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Discontinuous Galerkin time integration for second-order differential problems: formulations, analysis, and analogies

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Abstract

We thoroughly investigate Discontinuous Galerkin (DG) discretizations as time integrators for second-order oscillatory systems, considering both second-order and first-order formulations of the original problem. Key contributions include new convergence analyses for the second-order formulation and equivalence proofs between DG and classical time-stepping schemes (such as Newmark schemes and general linear methods). In addition, the chapter provides a detailed review and convergence analysis for the first-order formulation, alongside comparisons of the proposed schemes in terms of accuracy, consistency, and computational cost.

1 Introduction

We review in this chapter time-discontinuous Galerkin finite element methods for the numerical approximation of linear systems of second-order ordinary differential equations (ODEs). Such systems of ODEs typically come from space discretizations of wave propagation problems in acoustics, elastodynamics, or electromagnetism. Conventional techniques for numerically integrating second-order ODEs usually use implicit or explicit finite differences, Runge-Kutta, and Newmark methods (see [24, 29, 10] for a comprehensive overview). In various engineering scenarios, explicit methods are generally favored over implicit ones due to their computational efficiency. Despite implicit methods being (in general) unconditionally stable, explicit methods are computationally cheaper. However, explicit methods are limited by the time step constraints imposed by the Courant–Friedrichs–Lewy (CFL) condition. This constraint, determined by the spatial discretization parameters and media properties, can significantly impact the computational effectiveness. One potential approach to mitigate the CFL constraint is implementing suitable local time stepping (LTS) algorithms [12, 14, 18], where a smaller time step is used only when necessary according to the local CFL condition. Alternatively, explicit LTS methods could be adopted, by using either high-order derivative discontinuous Galerkin approaches [35], or Tent Pitching algorithms [17, 11].

In contrast to the aforementioned methods, we analyze implicit time-integration schemes proposed in [4, 5] that achieve arbitrarily high accuracy, based on a discontinuous Galerkin (DG) approach. DG methods were initially introduced to approximate hyperbolic problems in space [31] and subsequently extended to handle elliptic and parabolic equations [39, 6]. The DG method was also employed to tackle initial-value problems, cf. the pioneering work in [25]. In time-dependent scenarios, information flows forward in time, and solutions exhibit a causal behavior, relying on past events but not future ones. Unlike finite difference time integration schemes, where the current solution is influenced by previous steps, time discontinuous Galerkin methods, operating over time intervals $I_n = [t_n, t_{n+1}]$, establish a causal system where the solution in I_n depends exclusively on the solution in I_{n-1} . The use of DG schemes as time integrators is motivated by the fact that they can be easily combined with DG (or other) space discretization methods to obtain a high-order space-time finite element formulation. Space-time finite elements for hyperbolic problems are typically constructed based on a system of firstorder differential equations, as shown for example in [19, 7, 22, 27, 23, 15, 17, 28]. On the other hand, only a limited number of recent studies have addressed finite element approximations of second-order differential systems [20, 36, 2, 40, 37, 33, 34].

We review here the main aspects of DG discretizations as time integrators of second-order oscillatory systems. In particular, we consider two different formulations that arise by considering the original problem in its second-order (i) or first-order formulation (ii). The main novelties of the chapter can be summarized as follows. For (i), we present a new framework for convergence analysis, addressing issues related to consistency and accuracy. We show the equivalence between the DG formulation with linear finite elements and the Newmark scheme, and propose optimization strategies for the DG method. Moreover, due to the unconventional algebraic structure of DG for second-order in time problems, the spectral properties of the time-stepping matrix have to be taken into account. This leads to a new analysis for this class of problems. Finally, by introducing a new theoretical framework for Generalized Linear Methods (GLMs) we show the equivalence between general order DG schemes and GLMs. For (ii) we present an in-depth review from [25], supplemented with a new algebraic proof. Moreover, we introduce the Lobatto IIIC scheme [21] and present a related convergence analysis. Finally, we refer the reader to [3] for a review of DG time-stepping methods applied to different wave propagation problems.

Our chapter is organized as follows. In Section 2 we review some classical integrators, e.g., the Newmark and the Runge-Kutta schemes, and briefly recall the classical GLM framework. Next, we analyze the time DG method for the second-order equation in Section 3. We discuss its property of accuracy and consistency and show the analogy with the Newmark or the GLM schemes. In Section 4, we consider the DG schemes as time integrators of first-order systems and show their equivalence with special implicit Runge-Kutta schemes. Finally, in Section 5 we compare the proposed schemes from the point of view of accuracy, consistency, and computational cost, and in Section 6 we draw some conclusions.

Throughout the chapter, we denote by $||\boldsymbol{a}||$ the Euclidean norm of a vector $\boldsymbol{a} \in \mathbb{R}^d$, $d \geq 1$ and by $||\boldsymbol{A}||$ the ℓ^2 -norm of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m, n \geq 1$. For a given $I \subset \mathbb{R}$ and $v : I \to \mathbb{R}$ we denote by $L^p(I)$ and $H^p(I)$, $p \in \mathbb{N}_0$, the classical Lebesgue and Hilbert spaces, and endow them with the usual norms, see [1]. Finally, we indicate the Lebesgue and Hilbert spaces for vector-valued functions as $\boldsymbol{L}^p(I) = [L^p(I)]^d$ and $\boldsymbol{H}^p(I) = [H^p(I)]^d$, $d \geq 1$.

2 A brief review of some integrators

We now briefly review three different classes of numerical time-stepping methods: the Newmark methods in section 2.1, the implicit Runge-Kutta methods in section 2.2, and the general linear methods in section 2.3. For this purpose, we consider the second-order model problem

$$\begin{cases} \ddot{u}(t) = f(u(t)) & \forall t \in (0,T], \\ u(0) = \widehat{u}_0, \\ \dot{u}(0) = \widehat{u}_1, \end{cases}$$
(1)

which can be written as a first-order system,

$$\begin{cases} \dot{v}(t) = f(u(t)) & \forall t \in (0,T], \\ \dot{u}(t) = v(t) & \forall t \in (0,T], \\ u(0) = \hat{u}_{0}, \\ v(0) = \hat{u}_{1}. \end{cases}$$
(2)

We consider a discrete time grid of N + 1 points $t_n = n\Delta t$, $n = 0, \ldots, N$, $\Delta t = \frac{T}{N}$, and denote by u_n and v_n the discrete approximations to $u(t_n)$ and $v(t_n)$. Finally, to keep the discussion simple, we assume that f is sufficiently regular.

2.1 The Newmark method

The derivation of the Newmark scheme is based on Taylor's theorem,

$$u(t_n + \Delta t) = u(t_n) + \Delta t \ \dot{u}(t_n) + \int_{t_n}^{t_{n+1}} \ddot{u}(\tau)(t_{n+1} - \tau) \ d\tau,$$

$$\dot{u}(t_n + \Delta t) = \dot{u}(t_n) + \int_{t_n}^{t_{n+1}} \ddot{u}(\tau) \ d\tau.$$
(3)

The Newmark scheme is obtained by a discretization of the integrals (remainder terms) in (3). This is achieved by the approximation $\ddot{u}(\tau) \approx (1 - \gamma)\ddot{u}(t_n) + \gamma\ddot{u}(t_{n+1})$, which inserted in the integrals above leads to

$$\int_{t_n}^{t_{n+1}} \ddot{u}(\tau)(t_{n+1}-\tau) \ d\tau \approx \Delta t^2 \ \left[\left(\frac{1}{2} - \frac{\gamma}{2}\right) \ddot{u}(t_n) + \frac{\gamma}{2} \ddot{u}(t_{n+1}) \right],$$

$$\int_{t_n}^{t_{n+1}} \ddot{u}(\tau) \ d\tau \approx \Delta t \ \left[(1-\gamma) \ddot{u}(t_n) + \gamma \ddot{u}(t_{n+1}) \right].$$
(4)

Even though a natural choice for γ would be $\gamma = \frac{1}{2}$ for both integrals (corresponding to the trapezoidal rule) in the Newmark framework, the term $\frac{\gamma}{2}$ is usually replaced by a second parameter β that can be chosen independently of γ . Now, recalling that $\ddot{u}(t) = f(u(t))$, we define $f_n := f(u_n)$. Thus, by inserting the approximation (4) into (3), we get

$$u_{n+1} = u_n + \Delta t \ v_n + \Delta t^2 \left[\left(\frac{1}{2} - \beta \right) f_n + \beta f_{n+1} \right],$$

$$v_{n+1} = v_n + \Delta t \ [(1 - \gamma) f_n + \gamma f_{n+1}].$$
(5)

Equation (5) represents the Newmark family of numerical integrators. As we are going to see, the choice of γ and β affects the properties of the numerical integrator. Thus, each pair (γ, β) gives rise to a different method.

The Newmark scheme is first-order accurate (an analysis is provided in Section 3.1.1) if $\gamma \neq \frac{1}{2}$. The only choice that makes it second-order accurate is $\gamma = \frac{1}{2}$, corresponding to a trapezoidal rule for the approximation of the integral in the second equation in (3). Notice that the choice of β does not affect the order of consistency, but the stability features of the method. The choice $\beta = 0$ leads to the leapfrog scheme, which is an explicit scheme and hence conditionally stable [30]. If $\beta > 0$ is chosen, then the Newmark scheme (5) becomes implicit, and the particular choice $\beta = \frac{1}{4}$ makes the method conservative: if $f(u) = \lambda$, with $\lambda < 0$, then it is possible to show that $|\dot{u}(t)|^2 + \lambda |u(t)|^2$ is constant for all $t \ge 0$, and the choice $\beta = \frac{1}{4}$ guarantees that $|v_n|^2 + \lambda |u_n|^2$ remains also constant for all n. Notice that the choice $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ corresponds to the trapezoidal rule for the discretization of both integrals in (4). It is the only choice making the method conservative, and one can show that it leads exactly to the Crank-Nicolson scheme for (2). A general treatment for different values of γ and β can be found in [30].

2.2 Implicit Runge-Kutta methods

An s-stage Runge-Kutta scheme applied to system (2) is characterized by the Butcher tableau

$$\begin{array}{c|c} c & A \\ \hline & b \end{array} \tag{6}$$

with matrix $A = \{a_{ij}\} \in \mathbb{R}^{s \times s}$, weight vector $\boldsymbol{b} = (b_1, \dots, b_s)$, and quadrature nodes $\boldsymbol{c} = (c_1, \dots, c_s)^\top$. Runge-Kutta methods update the solution using a sum over stage vectors as follows:

$$\begin{cases} \boldsymbol{z}_{n+1} = \boldsymbol{z}_n + \Delta t \sum_{i=1}^s b_i \boldsymbol{k}_{i,n}, \\ \boldsymbol{k}_{i,n} = f \left(\boldsymbol{z}_n + \Delta t \sum_{j=1}^s a_{ij} \boldsymbol{k}_{j,n} \right). \end{cases}$$
(7)

If the coefficients $\{a_{ij}\}$ in A are nonzero for $j \ge i$, for i = 1, 2, ..., s, then every $\mathbf{k}_{i,n}$ can be obtained explicitly as a function of the i-1 stages $\mathbf{k}_{1,n}, ..., \mathbf{k}_{i-1,n}$ computed previously. In this case, the scheme is called *explicit*. Otherwise, the Runge-Kutta scheme is called *implcit* and a non-linear system has to be solved to compute $\mathbf{k}_{i,n}$. When f is a linear operator represented by a matrix L, the stages $\mathbf{k}_{i,n}$, i = 1, ..., s, can be expressed as the solution of the linear system

$$\left(\begin{bmatrix} I & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & I \end{bmatrix} - \Delta t \begin{bmatrix} a_{11}L & \dots & a_{1s}L \\ \vdots & \ddots & \vdots \\ a_{s1}L & \dots & a_{ss}L \end{bmatrix} \right) \begin{bmatrix} \mathbf{k}_{1,n} \\ \vdots \\ \mathbf{k}_{s,n} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_s \end{bmatrix},$$
(8)

with $f_i = L z_n$, for i = 1, ..., s. System (8) can be expressed in the equivalent compact Kronecker product form

$$(I_s \otimes I_2 - \Delta t A \otimes L) \boldsymbol{k}_n = \boldsymbol{f}.$$
(9)

Lobatto methods are fully implicit RK methods for which the coefficients b_j and c_j in (6) are determined based on the Lobatto quadrature formula with s node coefficients c_1, \ldots, c_s , and s weight coefficients b_1, \ldots, b_s .

