$. This selection should coincide with the prescription introduced in equation (3) at the continuous level:

$$H(v_h) = \min\{1, r(u_h)\} \quad \text{if } v_h = 0. \tag{5}$$

We use the results coming from the theory of Filippov[11], and exploited for the numerical solution in [7, 8], in the context of *event-driven* methods. The reader may refer to the quoted references for details, which we do not report here for the sake of brevity.

2.4 Coupling with a Darcy model

We consider the coupling of the precipitation-dissolution model (1) - (2) with the Darcy equations for a single phase fluid with constant density (water, in the case of our interest). Namely, the advection velocity field for the cation transport process in (2) is the solution of the problem for $\mathbf{q}$ and p given by:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \mathrm{div}\mathbf{q} = 0 & \text{in } \Omega^{\mathrm{T}}, \\ \mathbf{q} = -\dfrac{k(\phi)}{\mu}\nabla p & \text{in } \Omega^{\mathrm{T}}, \\ p = p_D & \text{on } \Gamma_{\mathrm{D}}^{\mathrm{T}}, \\ \mathbf{q} \cdot \mathbf{n} = \eta & \text{on } \Gamma_{\mathrm{N}}^{\mathrm{T}}, \end{cases} \tag{6}$$

where we have indicated the essential and the natural boundary conditions with data η and p_D, respectively. Here ϕ is the porosity, $\mathbf{q}$ is the macroscopic velocity, p is the fluid pressure, $\mu > 0$ is the dynamic viscosity and k is a scalar permeability, i.e. we are considering the isotropic case.

The Darcy model is coupled to the cation dynamics by the fact that the precipitation and dissolution processes influence the porosity and thus also the permeability of the medium. In particular, with the increase of the precipitate concentration there is a consequent reduction of porosity; an empirical law for the variation of porosity with varying precipitate concentration is given by [14]:

$$\frac{d\phi}{dt} = -\frac{\partial v}{\partial t}.$$

At each time we have $\phi = \phi_0 - v$, with ϕ_0 a constant which we take, for the sake of simplicity, equal to 1. The permeability coefficient is modeled as a positive Lipschitz continuous function of porosity. A possible empirical law is [14]:

$$k(\phi) = (\phi)^2 \to K(\phi(v)) = (\phi_0 - v)^2 = (1-v)^2 + \epsilon,$$

for $v \in [0,1]$, where $\epsilon > 0$ is a small parameter that prevents the permeability to reach the zero value. In the following we will indicate the permeability as $k(v)$ to highlight the dependence on the precipitate concentration.

We derive now the weak formulation of the Darcy problem (6) coupled to the precipitation and dissolution problem. For the analysis we consider $\eta = 0$, $g = 0$ and $h = 0$. We also assume throughout that $|\Gamma_D| > 0$. Let us define the following functional spaces: $\mathcal{U} := \{u \in L^2((0,T); H_0^1(\Omega)) : \partial_t u \in L^2((0,T); H^{-1}(\Omega))\}$, $\mathcal{V} := \{v \in H^1((0,T); L^2(\Omega))\}$, $\mathcal{Q} := L^2(\Omega)$, $\mathcal{Z} := L^2((0,T); H_{div}(\Omega))$.

We can now state:

Problem $\mathbf{P_2}$. Find $(\mathbf{q}, p) \in \mathcal{Z} \times \mathcal{Q}$ and $(u, v) \in \mathcal{U} \times \mathcal{V}$, with $(u(0), v(0)) = (u_0, v_0)$, such that for all $(\boldsymbol{\tau}, \psi) \in \mathcal{Z} \times \mathcal{Q}$ and $(\omega, \theta) \in H_0^1 \times L^2(\Omega)$, such that

$$\begin{cases} \left(\frac{\mu}{k(v)}\mathbf{q}(t), \boldsymbol{\tau}\right) - (p(t), \mathrm{div}\boldsymbol{\tau}) = -(\mathrm{p_D}, \boldsymbol{\tau}\cdot\mathbf{n})_{\Gamma_\mathrm{D}} \\ (\mathrm{div}\mathbf{q}(\mathrm{t}), \psi) = (-\frac{\partial\phi(\mathrm{v})}{\partial \mathrm{t}}, \psi) = (\partial_\mathrm{t}\mathrm{v}(\mathrm{t}), \psi) \\ (\partial_t u(t), \omega) + (\nabla u(t), \nabla\omega) - (\mathbf{q}(t)u(t), \nabla\omega) \in (H(v(t)) - r(u(t)), \omega) \\ (\partial_t v(t), \theta) \in (r(u(t)) - H(v(t)), \theta) \end{cases} \tag{7}$$

for $t \in (0,T)$, and with $H(v)$ satisfying (3) and $u(0) = u_0 \in L^2(\Omega)$ and $v(0) = v_0 \in L^2(\Omega)$.

Our main results are stated in the following two theorems.

Theorem 2.2 *Let Ω be a convex polygonal domain, and let $p_D \in H^{1/2}(\Gamma_D)$. Assume moreover that $0 \le u_0 \le 1$ and $0 \le v_0 < 1$, and that $\mathrm{supp}(v_0)$ is formed by the union of a finite number of subsets contained in Ω and with Lipschitz continuous boundaries. Then there exists a solution of* (7)*.*

This theorem will be proven in the next section through the convergence result of the fully discrete problem employing the Euler method for time integration.

Theorem 2.3 *Under the same hypothesis of Theorem 2.2 and the additional assumptions that Ω be a convex polygonal domain with convex angles given by a rational fraction of π (for example $\Omega = [-L, L] \times [-l, l]$), and that the Dirichlet and Neumann data are imposed on whole polygon edges, if $\mathbf{q} \in [L^\infty(\Omega)]^2$, then the solution of* (7) *is unique.*

Remark 1 *The hypothesis of* $\mathbf{q} \in [L^\infty(\Omega)]^2$ *is perfectly reasonable due to the additional assumption of Theorem 2.3. Indeed, it is possible to prove, by using some results on the regularity of elliptic equations with rough coefficients in polygonal domains, that* $p \in C^{0,\beta}(\bar{\Omega})$*, for a* $\beta \in (0,1)$ *[15], see also [13, 21], and this provides the wanted regularity for the velocity. For the sake of brevity we omit the proof.*

Remark 2 *By compactness arguments it may be shown that if, in addition to the stated hypotheses,* $u_0 \in H_0^1(\Omega)$*, then* $u \in C^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$ *is a global strong solution of* (7).

In order to proceed with the proof of Theorem 2.3 we need the following

Lemma 2.2 *Under the stated hypotheses on the initial data and on* k*,* $0 \le v(t) < 1$ *and* $0 \le u(t) \le 1$ *a.e. in* $L^2(\Omega)$ *for all* $t \in (0,T]$*, and* $\mu/k(v(t)) \in L^\infty(\Omega)$ *and is positive. Moreover, the solution* u *belongs to* $L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$*,* $v \in L^\infty(0,T; L^2(\Omega))$*,* $\mathbf{q} \in H_{div}(\Omega)$*,* $p \in L^2(\Omega)$*,* $\partial_t v(t) \in L^\infty(\Omega)$ *and* $\partial_t u(t) \in L^2(0,T; H^{-1}(\Omega))$*. Hence, Problem* $\mathbf{P_2}$ *has meaning.*

Proof. Let us set $R(u,v) = r(u) - H(v)$. We may note that

- $R(u,v) \ge 0$ if $v \le 0$, for all u;
- $R(u,v)$ is either 0 or -1 if $u \le 0$;
- $R(u,v) \le 0$ if $v \ge 1$ and $0 \le u \le 1$;
- $R(u,v) \ge 0$ if $u \ge u^*$, so in particular when $u \ge 1$.

To prove that v is not negative it is sufficient to take $\theta = v^-(t) = \frac{1}{2}(v(t) - |v(t)|)$ as test function in (7) to get

$$||v^-(t)||^2 \le ||v_0^-||^2 + 2\int_0^t (R(u(\tau), v^-(\tau)), v^-(\tau))d\tau \le 0.$$

By which $v^-(t)$ is a zero element of $L^2(\Omega)$.

We now take $\omega = u^-(t)$ and $\psi = (u^-(t))^2$ in (7), and we exploit the non-negativity of v proven above and the properties of $R(u,v)$ to get, after integration by parts of the divergence term,

$$\frac{1}{2}\frac{d}{dt}||u^-(t)||^2 + ||\nabla u^-(t)||^2 + \frac{1}{2}(\partial_t v, u^-(t)^2) \in -(R(u^-, v), u^-(t)) \le 0,$$

by which, since $||u_0^-|| = 0$,

$$\begin{aligned} ||u^-(t)||^2 + 2\int_0^t ||\nabla u^-(\tau)||^2 d\tau &\le ||u_0^-||^2 + \frac{1}{2}\int_0^t (R(u^-(\tau), v(\tau)), u^-(\tau)^2)d\tau \\ &\le \frac{1}{2}\int_0^t ||u^-(\tau)||^2 d\tau. \end{aligned}$$

By Gronwall inequality we get $||u^-(t)|| = 0$. The fact that u cannot exceed 1 is obtained in a similar way, by looking at the negative part of $1-u$. Using these bounds for u we may prove that $v \le 1$ by looking at the negative part of $1-v$. Note that, since the reaction rate $R(u,v)$

is zero (for $v = 0$) or negative (for $v \neq 0$) for $0 \leq u \leq 1$, v can only decrease from $v_0 < 1$ to zero, so that $v < 1$ at all times. Consequently, $\mu/k(v(t)) \leq \mu/k(v_0) \in L^\infty(\Omega)$ and is positive.

For what concerns the regularity of the solutions, we take $\omega = u(t)$ in (7), to obtain

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2 + ||\nabla u(t)||^2 \in -\frac{1}{2}\Big(\partial_t v(t), u^2\Big) - \Big(\partial_t v(t), u\Big) =$$
$$-\frac{1}{2}\Big(r(u) - H(v), u^2\Big) - \Big(r(u) - H(v), u\Big),$$

after integrating by parts and using $(7)_2$ with $\psi = u^2$.

From the fact that $r(u)$ is non-negative and $H(v)$ takes values between 0 and 1, the application of Cauchy-Schwartz and Young inequalities gives

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2 + ||\nabla u(t)||^2 \leq \frac{1}{2}||u(t)||^2 + |\Omega|^{1/2}||u(t)|| \leq \frac{|\Omega|}{2} + ||u(t)||^2.$$

Then, by applying Gronwall inequality we have

$$||u(t)||^2 + 2\int_0^t ||\nabla u(s)||^2 ds \leq (||u_0||^2 + T|\Omega|)(1 + Te^{2T}) \quad \text{for } t \in (0, T]. \tag{8}$$

As for v, we take $\theta = v$ in (7), we use the local Lipschitz continuity of $r(u)$ and (8) to obtain that $||v(t)||$ is bounded uniformly in $(0, T]$.

The boundedness of $\partial_t v$, and consequently that of $\text{div}\mathbf{q}$, derive from that of $r(u) - H(v)$. If we choose $\boldsymbol{\tau} = \nabla\eta$ in (7), where η satisfies

$$\begin{cases} -\Delta\eta = p \\ \eta|_{\Gamma_D} = 0, \quad \frac{\partial\eta}{\partial n} = 0 \text{ on } \Gamma_N, \end{cases}$$

we obtain $||p||^2 \leq C + D||\mathbf{q}||^2$, thanks to the boundedness of $\mu/k(v)$ and using the Lax-Milgram estimate $||\nabla\eta|| \leq ||p||$.

We now take $\boldsymbol{\tau} = \mathbf{q}$ and $\psi = p$ in (7), exploit the facts that $0 \leq H(v) \leq 1$, $r(u)$ is bounded and $k(v)$ is positively bounded away from zero, and use the estimate for $||p||^2$, to obtain $||\mathbf{q}||^2 \leq C$.