The s nodes c_i are the roots of the polynomial of degree s

$$\frac{d^{s-2}}{dt^{s-2}}(t^{s-1}(1-t)^{s-1}),$$

and satisfy $c_1 < c_2 < \cdots < c_s$. The weights b_j and nodes c_j satisfy the order conditions B(2s-2), which are for generic argument p given by

$$B(p): \sum_{j=1}^{s} b_j c_j^{k-1} = \frac{1}{k}, \quad k = 1, \dots, p,$$

implying that the quadrature formula is of order 2s - 2. The families of Lobatto RK methods differ only in the values of their coefficients a_{ij} . Here, we recall the following order conditions that can be used to determine their expressions:

$$C(q): \quad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \quad i = 1, \dots, s \text{ and } k = 1, \dots, q,$$
(10)

$$D(r): \sum_{i=1}^{s} b_i c_i^{k-1} a_{ij} = \frac{b_j}{k} (1 - c_i^k), \quad j = 1, \dots, s \text{ and } k = 1, \dots, r.$$
(11)

The importance of these order conditions comes from a fundamental result due to Butcher, cf. also [38, Therem 5.1]: if the coefficients b_i , c_i , a_{ij} of a Runge-Kutta method satisfy B(p), C(q), and D(r), with $p \leq q + r + 1$ and $p \leq 2q + 2$, then the method is of order p.

Four main families belong to the set of Lobatto RK methods, namely Lobatto IIIA, Lobatto IIIB, Lobatto IIIC, and Lobatto IIIC^{*} methods, cf. [38]. We briefly recall some results for Lobatto IIIC: their coefficients a_{ij} are defined by $a_{i1} = b_1$ for i = 1, ..., s and the order conditions C(s - 1). They satisfy the order conditions D(s - 1) and $a_{sj} = b_j$, for j = 1, ..., s. Lobatto IIIC methods are of order 2s - 2. The order of convergence is evaluated for the last stage (not intermediate stages for which the order is s; see [38]). They are not symmetric. Their stability function R(z) is given by the (s - 2, s)-Padé approximation to e^z , e.g.,

$$R(z) = \frac{1}{1 - z + \frac{z^2}{2}}, \quad s = 2,$$

$$R(z) = \frac{1 + \frac{1}{4}z}{1 - \frac{1}{4}z + \frac{1}{2}\frac{z^2}{2} - \frac{1}{4}\frac{z^3}{3}}, \quad s = 3$$

They are L-stable, i.e., they are A-stable and $R(z) \to 0$ as $z \to \infty$. They are algebraically stable, i.e.,

$$B := \operatorname{diag}(b_1, \ldots, b_s), \text{ and } M := BA + A^{\top}B - \boldsymbol{b}\boldsymbol{b}^{\top}$$

are positive semi-definite, [21]. The above condition implies that the Lobatto IIIC methods are B-stable. Thus, they are excellent methods for stiff problems.

2.3 General linear methods

General linear methods (GLMs) were introduced by Butcher in [8] and further developed by several authors; see, e.g., [9] for a general survey. For our short review, we start from the point of view given by [16], which is a direct application of Butcher's original idea to second-order problems.

The essential idea of GLMs is to consider a vector $\mathbf{y}^{[n]} \in \mathbb{R}^r$, $r \in \mathbb{N}^+$, which does not only contain the approximation u_n to $u(t_n)$, but also additional information in components needed to build the integration step, and a stage-like vector (in the spirit of RK methods), denoted by $Y \in \mathbb{R}^s$, $s \in \mathbb{N}^+$, accounting for implicit parts of the integration step. Thus, a single step of a GLM reads as

$$\begin{bmatrix} Y \\ \mathbf{y}^{[n+1]} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} \Delta t^2 f(Y) \\ \mathbf{y}^{[n]} \end{bmatrix},$$
(12)

where the entries of the matrices $A \in \mathbb{R}^{s \times s}$, $U \in \mathbb{R}^{s \times r}$, $B \in \mathbb{R}^{r \times s}$, and $V \in \mathbb{R}^{r \times r}$ are used to define the GLM. The system (12) can be written explicitly as

$$Y_i = \Delta t^2 \sum_{j=1}^s a_{i,j} f(Y_j) + \sum_{j=1}^r u_{i,j} y_j^{[n]}, \qquad \text{for } i = 1, \dots, s, \qquad (13)$$

$$y_i^{[n+1]} = \Delta t^2 \sum_{j=1}^s b_{i,j} f(Y_j) + \sum_{j=1}^r v_{i,j} y_j^{[n]}, \qquad \text{for } i = 1, \dots, r.$$
(14)

Notice that (12) can be implicit in the stage vector Y, while it is explicit in \mathbf{y} . Let us give a simple example to better clarify the idea behind a GLM.

Example 1 (Newmark as a GLM). Consider the Newmark scheme (5) for $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$:

$$u_{n+1} = u_n + \Delta t \ v_n + \frac{\Delta t^2}{4} \left[f(u_{n+1}) + f(u_n) \right],$$

$$v_{n+1} = v_n + \frac{\Delta t}{2} \left[f(u_{n+1}) + f(u_n) \right].$$
(15)

To write this scheme in the form (12), we introduce the variables $a_n := \Delta t v_n$, $b_n := \Delta t^2 f(u_n)$, and rewrite the first equation in (15) as

$$Y = u_n + a_n + \frac{\Delta t^2}{4} f(Y) + \frac{1}{4} b_n,$$
(16)

and the second equation in (15) as

$$a_{n+1} = a_n + \frac{1}{2}b_n + \frac{\Delta t^2}{2}f(Y).$$
(17)

Notice that $Y = u_{n+1}$. This also implies that

$$b_{n+1} = \Delta t^2 \ f(u_{n+1}) = \Delta t^2 \ f(Y).$$
(18)

Collecting (16), (17), (18) and the first equation of (15), we obtain

$\begin{bmatrix} Y \end{bmatrix}$		1/4	1	1/4	1]	$\left[\Delta t^2 f(Y)\right]$	
a_{n+1}	=	1/2	1	1/2	0	a_n	
$ b_{n+1} ^{=}$		1	0	0	0	b_n	•
u_{n+1}		1/4	1	1/4	1	u_n	

This is exactly in the form (12) with $\mathbf{y}^{[n]} = [a_n, b_n, u_n]^{\top}$. Notice that this system is only implicit in the variable Y, and a_n and b_n contain auxiliary information, and form the vector $\mathbf{y}^{[n]}$ together with the approximation u_n .

A GLM of the form (12) is convergent if and only if it is consistent and zero stable, (see, e.g., [9, 13]). Thus, we need to clarify the notions of convergence, consistency and zero stability for GLMs.

Neglecting the issue of a starting procedure (see [9, 13]), we say that a GLM converges if $\mathbf{y}^{[n]} \rightarrow \widehat{\mathbf{y}}^{[n]} = \mathbf{q}_0 u(t_n)$, for all n and some vector \mathbf{q}_0 , as $\Delta t \rightarrow 0$.

The zero stability is the property of the method to well represent the solution of $\ddot{u}(t) = 0$. For the case f = 0, the GLM (12) becomes $\mathbf{y}^{[n+1]} = V \mathbf{y}^{[n]}$. Therefore, the zero stability is related to the spectral properties of the matrix V [13, Theorem 4.1]: a GLM of the form (12) is zero stable if the eigenvalues of V lie within or on the unit circle and the multiplicity of those lying on the unit circle is at most two.¹

The consistency of GLMs is usually accomplished by the so-called *order conditions* (see, e.g., [9, 13, 13, 26]). To obtain them, consider the solution u(t) to (1) and an input vector $\mathbf{y}^{[n]}$, whose components are written in a generalized Taylor form:

$$y_i^{[n]} := \sum_{k=0}^p q_{i,k} \Delta t^k u^{(k)}(t_n) + O(\Delta t^{p+1}), \quad \text{for } i = 1, \dots, r,$$
(19)

where $q_{i,k}$ are given coefficients, and

$$Y_i = u(t_n + c_i \Delta t) + O(\Delta t^{p+1}) \quad \text{for } i = 1, \dots, s,$$

$$(20)$$

where c_i are stage coefficients. Notice that this definition is consistent with the notion of convergence we stated above. Now, the driving question is: under which conditions on the matrices A, B, U, and V and on the coefficients $q_{i,k}$ and c_i does one step of a GLM (12) produce exactly the output

$$y_i^{[n+1]} := \sum_{k=0}^p q_{i,k} \Delta t^k u^{(k)}(t_n + \Delta t) + O(\Delta t^{p+1}), \quad \text{for } i = 1, \dots, r,$$
(21)

if the input (19) is considered? More precisely: which conditions on $A, B, U, V, q_{i,k}$, and c_i guarantee that the output is exact in all the terms till order p?

The answer is provided by the order conditions that are algebraic conditions on A, B, U, V, and $q_{i,k}$. Notice that we consider here the same order p in (21) and (20). This choice is made only for the sake of simplicity, and more general results can be found in the literature (see, e.g., [13]). The derivation proceeds by Taylor expansions, and we obtain the order conditions as necessary conditions for (19), (21), and (20). Recalling that $\ddot{u}(t) = f(u(t))$ and using (20) we get

$$\Delta t^2 f(Y_i) = \Delta t^2 \ddot{u}(t_n + c_i \Delta t) + O(\Delta t^{p+3}) = \sum_{k=2}^p \Delta t^k u^{(k)}(t_n) \frac{c_i^{k-2}}{(k-2)!} + O(\Delta t^{p+1}).$$
(22)

Now, using (13), (19), (20), and (22), we get

$$u(t_{n} + c_{i}\Delta t) \stackrel{(20)}{=} Y_{i} + O(\Delta t^{p+1}) \stackrel{(13)}{=} \Delta t^{2} \sum_{j=1}^{s} a_{i,j} f(Y_{j}) + \sum_{j=1}^{r} u_{i,j} y_{j}^{[n]} + O(\Delta t^{p+1})$$

$$\stackrel{(22)}{=} \sum_{j=1}^{s} a_{i,j} \sum_{k=2}^{p} \Delta t^{k} u^{(k)}(t_{n}) \frac{c_{i}^{k-2}}{(k-2)!} + \sum_{j=1}^{r} u_{i,j} y_{j}^{[n]} + O(\Delta t^{p+1})$$

$$\stackrel{(19)}{=} \sum_{j=1}^{s} a_{i,j} \sum_{k=2}^{p} \Delta t^{k} u^{(k)}(t_{n}) \frac{c_{i}^{k-2}}{(k-2)!} + \sum_{j=1}^{r} u_{i,j} \sum_{k=0}^{p} q_{j,k} \Delta t^{k} u^{(k)}(t_{n})$$

$$+ O(\Delta t^{p+1}).$$

Rearranging the sums in this expression, we get

$$u(t_n + c_i \Delta t) = \sum_{j=1}^r u_{i,j} q_{j,0} u(t_n) + \sum_{j=1}^r u_{i,j} q_{j,1} \Delta t \ u^{(1)}(t_n) + \sum_{k=2}^p \Delta t^k \left[\sum_{j=1}^r u_{i,j} q_{j,k} u^{(k)}(t_k) + \sum_{j=1}^s a_{i,j} \frac{c_j^{(k-2)}}{(k-2)!} \right] + O(\Delta t^{p+1}).$$
(23)

¹Note that this definition of zero stability is the one given in [13] (see also [38, Definition 10.1]) for second-order equations. The meaning is that, when applied to (1) with f = 0, and $\hat{u}_1 = 0$, a zero-stable method preserves the second-order derivative. A different definition is provided in [38, Definition 3.2] for first-order systems.