Since $u \in L^2(0, T; H_0^1(\Omega))$ and $\mathbf{q}u \in L^\infty(0, T; L^2(\Omega))$, we have that $\partial_t u \in L^2(0, T; H^{-1}(\Omega))$. Note that, if $u_0 \in H_0^1(\Omega)$, since $\mathbf{q} \in [L^\infty(\Omega)]^2$, by choosing $\omega = \partial_t u$, we obtain that $u \in L^\infty(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$. □

Remark 3 *We observe that, since $v = 0$ and $u = 1$ are stable equilibrium points of the precipitation-dissolution dynamical system, if $v_0 < 1$ and $u_0 < 1$, the lines of discontinuity which separate regions where $v_0 > 0$ from regions where $v_0 < 0$ remain fixed in time, so they are regular if they are regular at $t = 0$.*

We are now in the position to prove Theorem 2.3: **Proof.** Assume there exists two $(u_1, v_1, \mathbf{q}_1, p_1)$ and $(u_2, v_2, \mathbf{q}_2, p_2)$, according to Theorem 2.2, and define: $\bar{u} = u_1 - u_2$, $\bar{v} = v_1 - v_2$, $\bar{\mathbf{q}} = \mathbf{q}_1 - \mathbf{q}_2$, $\bar{p} = p_1 - p_2$. We have at $t = 0$ that $\bar{u}(0) = 0$, $\bar{v}(0) = 0$, $\bar{\mathbf{q}}(0) = \mathbf{0}$ and $\bar{p}(0) = 0$. Taking $\theta = \bar{v}(t, \mathbf{x})$ from the fourth equation of system (7) we obtain:

$$\frac{1}{2}\frac{d}{dt}||\bar{v}||^2 \in (r(u_1) - r(u_2), \bar{v}) - (H(v_1) - H(v_2), \bar{v}).$$

Now, using the monotonicity property of the set valued map $H(v)$ (note that this property is still valid with the prescription (3)), the Lipschitz continuity of $r(u)$ (with constant L_r) and the Schwartz and the Young inequalities, we obtain:

$$\frac{d}{dt}||\bar{v}(t,\mathbf{x})||^2 \leq L_r^2||\bar{u}(t,\mathbf{x})||^2 + ||\bar{v}(t,\mathbf{x})||^2.$$

We then have, from the Gronwall inequality:

$$||\bar{v}(t,\mathbf{x})||^2 \leq C\, e^T \int_0^t ||\bar{u}(s,\mathbf{x})||^2 ds. \tag{9}$$

Taking now $\omega = \bar{u}(t,\mathbf{x})$ in the difference between equations $(7)_3$ for $(u_1, v_1, \mathbf{q}_1, p_1)$ and $(u_2, v_2, \mathbf{q}_2, p_2)$ we have:

$$(\partial_t \bar{u}(t), \bar{u}(t)) + ||\nabla \bar{u}(t)||^2 - (\bar{\mathbf{q}} u_1(t), \nabla \bar{u}(t)) - (\mathbf{q}_2 \bar{u}(t), \nabla \bar{u}(t)) \in (\partial_t \bar{v}(t), \bar{u}(t)), \tag{10}$$

which we rewrite, after integration by parts of the fourth term and integration in time from 0 to $t < T$, as the following differential inclusion:

$$\frac{1}{2}||\bar{u}(t)||^2 + \int_0^t ||\nabla \bar{u}(s)||^2 ds \in \int_0^t (\bar{\mathbf{q}} u_1(s), \nabla \bar{u}(s)) ds - \frac{1}{2}\int_0^t \left(\frac{\partial v_2(s)}{\partial t}, \bar{u}^2(s)\right) ds +$$
$$\int_0^t (H(v_1(s)) - H(v_2(s)), \bar{u}(s)) ds - \int_0^t (r(u_1(s)) - r(u_2(s)), \bar{u}(t)). \tag{11}$$

We now observe that we can write the following inequality:

$$||H(v_1) - H(v_2)|| \leq C||\bar{u}|| + D||\bar{v}||. \tag{12}$$

Indeed, if $v_1 = v_2$, from (3), Lemma (2.2) and from the Lipschitz continuity of $r(u)$, we get the estimate $||H(v_1) - H(v_2)|| \leq C||\bar{u}||$; if $v_1, v_2 \neq 0$, we get the estimate $||H(v_1) - H(v_2)|| \leq D||\bar{v}||$; if $v_1 \neq 0$ and $v_2 = 0$ (or viceversa), from estimate (9) we have that $u_1 \neq u_2$ (almost everywhere in Ω), and from (3) we have $H(v_1) - H(v_2) \leq C||u^\star - u_2|| \leq C||\bar{u}||$.

Using Lemma 2.2, Cauchy-Schwartz and Young inequalities, the monotonicity and the Lipschitz continuity of $r(u)$ we can then write the following inequality:

$$||\bar{u}(t)||^2 + \int_0^t ||\nabla \bar{u}(s)||^2 ds \leq C \int_0^t ||\bar{\mathbf{q}}(s)||^2 ds + D\int_0^t ||\bar{u}(s)||^2 ds + E\int_0^t \big(\int_0^s ||\bar{u}(r)||^2 dr\big) ds.$$

This is an integral inequality with a double integral. It can be shown (see for instance [19]) that it implies:

$$||\bar{u}(t)||^2 \leq C\int_0^t ||\bar{\mathbf{q}}(\mathbf{s})||^2 ds. \tag{13}$$

Subtracting the first and the second equations of system (7) for $(u_2, v_2, \mathbf{q}_2, p_2)$ from the equations for $(u_1, v_1, \mathbf{q}_1, p_1)$, we obtain:

$$\begin{cases} \left(\left[\frac{\mu}{k(v_1)} - \frac{\mu}{k(v_2)}\right]\mathbf{q_1}, \boldsymbol{\tau}\right) + \left(\frac{\mu}{k(v_2)}\bar{\mathbf{q}}, \boldsymbol{\tau}\right) - (\bar{p}, \mathrm{div}\boldsymbol{\tau}) = 0, \\ (\mathrm{div}\bar{\mathbf{q}}, \psi) = (\partial_{\mathrm{t}}\bar{\mathrm{v}}, \psi). \end{cases} \tag{14}$$

If we choose $\boldsymbol{\tau} = \nabla\eta$, where η satisfies

$$\begin{cases} -\Delta\eta = \bar{p} \\ \eta|_{\Gamma_D} = 0, \quad \frac{\partial\eta}{\partial n} = 0 \text{ on } \Gamma_N, \end{cases}$$

we obtain from the first equation of (14) that:

$$||\bar{p}||^2 + \left(\left[\frac{\mu}{k(v_1)} - \frac{\mu}{k(v_2)} \right] \mathbf{q_1}, \nabla \eta \right) + \left(\frac{\mu}{k(v_2)} \bar{\mathbf{q}}, \nabla \eta \right) = 0.$$

Applying Lemma (2.2), the Lax-Milgram estimate $||\nabla \eta|| \leq ||\bar{p}||$, the Lipschitz continuity property of $k(v)$, estimate (9), the hypothesis $\mathbf{q_1} \in [L^\infty(\Omega)]^2$ and the Young inequality, we finally obtain:

$$||\bar{p}||^2 \leq C||\bar{\mathbf{q}}||\,||\bar{p}|| + D||\bar{p}||\,||\bar{v}|| \to ||\bar{p}||^2 \leq C||\bar{\mathbf{q}}||^2 + D\int_0^t ||\bar{u}(s)||^2 ds. \tag{15}$$

We now take $\boldsymbol{\tau} = \bar{\mathbf{q}}$ and $\psi = \bar{q}$ in (14), obtaining:

$$\left(\left[\frac{\mu}{k(v_1)} - \frac{\mu}{k(v_2)} \right] \mathbf{q_1}, \bar{\mathbf{q}} \right) + \left(\frac{\mu}{k(v_2)} \bar{\mathbf{q}}, \bar{\mathbf{q}} \right) \in \big(r(u_1) - r(u_2), \bar{p} \big) - \big(H(v_1) - H(v_2), \bar{p} \big). \tag{16}$$

Since $||H(v_1) - H(v_2)|| \leq C||\bar{u}|| + D||\bar{v}||$, $r(u)$ and $k(v)$ are Lipschitz and $\mathbf{q}$ is bounded, considering that the function $1/k(v)$ is positively bounded away from zero, we have that

$$||\bar{\mathbf{q}}||^2 \leq C||\bar{u}||\,||\bar{p}|| + D||\bar{v}||\,||\bar{p}|| + E||\bar{v}||\,||\bar{\mathbf{q}}||. \tag{17}$$

Then, thanks to Young inequality and (15), we get:

$$||\bar{\mathbf{q}}||^2 \leq C||\bar{u}||^2 + D\int_0^t ||\bar{u}(s)||^2 ds, \tag{18}$$

and, by substitution into (13),

$$||\bar{u}(t)||^2 \leq C\int_0^t ||\bar{u}(s)||^2 ds + D\int_0^t \big(\int_0^s ||\bar{u}(r)||^2 dr \big) ds. \tag{19}$$

This integral inequality with a double integral implies that [19] $||\bar{u}(t)||^2 = 0$. As a consequence of (9), (18) and (15), we then have: $||\bar{v}(t)|| = 0$, $||\bar{\mathbf{q}}(\mathbf{t})|| = 0$ and $||\bar{p}(t)|| = 0$.

We have thus shown that $(u_2, v_2, \mathbf{q}_2, p_2)$ is equal to $(u_1, v_1, \mathbf{q}_1, p_1)$ almost everywhere, i.e. the solution is unique. □

2.5 Existence of solutions of the coupled problem

The existence of solutions is proved through a Faedo-Galerkin approach using a discretized problem. We write a full discrete approximation of Problem $\mathbf{P}_2$ by a finite difference scheme in time, a dual mixed hybridized finite element discretization in space for the Darcy equation and a primal hybrid finite element discretization in space for the species transport equations. Since we will use event-driven methods [9] that make use of the Filippov prescriptions for the numerical solution of the discrete ODE DRH, we have chosen an explicit Euler time discretization. The Euler semi-implicit method used in [16] (in the case of a given Darcy field) is not feasible in our case, since it updates the v variable at a given time step using a value of the u variable at the next time step, which makes event-localization techniques impracticable. Implicit time advancing schemes that use techniques from fixed point theorems for set valued functions are currently under the investigation of the authors.

Let $\mathcal{T}_h$ be a regular conforming decomposition of Ω into triangles, and let us introduce the following finite element spaces:

$$\mathcal{Z}_h := \{\mathbf{q}_h \in \prod_{K\in\mathcal{T}_h} H(\mathrm{div},\mathrm{K}) | \mathbf{q}_{\mathrm{h}|\mathrm{K}} \in \mathbb{RT}_0(\mathrm{K})\ \forall \mathrm{K} \in \mathcal{T}_\mathrm{h}\},$$
$$\mathcal{V}_h := \{p_h \in L^2(\Omega) | p_{h|K} \in \mathbb{P}_0(K)\ \forall K \in \mathcal{T}_h\},$$
$$\mathcal{P}_h := \{\lambda_h \in \prod_{K\in\mathcal{T}_h} H^{1/2}(\partial K) | \lambda_{h|\partial K} \in \mathbb{P}_0(\partial K)\ \forall K \in \mathcal{T}_h, \lambda_{h|\partial\Omega} = 0\}\},$$
$$\mathcal{W}_h := \{v_h \in \prod_{K\in\mathcal{T}_h} H^1(K) | v_{h|K} \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h\},$$
$$\mathcal{Q}_h := \{\mu_h \in \prod_{K\in\mathcal{T}_h} H^{-1/2}(\partial K) | \mu_{h|\partial K} \in \mathbb{P}_0(\partial K)\ \forall K \in \mathcal{T}_h\},$$

where $\mathbb{P}_i(K)$ indicates the space of polynomials of order i on K, $\mathbb{RT}_0(K)$ is the zero order Raviart-Thomas space, and $\mathbb{P}_0(\partial K)$ is the zero order polynomial on each edge of ∂K. Introducing the local projection operators $\Pi_K : H(\mathrm{div},\mathrm{K}) \to \mathbb{RT}_0(\mathrm{K})$; $P_K^0 : L^2(K) \to \mathbb{P}_0(K)$; $P_K^1 : H^1(K) \to \mathbb{P}_1(K)$; $\rho_K : \prod_{K\in\mathcal{T}_h} L^2(\partial K) \to \mathbb{R}_0(\partial K)$; we have the well known results from interpolation theory [5]:

$$\begin{gathered}
||\boldsymbol{\tau} - \Pi_K\boldsymbol{\tau}||_{0,K} \le Ch_K||\boldsymbol{\tau}||_{1,K}, \quad \text{for} \quad \boldsymbol{\tau} \in \left[\mathrm{H}^1(\mathrm{K})\right]^2, \\
||\mathrm{div}\big(\boldsymbol{\tau} - \Pi_\mathrm{K}\boldsymbol{\tau}\big)||_{0,\mathrm{K}} \le \mathrm{Ch}_\mathrm{K}||\mathrm{div}\boldsymbol{\tau}||_{1,\mathrm{K}}, \quad \text{for} \quad \mathrm{div}\boldsymbol{\tau} \in \mathrm{H}^1(\mathrm{K}), \\
||v - P_K^0 v||_{0,K} \le Ch_K||v||_{1,K}, \quad \text{for} \quad \mathrm{v} \in \mathrm{H}^1(\mathrm{K}), \\
||v - P_K^1 v||_{0,K} \le Ch_K||v||_{1,K}, \quad \text{for} \quad \mathrm{v} \in \mathrm{H}^1(\mathrm{K}), \\
||\lambda - \rho_K\lambda||_{1/2,e_h} \le Ch_K||w||_{2,K}, \quad \text{for} \quad \mathrm{w} \in \mathrm{H}^2(\mathrm{K}) | \mathrm{w}_{|\partial\mathrm{K}} = \lambda, \\
||\lambda - \rho_K\lambda||_{-1/2,e_h} \le Ch_K||\mathrm{div}\mathbf{q}||_{1,\mathrm{K}}, \quad \text{for} \quad \mathbf{q} \in \left[\mathrm{H}^2(\mathrm{K})\right]^2 | \, \mathbf{q}\cdot\mathbf{n}_{|\partial\mathrm{K}} = \lambda,
\end{gathered} \tag{20}$$

where e_h is the set of edges of K. A fully discrete scheme of Problem $\mathbf{P_2}$ is obtained from a Euler explicit time discretization. Starting with $u_h^0 = P_h u_0$ and $v_h^0 = P_h v_0$, where P_h is the global interpolation operator, with $u_0 \in H_0^1(\Omega)$ and $u_0 \in L^2(\Omega)$, given $\tau = T/N$, $N \in \mathbb{N}$, and $t_n = n\tau$, $n = 1, ..., N$, we set:

Problem $\mathbf{P}_2^h$. Find $(\mathbf{q}_h^n, p_h^n, \lambda_h^n) \in \mathcal{Z}_h \times \mathcal{V}_h \times \mathcal{P}_h$ and $(u_h^n, v_h^n, \mu_h^n) \in \mathcal{W}_h \times \mathcal{W}_h \times \mathcal{Q}_h$, given $(u_h^{n-1}, v_h^{n-1}) \in \mathcal{U}_h \times \mathcal{U}_h$, such that for all $(\boldsymbol{\tau}_h, \psi_h, \rho_h) \in \mathcal{Z}_h \times \mathcal{V}_h \times \mathcal{P}_h$ and

$(\omega_h, \theta_h, \nu_h) \in \mathcal{W}_h \times \mathcal{W}_h \times \mathcal{Q}_h$:

$$
\begin{cases}
\sum_{K\in\mathcal{T}_h} \left[\int_K \frac{\mu}{k(v_h^{n-1})} \mathbf{q}_h^n \cdot \boldsymbol{\tau}_h - \int_K p_h^n \mathrm{div} \boldsymbol{\tau}_\mathrm{h} + \int_{\partial \mathrm{K}} \lambda_\mathrm{h}^\mathrm{n} \boldsymbol{\tau}_\mathrm{h} \cdot \mathbf{n} + \int_{\partial \mathrm{K} \cap \Gamma_\mathrm{D}} \mathrm{p_D} \boldsymbol{\tau}_\mathrm{h} \cdot \mathbf{n} \right] = 0, \\
\sum_{K\in\mathcal{T}_h} \left[\int_K \psi_h \mathrm{div} \mathbf{q}_\mathrm{h}^\mathrm{n} \right] = \int_\Omega \big(\mathrm{r}(\mathrm{u}_\mathrm{h}^{\mathrm{n}-1}) - \mathrm{H}(\mathrm{v}_\mathrm{h}^{\mathrm{n}-1}) \big) \psi_\mathrm{h}, \\
\sum_{K\in\mathcal{T}_h} \int_{\partial K} \rho_h \mathbf{q}_h^n \cdot \mathbf{n} = 0, \\
\sum_{K\in\mathcal{T}_h} \left[\int_K (u_h^n - u_h^{n-1}) \omega_h + \tau \int_K \nabla u_h^n \nabla \omega_h - \tau \int_K \mathbf{q}_h^n u_h^n \nabla \omega_h - \tau \int_{\partial K} \mu_h^n \omega_h \right] = \\
-\tau \int_\Omega (r(u_h^{n-1}) - H(v_h^{n-1})) \omega_h, \\
\int_\Omega (v_h^n - v_h^{n-1}) \theta_h \in \tau \int_\Omega (r(u_h^{n-1}) - H(v_h^{n-1})) \theta_h, \\
\sum_{K\in\mathcal{T}_h} \int_{\partial K} u_h^n \nu_h = 0.
\end{cases}
\tag{21}
$$

Remark 4 *Defining the operator* $B := (B\mathbf{q} \cdot \mathbf{n}, \rho) = \sum_K \int_{\partial K} \rho \mathbf{q} \cdot \mathbf{n}, \forall \rho \in \prod_{K\in\mathcal{T}_h} H^{1/2}(\partial K), \rho_{\partial\Omega} = 0$, *we can identify* $H(\mathrm{div}, \Omega) = \ker \mathrm{B}$. *Having introduced a finite element space* $\mathcal{P}_h$ *of functions in* $\mathbb{P}_0(\partial K)$*, which are discontinuous at the edge vertices of* ∂K*, the method is an hybridized dual mixed one, which enforces the local reciprocity constraint [5].*

Analogously, by defining the operator $C := (Cv, \mu) = \sum_K \int_K v\mu, \forall \mu \in \prod_{K\in\mathcal{T}_h} H^{-1/2}(\partial K)$ *we can identify* $H_0^1(\Omega) = \ker C$. *Since* $\mathcal{Q}_{h|K} = \mathbb{P}_0(\partial K)$*, the space* $\ker C_h$ *is given by functions in* $\mathcal{W}_h$ *which are continuous at the middle point of each edge of* ∂K*; the primal hybrid formulation is thus equivalent to a non conforming primal formulation on a Crouzier - Raviart finite element space [5].*

Lemma 2.3 *If* u_h^{n-1} *and* v_h^{n-1} *are non negative, there exists a value* $\bar{\tau}$ *such that* u_h^n *and* v_h^n *are non negative for* $\tau < \bar{\tau}$. *Moreover,* $u_h^n, v_h^n \in L^\infty(K)$. *In particular, if* $u_h^{n-1} \leq u^\star$ *and* $v_h^{n-1} < 1$*, then* $u_h^n \leq u^\star$ *and* $v_h^n < 1$ *almost everywhere in* $L^2(\Omega)$.

Proof. Let us firstly introduce the operator $[\cdot]_{-|K}$, which takes the negative part of its argument inside an element K. A global operator $[\cdot]_-$ can be defined on $\mathcal{W}_h$ so that $[v_h]_-|_K = [v_h]_{-|K}$. Clearly, the image of this operator is not $\mathcal{W}_h$, since $H(-v_{h|K})v_{h|K} \notin \mathbb{P}_1(K)$ (note that this would be the case if $\mathcal{W}_h = \mathbb{P}_0$, but in this case the discrete problem is unstable). To circumvent this difficulty, we assume to have a partition $\mathcal{T}_h$ in which there are no triangles which cross lines (or contain points) where the function v_h^n changes sign, but all the triangles near these lines have one or two vertices on them and are on one side with respect to them. To obtain a regular conforming mesh, the lines on which the solutions change sign must be Lipschitz continuous. Recalling Remark 3, the regions of the domain where $v_h^n, u_h^n, v_h^n - 1, u_h^n - 1$ change sign remain constant in time. Hence we require an initial condition v_h^0 which is positive inside subdomains with Lipschitz continuous boundary. Then $[v_h]_- = v_h$ in the elements contained in the regions where v_h is negative, and is extended to zero in the other elements. This choice implies that

$[v_h]_- \in \mathcal{W}_h$. We can apply the same argument to u_h^n. Note that we cannot exploit a maximum principle, since, for our kind of problem and for the finite element spaces we are using, there are no standard discrete maximum principles.

We now start by taking $\theta_h = [v_h^n]_-$ in the fifth equation of system (21), obtaining:

$$\sum_K ||[v_h^n]_-||_K^2 \in \sum_K (v_h^{n-1}, [v_h^n]_-)_K + \tau \sum_K (r(u_h^{n-1}) - H(v_h^{n-1}), [v_h^n]_-)_K.$$

If $v_h^{n-1} = 0$, thanks to (5), the only value of the set on the right hand side is zero, so $v_h^n = 0$ in $L^2(\Omega)$. If $v_h^{n-1} \neq 0$, by defining

$$\bar{\tau}_1 = \min_K \left\{ \frac{v_h^{n-1}}{|r(u_h^{n-1}) - H(v_h^{n-1})|} \right\},$$

we have that $v_h^n > 0$ for $\tau < \bar{\tau}_1$, since in that case the value on the right hand side would be negative. Note that, if $v_h^{n-1} \ll 1$ and $v_h^{n-1} \neq 0$, and if $r(u_h^{n-1}) \ll 1$, $\bar{\tau}_1 \ll 1$. If $\tau > \bar{\tau}_1$, the solution v_h^n could become negative, but, since in that case the term $r(u_h^n) - H(v_h^n) \geq 0$, at the successive time steps it would eventually increase to the threshold value $v_h = 0$ and continue sliding on it. The problem of negative concentration is avoided by localizing the threshold $v = 0$ with an event driven strategy.

We take $\omega_h = [u_h^n]_-$ and $\nu_h = \mu_h^n$ in the fourth and the sixth equations of system (21), respectively, to obtain:

$$\sum_K \Bigg(||[u_h^n]_-||_K^2 + \tau ||[\nabla u_h^n]_-||_K^2 + \frac{\tau}{2}(\mathrm{div}\mathbf{q}_\mathrm{h}^\mathrm{n}, [\mathrm{u}_\mathrm{h}^\mathrm{n}]_-^2)_\mathrm{K} - \frac{\tau}{2}(\mathbf{q}_\mathrm{h}^\mathrm{n} \cdot \mathbf{n}, [\mathrm{u}_\mathrm{h}^\mathrm{n}]_-^2)_{\partial\mathrm{K}} +,$$
$$\tau (r(u_h^{n-1}) - H(v_h^{n-1}), [u_h^n]_-)_K \Bigg) = \sum_K (u_h^{n-1}, [u_h^n]_-)_K.$$

We show now that the fourth term on the left hand side is equal to zero. First of all, we note that $\mathbf{q}_h^n \cdot \mathbf{n} \in \mathbb{P}_0(\partial K)$, and, thanks to the local reciprocity at the midpoint of each edge of ∂K provided by the hybridization of the dual mixed method, we have that $\mathbf{q}_h^n \cdot \mathbf{n} \in \mathcal{Q}_h$. Now, since $[u_h^n]_- \in \ker C_h$ is continuous at the midpoints of each edge of the triangulation, we have effectively that $\sum_K (\mathbf{q}_h^n \cdot \mathbf{n}, [u_h^n]_-)_K = \sum_K (\mathbf{q}_h^n \cdot \mathbf{n}, [u_h^n]_-^2)_K = 0$. We thus obtain:

$$\sum_K \Bigg(||[u_h^n]_-||_K^2 + \tau ||[\nabla u_h^n]_-||_K^2 + \frac{\tau}{2}(r(u_h^{n-1}) - H(v_h^{n-1}), [u_h^n]_-^2)_K +$$
$$\tau (r(u_h^{n-1}) - H(v_h^{n-1}), [u_h^n]_-)_K \Bigg) = \sum_K (u_h^{n-1}, [u_h^n]_-)_K,$$

where the first two terms are positive. For what concerns the term

$$\frac{\tau}{2}(r(u_h^{n-1}) - H(v_h^{n-1}), [u_h^n]_-^2)_K + \tau (r(u_h^{n-1}) - H(v_h^{n-1}), [u_h^n]_-)_K,$$

we have, if $u_h^{n-1} \leq u^\star$, the following cases:

- if $|u_h^n| < u^\star$, $\frac{1}{2}[u_h^n]_-^2 + [u_h^n]_- < 0$ and $r(u_h^{n-1}) - H(v_h^{n-1}) \leq 0$;
- if $|u_h^n| > u^\star$, we have to bound τ in order to have a positive solution.

Now we show the following: if $u_h^{n-1} \leq u^\star$, than there exists a $\bar{\tau}_2$ such that $u_h^n \leq u^\star$ for $\tau < \bar{\tau}_2$. Take $\omega_h = [u_h^n - u^\star]_+$ and $\nu_h = \mu_h^n$ in the fourth and the sixth equations of the system (21) respectively, where $[\cdot]_+$ is defined analogously to $[\cdot]_-$. We are now in the position of writing that

$$\sum_K \Bigg(||[u_h^n - u^\star]_+||_K^2 + \tau||\nabla[(u_h^n - u^\star)]_+||_K^2 + \frac{\tau}{2}(r(u_h^{n-1}) - H(v_h^{n-1}), [u_h^n - u^\star]_+^2)_K$$
$$- \frac{\tau}{2}(u^\star)^2 \int_{\partial K} \mathbf{q}_h^n \cdot \mathbf{n} + \tau(r(u_h^{n-1}) - H(v_h^{n-1}), [u_h^n - u^\star]_+)_K \Bigg) =$$
$$\sum_K (u_h^{n-1} - u^\star, [u_h^n - u^\star]_+)_K.$$

Since $r(u_h^{n-1}) - H(v_h^{n-1}) \leq 0$, by applying the divergence theorem to the fourth term we can see that, if $u_h^n < u^\star - 1 + \sqrt{1 + (u^\star)^2}$, the sum of the third, the fourth and the fifth terms in the left hand side is positive, thus $u_h^n \leq u^\star$. If $u_h^n < 2u^\star$, the sum of the third and the fourth terms in the left hand side is positive. Since both the fifth term on the left hand side and the term on the right hand side are negative, there exists a bound $\bar{\tau}_2$ for which $u_h^n \leq u^\star$ if $\tau < \bar{\tau}_2$. For example, when $u^\star = 1$ and $r(u_h^{n-1}) := u_h^{n-1}$, we have $\bar{\tau}_2 = 1$ if $u_h^n < 2u^\star$, and $\bar{\tau}_2 = 2/3$ if $u_h^n > 2u^\star$, hence we choose $\bar{\tau}_2 = 2/3$; when $u^\star = 1$ and $r(u_h^{n-1}) := (u_h^{n-1})^2$, we have $\bar{\tau}_2 = \dfrac{1}{2}$ if $u_h^n < 2u^\star$, $\bar{\tau}_2 = \dfrac{1}{3}$ if $u_h^n > 2u^\star$, hence we choose $\bar{\tau}_2 = \dfrac{1}{3}$. By choosing $\bar{\tau} = \min[\bar{\tau}_1, \bar{\tau}_2]$, we obtain the thesis. When $u_h^{n-1} \leq u^\star$ and $v_h^{n-1} < 1$, from the fifth equation of system (21) we get $v_h^n < 1$, since $r(u_h^{n-1}) - H(v_h^{n-1}) < 0$. □

Existence and uniqueness of the solution to **Problem $\mathbf{P}_2^h$** derive from the following facts:

- The term $\mu/K(v_h^{n-1})$ is bounded and always positive, the quadratic form $\left(\dfrac{\mu}{K(v_h^{n-1})}\mathbf{q}_h^n, \mathbf{q}_h^n\right)_K$ is continuous and coercive over $\ker B_h$, the finite dimensional spaces satisfy the discrete inf-sup condition and the forcing terms are well defined. Therefore the mixed hybridized formulation of the Darcy equations has a unique solution [5].

- The bilinear form associated to the transport equation for u_h is weakly coercive over $\ker C_h$, the finite dimensional spaces satisfy the discrete inf-sup condition and the forcing term is well defined. Therefore, the primal hybrid formulation for u_h^n has a unique solution [5].

- The application of the Filippov selection procedure ensures that there exists a unique sequence of solutions v_h^n, which converges, for $N \to \infty$, to the unique solution of the continuous in time inclusion problem uniformly in $C^0([0,T], \mathbb{R}^2)$ [11].