Now, we expand $u(t_n + c_i \Delta t)$ and write

$$u(t_n + c_i \Delta t) = u(t_n) + c_i \Delta t \ u^{(1)}(t_n) + \sum_{k=2}^p \Delta t^k \frac{c_i^k}{k!} u^{(k)}(t_n) + O(\Delta t^{p+1}).$$
(24)

Now, by a direct comparison of (23) and (24), matching term by term for each i, we obtain the conditions

$$U\mathbf{q}_0 = \mathbf{1}$$
 $k = 0$ (preconsistency), (25a)

$$U\mathbf{q}_1 = \mathbf{c}$$
 $k = 1$ (preconsistency), (25b)

$$U\mathbf{q}_2 + A\mathbf{1} = \frac{1}{2}\mathbf{c}^2 \qquad \qquad k = 2 \text{ (stage consistency)}, \qquad (25c)$$

$$U\mathbf{q}_{k} + \frac{1}{(k-2)!}A\mathbf{c}^{k-2} = \frac{1}{k!}\mathbf{c}^{k} \qquad k > 2,$$
(25d)

where $\mathbf{1} \in \mathbb{R}^s$ denotes a vector whose components are all equal to 1, and $\mathbf{c}^k = [c_1^k, \dots, c_s^k]^\top$. Now, using (14), (19), (20), (21), and (22), and proceeding as before, we get

$$V\mathbf{q}_0 = \mathbf{q}_0$$
 $k = 0$ (preconsistency), (26a)

$$V\mathbf{q}_1 = \mathbf{q}_0 + \mathbf{q}_1$$
 $k = 1$ (preconsistency), (26b)

$$V\mathbf{q}_2 + B\mathbf{1} = \frac{1}{2}\mathbf{q}_0 + \mathbf{q}_1 + \mathbf{q}_2$$
 $k = 2,$ (26c)

$$V\mathbf{q}_{k} + \frac{1}{(k-2)!}B\mathbf{c}^{k-2} = \sum_{\ell=0}^{k} \frac{1}{\ell!}\mathbf{q}_{k-\ell} \qquad k > 2.$$
(26d)

Equations (25a)-(26d) are the order conditions. Even though we derived them as necessary conditions for (19), (21), and (20), they can be shown to be also sufficient conditions; see [9, 13]. The equations (25a), (25b), (26a), and (26b) are also known as preconsistency conditions. A GLM is said to be consistent if it is preconsistent and (26c) holds. Moreover, a GLM is stage consistent if it satisfies equation (25c). The conditions (26a) and (26b) can be interpreted as the property of the GLM to properly represent solutions to $\ddot{y}(t) = 0$. If this equation is complemented by initial conditions $y(0) \neq 0$ and $\dot{y}(0) = 0$, then y(t) is constant. This is clearly represented by the condition $\mathbf{q}_0 = V\mathbf{q}_0$. Moreover, if $\ddot{y}(t) = 0$ is complemented by initial conditions $y(0) \neq 0$ and $\dot{y}(0) \neq 0$, then y(t) is linear-affine. This structure is represented by the condition $\mathbf{q}_0 + \mathbf{q}_1 = V\mathbf{q}_1$.

Now a GLM is convergent if and only if it is zero stable and it satisfies (25a), (25b), (26a), (26b), and (26c). Clearly, in this case the order of convergence is 1. If in addition, (25c), (25d) and (26d) are satisfied for k = 2, ..., p, then the GLM is convergent of order p.

Example 2 (Order conditions of Newmark as a GLM). Recalling the Newmark scheme written as a

GLM in Example 1, we have that $V = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \\ 1 & 1/4 & 1 \end{bmatrix}$, which has eigenvalues equal to 0 and 1. Thus,

the Newmark scheme as a GLM is zero stable. A direct calculation shows that the vectors

$$\mathbf{q}_0 = \begin{bmatrix} 0\\0\\1 \end{bmatrix}, \quad \mathbf{q}_1 = \begin{bmatrix} 1\\0\\0 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} -\frac{1}{12}\\0\\0 \end{bmatrix}$$

satisfy the order conditions for k = 0, 1, 2, 3. However, it is also possible to show that there does not exist a vector \mathbf{q}_4 such that the order conditions hold for k = 4. Thus, p = 3 and we can conclude that the Newmark scheme as a GLM is second-order convergent as expected.

3 Time DG methods for second-order equations: DG2

Let us consider T > 0 and the model problem

$$\begin{cases} \ddot{u}(t) + \lambda u(t) = 0 \qquad \forall t \in (0, T], \\ u(0) = \hat{u}_0, \\ \dot{u}(0) = \hat{u}_1, \end{cases}$$
(27)



Figure 1: Example of a time domain partition and a zoom with the values t_n^+ and t_n^- .

where $\lambda \in \mathbb{R}^+$ is a positive coefficient and $\hat{u}_0, \hat{u}_1 \in \mathbb{R}$. We consider a partition for the interval I = (0, T]into N sub-intervals (time slabs) $I_n = (t_{n-1}, t_n]$ such that $\Delta t_n = t_n - t_{n-1}$, for n = 1, ..., N, with $t_0 = 0$ and $t_N = T$, as shown in Figure 1. We suppose in what follows that all time slabs have the same size, that is $\Delta t_n = \Delta t$ for all n. To build an approximation to the solution u on each I_n , we multiply (27) by $\dot{v}(t)$ and integrate over I_n ,

$$(\ddot{u}, \dot{v})_{I_n} + \lambda(u, \dot{v})_{I_n} = 0.$$
 (28)

Next, assuming sufficient regularity of u (namely $u \in H^2(0,T) \in C^1([0,T])$), and observing that $[u]_n = [\dot{u}]_n = 0$, for n = 1, ..., N, we can rewrite (28) by adding suitable consistent terms for any $s \ge 0$ as follows:

$$(\ddot{u},\dot{v})_{I_n} + \lambda(u,\dot{v})_{I_n} + [\dot{u}]_{n-1}\dot{v}_{n-1}^+ + \lambda[u]_{n-1}v_{n-1}^+ - s\lambda\{\dot{u}\}_{n-1}\dot{v}_{n-1}^+ = 0.$$
(29)

For s = 0, this formulation is equivalent to the one shown in [4]. The term $-s\lambda\{\dot{u}\}_{n-1}\dot{v}_{n-1}^+$ is introduced as a correction term to improve the accuracy of the scheme. Moreover, to guarantee that this is a consistent correction, the parameter s will be chosen proportional to Δt^p , with p to be determined. Next, we introduce the local finite-dimensional space $\mathbb{V}_n^r = \{v : I_n \to \mathbb{R} : v \in \mathbb{P}^r(I_n)\}$, where $\mathbb{P}^r(I_n)$ is the space of polynomials of degree $r \geq 1$ on I_n and define the DG space as

$$\mathbb{V}_{DG} = \{ v \in L^2(0,T) : v |_{I_n} \in \mathbb{V}_n^r \quad \forall n = 1, \dots, N \},\$$

having finite dimension (r+1)N. Summing over all time slabs in (29), we obtain the problem: find $u_{DG} \in \mathbb{V}_{DG}$ such that

$$\mathcal{A}(u_{DG}, v) = F(v) \qquad \forall v \in \mathbb{V}_{DG},\tag{30}$$

where $\mathcal{A}: \mathbb{V}_{DG} \times \mathbb{V}_{DG} \to \mathbb{R}$ is defined by

$$\mathcal{A}(u,v) := \sum_{n=1}^{N} \left((\ddot{u}, \dot{v})_{I_n} + \lambda (u, \dot{v})_{I_n} \right) + \sum_{n=1}^{N-1} \left(\dot{[u]}_n \dot{v}_n^+ + \lambda [u]_n v_n^+ - s\lambda \{\dot{u}\}_n \dot{v}_n^+ \right) + \dot{u}_0^+ \dot{v}_0^+ + \lambda u_0^+ v_0^+ - \frac{s}{2} \lambda \dot{u}_0^+ \dot{v}_0^+,$$
(31)

and the linear functional $F : \mathbb{V}_{DG} \to \mathbb{R}$ is defined as

$$F(v) := \hat{u}_1 \dot{v}_0^+ + \lambda \hat{u}_0 v_0^+ + \frac{s}{2} \lambda \hat{u}_1 \dot{v}_0^+.$$
(32)

For n = 1, we adopted the convention $u_0^- = \hat{u}_0$ and $\dot{u}_0^- = \hat{u}_1$. Moreover, we point out that the definition of the bilinear form $\mathcal{A}(\cdot, \cdot)$ makes sense whenever its arguments are, at least, $H^2(I_n)$ functions for any $n = 1, \ldots, N$. The existence and uniqueness of the discrete solution as well as stability bounds in a suitable mesh-dependent norm can be found in [4] for the case s = 0.

3.1 Algebraic formulation for the DG2 discretization

We focus on the algebraic formulation of problem (29) by considering a generic time slab I_n , where a local polynomial degree r is used. With the DG technique (29), it is possible to compute the solution of the problem separately for one time slab at a time, by using the solution computed at the previous time slab. To do so, we introduce a basis $\{\psi_{n,j}(t)\}_{j=1,\dots,r+1}$ for the polynomial space $\mathbb{P}^r(I_n), r+1$

being the dimension of the local finite-dimensional space \mathbb{V}_n^r . Next, we write the trial function u_{DG} as a linear combination of the basis functions, i.e.,

$$u(t) = \sum_{j=1}^{r+1} u_{n,j} \psi_{n,j}(t), \text{ for } t \in I_n,$$

where $u_{n,j} \in \mathbb{R}$, for j = 1, ..., r+1. By writing equation (29) for any test function $\psi_{n,i}$, i = 1, ..., r+1and n = 0, ..., N, we obtain the system

$$A\boldsymbol{u} = \boldsymbol{b},\tag{33}$$

where

$$A = \begin{bmatrix} I & & & \\ A_{-} & A_{+} & & \\ & A_{-} & A_{+} & \\ & & \ddots & \ddots & \\ & & & A_{-} & A_{+} \end{bmatrix}, \quad u = \begin{bmatrix} u_{0} \\ u_{1} \\ u_{2} \\ \vdots \\ u_{N} \end{bmatrix}, \quad b = \begin{bmatrix} b_{0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Here, $\boldsymbol{u} \in \mathbb{R}^{(r+1)(N+1)}$ is the vector containing the slab coefficients $\boldsymbol{u}_n = (u_{n,1}, \ldots, u_{n,r+1})^\top \in \mathbb{R}^{(r+1)}$, $n = 1, \ldots, N$, \boldsymbol{u}_0 and \boldsymbol{b}_0 contain the initial conditions, and the local matrices A_- and A_+ are given by

$$A_{-,ij} = -\langle \dot{\psi}_{n,j}, \dot{\psi}_{n+1,i} \rangle_{t_n} - \lambda \langle \psi_{n,j}, \psi_{n+1,i} \rangle_{t_n} - \frac{s}{2} \lambda \langle \dot{\psi}_{n,j}, \dot{\psi}_{n+1,i} \rangle_{t_n},$$
(34)

$$A_{+,ij} = (\ddot{\psi}_{n+1,j}, \dot{\psi}_{n+1,i})_{I_n} + \lambda(\psi_{n+1,j}, \dot{\psi}_{n+1,i})_{I_n} + \langle \dot{\psi}_{n+1,j}, \dot{\psi}_{n+1,i} \rangle_{t_n}$$

$$+ \lambda \langle \psi_{n+1,j}, \psi_{n+1,i} \rangle_{t_n} - \frac{s}{2} \lambda \langle \dot{\psi}_{n+1,j}, \dot{\psi}_{n+1,i} \rangle_{t_n}$$
(35)

for $i, j = 1, \ldots, r + 1$. Clearly, defining $G := -A_+^{-1}A_-$, (33) can be written as

$$A_{+}\boldsymbol{u}_{n+1} + A_{-}\boldsymbol{u}_{n} = \boldsymbol{0} \iff \boldsymbol{u}_{n+1} = G\boldsymbol{u}_{n} \text{ for } n = 0, 1, \dots, N-1.$$
(36)

To study the accuracy and consistency of a scheme of the form (33)-(36), we define error and truncation error as $\boldsymbol{e}_n := \boldsymbol{u}_n^{ex} - \boldsymbol{u}_n$ and $\boldsymbol{\theta}_n := \boldsymbol{u}_{n+1}^{ex} - G\boldsymbol{u}_n^{ex}$, for all n, where \boldsymbol{u}_n^{ex} denotes the vector containing the exact solution evaluated at the nodes of the slab I_n . The definitions of \boldsymbol{e}_n and $\boldsymbol{\theta}_n$ yield immediately

$$\boldsymbol{e}_{n+1} = \boldsymbol{G}\boldsymbol{e}_n + \boldsymbol{\theta}_n. \tag{37}$$

We now have the following result.