We proceed now to obtain energy estimates, which will be used later to show the convergence of a time continuous approximation of the discrete solutions to the weak solution of the continuous problem.

Lemma 2.4 (Energy estimates) *There exist constants $C > 0$ independent of τ and h such that the following estimates hold:*

$$\sup_{k=1,\dots,N} ||v_h^k - v_h^{k-1}||_K + \sum_{n=1}^{N} ||u_h^n - u_h^{n-1}||_K^2 \le C\tau, \tag{22}$$

$$\sum_{n=1}^{N} ||\nabla(u_h^n - u_h^{n-1})||_K^2 + \tau \sum_{n=1}^{N} ||\nabla u_h^n||_K^2 \le C, \tag{23}$$

$$\sup_{k=1,\dots,N} ||\mathbf{q}_h^k||_K \le C, \tag{24}$$

$$\sup_{k=1,\dots,N} ||p_h^k||_K + \sup_{k=1,\dots,N} ||\lambda_h^k||_{\partial K} \le C, \tag{25}$$

$$\sum_{n=1}^{k} ||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||_K^2 \le C\tau, \tag{26}$$

$$\sum_{n=1}^{N} ||p_h^n - p_h^{n-1}||_K^2 + \sum_{n=1}^{N} ||\lambda_h^n - \lambda_h^{n-1}||_{\partial K}^2 \le C\tau, \tag{27}$$

$$\tau \sum_{n=1}^{N} ||\mathrm{div}\mathbf{q}_\mathrm{h}^\mathrm{n}||_\mathrm{K}^2 \le \mathrm{C}, \tag{28}$$

$$\sum_{n=1}^{N} ||\mathrm{div}(\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^\mathrm{n-1})||_\mathrm{K}^2 \le \mathrm{C}\tau. \tag{29}$$

Proof. We take $\omega_h = u_h^n$ and $\nu_h = \mu_h^n$ in the fourth and in the sixth equations of system (21), respectively, to have:

$$\sum_K \left\{ \frac{1}{2}\left[||u_h^n||_K^2 - ||u_h^{n-1}||_K^2 + ||u_h^n - u_h^{n-1}||_K^2 \right] + \tau||\nabla u_h^n||_K^2 + \tau(r(u_h^{n-1}) - H(v_h^{n-1}), (u_h^n)^2)_K \right\},$$
$$= -\tau(r(u_h^{n-1}) - H(v_h^{n-1}), u_h^n)_K.$$

Since $r(\cdot)$ is positive, the term $(r(u_h^{n-1}), (u_h^n)^2)$ is positive. Using the boundedness of $H(\cdot)$, the Lipschitz continuity of $r(\cdot)$, the Cauchy-Schwartz and the Young inequalities, we obtain:

$$\frac{1}{2}\big(||u_h^n||_K^2 - ||u_h^{n-1}||_K^2 + ||u_h^n - u_h^{n-1}||_K^2\big) + \tau||\nabla u_h^n||_K^2 \le C\tau||u_h^n||_K^2 + \tau L_r||u_h^{n-1}||_K||u_h^n||_K$$
$$+ C\tau||u_h^n||_K \le C\tau||u_h^n||_K^2 + C\tau||u_h^{n-1}||_K^2 + C\tau + \frac{1}{2}\tau||u_h^n||_K^2,$$

and, by summing over $n = 1, \dots, k$, for an arbitrary $k \le N$,

$$\frac{1}{2}||u_h^k||_K^2 + \frac{1}{2}\sum_{n=1}^{k} ||u_h^n - u_h^{n-1}||_K^2 + \tau \sum_{n=1}^{k} ||\nabla u_h^n||_K^2 \le \frac{1}{2}||u_h^I||_K^2 + C.$$

Here we have used the fact that $u_h^n \in L^\infty(K) \subset L^2(K)$. This result implies the estimate in the second part of (23).

Remark 5 *The same result could be obtained by using the discrete Gronwall inequality, in this case we do not need $u_h^n \in L^\infty(K)$. Moreover, we could obtain the same result without taking an integration by parts of the advection term in the fourth equation of system* (21)*, by using the fact that $\mathbf{q}_h^n \in [L^\infty(K)]^2$, or using the estimate of $||\mathbf{q}_h^n||$ dependent on $||u_h^n||$ given by* (30)*, and applying the discrete Gronwall inequality.*

We take now $\theta_h = v_h^n - v_h^{n-1}$ in the fifth equation of system (21), consequently the first part of (22) gives:

$$||v_h^n - v_h^{n-1}||_K^2 \leq \tau ||r(u^\star)||_K \, ||v_h^n - v_h^{n-1}||_K + \tau C ||v_h^n - v_h^{n-1}||_K,$$

thanks to Schwartz inequality. To obtain (24) and (25), we take $\boldsymbol{\tau}_h = \Pi_K \nabla \eta_h$ in the first equation of system (21), where η_h is a solution of

$$\begin{cases} -\mathrm{div} \nabla \eta_\mathrm{h} = \mathrm{p}_\mathrm{h}^\mathrm{n} \ \text{ in } \ \mathrm{K} \\ \nabla \eta_h \cdot \mathbf{n}|_{\partial K} = \lambda_h^n. \end{cases}$$

Note that $\mathrm{div}[\Pi_\mathrm{K} \nabla \eta_\mathrm{h}] = \mathrm{P}_{\mathrm{k},0} \mathrm{div} \nabla \eta_\mathrm{h} = -\mathrm{P}_{\mathrm{k},0} \mathrm{p}_\mathrm{h}^\mathrm{n} = -\mathrm{p}_\mathrm{h}^\mathrm{n}$, and that $\Pi_K \nabla \eta_h \in \mathbb{RT}_0(K)$. Besides, $||\Pi_K \nabla \eta_h||_K \leq C ||p_h^n||_k$. Then:

$$\begin{aligned} &||p_h^n||_K^2 + ||\lambda_h^n||_{\partial K}^2 = -(p_D, \lambda_h^n)_{\partial K \cap \Gamma_D} - \left(\frac{\mu}{k(v_h^{n-1})} \mathbf{q}_h^n, \Pi_K \nabla \eta_h \right)_K = \\ &-(\tilde{p}_D, p_h^n)_K - (\nabla \tilde{p}_D, \Pi_K \nabla \eta_h)_K - \left(\frac{\mu}{k(v_h^{n-1})} \mathbf{q}_h^n, \Pi_K \nabla \eta_h \right)_K \leq \\ &\frac{1}{4} ||p_h^n||_K^2 + C + C \sup_K \left[\frac{\mu}{k(v_h^{n-1})} \right] ||\mathbf{q}_h^n|| \, ||p_h^n||_K \leq \frac{1}{2} ||p_h^n||_K^2 + C + C ||\mathbf{q}_h^n||^2, \end{aligned}$$

where $\tilde{p}_D$ is an harmonic lifting of the boundary data. Hence, we may write:

$$||p_h^n||_K^2 + ||\lambda_h^n||_{\partial K}^2 \leq C(1 + ||\mathbf{q}_h^n||^2).$$

Now, let us take $\boldsymbol{\tau}_h = \mathbf{q}_h^n$, $\psi_h = p_h^n$ and $\rho_h = \lambda_h^n$ in the first, the second and the third equations of system (21), respectively. We obtain:

$$\left(\frac{\mu}{k(v_h^{n-1})} \mathbf{q}_h^n, \mathbf{q}_h^n \right)_K - (r(u_h^{n-1}) - H(v_h^{n-1}), p_h^n)_K = -(\tilde{p}_D, r(u_h^{n-1}) - H(v_h^{n-1}))_K - (\nabla \tilde{p}_D, \mathbf{q}_h^n)_K \tag{30}$$

Since $1/k(v_h^{n-1})$ is positive and bounded away from zero, we can write:

$$||\mathbf{q}_h^n||_K^2 \leq \frac{1}{4C_p} ||p_h^n||_K^2 + C + \frac{1}{4} ||\mathbf{q}_h^n||_K^2 \leq \frac{1}{2} ||\mathbf{q}_h^n||_K^2 + C,$$

where C_p is the constant C in the inequality $||p_h^n||_K^2 \leq D + C||\mathbf{q}_h^n||^2$. We thus obtain estimates (24) and (25).

If we take now $\psi_h = \mathrm{div} \mathbf{q}_\mathrm{h}^\mathrm{n}$ in the second equation of system (21), we get:

$$\tau ||\mathrm{div} \mathbf{q}_\mathrm{h}^\mathrm{n}||_\mathrm{K}^2 \leq \tau ||\mathrm{r}(\mathrm{u}_\mathrm{h}^{\mathrm{n}-1}) - \mathrm{H}(\mathrm{v}_\mathrm{h}^{\mathrm{n}-1})||_\mathrm{K} ||\mathrm{div} \mathbf{q}_\mathrm{h}^\mathrm{n}||_\mathrm{K} \leq \mathrm{C}\tau + \frac{1}{2} \tau ||\mathrm{div} \mathbf{q}_\mathrm{h}^\mathrm{n}||_\mathrm{K}^2.$$

and, by summing over $n = 1, \ldots, k$, for a $k \leq N$, we are able to obtain (28).

Taking $\omega_h = u_h^n - u_h^{n-1}$ and $\nu_h = \mu_h^n$ in the fourth and in the sixth equations of system (21), respectively, allows us to write that

$$||u_h^n - u_h^{n-1}||_K^2 + \tau(\nabla u_h^n, \nabla(u_h^n - u_h^{n-1}))_K - \tau(\mathbf{q}_h^n u_h^n, \nabla(u_h^n - u_h^{n-1}))_K = -(v_h^n - v_h^{n-1}, u_h^n - u_h^{n-1})_K.$$

Then, by applying integration by parts to the term $(\mathbf{q}_h^n u_h^n, \nabla(u_h^n - u_h^{n-1}))_K$, using the Cauchy and Young inequalities, Lemma (2.3), equations (22), (24) and (28) and the fact that $\nabla u_h^n \in L^\infty(K)$ (since $u_h^n \in L^\infty(K) \cap \mathbb{P}_1(K)$), we get:

$$\begin{aligned}
&||u_h^n - u_h^{n-1}||_K^2 + \frac{1}{2}\tau\big(||\nabla u_h^n||_K^2 - ||\nabla u_h^{n-1}||_K^2 + ||\nabla(u_h^n - u_h^{n-1})||_K^2\big) \\
&= -\tau(u_h^n \mathrm{div}\mathbf{q}_\mathrm{h}^\mathrm{n}, (\mathrm{u}_\mathrm{h}^\mathrm{n} - \mathrm{u}_\mathrm{h}^{\mathrm{n}-1}))_\mathrm{K} - \tau(\mathbf{q}_\mathrm{h}^\mathrm{n} \nabla \mathrm{u}_\mathrm{h}^\mathrm{n}, (\mathrm{u}_\mathrm{h}^\mathrm{n} - \mathrm{u}_\mathrm{h}^{\mathrm{n}-1}))_\mathrm{K} - (\mathrm{v}_\mathrm{h}^\mathrm{n} - \mathrm{v}_\mathrm{h}^{\mathrm{n}-1}, \mathrm{u}_\mathrm{h}^\mathrm{n} - \mathrm{u}_\mathrm{h}^{\mathrm{n}-1})_\mathrm{K} \\
&\le C\tau^2 + \frac{1}{6}||u_h^n - u_h^{n-1}||_K^2 + C\tau^2 + \frac{1}{6}||u_h^n - u_h^{n-1}||_K^2 + C\tau^2 + \frac{1}{6}||u_h^n - u_h^{n-1}||_K^2,
\end{aligned}$$

By which we can write:

$$\frac{1}{2}\sum_{n=1}^{k} ||u_h^n - u_h^{n-1}||_K^2 + \frac{1}{2}\tau||\nabla u_h^k||_K^2 + \frac{1}{2}\tau\sum_{n=1}^{k} ||\nabla u_h^n - \nabla u_h^{n-1}||_K^2 \le ||\nabla u_h^0||^2\tau + C\tau,$$

and obtain the second part of (22), and the first part of (23).

We now set $\psi_h = \mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]$ and take the difference between the equations written at time n and $n-1$. Thanks to (12) and the Young and Schwartz inequalities, we obtain:

$$\tau||\mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]||_\mathrm{K}^2 \le \mathrm{C}\tau||\mathrm{u}_\mathrm{h}^{\mathrm{n}-1} - \mathrm{u}_\mathrm{h}^{\mathrm{n}-2}||_\mathrm{K}^2 + \frac{1}{4}\tau||\mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]||_\mathrm{K}^2 + \mathrm{D}\tau||\mathrm{v}_\mathrm{h}^{\mathrm{n}-1} - \mathrm{v}_\mathrm{h}^{\mathrm{n}-2}||_\mathrm{K}^2 + \frac{1}{4}\tau||\mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]||_\mathrm{K}^2. \quad (31)$$

Estimate (29) is thus obtained by summing over $n = 1, \ldots, k$, for a $k \le N$, choosing $u_h^{-1} = u_h^0$, $v_h^{-1} = v_h^0$, and using (22).