Lemma 1 (accuracy and consistency). Consider the scheme (37) for n = 0, ..., N - 1. Let $G = WDW^{-1}$ be the eigen-decomposition of G, with D denoting the diagonal matrix whose entries are the eigenvalues of G. Assume that the spectral radius of G satisfies $\rho(G) \leq 1$. If $\mathbf{e}_0 = \mathbf{0}$, $W = O(\Delta t^q)$, and $W^{-1}\boldsymbol{\theta}_n = O(\Delta t^p)$, for some $q, p \in \mathbb{N}$, then $\mathbf{e}_n = O(\Delta t^{q+p-1})$.

Proof. Using (37) recursively, we obtain $e_{n+1} = G^n e_0 + \sum_{\ell=0}^n G^\ell \theta_{n-\ell}$. Now, using that $e_0 = \mathbf{0}$ and the decomposition $G = WDW^{-1}$, we get

$$\boldsymbol{e}_{n+1} = W \sum_{\ell=0}^{n} D^{\ell} W^{-1} \boldsymbol{\theta}_{n-\ell}.$$
 (38)

Equation (38) allows us to estimate

$$\|\boldsymbol{e}_{n+1}\| \le \|W\| \sum_{\ell=0}^{n} \rho(G)^{\ell} \|W^{-1}\boldsymbol{\theta}_{n-\ell}\| \le \|W\|N \max_{\ell=0,\dots,N-1} \|W^{-1}\boldsymbol{\theta}_{\ell}\|,$$

where we used that $\rho(G) \leq 1$. The result follows by noticing that $N = O(\Delta t^{-1})$.

According to Lemma 1, the order of the method can be obtained directly from the orders of W and $W^{-1}\boldsymbol{\theta}_{\ell}$. However, one could think that the order of $\boldsymbol{\theta}_{\ell}$ alone is enough to determine the order of the method. Indeed, this is the case for, e.g., the explicit Euler method which can be written in the form $\boldsymbol{u}_{n+1} = G\boldsymbol{u}_n$ with $G = I + \Delta t G_1$. This particular structure of G allows one to follow a standard proof where, from $\boldsymbol{e}_{n+1} = \sum_{\ell=0}^n G^{\ell} \boldsymbol{\theta}_{n-\ell}$, it is possible to obtain $\|\boldsymbol{e}_{n+1}\| \leq N(1 + \Delta t \|G_1\|)^N \max_{\ell=0,\dots,N-1} \|\boldsymbol{\theta}_{\ell}\|$. Then, one can estimate $(1 + \Delta t \|G_1\|)^N \leq \exp(\Delta t N \|G_1\|) \leq \exp(T \|G_1\|) =: C$ to get $\|\boldsymbol{e}_{n+1}\| \leq CN \max_{\ell=0,\dots,N-1} \|\boldsymbol{\theta}_{\ell}\|$, and hence

$$\boldsymbol{\theta}_n = O(\Delta t^p) \Rightarrow \boldsymbol{e}_n = O(\Delta t^{p-1}).$$

Thus, the order of the method is obtained directly from the order of θ_{ℓ} . However, if G has a more general structure, like e.g., $G = G_0 + \Delta t G_1$ with $G_0 \neq I$, the above argument cannot be applied. In this case, Lemma 1 says that the orders of $W^{-1}\theta_{\ell}$ and W are sufficient to obtain the order of the method. Moreover, as we are going to see in the next sections, the order of the method cannot be determined from the order of θ_{ℓ} only, and the effect of W and W^{-1} will become apparent. Finally, we wish to remark that the orders Δt^q and Δt^p of W and W^{-1} are generally related. In fact, a direct calculation leads to $||W^{-1}|| = \sup_{||\mathbf{x}||=1} \frac{1}{||W\mathbf{x}||}$. Thus, if $W = O(\Delta t^p)$ then $W^{-1} = O(\Delta t^q)$ with q at least equal to -p. However, assuming that $W = \Delta t^p \sum_j \Delta t^j W_j$ with ker $W_0 \cap \ker W_1 \cap \cdots \cap \ker W_{\ell} \neq \emptyset$, for some $\ell \in \mathbb{N}$, then one can find a vector \mathbf{y} in this intersection and obtain for Δt small that $||W^{-1}|| = \sup_{||\mathbf{x}||=1} \frac{1}{||W\mathbf{x}||} = \frac{1}{||W\mathbf{x}||} = O(\Delta t^{-(p+\ell)})$. Then, it is crucial to study the dependence of W and W^{-1} on Δt .

In the following subsections, we investigate the structure of the matrices A_+ and A_- for different polynomial degrees and study the corresponding accuracy and consistency.

3.1.1 Polynomial degree r = 1

For \mathbb{V}_n^1 we define the basis functions

$$\psi_{n,1}(t) = \frac{t_n - t}{\Delta t}, \quad \psi_{n,2}(t) = \frac{t - t_{n-1}}{\Delta t}, \quad t \in [t_{n-1}, t_n].$$
(39)

By substituting the expressions (39) into (34)-(35) we obtain the matrices

$$A_{-} = \frac{1}{4\Delta t^2} \begin{bmatrix} -2s\lambda - 4 & -4\lambda\Delta t^2 + 2s\lambda + 4\\ 2s\lambda + 4 & -2s\lambda - 4 \end{bmatrix},$$
$$A_{+} = \frac{1}{4\Delta t^2} \begin{bmatrix} 2\lambda\Delta t^2 - 2s\lambda + 4 & -2\lambda\Delta t^2 + 2s\lambda - 4\\ 2\lambda\Delta t^2 + 2s\lambda - 4 & 2\lambda\Delta t^2 - 2s\lambda + 4 \end{bmatrix}$$

Thus, the matrix $G = -A_+^{-1}A_-$ is

$$G = \begin{bmatrix} 0 & 1\\ \frac{2+\lambda s}{\lambda s - \Delta t^2 \lambda - 2} & \frac{\Delta t^2 \lambda - 4}{\lambda s - \Delta t^2 \lambda - 2} \end{bmatrix}.$$
 (40)

Hence, (36) implies that

$$u_{n,2} = u_{n+1,1},$$

$$(2+s\lambda)u_{n,1} + (\lambda\Delta t^2 - 4)u_{n,2} = (s\lambda - \lambda\Delta t^2 - 2)u_{n+1,2}.$$
(41)

Notice that the equation $u_{n,2} = u_{n+1,1}$ represents the continuity of the DG solution between the adjacent slabs I_n and I_{n+1} . The meaning of the second equation in (41) will be clarified in what follows (see Theorem 1).

Let us now, study consistency and accuracy. A direct calculation using a Taylor expansion allows us to obtain

$$\boldsymbol{\theta}_{n} = \begin{bmatrix} 0\\ \frac{48\Delta t^{2}(\ddot{u}(t_{n})+\lambda u(t_{n}))-8\Delta t^{3}\lambda(\dddot{u}(t_{n})s-3\dot{u}(t_{n}))-48\dot{u}(t_{n})\Delta t\lambda s+4\Delta t^{4}(3\ddot{u}(t_{n})\lambda+\dddot{u}(t_{n}))+O(\Delta t^{5})}{24\lambda\Delta t^{2}-24(\lambda s-2)} \end{bmatrix}.$$
(42)

Now, recalling (27), we have that $\ddot{u}(t_n) + \lambda u(t_n) = 0$. Thus, it follows that $\theta_n = O(\Delta t)$ for s = O(1), $\theta_n = O(\Delta t^2)$ for $s = O(\Delta t)$, and $\theta_n = O(\Delta t^3)$ for $s = O(\Delta t^2)$. Notice also that choosing s with higher order does not increase the consistency order. Thus, we focus on the case $s = O(\Delta t^2)$ and set $s = a\Delta t^2$. Using $s = a\Delta t^2$ in (41), we get

$$u_{n,2} = u_{n+1,1},$$

$$(2 + a\Delta t^2\lambda)u_{n,1} + (\lambda\Delta t^2 - 4)u_{n,2} = (a\Delta t^2\lambda - \lambda\Delta t^2 - 2)u_{n+1,2}.$$
(43)

Moreover, we notice the following two particular cases:

$$\|\boldsymbol{\theta}_{n}\| = \begin{cases} \frac{|6\Delta t^{3}\lambda\dot{u}(t_{n}) + \Delta t^{4}(3\ddot{u}(t_{n})\lambda + \ddot{u}^{*}(t_{n})) + O(\Delta t^{5})|}{12 + 6\lambda\Delta t^{2}} & \text{if } s = 0 \ (a = 0), \\ \frac{|\Delta t^{4}(3\ddot{u}(t_{n})\lambda + \ddot{u}^{*}(t_{n})) + O(\Delta t^{6})|}{12 + 3\lambda\Delta t^{2}} & \text{if } s = \frac{\Delta t^{2}}{2} \ (a = \frac{1}{2}). \end{cases}$$
(44)

This implies that the method is first order if a = 0, and second order if $a = \frac{1}{2}$.

Now, we show an equivalence result between \mathbb{P}^1 -DG and the Newmark methods reviewed in section 2.1. For this purpose, we notice that choosing $f(u) = \lambda u$ the Newmark formula (5) can be written as $\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = G_{\text{New}} \begin{bmatrix} u_n \\ v_n \end{bmatrix}$, where

$$G_{\rm New} = \begin{bmatrix} \frac{-(1-2\beta)\Delta t^2\lambda + 2}{2\beta\Delta t^2\lambda + 2} & \frac{2\Delta t}{2\beta\Delta t^2\lambda + 2} \\ \frac{(\gamma - 2\beta)\Delta t^3\lambda^2 - 2\Delta t\lambda}{2\beta\Delta t^2\lambda + 2} & \frac{((\gamma - 2\beta)\Delta t^3\lambda + (2\beta - 2\gamma)\Delta t^2\lambda + 2}{2\beta\Delta t^2\lambda + 2} \end{bmatrix}$$

We can now prove our first result.

Theorem 1 (\mathbb{P}^1 -DG is a Newmark scheme). Let $s = a\Delta t^2$. The scheme (41), equivalently (43), is a Newmark scheme with $\beta = (1-a)/2$ and $\gamma = 1-a$. In particular, the DG vector $(u_{n,1}, u_{n,2})^{\top}$ can be mapped into the Newmark vector $(u_n, v_n)^{\top}$ by the matrix

$$T = \begin{bmatrix} 0 & 1\\ \frac{-(a+(1-a)/2-1/2)\Delta t^2\lambda - 1}{\Delta t} & \frac{(a+(1-a)/2-1)\Delta t^2\lambda + 1}{\Delta t} \end{bmatrix},$$

and the similarity relation $G = T^{-1}G_{\text{New}}T$ holds.