We proceed by taking $\boldsymbol{\tau}_h = \mathbf{q}_h^n - \mathbf{q}_h^{n-1}$ and $\rho_h = \lambda_h^n$ in the first and the third equations of system (21), respectively, and by taking the difference between the equations written at times n and $(n-1)$:

$$\tau\left(\frac{\mu}{k(v_h^{n-2})}[\mathbf{q}_h^n - \mathbf{q}_h^{n-1}], [\mathbf{q}_h^n - \mathbf{q}_h^{n-1}]\right)_K = -\tau\left(\left[\frac{\mu}{k(v_h^{n-1})} - \frac{\mu}{k(v_h^{n-2})}\right]\mathbf{q}_h^n, [\mathbf{q}_h^n - \mathbf{q}_h^{n-1}]\right)_K + \tau(p_h^n - p_h^{n-1}, \mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}])_\mathrm{K}. \quad (32)$$

We estimate the first term on the right hand side using the fact that $\mathbf{q}_h^n \in [L^\infty(K)]^2$ and that the function $[k(v_h)]^{-1}$ is Lipschitz continuous for $v_h \in [0, 1)$. The application of Young inequality and (22) gives:

$$\tau\left(\left[\frac{\mu}{k(v_h^{n-1})} - \frac{\mu}{k(v_h^{n-2})}\right]\mathbf{q}_h^n, [\mathbf{q}_h^n - \mathbf{q}_h^{n-1}]\right)_K \le \frac{1}{2}\tau||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||^2 + C\tau^3.$$

Hence, since $[k(v_h^{n-2})]^{-1}$ is positive and bounded away from zero, and thanks to (31), we get from (32) that

$$\frac{1}{2}\tau||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||_K^2 \le C\tau^3 + D\tau||\mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]||_\mathrm{K}||\mathrm{p}_\mathrm{h}^\mathrm{n} - \mathrm{p}_\mathrm{h}^{\mathrm{n}-1}||_\mathrm{K} \le \mathrm{C}\tau^3 + \mathrm{D}\tau^2||\mathrm{p}_\mathrm{h}^\mathrm{n} - \mathrm{p}_\mathrm{h}^{\mathrm{n}-1}||_\mathrm{K}.$$

Summing over $n = 1, \ldots, k$, for a $k \le N$, it is now possible to show that

$$\tau \sum_{n=1}^{k} ||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||_K^2 \le C\tau^2 + D\tau^2 \sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K. \tag{33}$$

Now, we set $\boldsymbol{\tau}_h = \Pi_K \nabla \eta_h$ in the first equation of system (21), where

$$\begin{cases} -\mathrm{div}\nabla\eta_\mathrm{h} = \mathrm{p}_\mathrm{h}^\mathrm{n} - \mathrm{p}_\mathrm{h}^{\mathrm{n}-1} \ \text{ in } \ \mathrm{K} \\ \nabla\eta_h \cdot \mathbf{n}|_{\partial K} = \lambda_h^n - \lambda_h^{n-1} \end{cases}$$

and take the difference between the equations written at times n and $(n-1)$:

$$\begin{aligned} &||p_h^n - p_h^{n-1}||_K^2 + ||\lambda_h^n - \lambda_h^{n-1}||_K^2 = \\ &\left(\left[\frac{\mu}{k(v_h^{n-1})} - \frac{\mu}{k(v_h^{n-2})}\right]\mathbf{q}_h^n, \Pi_K \nabla\eta_h\right)_K + \left(\frac{\mu}{k(v_h^{n-2})}[\mathbf{q}_h^n - \mathbf{q}_h^{n-1}], \Pi_K\nabla\eta_h\right)_K. \end{aligned} \tag{34}$$

The first term on the right hand side can be bounded using a trilinear Holder inequality, (or using the fact that $\mathbf{q}_h^n \in [L^\infty(K)]^2$), noting that, at the discrete level, $||\nabla \mathbf{q}_h^n||_K = ||\mathrm{div}\mathbf{q}_\mathrm{h}^\mathrm{n}||_\mathrm{K}$, since $\mathbf{q}_h^n \in \mathbb{RT}_0(K)$. Hence:

$$\begin{aligned} &\left(\left[\frac{\mu}{k(v_h^{n-1})} - \frac{\mu}{k(v_h^{n-2})}\right]\mathbf{q}_h^n, \Pi_K \nabla\eta_h\right)_K \le C\tau ||\Pi_K \nabla \eta_h||_{[H^1(K)]^2} ||\mathbf{q}_h^n||_{[H^1(K)]^2} \\ &\le \frac{1}{4}||p_h^n - p_h^{n-1}||_K^2 + C\tau^2 + D\tau^2 ||\mathrm{div}\mathbf{q}_\mathrm{h}^\mathrm{n}||_\mathrm{K}^2. \end{aligned}$$

The second term in the right hand side can be bounded using the Cauchy-Schwartz inequality and Lemma 2.3:

$$\left(\frac{\mu}{k(v_h^{n-2})}[\mathbf{q}_h^n - \mathbf{q}_h^{n-1}], \Pi_K\nabla\eta_h\right)_K \le \frac{1}{4}||p_h^n - p_h^{n-1}||_K^2 + C||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||_K^2. \tag{35}$$

Thanks to equation (28), we may write:

$$\frac{1}{2}\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K^2 + \sum_{n=1}^{k} ||\lambda_h^n - \lambda_h^{n-1}||_{\partial K}^2 \le C\tau + D\sum_{n=1}^{k} ||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||_K^2, \tag{36}$$

and, by substituting (33) into (36), we get:

$$\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K^2 \le C\tau + D\tau \sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K. \tag{37}$$

This inequality can be refined starting from the following identity:

$$\left(\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K\right)^2 + \sum_{n=1}^{k}\sum_{m>n}^{k} ([||p_h^n - p_h^{n-1}||_K - ||p_h^m - p_h^{m-1}||_K])^2 = k\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K^2,$$

which, substituted into (37), gives:

$$\left(\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K\right)^2 \le \frac{C}{\tau}\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K^2 \le C + D\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K. \tag{38}$$

This quadratic inequality implies that

$$\sum_{n=1}^{k} ||p_h^n - p_h^{n-1}||_K \leq C. \tag{39}$$

Using (39) in (33), we get estimate (26), while using (26) in (36), we get estimate (27). $\square$

We now associate to the sequence of discrete solutions $(\mathbf{q}_h^n, p_h^n, \lambda_h^n, u_h^n, v_h^n)$ of Problem $\mathbf{P}_2^h$ the following time continuous approximation:

$$\begin{aligned}
&\mathbf{Q}_h^\tau(t) := \mathbf{q}_h^n \frac{t-t_{n-1}}{\tau} + \mathbf{q}_h^{n-1}\frac{t_n - t}{\tau}, \quad P_h^\tau(t) = p_h^n \frac{t-t_{n-1}}{\tau} + p_h^{n-1}\frac{t_n - t}{\tau}, \\
&\Lambda_h^\tau(t) = \lambda_h^n \frac{t-t_{n-1}}{\tau} + \lambda_h^{n-1}\frac{t_n - t}{\tau}, \\
&U_h^\tau(t) := u_h^n \frac{t-t_{n-1}}{\tau} + u_h^{n-1}\frac{t_n - t}{\tau}, \quad V_h^\tau(t) := v_h^n \frac{t-t_{n-1}}{\tau} + v_h^{n-1}\frac{t_n - t}{\tau},
\end{aligned} \tag{40}$$

for $t \in [t_{n-1}, t_n]$, $n = 1, \cdots, N$. They are a family of linear time interpolants that depend on the parameters h and τ.

To simplify the equations we introduce the following notations for functions f and g: $(f,g)_A^T = \int_0^T (f(t), g(t))_A dt$ for a given $L^2(A)$ product $(f,g)_A$, omitting A if $A = \Omega$. It effectively indicates the $L^2(A \times (0,T))$ product. While $(f,g)_A^{t_n} = \int_{t_{n-1}}^{t_n} (f(t), g(t))_A dt$ is used to indicate the $L^2(A \times (t_{n-1}, t_n))$ product.

We consider system (21), by multiplying it by a $C^1([0,T])$ function which is zero at T and integrating in time from 0 to T, we obtain that $(\mathbf{Q}_h^\tau, P_h^\tau, \Lambda_h^\tau, U_h^\tau, V_h^\tau)$ satisfy the following weak formulation:

For any $\boldsymbol{\tau} \in L^2(0,T;\mathcal{Z}), \psi \in L^2(0,T;\mathcal{V}), \rho \in L^2(0,T;\mathcal{P}), \omega \in L^2(0,T;H_0^1(\Omega))$, $\theta \in L^2(0,T;L^2(\Omega))$, given $\boldsymbol{\tau}_h = \Pi_K \tau$, $\psi_h = P_{k,0}\psi$, $\rho_h = \rho_K \rho$, $\omega_h = P_{k,1}\omega$, $\theta_h = P_{k,1}\theta$:

$$
\begin{cases}
(\frac{\mu}{k(V_h^\tau)}\mathbf{Q}_h^\tau, \boldsymbol{\tau})_K^T - (P_h^\tau, \mathrm{div}\boldsymbol{\tau})_{\mathrm{K}}^{\mathrm{T}} + (\Lambda_{\mathrm{h}}^\tau, \boldsymbol{\tau}\cdot\mathbf{n})_{\partial \mathrm{K}}^{\mathrm{T}} + (\mathrm{p_D}, \boldsymbol{\tau}\cdot\mathbf{n})_{\partial\mathrm{K}\cap\Gamma_{\mathrm{D}}}^{\mathrm{T}} = \\
\sum_{n=1}^N \Bigg[(\frac{\mu}{k(V_h^\tau)}\mathbf{Q}_h^\tau, [\boldsymbol{\tau}-\boldsymbol{\tau}_h])_K^{t_n} + ([\frac{\mu}{k(V_h^\tau)} - \frac{\mu}{k(v_h^n)}]\mathbf{Q}_h^\tau, \boldsymbol{\tau}_h)_K^{t_n} + (\frac{\mu}{k(V_h^\tau)}[\mathbf{Q}_h^\tau - \mathbf{q}_h^n], \boldsymbol{\tau}_h)_K^{t_n} \\
-([\frac{\mu}{k(V_h^\tau)} - \frac{\mu}{k(v_h^n)}][\mathbf{Q}_h^\tau - \mathbf{q}_h^n], \boldsymbol{\tau}_h)_K^{t_n} - (P_h^\tau, \mathrm{div}[\boldsymbol{\tau}-\boldsymbol{\tau}_{\mathrm{h}}])_{\mathrm{K}}^{\mathrm{t_n}} - ([\mathrm{P}_{\mathrm{h}}^\tau - \mathrm{p}_{\mathrm{h}}^{\mathrm{n}}], \mathrm{div}\boldsymbol{\tau}_{\mathrm{h}})_{\mathrm{K}}^{\mathrm{t_n}} \\
+ \sum_{n=1}^N (\Lambda_h^\tau, [\boldsymbol{\tau}-\boldsymbol{\tau}_h]\cdot\mathbf{n})_{\partial K}^{t_n} + \sum_{n=1}^N ([\Lambda_h^\tau - \lambda_h^n], \boldsymbol{\tau}_h\cdot\mathbf{n})_{\partial K}^{t_n} + \sum_{n=1}^N (p_D, [\boldsymbol{\tau}-\boldsymbol{\tau}_h]\cdot\mathbf{n})_{\partial K\cap\Gamma_D}^{t_n} \Bigg], \\
(\mathrm{div}\mathbf{Q}_{\mathrm{h}}^\tau, \psi)_{\mathrm{K}}^{\mathrm{T}} - (\partial_{\mathrm{t}} \mathrm{V}_{\mathrm{h}}^\tau, \psi)_{\mathrm{K}}^{\mathrm{T}} = \sum_{\mathrm{n}=1}^{\mathrm{N}} \Bigg[(\mathrm{div}\mathbf{Q}_{\mathrm{h}}^\tau, [\psi - \psi_{\mathrm{h}}])_{\mathrm{K}}^{\mathrm{t_n}} + (\mathrm{div}[\mathbf{Q}_{\mathrm{h}}^\tau - \mathbf{q}_{\mathrm{h}}^{\mathrm{n}}], \psi_{\mathrm{h}})_{\mathrm{K}}^{\mathrm{t_n}} \\
-(\partial_t V_h^\tau, [\psi - \psi_h])_K^{t_n} \Bigg], \\
(\mathbf{Q}_h^\tau\cdot\mathbf{n}, \rho)_{\partial K}^T = \sum_{n=1}^N \Bigg[(\mathbf{Q}_h^\tau\cdot\mathbf{n}, [\rho-\rho_h])_{\partial K}^{t_n} + \int_{t_{n-1}}^{t_n} ([\mathbf{Q}_h^\tau - \mathbf{q}_h^n]\cdot\mathbf{n}, \rho_h)_{\partial K}^{t_n} \Bigg], \\
(\partial_t U_h^\tau, \omega)^T + (\nabla U_h^\tau, \nabla\omega)^T - (\mathbf{Q}_h^\tau U_h^\tau, \nabla\omega)^T + (\partial_t V_h^\tau, \omega)^T = \\
\sum_{n=1}^N \Bigg[(\partial_t U_h^\tau, [\omega-\omega_h])^{t_n} + \int_{t_{n-1}}^{t_n} (\partial_t V_h^\tau, [\omega-\omega_h])^{t_n} + \int_{t_{n-1}}^{t_n} (\nabla U_h^\tau, [\nabla\omega - \nabla\omega_h])^{t_n} \\
+([\nabla U_h^\tau - \nabla u_h^n], \nabla\omega_h)^{t_n} - (\mathbf{Q}_h^\tau U_h^\tau, [\nabla\omega-\nabla\omega_h])^{t_n} - ([\mathbf{Q}_h^\tau - \mathbf{q}_h^n] U_h^\tau, \nabla\omega_h)^{t_n} \\
-(\mathbf{Q}_h^\tau [U_h^\tau - u_h^n], \nabla\omega_h)^{t_n} + ([\mathbf{Q}_h^\tau - \mathbf{q}_h^n][U_h^\tau - u_h^n], \nabla\omega_h)^{t_n} \Bigg], \\
(\partial_t V_h^\tau, \theta)^T \in (r(U_h^\tau) - H(V_h^\tau), \theta)^T + \sum_{n=1}^N \Bigg[(\partial_t V_h^\tau, [\theta-\theta_h])^{t_n} - (r(U_h^\tau), [\theta-\theta_h])^{t_n} \\
-([r(U_h^\tau) - r(u_h^{n-1})], \theta_h)^{t_n} + (H(V_h^\tau), [\theta-\theta_h])^{t_n} + ([H(V_h^\tau) - H(v_h^{n-1})], \theta_h)^{t_n} \Bigg].
\end{cases}
\tag{41}
$$

Note that, since $\mathcal{W}_h \in \ker C_h$, the terms in the species transport equations corresponding to the hybrid variables at inter-element variables can be eliminated, and the equations for (U_h^τ, V_h^τ) correspond to a primal formulation eventually converging to the continuous weak formulation (7) of problem P_2. The equation for $(\mathbf{Q}_h^\tau, P_h^\tau, \Lambda_h^\tau)$ is an hybrid formulation on each element K, which eventually converges to a continuous dual mixed hybrid formulation, which is equivalent to the dual mixed formulation (7) of problem P_2, since the continuous and the discrete quadratic forms are continuous and coercive over $\ker B$ and $\ker B_h$, and the continuous and the discrete inf-sup conditions are satisfied.