Proof. Because of the continuity condition $u_{n,2} = u_{n+1,1}$, we can introduce the notation $w_n = u_{n,2} = u_{n+1,1}$, $w_{n-1} = u_{n,1}$, and $w_{n+1} = u_{n+1,2}$. Therefore, by a simple manipulation, the second equation in (43) is equivalent to

$$w_{n+1} - 2w_n + w_{n-1} + \lambda \Delta t^2 \left(\frac{1-a}{2}w_{n+1} + \frac{1}{2}w_n + \frac{a}{2}w_{n-1}\right) = 0.$$
(45)

Now, we consider the Newmark scheme (5) for (27), and rewrite it as a single equation in only the u_n variables. To do so, we first write the first equation in (5) for two consecutive time steps as

$$u_{n+1} - u_n = \Delta t \ v_n + \Delta t^2 \ \left[-\left(\frac{1}{2} - \beta\right) \lambda u_n - \beta \lambda u_{n+1} \right],$$
$$u_n - u_{n-1} = \Delta t \ v_{n-1} + \Delta t^2 \ \left[-\left(\frac{1}{2} - \beta\right) \lambda u_{n-1} - \beta \lambda u_n \right].$$

By subtracting these two equations we get

$$u_{n+1} - 2u_n + u_{n-1} = \Delta t \ (v_n - v_{n-1}) + \Delta t^2 \ \left[-\beta \lambda u_{n+1} - \left(\frac{1}{2} - 2\beta\right) \lambda u_n + \left(\frac{1}{2} - \beta\right) \lambda u_{n-1} \right].$$

Now, using the second equation in (5), written as

$$v_n - v_{n-1} = \Delta t \ [-(1 - \gamma)\lambda u_{n-1} - \gamma \lambda u_n],$$

we obtain

$$u_{n+1} - 2u_n + u_{n-1} + \lambda \Delta t^2 \left[\beta u_{n+1} + \left(\frac{1}{2} - 2\beta + \gamma\right) u_n + \left(\frac{1}{2} + \beta - \gamma\right) u_{n-1} \right] = 0.$$
(46)

By comparing (46) with (45), we see that (45) is a Newmark scheme with $\beta = (1-a)/2$ and $\gamma = 1-a$. In particular, we proved that $u_n = w_n = u_{n,2}$ for all n. This corresponds to the first equation in $T(u_{n,1}, u_{n,2})^{\top} = (u_n, v_n)^{\top}$. To obtain the second equation of $T(u_{n,1}, u_{n,2})^{\top} = (u_n, v_n)^{\top}$, one can solve the first equation in (5) for v_n , replace the result into the second equation and use the relations $u_n = u_{n,2}$ and $u_{n+1} = u_{n,1}$. Finally, the similarity relation $G = T^{-1}G_{\text{New}}T$ can be obtained by a direct calculation.

According to the discussion provided in Section 2.1, the Newmark scheme is second-order accurate if and only if $\gamma = \frac{1}{2}$, it is stable if $\gamma \geq \frac{1}{2}$, while the choice of $\beta = \frac{1}{4}$ provides conservation properties. Using Theorem 1, we can immediately transfer this properties to our DG scheme. Since $\gamma = \frac{1}{2}$ is obtained only for $a = \frac{1}{2}$ ($s = \frac{\Delta t^2}{2}$), which also corresponds to $\beta = \frac{1}{4}$, we obtain that our DG scheme is first-order accurate for all values of a different from $\frac{1}{2}$. This is also in agreement with (44). Thus, the choice a = 0 (that is s = 0) corresponds to a first-order convergent method.

Even though one can benefit from the analogy with the class of Newmark methods, we prove convergence of our DG scheme using the general results obtained in Section 3.1. This will be useful to get a detailed analysis and compare the \mathbb{P}^1 and \mathbb{P}^2 cases.

Lemma 2 (Eigen-decomposition of G for \mathbb{P}^1). Consider the matrix G defined in (40) and let $s := a\Delta t^2$. The corresponding eigen-decomposition is $G = WDW^{-1}$ with

$$W = \begin{bmatrix} 1 & 1\\ \lambda_1 & \lambda_2 \end{bmatrix} \quad and \quad D = \begin{bmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{bmatrix}, \tag{47}$$

with $\lambda_{1,2} = \frac{\pm \Delta t \sqrt{(4a^2 - 4a + 1)\Delta t^2 \lambda^2 - 16\lambda} + \Delta t^2 \lambda - 4}{2(a - 1)\Delta t^2 \lambda - 4}$. Moreover, it holds that

- (a) If $a = \frac{1}{2}$, then $\rho(G) = 1$ for all Δt and λ .
- (b) If a = 0, then $\rho(G) < 1$ for all Δt and λ .

(c) If
$$a \in \left(\frac{1}{2} - \frac{2}{\Delta t \sqrt{\lambda}}, \frac{1}{2}\right)$$
, then $\rho(G) < 1$. If $a \in \left(\frac{1}{2}, \frac{1}{2} + \frac{2}{\Delta t \sqrt{\lambda}}\right)$, then $\rho(G) > 1$.

(d)
$$W = W_0 + O(\Delta t) = O(1)$$
, with $W_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Proof. The matrices V and D can be obtained by a direct calculation. Let us now prove point (a). If we set $a = \frac{1}{2}$, then $\lambda_{1,2} = \frac{4-\lambda\Delta t^2 \pm i4\Delta t\sqrt{\lambda}}{4+\lambda\Delta t^2}$, which have both unit modulus. Let us now focus on point (b). By setting a = 0 and using the change of variables $x = \Delta t\sqrt{\lambda}$, the eigenvalues of G become $\lambda_{1,2} = \frac{\pm x\sqrt{x^2-16}+x-4}{-2x-4}$. We distinguish two cases: $x \leq 4$ and x > 4. In the first case, the two eigenvalues are complex conjugate and thus have the same modulus. A direct calculation reveals that $|\lambda_{1,2}| < 1$ if and only if $-4x^4 - 8x^2 < 0$, which holds for all x > 0 (and thus for all Δt and λ). In the second case, one has $\rho(G) = \frac{|x\sqrt{x^2-16}+x^2-4|}{|2x^2+4|}$. Taking the square, one can compute that $\rho(G)^2 = 1/9 < 1$ for x = 4 and that $\lim_{x\to\infty} \rho(G)^2 = 1$. Moreover, a direct calculation allows us to get

$$\frac{d(\rho(G)^2)}{dx} = \frac{(x\sqrt{x^2 - 16} + x^2 - 4)^2}{(-2x^2 - 4)^2} \frac{\frac{2x^4 + 4x^2}{\sqrt{x^2 - 16}} + 4x^3 + \sqrt{x^2 - 16}(2x^2 + 4) + 4x}{(x^2 + 2)}$$

which is clearly positive for all x > 4. Thus, the spectral radius $\rho(G)$ is monotonically increasing and bounded by 1. Let us now focus on point (c). We consider only the case $(4a^2 - 4a + 1)x \le 16$, which is equivalent to $a \in \left[\frac{1}{2} - \frac{2}{x}, \frac{1}{2} + \frac{2}{x}\right]$. In this case, the two eigenvalues are complex conjugate and have the same modulus, and a direct calculation reveals that $|\lambda_{1,2}| < 1$ if and only if $(-8a^2 + 12a - 4)x^2 + 16a - 8 <$ 1, which holds for all $a \in \left(\frac{1}{2} - \frac{2}{x}, \frac{1}{2}\right)$. Thus, $\rho(G) < 1$ for $a \in \left(\frac{1}{2} - \frac{2}{x}, \frac{1}{2}\right)$ and $\rho(G) > 1$ for $a \in \left(\frac{1}{2}, \frac{1}{2} + \frac{2}{x}\right)$. Finally, point (d) follows by a direct Taylor expansion of W.

Now, we can finally come back to the consistency and accuracy of our DG method.

Theorem 2 (Consistency and accuracy for \mathbb{P}^1). Consider the DG scheme (37)-(36) with $s = a\Delta t^2$. Then $\theta_n = O(\Delta t^{p+1})$, $W^{-1}\theta_n = O(\Delta t^p)$, and $e_n = O(\Delta t^{p-1})$ with p = 2 for $a \in \{0\} \cup (\frac{1}{2} - \frac{2}{\Delta t\sqrt{\lambda}}, \frac{1}{2})$ and p = 3 for $a = \frac{1}{2}$. *Proof.* The result follows from Lemma 1 and Lemma 2 once the estimates for the consistency errors are obtained. Using (42) with $s = a\Delta t^2$ (and recalling that $\ddot{u}(t_n) + \lambda u(t_n) = 0$), one obtains

$$\boldsymbol{\theta}_n = \begin{bmatrix} 0\\ \frac{8\Delta t^3 \lambda (3-6a)\dot{u}(t_n) + 4\Delta t^4 (3\ddot{u}(t_n)\lambda + \ddot{u}(t_n)) + O(\Delta t^5)}{24\lambda\Delta t^2 (1-a) + 48} \end{bmatrix}.$$

Therefore, $\theta_n = O(\Delta t^3)$ for all considered values of a, and $\theta_n = O(\Delta t^4)$ for $a = \frac{1}{2}$. Similarly, a direct calculation using a Taylor expansion leads to

$$W^{-1}\boldsymbol{\theta}_n = \begin{bmatrix} \frac{(2a-1)\dot{u}(t_n)\Delta t^2\sqrt{\lambda}}{-4i} + \frac{\Delta t^3}{192} \left(\frac{24\ddot{x}(t_n)\lambda + 8\,\ddot{x}'(t_n)}{i\sqrt{\lambda}} + C_1(a)\lambda^3 + C_2(a)\lambda^2\right) + O(\Delta t^4) \\ \frac{(2a-1)\dot{u}(t_n)\Delta t^2\sqrt{\lambda}}{4i} - \frac{\Delta t^3}{192} \left(\frac{24\ddot{x}(t_n)\lambda + 8\,\ddot{x}'(t_n)}{i\sqrt{\lambda}} + C_1(a)\lambda^3 + C_2(a)\lambda^2\right) + O(\Delta t^4) \end{bmatrix},$$

where i is the imaginary unit, and

$$C_1(a) = \frac{12u(t_n)(a-a^2) - 3u(t_n)}{i\lambda^{3/2}} \quad \text{and} \quad C_2(a) = \frac{12\ddot{u}(t_n)(a-a^2) - 3\ddot{u}(t_n)}{i\lambda^{3/2}}.$$
$$\boldsymbol{\theta}_n = O(\Delta t^2) \text{ and } W^{-1}\boldsymbol{\theta}_n = O(\Delta t^3) \text{ for } a = \frac{1}{2}.$$

Thus, $W^{-1}\boldsymbol{\theta}_n = O(\Delta t^2)$ and $W^{-1}\boldsymbol{\theta}_n = O(\Delta t^3)$ for $a = \frac{1}{2}$.

3.1.2 Polynomial degree r = 2

For \mathbb{V}_n^2 we consider the basis functions

$$\psi_{n,1}(t) = -\frac{2(t - t_{n-\frac{1}{2}})(t_n - t)}{\Delta t^2}, \quad \psi_{n,2}(t) = \frac{4(t - t_{n-1})(t_n - t)}{\Delta t^2},$$

$$\psi_{n,3}(t) = \frac{2(t_{n-1} - t)(t_{n-\frac{1}{2}} - t)}{\Delta t^2}, \quad t \in [t_{n-1}, t_n].$$
(48)

Replacing (39) into (34)-(35), we get

$$A_{-} = \frac{1}{12\Delta t^2} \begin{bmatrix} 3(6s\lambda + 12) & -12(6s\lambda + 12) & -3(4\lambda\Delta t^2 - 18s\lambda - 36) \\ -12(2s\lambda + 4) & 48(2s\lambda + 4) & -36(2s\lambda + 4) \\ 3(2s\lambda + 4) & -12(2s\lambda + 4) & 9(2s\lambda + 4) \end{bmatrix},$$

$$A_{+} = \frac{1}{12\Delta t^{2}} \begin{bmatrix} 3(2\lambda\Delta t^{2} - 18s\lambda + 20) & -4(2\lambda\Delta t^{2} - 18s\lambda + 12) & (2\lambda\Delta t^{2} - 18s\lambda - 12) \\ 4(2\lambda\Delta t^{2} + 18s\lambda - 36) & -48(2s\lambda - 4) & -4(2\lambda\Delta t^{2} - 6s\lambda + 12) \\ -(2\lambda\Delta t^{2} + 18s\lambda - 84) & 4(2\lambda\Delta t^{2} + 6s\lambda - 36) & 3(2\lambda\Delta t^{2} - 2s\lambda + 20) \end{bmatrix}.$$

Thus, the matrix $G = -A_+^{-1}A_-$ is given by

$$G = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{(3\Delta t^2\lambda^2 + 36\lambda)s + 6\Delta t^2\lambda + 72}{2d} & \frac{s + 12\Delta t^2\lambda + 144}{d} & -\frac{36\lambda s - \Delta t^4\lambda^2 + 12\Delta t^2\lambda + 360}{2d} \\ \frac{(3\Delta t^2\lambda^2 - 36\lambda)s + 6\Delta t^2\lambda - 72}{d} & -\frac{(12\Delta t^2\lambda^2 - 144\lambda)s + 24\Delta t^2\lambda - 288}{d} & -\frac{72\lambda s + \Delta t^4\lambda^2 - 48\Delta t^2\lambda + 288}{d} \end{bmatrix},$$

where $d = (9\Delta t^2\lambda^2 + 36\lambda)s - \Delta t^4\lambda^2 - 6\Delta t^2\lambda - 72$. As for the \mathbb{P}^1 case, we notice that the first row of G corresponds to the continuity condition $(u_{n,3} = u_{n+1,1})$.

Now, we proceed as in Section 3.1.1 and study $\boldsymbol{\theta}_n$ by a Taylor expansion of u(t):

$$\boldsymbol{\theta}_n = \begin{bmatrix} 0 \\ \frac{\dot{u}(t_n)s\lambda\Delta t}{s\lambda - 2} + O(\Delta t^3) \\ \frac{2\dot{u}(t_n)s\lambda\Delta t}{s\lambda - 2} + O(\Delta t^3) \end{bmatrix}.$$

Thus, to get that θ_n is at least of order Δt^3 , one needs that $s = O(\Delta t^2)$. Hence, as for the \mathbb{P}^1 case, we set $s = a\Delta t^2$ and study the method with respect to the parameter a. In this case, the consistency error $\boldsymbol{\theta}_n$ becomes

$$\boldsymbol{\theta}_{n} = \begin{bmatrix} 0 \\ -\frac{(8\dot{u}(t_{n})a\lambda - \ddot{u}^{*}(t_{n}))\Delta t^{3}}{16} + \frac{(4u(t_{n})\lambda^{2} - 5\,\ddot{u}^{*}(t_{n}))\Delta t^{4}}{384} + O(\Delta t^{5}) \\ -\frac{((12a-2)\dot{u}(t_{n})\lambda - 3\,\ddot{u}^{*}(t_{n}))\Delta t^{3}}{16} + \frac{(4u(t_{n})\lambda^{2} - \ddot{u}^{*}(t_{n}))\Delta t^{4}}{96} + O(\Delta t^{5}) \end{bmatrix},$$
(49)

which shows that there is no value of a eliminating simultaneously the third-order terms of θ_n . In particular, in view of the relation $\frac{d}{dt}(u(t_n)\lambda + 3\ddot{u}(t_n)) = 0$, the choice a = -1/8 cancels the third-order term in the second component of θ_n , but not the one in the third component. Similarly, the choice a = -1/12 cancels the third-order term in the third component of θ_n , but not the one in the third component. Similarly, the choice a = -1/12 cancels the third-order term in the third component of θ_n , but not the one in the second component. Therefore, the analysis of θ_n does not provide any information on the choice of a: for all values of a one gets that $\theta_n = O(\Delta t^3)$.

To better understand the role of the parameter a, we recall the framework of Lemma 1 (used already in Section 3.1.1) and study $W^{-1}\theta_n$. To do so, we first need to compute the eigen-decomposition $G = WDW^{-1}$ obtained by the matrices

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{4} & \widehat{\lambda}_1 & \widehat{\lambda}_2 \\ 0 & \lambda_1 & \lambda_2 \end{bmatrix}$$
(50)

with

$$\lambda_{1,2} = \frac{(6a-1)\Delta t^4 \lambda^2 + 60\Delta t^2 \lambda - 144}{(18a-2)\Delta t^4 \lambda^2 + (72a-12)\Delta t^2 \lambda - 144} \\ \pm \frac{\Delta t \sqrt{a_4 \Delta t^6 \lambda^4 + a_3 \Delta t^4 \lambda^3 + a_2 \Delta t^2 \lambda^2 + a_1 \lambda}}{(18a-2)\Delta t^4 \lambda^2 + (72a-12)\Delta t^2 \lambda - 144},$$

$$\hat{\lambda}_{1,2} = \frac{(1-6a)\Delta t^6 \lambda^3 - (288a+24)\Delta t^4 \lambda^2 - (864a+432)\Delta t^2 \lambda - 3456}{(36a-4)\Delta t^6 \lambda^3 + (24-288a)\Delta t^4 \lambda^2 - 1728a\Delta t^2 \lambda + 3456} \\ \mp \frac{(\Delta t^3 \lambda + 12\Delta t)\sqrt{a_4 \Delta t^6 \lambda^4 + a_3 \Delta t^4 \lambda^3 + a_2 \Delta t^2 \lambda^2 + a_1 \lambda}}{(36a-4)\Delta t^6 \lambda^3 + (24-288a)\Delta t^4 \lambda^2 - 1728a\Delta t^2 \lambda + 3456},$$
(51)

where

$$a_4 = 144a^2 - 24a + 1, a_3 = 1728a^2 + 720a - 144, a_2 = 5184a^2 + 3456, a_1 = -20736.$$

Now, using again a Taylor expansion, we obtain by a direct calculation that

$$W^{-1}\boldsymbol{\theta}_{n} = \begin{bmatrix} \frac{(4\dot{u}(t_{n})+3\ddot{u}(t_{n}))\Delta t^{3}}{12} + O(\Delta t^{4}) \\ -\frac{i\sqrt{\lambda}((2a-1)\lambda\dot{u}(t_{n})-\ddot{u}(t_{n}))\Delta t^{2}}{4\lambda} - \frac{C_{1}(a)\Delta t^{3}}{48\lambda} + O(\Delta t^{4}) \\ -\frac{i\sqrt{\lambda}((-2a+1)\lambda\dot{u}(t_{n})+\ddot{u}(t_{n}))\Delta t^{2}}{4\lambda} - \frac{C_{2}(a)\Delta t^{3}}{48\lambda} + O(\Delta t^{4}) \end{bmatrix},$$
(52)

where

$$C_{1}(a) = [8\dot{u}(t_{n}) + (-3\sqrt{\lambda}ia^{2} + 6\sqrt{\lambda}ia + 4\sqrt{\lambda}i)u(t_{n})]\lambda^{2}$$

+ $[(-3\sqrt{\lambda}ia^{2} + 6\sqrt{\lambda}ia)\ddot{u}(t_{n}) + 6\ddot{u}(t_{n})]\lambda - 2\sqrt{\lambda}i\ddot{u}(t_{n}),$
$$C_{2}(a) = [8\dot{u}(t_{n}) - (-3\sqrt{\lambda}ia^{2} + 6\sqrt{\lambda}ia + 4\sqrt{\lambda}i)u(t_{n})]\lambda^{2}$$

+ $[(3\sqrt{\lambda}ia^{2} - 6\sqrt{\lambda}ia)\ddot{u}(t_{n}) + 6\ddot{u}(t_{n})]\lambda + 2\sqrt{\lambda}i\ddot{u}(t_{n}).$

In view of the relation $\frac{d}{dt}(u(t_n)\lambda + 3\ddot{u}(t_n)) = 0$, we observe from (52) that a = 0 is the only value that can make $W^{-1}\theta_n$ of order Δt^3 . This is in contrast with the fact that some nonzero values of a can improve the order of some components of θ_n . This observation shows very clearly that, to study our DG methods the key quantity to look at is $W^{-1}\theta_n$, according to the framework of Lemma 1. A study of θ_n only can not be sufficient for fully understanding the behavior of the method. We next summarize our findings, where we focus on three different choices of a: a = 0, making $W^{-1}\theta_n$ of order Δt^3 , and a = -1/8 and a = -1/12, making some components of θ_n of order Δt^4 .

Lemma 3 (Eigen-decomposition of G for \mathbb{P}^2). The eigen-decomposition of G is $G = WDW^{-1}$ obtained by the matrices given in (50). For all values a = 0, -1/8, -1/12 it holds that $\rho(G) \leq 1$ for all small enough values of $\sqrt{\lambda}\Delta t$, and W = O(1).

Proof. The eigen-decomposition $G = WDW^{-1}$ and the fact that W = O(1) can be obtained by direct calculations (and a Taylor expansion). Let us now study the eigenvalues of a. For a = 0, and introducing the (positive) variable $x = \Delta t \sqrt{\lambda}$, we have

$$\lambda_{1,2} = \frac{-x^4 + 60x^2 - 144 \pm x\sqrt{x^6 - 144x^4 + 3456x^2 - 20736}}{-2x^4 - 12x^2 - 144}$$



Figure 2: Spectral radius of G as a function of $\sqrt{\lambda \Delta t}$ for different values of a.

Now, for x small enough, it holds that $x^6 - 144x^4 + 3456x^2 - 20736 \le 0$. In this case, the eigenvalues λ_1 and λ_2 are complex conjugates and have the same modulus. A direct calculation shows that $|\lambda_{1,2}| \le 1$ if and only if $-4x^8 - 24x^6 - 288x^4 \le 0$, which holds for all $x \ge 0$. The same arguments can be used to obtain the result for a = -1/12 and a = -1/8.

Remark 1 (Spectral radius of $\rho(G)$). The result of Lemma 3 on the spectral radius $\rho(G)$ could be improved. In particular, one has that $\rho(G) \leq 1$ for $a \leq 0$ and any values of Δt and λ , as shown in Figure 2 (left). Instead, if a > 0 the spectral radius $\rho(G)$ becomes positive, meaning that the method is unstable. However, proving these results requires the study of third-order polynomials, like $x^6 - 144x^4 + 3456x^2 - 20736$ shown in the proof of Lemma 3. In this case, for example, the corresponding roots can be obtained by using Cardano's formula. However, this approach leads to complicated formulas and their study is beyond the scope of this work.

Theorem 3 (Consistency and accuracy for \mathbb{P}^2). Consider the DG scheme (37)-(36) with $s = a\Delta t^2$. Then, under the assumptions of Lemma 3, it holds that $\theta_n = O(\Delta t^3)$ for any value of $a \in \mathbb{R}$, $W^{-1}\theta_n = O(\Delta t^2)$ for any $a \in \mathbb{R}$ and $W^{-1}\theta_n = O(\Delta t^3)$ for a = 0. Correspondingly, $e_n = O(\Delta t^p)$ with p = 1 for a = -1/12 and a = -1/8 and p = 2 for a = 0.

Proof. The proof follows from Lemma 3, Lemma 1 and the discussion above.