In order to pass to the limit in (41) for $h, \tau \to 0$ and to identify the system that the limit points satisfy, we need the following estimates and convergence results.

Lemma 2.5 *The continuous interpolants satisfy:*

$$\mathbf{Q}_h^\tau \in L^2(0,T;H(\mathrm{div},\mathrm{K})) \cap \mathrm{L}^\infty(0,\mathrm{T};\mathrm{L}^2(\mathrm{K})), \tag{42}$$
$$P_h^\tau \in L^\infty(0,T;L^2(K)), \tag{43}$$
$$\Lambda_h^\tau \in L^\infty(0,T;L^2(\partial K)), \tag{44}$$
$$U_h^\tau \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \tag{45}$$
$$\partial_t U_h^\tau \in L^2(0,T;L^2(\Omega)), \tag{46}$$
$$V_h^\tau \in L^\infty(0,T;L^2(\Omega)), \tag{47}$$
$$\partial_t V_h^\tau \in L^2(0,T;L^2(\Omega)). \tag{48}$$

Proof. . Consider the equation

$$||\nabla U_h^\tau||^2 = ||\nabla u_h^{n-1} + \nabla[u_h^n - u_h^{n-1}]\frac{t-t_{n-1}}{\tau}||^2 \leq 2||\nabla u_h^{n-1}||^2 + 2\frac{(t-t_{n-1})^2}{\tau^2}||\nabla[u_h^n - u_h^{n-1}]||^2.$$

We have, by integration it over t and using estimate (23), that:

$$\int_0^T ||\nabla U_h^\tau||^2 dt \leq 2\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\nabla u_h^{n-1}||^2 + 2\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^2}{\tau^2}||\nabla[u_h^n - u_h^{n-1}]||^2$$
$$\leq 2\tau\sum_{n=1}^N ||\nabla u_h^{n-1}||^2 + \frac{2}{3}\tau\sum_{n=1}^N ||\nabla[u_h^n - u_h^{n-1}]||^2 \leq C.$$

From this estimate and Lemma (2.3) we obtain (45) and (47). For what concerns the derivative estimates, we have that

$$\int_0^T ||\partial_t V_h^\tau||^2 dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\frac{v_h^n - v_h^{n-1}}{\tau}||^2 dt \leq \sum_{n=1}^N \tau||\frac{v_h^n - v_h^{n-1}}{\tau}||^2 \leq C,$$

thanks to (22). The estimate for $||\partial_t U_h^\tau||^2_{\Omega^T}$ is obtained similarly. Hence, we can obtain (46) and (48) as well. We consider now the expression

$$||\mathrm{div}\mathbf{Q}_\mathrm{h}^\tau||^2 = ||\mathrm{div}\mathbf{Q}_\mathrm{h}^{\mathrm{n}-1} + \mathrm{div}[\mathbf{Q}_\mathrm{h}^\mathrm{n} - \mathbf{Q}_\mathrm{h}^{\mathrm{n}-1}]\tfrac{\mathrm{t}-\mathrm{t}_{\mathrm{n}-1}}{\tau}||^2 \leq 2||\mathrm{div}\mathbf{Q}_\mathrm{h}^{\mathrm{n}-1}||^2$$
$$+2\tfrac{(t-t_{n-1})^2}{\tau^2}||\mathrm{div}[\mathbf{Q}_\mathrm{h}^\mathrm{n} - \mathbf{Q}_\mathrm{h}^{\mathrm{n}-1}]||^2,$$

which integrated in time, thanks to (28) and (29), provide:

$$\int_0^T ||\mathrm{div}\mathbf{Q}_\mathrm{h}^\tau||^2 \mathrm{dt} \leq \mathrm{C}.$$

From this estimate and (24) and (25) we obtain (42), (43) and (44). □ We are now in the position of deriving the following convergence result.

Lemma 2.6 (Convergence results) *There exists a subsequence of continuous inter-*

polants that, for $(h,\tau)\to 0$, satisfy

$$
\begin{aligned}
&U_h^\tau \rightharpoonup u \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{H}_0^1(\Omega)),\\
&\partial_t U_h^\tau \rightharpoonup \partial_t u \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{-1}(\Omega)),\\
&V_h^\tau \rightharpoonup v \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega)),\\
&\partial_t V_h^\tau \rightharpoonup \partial_t v \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega)),\\
&\mathbf{Q}_h^\tau \rightharpoonup \mathbf{q} \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{H}(\mathrm{div},\Omega)),\\
&p_h^\tau \rightharpoonup p \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega)),\\
&\Lambda_h^\tau \rightharpoonup \lambda \text{ in } \mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{1/2}(\Omega)) \hookrightarrow \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega)).
\end{aligned}
$$

While,

$$U_h^\tau \to u \text{ in } \mathrm{L}^\mathrm{q}(0,\mathrm{T};\mathrm{L}^2(\Omega)), \forall \mathrm{q}\geq 1.$$

Proof. The first set of results derive from (42) -(48), by the application of the Banach-Alaoglu theorem [23]. The last result is obtained thanks to compactness embedding, from the application of the method of the Hilbertian triad [23] and from the Lebesgue dominated convergence theorem. Indeed, thanks to (45) and (46) the set U_h^τ is relatively compact in $L^2(0,T;L^2(\Omega))$ and there exists a subsequence of U_h^τ which converges to the limit point u in $L^q(0,T;L^2(\Omega))$ $\forall q\geq 1$. □

The strong convergence of U_h^τ in $L^2(0,T;L^2(\Omega))$ makes it possible to pass to the limit in the nonlinear term $r(U_h^\tau)$ of equation (41).

Remark 6 *Note that, by estimates* (26) *and* (27)*, and exploiting the fact that translations in space of p_h^n are bounded, we can as well show strong convergence of $\mathbf{Q}_h^\tau$ and p_h^τ. This could in principle allow to extend the result on existence of solutions of problem $\mathbf{P_2}$ also to the case $H = H(v - v*(p))$, with $v*(p)\in[0,1)$ a Lipschitz function of the pressure which is zero for $p\leq 0$. However, we are not addressing here this case, which may be relevant for some applications.*

Note that the family of functions V_h^τ is only weakly convergent to a limit point in $L^2(0,T;L^2(\Omega))$. This is not a problem when passing to the limit in terms like $\int_0^T((H(V_h^\tau),\theta)\to\int_0^T(H(v),\theta)$. This is due to the properties of the multivalued map H. Namely, the set $H(V_h^\tau)$ is bounded and convex, so it is weakly closed. Hence it is weakly compact, and admits a weakly convergent subsequence: $\lim_{(h,\tau)}(H(V_h^\tau),\theta)\to(H(v),\theta)$. Moreover, since V_h^τ is weakly convergent in $L^2(0,T;L^2(\Omega))$ and using the upper semicontinuity and the maximal monotonicity property of the multivalued map H, we have that $\lim_{(h,\tau)}\int_0^T(H(V_h^\tau),\theta)=\int_0^T\lim_{(h,\tau)}(H(V_h^\tau),\theta)$. This would be sufficient if we were solving the problem with a given Darcy flux: solving directly the inclusion problem without regularization avoids the necessity of strong convergence of V_h^τ in $L^2(0,T;L^2(\Omega))$. Since however we are considering the coupling with a Darcy field, in order to pass to the limit in terms containing the non linear permeability factor $[k(V_h^\tau)]^{-1}$ we need strong convergence. We obtain it in the following lemma.

Lemma 2.7 $V_h^\tau\to v$ *in* $L^2(0,T;L^2(\Omega))$.

Proof. We start from the fifth equation of system (21). Since the discrete functions are in $P_1(K)$, taking the test function θ_h equal to the indicator function on a triangle $K \subset \mathcal{T}_h$, we have that on each triangle K:

$$\nabla v_h^n \in \tau \nabla P_{k,1} r(u_h^{n-1}) - \nabla v_h^{n-1} - \tau \nabla P_{k,1} H(v_h^{n-1}). \tag{49}$$

We now use the following result [6]: given a Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ with constant L_g, for any $v \in \mathcal{W}_h$ we have

$$||\nabla P_h g(v)|| \leq L_g ||\nabla v||.$$

This result can be easily generalized to a semicontinuous one-sided Lipschitz continuous function. Recalling the fact that the map $-H(\cdot)$ is OSLC with constant zero, observing that no triangle crosses the surface of discontinuity where $v_h = 0$, and using (5), we obtain

$$||\nabla P_h H(v)|| \leq L_g ||\nabla u||.$$

and hence, from (49),

$$||\nabla v_h^n||_K - ||\nabla v_h^{n-1}||_K \leq \tau 2 L_r ||\nabla u_h^{n-1}||_K.$$

Summing over $n = 1, \ldots, k$, we obtain:

$$||\nabla v_h^k||_K \leq ||\nabla v_h(0)||_K + \sum_{n=1}^{k} \tau 2 L_r ||\nabla u_h^{n-1}||_K.$$

Recalling (23) and since $v_h(0) \in H^1(K)$, we have:

$$\int_0^T ||\nabla V_h^\tau||^2 dt \leq C,$$

hence, $V_h^\tau \in L^2(0,T;H_0^1(\Omega))$. Now, since $\partial_t V_h^\tau \in L^2(0,T;L^2(\Omega))$, we have by applying the method of the Hilbertian triad [23] that there exists a subsequence V_h^τ which converges strongly to the limit point v in $L^2(0,T;L^2(\Omega))$. $\square$

We finally investigate the limit equations of system (41) for $(\tau, h) \to 0$.

Theorem 2.4 *The limit point $(\mathbf{q}, p, \lambda, u, v)$ is the weak solution of an hybrid formulation of the weak problem P^2, which is equivalent to the weak solution of problem P^2.*

Proof. Let us start from considering the first equation of system (41). The left hand side converges to the limit

$$\int_0^T \left(\frac{\mu}{k(v)} \mathbf{q}, \boldsymbol{\tau} \right)_K dt - \int_0^T (p, \mathrm{div}\boldsymbol{\tau})_\mathrm{K} \mathrm{dt} + \int_0^\mathrm{T} (\lambda, \boldsymbol{\tau} \cdot \mathbf{n})_{\partial\mathrm{K}} + \int_0^\mathrm{T} (\mathrm{p_D}, \boldsymbol{\tau} \cdot \mathbf{n})_{\partial\mathrm{K} \cap \Gamma_\mathrm{D}}.$$

For all but the first term this is a direct consequence of the convergence results (42), (43) and (44). The first term can be rewritten as:

$$\int_0^T \left(\frac{\mu}{k(V_h^\tau)} \mathbf{Q}_h^\tau, \boldsymbol{\tau} \right)_K dt = \int_0^T \left(\frac{\mu}{k(v)} \mathbf{Q}_h^\tau, \boldsymbol{\tau} \right)_K dt + \int_0^T \left(\left[\frac{\mu}{k(V_h^\tau)} - \frac{\mu}{k(v)} \right] \mathbf{Q}_h^\tau, \boldsymbol{\tau} \right)_K dt.$$

Since $V_h^\tau \to v$ strongly and is in $L^\infty(\Omega)$, and since $\mathbf{Q}_h^\tau$ is weakly convergent in $L^2(0,T;L^2(\Omega))$, the term on the right hand side converges to the desired limit. Choosing a test function $\boldsymbol{\tau} \in L^2(0,T;[C_0^\infty(\Omega)]^2)$, we can show that the second term on the right hand side is zero by bounding it using the estimate

$$\int_0^T \left(\left[\frac{\mu}{k(V_h^\tau)} - \frac{\mu}{k(v)} \right] \mathbf{Q}_h^\tau, \boldsymbol{\tau} \right)_K dt \leq \left|\left| \left[\frac{\mu}{k(V_h^\tau)} - \frac{\mu}{k(v)} \right] \mathbf{Q}_h^\tau \right|\right|_{[L^1(\Omega)]^2}$$
$$\leq C||V_h^\tau - v||_{[L^2(\Omega)]^2}||\mathbf{Q}_h^\tau||_{L^2(\Omega)} = 0,$$

where we have used the Lipschitz continuity of the function $[k(\cdot)]^{-1}$ and the fact that $V_h^\tau \to v$ in the $L^2(\Omega)$ norm. Hence, the left hand side of the first equation of system (41) converges in the distributional sense to the continuous hybrid formulation of the first equation of problem P_2.

We now show that the terms in the right hand side of the first equation of system (41) converge to zero for $(h,\tau) \to 0$. Let us denote these terms by the notation $\mathcal{I}_1, \cdots, \mathcal{I}_9$. Considering Lemma (2.3), estimate (42) and the first interpolation estimate of (20), we have that

$$|\mathcal{I}_1| \leq C\left(\int_0^T ||\mathbf{Q}_h^\tau||_K^2 dt\right)^{1/2}\left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\boldsymbol{\tau} - \boldsymbol{\tau}_h||_K^2 dt\right)^{1/2} \leq Ch||\boldsymbol{\tau}||_{L^2(0,T;[H^1(K)]^2)} \to 0.$$

Considering the first estimate in equation (22), the estimate (42), the Lipschitz continuity property of the function $[k(\cdot)]^{-1}$, the facts that $||\mathrm{div}\mathbf{Q}_\mathrm{h}^\tau|| = ||\nabla\mathbf{Q}_\mathrm{h}^\tau||$ and $||\mathrm{div}\boldsymbol{\tau}_\mathrm{h}|| = ||\nabla\boldsymbol{\tau}_\mathrm{h}||$, and applying a trilinear Holder inequality, we may write:

$$|\mathcal{I}_2| \leq C\left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||v_h^n - v_h^{n-1}||_K^2||\mathrm{div}\mathbf{Q}_\mathrm{h}^\tau||_\mathrm{K}^2\mathrm{dt}\right)^{1/2}\left(\sum_{\mathrm{n}=1}^\mathrm{N} \int_{\mathrm{t}_{\mathrm{n}-1}}^{\mathrm{t}_\mathrm{n}} ||\nabla\boldsymbol{\tau}_\mathrm{h}||_\mathrm{K}^2\mathrm{dt}\right)^{1/2}$$
$$\leq C\left(\sum_{n=1}^N \tau^3||\mathrm{div}\mathbf{Q}_\mathrm{h}^\tau||_\mathrm{K}^2\right)^{1/2}\left(\sum_{\mathrm{n}=1}^\mathrm{N} \tau||\nabla\boldsymbol{\tau}_\mathrm{h}||_\mathrm{K}^2\right)^{1/2} \to 0.$$

While, by considering Lemma (2.3) and estimate (26), we get:

$$|\mathcal{I}_3| \leq C\left(\sum_{n=1}^N \tau||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||_K^2\right)^{1/2}\left(\sum_{n=1}^N \tau||\boldsymbol{\tau}_h||_K^2\right)^{1/2} \to 0.$$

Considering the first estimate in equation (22), estimate (29), and applying a trilinear Holder inequality, we are able to write:

$$|\mathcal{I}_4| \leq C\left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||v_h^n - v_h^{n-1}||_K^2||\mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]||_\mathrm{K}^2\mathrm{dt}\right)^{1/2}\left(\sum_{\mathrm{n}=1}^\mathrm{N} \int_{\mathrm{t}_{\mathrm{n}-1}}^{\mathrm{t}_\mathrm{n}} ||\nabla\boldsymbol{\tau}_\mathrm{h}||_\mathrm{K}^2\mathrm{dt}\right)^{1/2}$$
$$\leq C\left(\sum_{n=1}^N \tau^3||\mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]||_\mathrm{K}^2\right)^{1/2}\left(\sum_{\mathrm{n}=1}^\mathrm{N} \tau||\nabla\boldsymbol{\tau}_\mathrm{h}||_\mathrm{K}^2\right)^{1/2} \to 0.$$

Considering (25) and the second interpolation estimate in (20), we get:

$$|\mathcal{I}_5| \leq C\left(\int_0^T ||P_h^\tau||_K^2 dt\right)^{1/2}\left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\mathrm{div}[\boldsymbol{\tau} - \boldsymbol{\tau}_\mathrm{h}]||_\mathrm{K}^2\mathrm{dt}\right)^{1/2} \leq \mathrm{Ch}||\boldsymbol{\tau}||_{\mathrm{L}^2(0,\mathrm{T};[\mathrm{H}^2(\mathrm{K})]^2)} \to 0.$$

Note that for this estimate we need to restrict the test function to be in $L^2(0,T;[H^2(K)]^2)$. Density arguments ensure that the limit points satisfy the continuous weak formulation also for

$\boldsymbol{\tau} \in L^2(0,T;[H^1(K)]^2)$. Thanks to the first estimate in (27), we have:

$$|\mathcal{I}_6| \leq C\bigg(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} ||p_h^n - p_h^{n-1}||_K^2 dt\bigg)^{1/2}\bigg(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} ||\mathrm{div}\boldsymbol{\tau}_\mathrm{h}||_\mathrm{K}^2 \mathrm{dt}\bigg)^{1/2}$$
$$\leq C\bigg(\sum_{n=1}^{N}\tau||p_h^n - p_h^{n-1}||_K^2\bigg)^{1/2}\bigg(\sum_{n=1}^{N}\tau||\mathrm{div}\boldsymbol{\tau}_\mathrm{h}||_\mathrm{K}^2\bigg)^{1/2} \to 0.$$

Considering the second estimate in equation (26) and the sixth interpolation estimate of (20) we obtain in a similar manner that $|\mathcal{I}_7| \to 0$, $|\mathcal{I}_8| \to 0$ and $|\mathcal{I}_9| \to 0$.

Let us consider now the second equation of system (41). The left hand side converges to:

$$\int_0^T (\mathrm{div}\mathbf{q},\psi)_\mathrm{K}\mathrm{dt} - \int_0^\mathrm{T} (\partial_\mathrm{t}\mathrm{v},\psi)_\mathrm{K}\mathrm{dt},$$

as a direct consequence of the convergence results in (42) and (48). We now show that the terms in the right hand side converge to zero for $(h,\tau) \to 0$. Let us denote these terms by the notation $\mathcal{I}_1,\cdots,\mathcal{I}_3$. Using (42) and the third interpolation estimate of (20), we have

$$|\mathcal{I}_1| \leq \bigg(\int_0^T ||\mathrm{div}\mathbf{Q}_\mathrm{h}^\tau||_\mathrm{K}^2\mathrm{dt}\bigg)^{1/2}\bigg(\sum_{\mathrm{n}=1}^{\mathrm{N}}\int_{\mathrm{t}_{\mathrm{n}-1}}^{\mathrm{t}_\mathrm{n}} ||\psi - \psi_\mathrm{h}||_\mathrm{K}^2\mathrm{dt}\bigg)^{1/2} \leq \mathrm{Ch}||\psi||_{\mathrm{L}^2(0,\mathrm{T};\mathrm{H}^1(\mathrm{K}))} \to 0.$$

Note that we are again restricting the test functions to $L^2(0,T;H^1(K))$; density arguments extend the result to $\psi \in L^2(0,T;L^2(\Omega))$. Estimate (29) allows us to write

$$|\mathcal{I}_2| \leq \bigg(\sum_{n=1}^{N}\tau||\mathrm{div}[\mathbf{q}_\mathrm{h}^\mathrm{n} - \mathbf{q}_\mathrm{h}^{\mathrm{n}-1}]||_\mathrm{K}^2\bigg)^{1/2}\bigg(\sum_{\mathrm{n}=1}^{\mathrm{N}}\tau||\psi_\mathrm{h}||_\mathrm{K}^2\bigg)^{1/2} \to 0.$$

Considering estimate (48) and the third interpolation estimate of (20), we also have that

$$|\mathcal{I}_3| \leq \bigg(\int_0^T ||\partial_t V_h^\tau||_K^2 dt\bigg)^{1/2}\bigg(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} ||\psi - \psi_h||_K^2 dt\bigg)^{1/2} \leq Ch||\psi||_{L^2(0,T;H^1(K))} \to 0.$$

For what concerns the third equation of system (41), the left hand side converges to the limit

$$\int_0^T (\mathbf{q}\cdot\mathbf{n},\rho)_{\partial K} dt.$$

This is a direct consequence of the convergence result in (42). We now show that the terms in the right hand side converge to zero for $(h,\tau) \to 0$. Let us denote these terms by the notation $\mathcal{I}_1, \mathcal{I}_2$. Using estimate (42) and the fifth interpolation estimate of (20) we have that

$$|\mathcal{I}_1| \leq \bigg(\int_0^T ||\mathbf{Q}_h^\tau\cdot\mathbf{n}||_K^2 dt\bigg)^{1/2}\bigg(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} ||\rho - \rho_h||_{\partial K}^2 dt\bigg)^{1/2} \leq Ch||\rho||_{L^2(0,T;H^{3/2}(K))} \to 0.$$

The term $|\mathcal{I}_2| = 0$, since $\rho_h \in \mathcal{P}_h$.

Let us proceed with the fourth equation of system (41). The left hand side converges to the limit:

$$\int_0^T (\partial_t u,\phi)dt + \int_0^T (\nabla u,\nabla\phi)dt - \int_0^T (\mathbf{q}u,\nabla\phi)dt + \int_0^T (\partial_t v,\phi)dt.$$

For all but the third term this is a direct consequence of the convergence results (45), (46) and (48). The third term can be rewritten as:

$$\int_0^T (\mathbf{Q}_h^\tau U_h^\tau, \nabla\phi)dt = \int_0^T (\mathbf{q}U_h^\tau, \nabla\phi)dt + \int_0^T ([\mathbf{Q}_h^\tau - \mathbf{q}]U_h^\tau, \nabla\phi)dt.$$

Since $\mathbf{Q}_h^\tau \to \mathbf{q}$ strongly in $[L^2(\Omega)]^2$, and since U_h^τ is weakly convergent in $L^2(0,T;H_0^1(\Omega))$, the term on the right hand side converges to the desired limit. The second term on the right hand side is zero, since $U_h^\tau \in L^\infty(\Omega)$ and $\mathbf{Q}_h^\tau \to \mathbf{q}$ strongly in $[L^2(\Omega)]^2$. Hence, the left hand side of the fourth equation of system (41) converges in the distributional sense to the fourth equation of problem P_2.

We now show that the terms in the right hand side converge to zero for $(h,\tau) \to 0$. Let us denote these terms by the notation $\mathcal{I}_1, \cdots, \mathcal{I}_8$. Thanks to (46) and the fourth interpolation estimate in (20), we may write:

$$|\mathcal{I}_1| \le \left(\int_0^T ||\partial_t U_h^\tau||^2 dt\right)^{1/2} \left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\phi - \phi_h||^2 dt\right)^{1/2} \le Ch||\phi||_{L^2(0,T;H_0^1(\Omega))} \to 0.$$

Similarly, we obtain $|\mathcal{I}_2| \to 0$ by considering estimate (48), and the fourth interpolation estimate of (20). Using (45) and the fourth interpolation estimate in (20) we can write:

$$|\mathcal{I}_3| \le \left(\int_0^T ||\nabla U_h^\tau||^2 dt\right)^{1/2} \left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\nabla\phi - \nabla\phi_h||^2 dt\right)^{1/2} \le Ch||\phi||_{L^2(0,T;H^2(\Omega))} \to 0.$$

Note that we have to restrict the test function to $\phi \in L^2(0,T;H^2(\Omega)^2)$. Density arguments ensure that the limit points satisfy the continuous weak formulation also for $\phi \in L^2(0,T;H_0^1(\Omega))$. Considering (23), we have that

$$|\mathcal{I}_4| \le \left(\sum_{n=1}^N \tau||\nabla u_h^n - \nabla u_h^{n-1}||^2\right)^{1/2} \left(\sum_{n=1}^N \tau||\nabla\phi_h||^2\right)^{1/2} \to 0.$$

Thanks to Lemma (2.3), estimate (42) and the fourth interpolation estimate in (20), we obtain

$$|\mathcal{I}_5| \le C\left(\int_0^T ||\mathbf{Q}_h^\tau||^2 dt\right)^{1/2} \left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\nabla\phi - \nabla\phi_h||^2 dt\right)^{1/2} \le Ch||\phi||_{L^2(0,T;H^2(\Omega))} \to 0.$$

While, using Lemma (2.3) and (26),

$$|\mathcal{I}_6| \le C\left(\sum_{n=1}^N \tau||\mathbf{q}_h^n - \mathbf{q}_h^{n-1}||^2\right)^{1/2} \left(\sum_{n=1}^N \tau||\nabla\phi_h||^2\right)^{1/2} \to 0.$$

This bound can be used in a similar way to show that $|\mathcal{I}_8| \to 0$. The second part of (22) and the estimate (42) allow us to state that

$$|\mathcal{I}_7| \le C\left(\sum_{n=1}^N \tau||u_h^n - u_h^{n-1}||^2\right)^{1/2} \left(\sum_{n=1}^N \tau||\nabla\phi_h||^2\right)^{1/2} \to 0.$$

Finally, we consider the fifth equation in (41). The left hand side converges to

$$\int_0^T (\partial_t v, \theta)dt \in \int_0^T (r(u) - H(v), \theta)dt.$$

This is a direct consequence of the convergence results (45) and (46) (which implies the strong convergence for U_h^τ), equation (48) and the properties of the map $H(\cdot)$. Hence, the left hand side of the fifth equation of system (41) converges in the distributional sense to the fifth equation of problem P_2. For what concerns the terms in the right hand side, let us denote them by the notation $\mathcal{I}_1, \cdots, \mathcal{I}_5$. We have already demonstrated that $\mathcal{I}_1 \to 0$ if $\theta \in L^2(0,T;H_0^1(\Omega))$. Exploiting the Lipschitz continuity of the function $r(\cdot)$, estimate (45) and the fourth interpolation estimate of (20), we get

$$|\mathcal{I}_2| \le L_r \left(\int_0^T ||U_h^\tau||^2 dt\right)^{1/2} \left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\theta - \theta_h||^2 dt\right)^{1/2} \le Ch||\theta||_{L^2(0,T;H_0^1(\Omega))} \to 0.$$