From the results obtained in this section, we can clearly state that, if one considers a second-order problem of the form (27), it is more appropriate to use a DG method with \mathbb{P}^1 elements, rather than \mathbb{P}^2 . In fact, an appropriate choice of the parameter *a* leads the \mathbb{P}^1 -DG to second-order accuracy, which is the best accuracy that one can obtain with \mathbb{P}^2 .

3.1.3 Polynomial degree r = 3

For \mathbb{V}_n^3 we consider the set of basis functions

$$\begin{split} \psi_{n,1}(t) &= \frac{(t_n - 3t + 2t_{n-1})(2t_n - 3t + t_{n-1})(t_n - t)}{2\Delta t^3}, \\ \psi_{n,2}(t) &= 9\frac{(t - t_{n-1})(2t_n - 3t + t_{n-1})(t_n - t)}{2\Delta t^3}, \\ \psi_{n,3}(t) &= -9\frac{(t - t_{n-1})(t_n - 3t + 2t_{n-1})(t_n - t)}{2\Delta t^3}, \\ \psi_{n,4}(t) &= \frac{(t - t_{n-1})(t_n - 3t + 2t_{n-1})(2t_n - 3t + t_{n-1})}{2\Delta t^3}, \quad t \in [t_{n-1}, t_n]. \end{split}$$

In this case, one can proceed as in the previous sections to estimate $\boldsymbol{\theta}_n$, $W^{-1}\boldsymbol{\theta}_n$, and \boldsymbol{e}_n . This leads to an estimate $\boldsymbol{\theta}_n = O(\Delta t^4)$, $W^{-1}\boldsymbol{\theta}_n = O(\Delta t^4)$ and thus by Lemma 1, to $\boldsymbol{e}_n = O(\Delta t^3)$. If one chooses $s = a\Delta t^2$ or $s = a\Delta t^3$, these orders cannot be improved by an appropriate choice of a, exactly as in the \mathbb{P}^2 case.

3.2 DG2 as GLMs

In this section, we discuss the relation between the DG method introduced at the beginning of section 3 and the GLMs of section 2.3. In particular, we have seen in section 3.1 that DG methods for the second-order equation (27) can be written in the form $\mathbf{u}_{n+1} = G\mathbf{u}_n$, where $G = -A_+^{-1}A_-$. Now, the matrix G generally has a special structure (see sections 3.1.1, 3.1.2 and 3.1.3), which is used to prove that the main result of this section is given by the following theorem.

Theorem 4 (DG and GLM). Assume that the determinant of A_+ has the form $d = \alpha_0 + \sum_{k=1}^N \lambda^k \Delta t^{2k} \alpha_k$, for some coefficients $\alpha_k \in \mathbb{R}$ independent of Δt and λ . Moreover, assume that $G \in \mathbb{R}^{(r+1)\times(r+1)}$ has the form

$$G = \frac{1}{d} \Big(G_0 - \sum_{k=1}^N \lambda^k \Delta t^{2k} G_k \Big), \tag{53}$$

where the G_k are independent of Δt and λ . Then the iteration $\mathbf{u}_{n+1} = G\mathbf{u}_n$ is equivalent to a GLM (12) with $\mathbf{y}^{[n]} \in \mathbb{R}^{2(r+1)}$ such that $(\mathbf{y}^{[n]})_{1:r+1} = \mathbf{u}_n$, and

$$A = \begin{bmatrix} \frac{\alpha_1}{\alpha_0} I_{r+1} & \frac{\alpha_2}{\alpha_0} I_{r+1} & \cdots & \frac{\alpha_{N-1}}{\alpha_0} I_{r+1} & \frac{\alpha_N}{\alpha_0} I_{r+1} \\ -I_{r+1} & 0 & \cdots & 0 & 0 \\ 0 & -I_{r+1} & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & -I_{r+1} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} \frac{1}{\alpha_0} G_0 & I_{r+1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$
(54)

$$B = \begin{bmatrix} \frac{\alpha_1}{\alpha_0}I & \frac{\alpha_2}{\alpha_0}I_{r+1} & \cdots & \frac{\alpha_{N-1}}{\alpha_0}I_{r+1} & \frac{\alpha_N}{\alpha_0}I\\ \frac{1}{\alpha_0}G_1 & \frac{1}{\alpha_0}G_2 & \cdots & \frac{1}{\alpha_0}G_{N-1} & \frac{1}{\alpha_0}G_N \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\alpha_0}G_0 & I_{r+1}\\ 0 & 0 \end{bmatrix},$$
(55)

where $I_{r+1} \in \mathbb{R}^{(r+1) \times (r+1)}$ is the identity matrix.

Proof. Using the expression of d and (53), the formula $\mathbf{u}_{n+1} = G\mathbf{u}_n$ becomes

$$\mathbf{u}_{n+1} + \sum_{k=1}^{N} \lambda^k \Delta t^{2k} \frac{\alpha_k}{\alpha_0} \mathbf{u}_{n+1} = \frac{1}{\alpha_0} G_0 \mathbf{u}_n - \sum_{k=1}^{N} \lambda^k \Delta t^{2k} \frac{1}{\alpha_0} G_k \mathbf{u}_n.$$
(56)

Now to write (56) in the form (12), we introduce the (explicit) variables $\mathbf{y}^{[n]} \in \mathbb{R}^{2(r+1)}$,

$$(\mathbf{y}^{[n]})_{1:r+1} := \mathbf{u}_n,$$

$$(\mathbf{y}^{[n]})_{r+1:2(r+1)} := -\sum_{k=1}^N \frac{\lambda^k \Delta t^{2k}}{\alpha_0} G_k \mathbf{u}_n,$$

(57)

and define the (implicit) variable Y_1 via the equation

$$Y_1 + \sum_{k=1}^{N} \lambda^k \Delta t^{2k} \frac{\alpha_k}{\alpha_0} Y_1 = \frac{1}{\alpha_0} G_0(\mathbf{y}^{[n]})_{1:r+1} + (\mathbf{y}^{[n]})_{r+1:2(r+1)}.$$
(58)

Next, we introduce the variables Y_k , k = 2, ..., N, as

$$Y_k := \lambda \Delta t^2 Y_{k-1}, \text{ for } k = 2, \dots, N,$$
(59)

which allows us to rewrite (58) as

$$Y_1 + \lambda \Delta t^2 \sum_{k=1}^{N} \frac{\alpha_k}{\alpha_0} Y_k = \frac{1}{\alpha_0} G_0(\mathbf{y}^{[n]})_{1:r+1} + (\mathbf{y}^{[n]})_{r+1:2(r+1)}.$$
(60)

Now, we notice that (56), (58), and (60) are equivalent and hence $Y_1 = \mathbf{u}_{n+1} = (\mathbf{y}^{[n+1]})_{1:r+1}$. This allows us to formulate (60) as

$$(\mathbf{y}^{[n+1]})_{1:r+1} + \lambda \Delta t^2 \sum_{k=1}^{N} \frac{\alpha_k}{\alpha_0} Y_k = \frac{1}{\alpha_0} G_0(\mathbf{y}^{[n]})_{1:r+1} + (\mathbf{y}^{[n]})_{r+1:2(r+1)}.$$
(61)

Now, using the second line of (57) and (59), we can write

$$(\mathbf{y}^{[n+1]})_{r+1:2(r+1)} = -\sum_{k=1}^{N} \frac{\lambda^k \Delta t^{2k}}{\alpha_0} G_k \mathbf{u}_{n+1} = -\sum_{k=1}^{N} \frac{\lambda \Delta t^2}{\alpha_0} G_k Y_k.$$
 (62)

The result follows by grouping together (60), (59), (61), and (62).

An important remark comes from the proof of the previous theorem. From (59) we get $Y_k = \lambda^k \Delta t^{2k} Y_1$. Thus, recalling that $Y_1 = \mathbf{u}_{n+1}$ is an approximation to u(t) at the interpolation nodes $t_n + c_i \Delta t$, we see that

$$(Y_k)_i = \lambda^k \Delta t^{2k} (Y_1)_i \approx \lambda^k \Delta t^{2k} u(t_n + c_i \Delta t) = \Delta t^{2k} u^{(2k)} (t_n + c_i \Delta t).$$

Therefore, one cannot consider the analysis approach reviewed in Section 2.3 (where $Y \approx u(t_{n+1})$; see (20)). For this reason, we present an approach that is a little different from the one considered in the literature (see, e.g., [13]), but that can be applied to our DG framework. The first step is to rewrite (12) in the equivalent form

$$Y^{n+1} + \Delta t^2 \lambda A Y^{n+1} = U \mathbf{y}^{[n]},$$

$$\mathbf{y}^{[n+1]} = G_{\text{GLM}} \mathbf{y}^{[n]},$$
(63)

where $G_{\text{GLM}} := V - \Delta t^2 \lambda B [I + \Delta t^2 \lambda A]^{-1} U$ and we introduced the superscript n + 1 in Y^{n+1} to make the dependence of Y on the time step explicit. We consider that Y^n and $\mathbf{y}^{[n]}$ are approximations to the solution vectors \widehat{Y}^n and $\widehat{\mathbf{y}}^{[n]}$. Now, it is natural to define the truncation errors T^n and $\boldsymbol{\theta}^{[n]}$ as the residuals of the two (block) equations in (63) when Y^n and $\mathbf{y}^{[n]}$ are replaced by \widehat{Y}^n and $\widehat{\mathbf{y}}^{[n]}$,

$$\widehat{Y}^{n+1} + \Delta t^2 \lambda A \widehat{Y}^{n+1} = U \widehat{\mathbf{y}}^{[n]} + T^n,$$

$$\widehat{\mathbf{y}}^{[n+1]} = G_{\text{GLM}} \widehat{\mathbf{y}}^{[n]} + \boldsymbol{\theta}^{[n]}.$$
(64)

Now, neglecting the issue of a starting procedure (see [9, 13]), namely $Y = \hat{Y}^0$ and $\mathbf{y}^{[0]} = \hat{\mathbf{y}}^{[0]}$, and introducing the errors

$$E^n := \widehat{Y}^n - Y^n$$
 and $\mathbf{e}^{[n]} := \widehat{\mathbf{y}}^{[n]} - \mathbf{y}^{[n]},$

we obtain

$$E^{n+1} + \Delta t^2 \lambda A E^{n+1} = U \mathbf{e}^{[n]} + T^n,$$

$$\mathbf{e}^{[n+1]} = G_{\text{GLM}} \mathbf{e}^{[n]} + \boldsymbol{\theta}^{[n]},$$
(65)

and the following result.

Theorem 5 (convergence of GLM). Consider (65) for n = 0, ..., N-1. Let $G_{\text{GLM}} = WDW^{-1}$ be the eigen-decomposition of G_{GLM} , with D denoting the diagonal matrix whose entries are the eigenvalues of G_{GLM} , and assume that $\rho(G_{\text{GLM}}) \leq 1$. If $\mathbf{e}^{[0]} = \mathbf{0}$, $E^0 = \mathbf{0}$, $W = O(\Delta t^q)$, and $W^{-1}\boldsymbol{\theta}_n = O(\Delta t^{p+1})$ and $T^n = O(\Delta t^{a+1})$, for some $a, p \in \mathbb{N}^+$, then $\mathbf{e}^{[n]} = O(\Delta t^{q+p})$ and $E^n = O(\Delta t^{\min(a+1,q+p)})$.

Proof. By the Neumann series and a simple manipulation of the first equation in (65) one gets

$$E^{n+1} = [I + \Delta t^2 \lambda A]^{-1} (U \mathbf{e}^{[n]} + T^n) = U \mathbf{e}^{[n]} + T^n + O(\Delta t^2).$$

Hence, if $T^n = O(\Delta t^{a+1})$ and $\mathbf{e}^{[n]} = O(\Delta t^{b+1})$, for some $a, b \in \mathbb{N}_+$, then $E^n = O(\Delta t^{\min(a+1,b+1)})$. Now, the second equation in (65) has exactly the form of (37), and the result follows from Lemma 1.