While, thanks to the Lipschitz continuity of $r(\cdot)$ and the second part of the estimate (22),

$$|\mathcal{I}_3| \le L_r \left(\sum_{n=1}^N \tau ||u_h^n - u_h^{n-1}||^2\right)^{1/2} \left(\sum_{n=1}^N \tau ||\theta_h||^2\right)^{1/2} \to 0.$$

The boundedness of the map $H(\cdot)$ allows us to write:

$$|\mathcal{I}_4| \le C \left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} ||\theta - \theta_h||^2 dt\right)^{1/2} \le Ch||\theta||_{L^2(0,T;H_0^1(\Omega))} \to 0.$$

Finally, thanks to the Filippov selection method, we have that $||H(V_h^\tau) - H(v_h^{n-1})|| \le L_r ||u_h^n - u_h^{n-1}||$ (note that if $v_h^{n-1} = 0$, than also $v_h^n = 0$). Hence, $|\mathcal{I}_5| \le |\mathcal{I}_3| \to 0$. Which provides the last result and completes the proof. □

3 A numerical example

In this section we present a test case concerning the numerical solution of **Problem** $\mathbf{P}_2^h$. We employ a numerical procedure based on the event driven method for DRH systems [9], applied to the Euler explicit scheme (10): this time-discrete scheme decouples the transport from the reaction and the Darcy terms. At each time step n we advance in time with the reaction term. If the trajectory meets the discontinuity surface at an instant t^* inside the current time step, we localize intersection between the trajectory and the surface and restart the integration from t^*, after having selected the corresponding element of the set $H(v_h)$ according to the Filippov prescription (5). The Darcy equations are then solved by static condensation, and the Darcy field is used inside the advection - diffusion - reaction equation for u_h^{n+1}. We compare the results obtained by the application of the event driven method with those obtained by the regularization approach of the right hand side introduced in [17], which is given by representing the Heaviside function through a linear interpolation

$$H_\delta(v) = \begin{cases} 0 & \text{if} \quad \mathrm{v} \le 0 \\ v/\delta & \text{if} \quad 0 \le \mathrm{v} \le \delta \\ 1 & \text{if} \quad \mathrm{v} > \delta \end{cases}$$

Here, δ is a small positive parameter that we set to 0.005. The problem is set in a square 2D domain $\Omega := (0,1) \times (0,1)$, with a Dirichlet boundary $\Gamma_D := \{y : \ x = 0, \ y \in$

$(0,1)\}$ and $\Gamma_N := \partial\Omega \setminus \Gamma_D$. Moreover, the viscosity is set to $\mu = 1$ and precipitation is modeled as $r(u) = u$. We set the following initial conditions

$$
\begin{aligned}
u|_{t=0} &= 1\text{in } (x,y) \in \Omega, \\
v|_{t=0} &= \begin{cases} 0.8 & \text{in } (x,y) \in \Omega_v, \\ 0 & \text{in } (x,y) \in \Omega \setminus \Omega_v. \end{cases}
\end{aligned}
$$

where $\Omega_v \subset \Omega, \quad \Omega_v := \{(x,y) :\ 0.4 \le x \le 0.6,\ 0.4 \le y \le 0.6\}$, and solve **Problem** $\mathbf{P}_2^h$ on the time interval $t \in [0,1]$, with a time step $\Delta t = 0.01$. We set $\Gamma_1 := \{y :\ x = 0,\ y \in (0,1)\}$, $\Gamma_2 := \{x :\ y = 0,\ x \in (0,1)\}$, $\Gamma_3 := \{y :\ x = 1,\ y \in (0,1)\}$ and $\Gamma_4 := \{x :\ y = 1,\ x \in (0,1)\}$, and $p = 0.5, u = 0$ on Γ_1, $p = 0, \nabla u \cdot \mathbf{n} = 0$ on Γ_3, $\mathbf{q} \cdot \mathbf{n} = 0, \nabla u \cdot \mathbf{n} = 0$ on $\Gamma_2 \cup \Gamma_4$. The domain is discretized with a structured triangular grid of 10000 elements.

The evolution of cation and precipitate concentrations and magnitude of the Darcy velocity are represented in Figure 1, both for the cases of the application of the event driven method and the regularization approach. The solution exhibits an attractive sliding motion on the discontinuity of the Heaviside function in $\Omega \setminus \Omega_v$, where, as a consequence, the precipitate concentration v remains constant and equal to 0. Note that the regularization approach fails at representing correctly the sliding motion on $v = 0$. This is due to the fact that, being $\Delta t > \delta$, the solution can exceed the threshold value and start oscillating around it. This causes spurious oscillations in the time evolution of the magnitude of the Darcy field as well. In Figure 2 we show the evolution of the difference between the cation concentration values calculated with the event driven and the regularization methods (u DRH - u Smooth), and between the precipitate concentration values calculated with the two methods (v DRH - v Smooth). We can observe that, after the first time step, (at time $t = 0.01$ [s]), the regularization approach exceeds the threshold value by a quantity equal to 2δ. The oscillations around the threshold value decrease in time, and the concentration values calculated with the regularization method approach those calculated with the event driven method as time advances.

The dependence of the numerical solutions on the ratio between Δt and δ, and the order of convergence of event driven and regularized methods for explicit and implicit first order and higher order schemes will be studied in more detail in a forthcoming work.

The magnitude of the Darcy velocity $\mathbf{q}$ and its streamlines are reported in Figure 3, for the case of the event driven method. The decrease of cation concentration u causes dissolution in Ω_v. The variation of v corresponds to a porosity change, and, consequently, to a change in the permeability in time. It can be observed that the velocity $\mathbf{q}$ increases in Ω_v as the precipitate dissolves. The streamlines superimposed in Figure 3 show that the precipitate concentration forms at the beginning of the simulation an obstacle for the flow that is then gradually eroded.

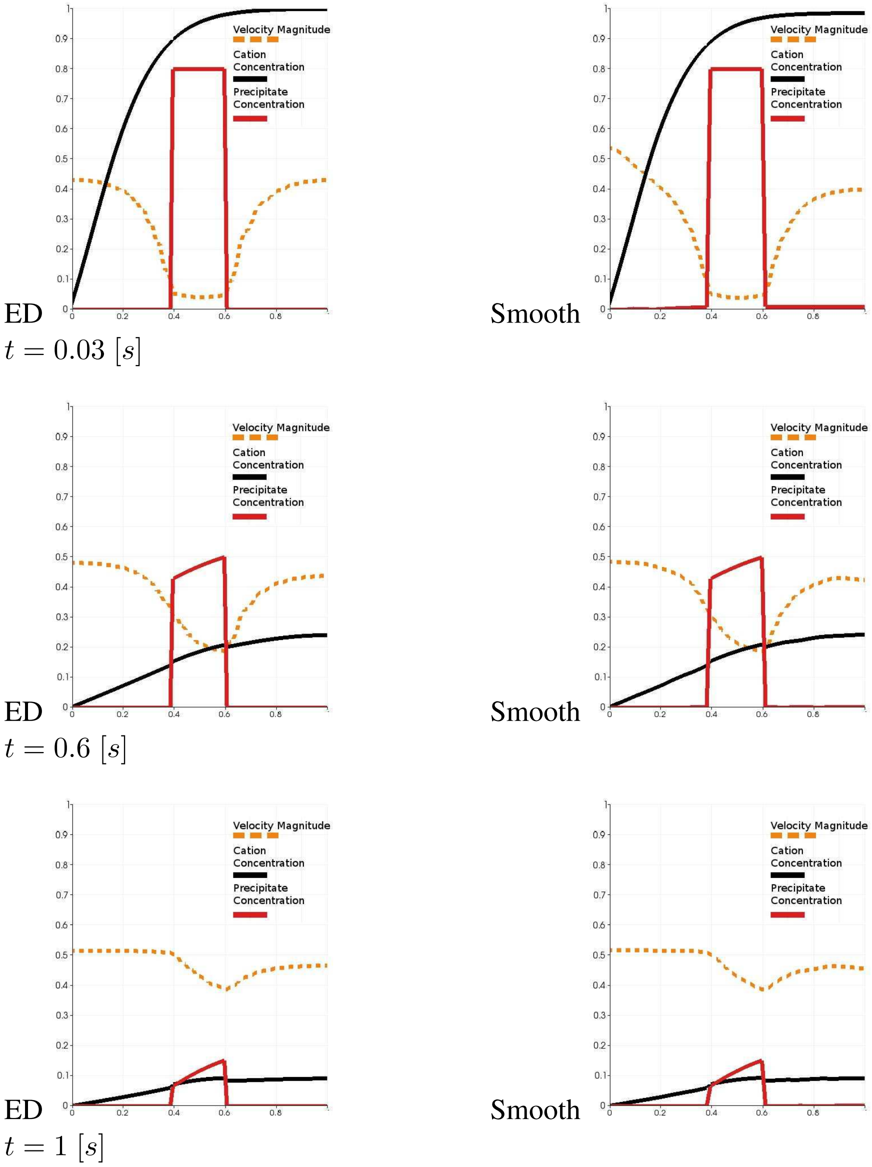
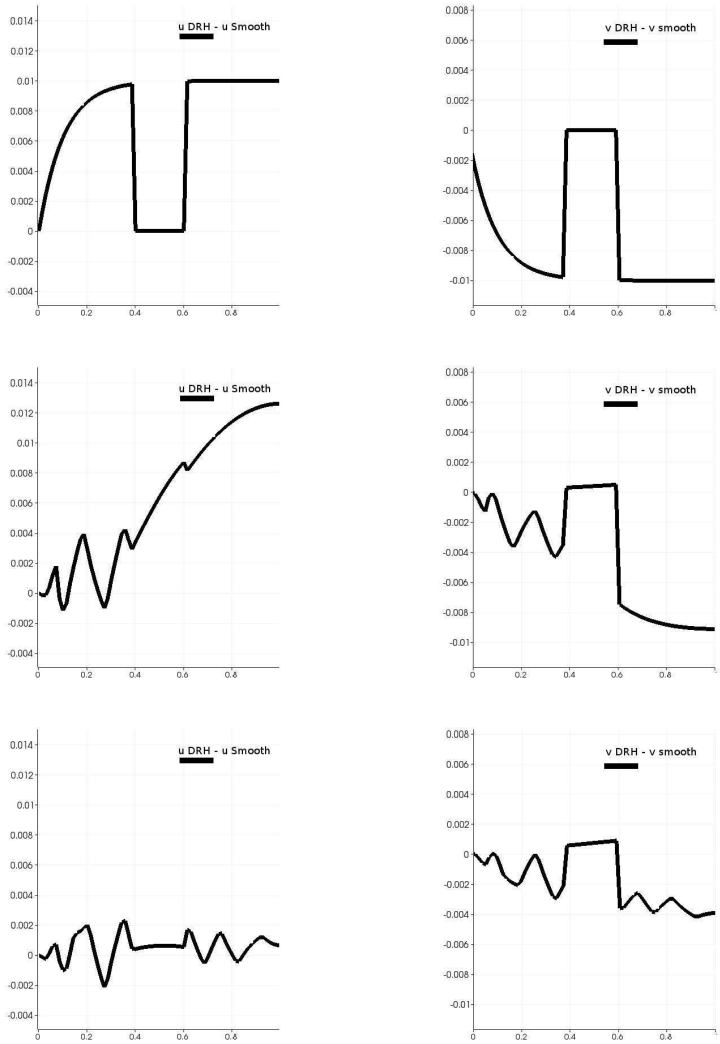


Figure 1: Cation concentration(u), precipitate concentration (v) and velocity magnitude $|\mathbf{q}|$ at times 0.03 $[s]$, 0.6 $[s]$, 1 $[s]$, in the case of the application of event driven method (ED) and regularization approach (Smooth).



$t = 0.01\ [s]$

$t = 0.06\ [s]$

$t = 0.2\ [s]$

Figure 2: Difference between the cation concentration values calculated with the event driven and the regularization methods (u DRH - u Smooth), and between the precipitate concentration values calculated with the event driven and the regularization methods (v DRH - v Smooth), at times $0.01\ [s]$, $0.06\ [s]$, $0.2\ [s]$.

$t = 0.3\ [s]$ $\quad$ $t = 0.6\ [s]$ $\quad$ $t = 1\ [s]$

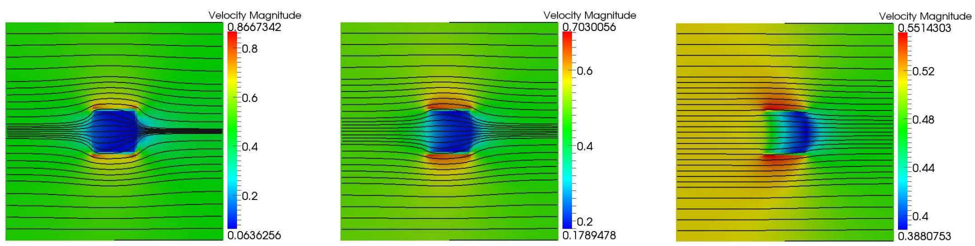


Figure 3: Streamlines and magnitude of the velocity field $\mathbf{q}$ at times $0.3\ [s]$, $0.6\ [s]$, $1\ [s]$, in the case of the application of event driven method.

4 Conclusion

In this paper we have shown the well-posedness of a simple model for dissolution-precipitation coupled with Darcy flow. We have treated the presence of thresholds in the reaction term without resorting to regularization. This has led to a problem that may be cast as differential inclusion.

We think that this result is rather important since it gives ground to event-driven numerical schemes for this class of problems that are able to properly treat the thresholds and allow to maintain the order in time of the basic scheme used for numerical integration. Even if we have used a very simple model for the dissolution process, numerical tests, which are the subject of forthcoming work, show that the technique of treating thresholds as discontinuities without regularization is not only a viable solution, but in several cases provides an effective numerical tool capable of giving accurate results in an efficient way.

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