Let us now turn our attention to the truncation errors T^n and θ^n . For this purpose, we proceed as in Section 2.3, consider the exact solution $\hat{Y}_k^n = \Delta t^{(2k-2)} y^{(2k-2)}(t_n + \mathbf{c}\Delta t)$ (namely, $\hat{Y}_{k,i}^n = \Delta t^{2k-2} y^{(2k-2)}(t_n + c_i\Delta t)$) and assume that the exact $\hat{\mathbf{y}}^{[n]}$ has the form

$$\widehat{\mathbf{y}}^{[n]} = \sum_{j=0}^{\infty} y^{(j)}(t_n) \Delta t^j \mathbf{q}_j,$$

for some vectors \mathbf{q}_j , $j = 0, 1, 2, \dots$ Replacing them into the first line of (64), and denoting by $\widehat{U} := \left[\frac{1}{\alpha_0}G_0 \ I_{r+1}\right]$, we can compute from the first block

$$T_{1}^{n} = \widehat{Y}_{1}^{n+1} + \Delta t^{2} \lambda \sum_{k=1}^{N} \frac{\alpha_{k}}{\alpha_{0}} \widehat{Y}_{k}^{n+1} - \widehat{U}\widehat{\mathbf{y}}^{[n]}$$

$$= y(t_{n} + \mathbf{c}\Delta t) + \Delta t^{2} \lambda \sum_{k=1}^{N} \frac{\alpha_{k}}{\alpha_{0}} \Delta t^{2k-2} y^{(2k-2)}(t_{n} + \mathbf{c}\Delta t) - \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \widehat{U} \mathbf{q}_{j}$$

$$= y(t_{n} + \mathbf{c}\Delta t) - \sum_{k=1}^{N} \frac{\alpha_{k}}{\alpha_{0}} \Delta t^{2k} y^{(2k)}(t_{n} + \mathbf{c}\Delta t) - \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \widehat{U} \mathbf{q}_{j}$$

$$= \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \frac{\Delta t^{j}}{j!} \mathbf{c}^{j} - \sum_{k=1}^{N} \frac{\alpha_{k}}{\alpha_{0}} \Delta t^{2k} \sum_{j=0}^{\infty} y^{(2k+j)}(t_{n}) \frac{\Delta t^{j}}{j!} \mathbf{c}^{j} - \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \widehat{U} \mathbf{q}_{j}$$

$$= \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \Big[\frac{1}{j!} \mathbf{c}^{j} - \widehat{U} \mathbf{q}_{j} \Big] - \sum_{j=0}^{\infty} \sum_{k=1}^{N} \frac{\alpha_{k}}{\alpha_{0}} y^{(2k+j)}(t_{n}) \frac{\Delta t^{j+2k}}{j!} \mathbf{c}^{j},$$
(66)

where $\mathbf{c}^{j} = [c_{1}^{j}, \dots, c_{N}^{j}]^{\top}$. Now, notice that

$$\sum_{j=0}^{\infty} \sum_{k=1}^{N} \frac{\alpha_k}{\alpha_0} y^{(2k+j)}(t_n) \frac{\Delta t^{j+2k}}{j!} \mathbf{c}^j = \sum_{j=2}^{\infty} \sum_{k=1}^{\lfloor j/2 \rfloor} \frac{\alpha_k}{\alpha_0} y^{(j)}(t_n) \frac{\Delta t^j}{(j-2k)!} \mathbf{c}^{j-2k},$$

which replaced into (66) leads to

$$T_{1}^{n} = \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \left[\frac{1}{j!} \mathbf{c}^{j} - \widehat{U} \mathbf{q}_{j} - \frac{1}{\alpha_{0}} \sum_{k=1}^{\lfloor j/2 \rfloor} \frac{\alpha_{k}}{(j-2k)!} \mathbf{c}^{j-2k} \right].$$
(67)

Thus, the truncation error T_1^n is of order $O(\Delta t^{a+1}),$ for $a\in\mathbb{N},$ if

$$\widehat{U}\mathbf{q}_{j} + \frac{1}{\alpha_{0}} \sum_{k=1}^{\lfloor j/2 \rfloor} \frac{\alpha_{k}}{(j-2k)!} \mathbf{c}^{j-2k} = \frac{1}{j!} \mathbf{c}^{j}, \text{ for } j = 0, 1, \dots, a.$$
(68)

Notice that, by means of (59), $T_1^n = O(\Delta t^{a+1})$ implies that $T_k^n = O(\Delta t^{a+1+2(k-1)})$ for k = 2, ..., N. Now, replacing the exact solution $\hat{\mathbf{y}}^{[n]}$ into the second block of (64), recalling the definition of $G_{\rm GLM}$, and using Neumann series and Taylor expansion, we get

$$\begin{aligned} \boldsymbol{\theta}^{[n]} &= \widehat{\mathbf{y}}^{[n+1]} - G_{\text{GLM}} \widehat{\mathbf{y}}^{[n]} \\ &= \sum_{j=0}^{\infty} y^{(j)}(t_{n+1}) \Delta t^{j} \mathbf{q}_{j} - \left[V - \Delta t^{2} \lambda B [I + \Delta t^{2} \lambda A]^{-1} U \right] \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \mathbf{q}_{j} \\ &= \sum_{j=0}^{\infty} y^{(j)}(t_{n+1}) \Delta t^{j} \mathbf{q}_{j} - \left[V + B \left[\sum_{\ell=0}^{\infty} (-1)^{\ell} \Delta t^{2\ell+2} \lambda^{\ell+1} A^{k} \right] U \right] \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \mathbf{q}_{j} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} y^{(j+\ell)}(t_{n}) \frac{1}{\ell!} \Delta t^{j+\ell} \mathbf{q}_{j} - \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} V \mathbf{q}_{j} \\ &- \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^{\ell} \Delta t^{j+2\ell+2} y^{(j+2\ell+2)}(t_{n}) B A^{k} U \mathbf{q}_{j} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} y^{(j)}(t_{n}) \frac{1}{\ell!} \Delta t^{j} \mathbf{q}_{j-\ell} - \sum_{j=0}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} V \mathbf{q}_{j} \\ &- \sum_{j=2}^{\infty} \sum_{\ell=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^{\ell} \Delta t^{j} y^{(j)}(t_{n}) B A^{\ell} U \mathbf{q}_{j-2\ell-2} \\ &= y(t_{n}) \left[\mathbf{q}_{0} - V \mathbf{q}_{0} \right] + y^{(1)}(t_{n}) \Delta t \left[\mathbf{q}_{0} + \mathbf{q}_{1} - V \mathbf{q}_{1} \right] \\ &+ \sum_{j=2}^{\infty} y^{(j)}(t_{n}) \Delta t^{j} \left[\sum_{\ell=0}^{j} \frac{1}{\ell!} \mathbf{q}_{j-\ell} - \sum_{\ell=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^{\ell} B A^{\ell} U \mathbf{q}_{j-2\ell-2} - V \mathbf{q}_{j} \right]. \end{aligned}$$

Thus, the truncation error $\boldsymbol{\theta}^{[n]}$ is of order $O(\Delta t^{p+1})$, for $p \in \mathbb{N}$, if

$$V \mathbf{q}_{0} = \mathbf{q}_{0},$$

$$V \mathbf{q}_{1} = \mathbf{q}_{0} + \mathbf{q}_{1},$$

$$V \mathbf{q}_{j} = \sum_{\ell=0}^{j} \frac{1}{\ell!} \mathbf{q}_{j-\ell} - \sum_{\ell=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^{\ell} B A^{\ell} U \mathbf{q}_{j-2\ell-2}, \text{ for } j = 2, \dots, p.$$
(70)

We summarize our findings in the following theorem.

Theorem 6 (Consistency of DG2 as GLM - order conditions). Let \mathbf{q}_j , $j = 0, 1, \ldots$, be vectors such that the order conditions (68) and (70) hold for some $p \in \mathbb{N}$ and $a \in \mathbb{N}$, respectively. Then, a DG method in the form (12) satisfies $T^n = O(\Delta t^{a+1})$ and $\boldsymbol{\theta}^{[n]} = O(\Delta t^{p+1})$.

In light of Theorem 5, the above result may not be sufficient to get an accurate estimate of the order of the method, since the role of W^{-1} applied to $\boldsymbol{\theta}^{[n]}$ may have a non-negligible impact. To estimate it, we still rely on the order conditions (70). To do so, we first consider an expansion $W^{-1} = \sum_{j=0}^{\infty} \Delta t^j W_j$, for some matrices W_j . Combining this expansion with (70), the relations

$$W_{\ell}V\mathbf{q}_{0} = W_{\ell}\mathbf{q}_{0},$$

$$W_{\ell}V\mathbf{q}_{1} = W_{\ell}(\mathbf{q}_{0} + \mathbf{q}_{1}),$$

$$W_{\ell}V\mathbf{q}_{j} = W_{\ell}\left[\sum_{\ell=0}^{j} \frac{1}{\ell!}\mathbf{q}_{j-\ell} - \sum_{\ell=0}^{\lfloor\frac{j-2}{2}\rfloor} (-1)^{\ell}BA^{\ell}U\mathbf{q}_{j-2\ell-2}\right], \text{ for } j = 2, \dots, p_{\ell},$$
(71)

arise naturally and the following result can be proved by a direct estimate of the term $W^{-1}\boldsymbol{\theta}^{[n]}$.

Theorem 7 (Consistency of DG2 as GLM - more order conditions). Assume that there exist vectors \mathbf{q}_{ℓ} , $\ell = 1, \ldots, p$, such that (70) holds. Further, assume that (71) holds for some $p_{\ell} \in \mathbb{N}$, for all $\ell = 0, 1, \ldots$ Then, $W^{-1}\boldsymbol{\theta}^{[n]} = O(\Delta t^{p_{\tilde{\ell}} + \tilde{\ell} + 1})$ with $\tilde{\ell} = \arg\min_{\ell} p_{\ell}$. Clearly, $p_{\tilde{\ell}} = \min_{\ell} p_{\ell} \ge p$. We now apply the theory developed in this section to the case \mathbb{P}^1 . The reader may appreciate how complicated the analysis can become in the GLM formalisms. For this reason, here we do not report the cases \mathbb{P}^2 and \mathbb{P}^3 .

Let us recall the matrix $G = -A_+^{-1}A_-$ (for $s = \Delta t^2/2$) computed in (40) and notice that $d = -2 - \lambda \Delta t^2/2$, with $\alpha_0 = -2$ and $\alpha_1 = -1/2$, and $G_0 = \begin{bmatrix} 0 & -2 \\ 2 & -4 \end{bmatrix}$ and $G_1 = \begin{bmatrix} 0 & 1/2 \\ -1/2 & -1 \end{bmatrix}$. Thus, the GLM matrices are

$$A = \frac{\alpha_1}{\alpha_0} I_2, \quad U = \begin{bmatrix} \frac{1}{\alpha_0 G_0} & I_2 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\alpha_1}{\alpha_0} I_2 \\ \frac{1}{\alpha_0} G_1 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\alpha_0} G_0 & I_2 \\ 0 & 0 \end{bmatrix}.$$

Now, direct calculations reveal that

$$G_{\rm GLM} = \begin{bmatrix} 0 & 1 - \frac{\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} & 1 - \frac{\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} & 0 \\ \frac{\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} - 1 & 2 - \frac{\lambda \Delta t^2}{2(\lambda \Delta t^2/4+1)} & 0 & 1 - \frac{\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} \\ - \frac{\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} & \frac{\lambda \Delta t^2}{2(\lambda \Delta t^2/4+1)} & 0 & - \frac{\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} \\ \frac{\lambda \Delta t^2}{2(\lambda \Delta t^2/4+1)} & - \frac{5\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} & - \frac{\lambda \Delta t^2}{4(\lambda \Delta t^2/4+1)} & - \frac{\lambda \Delta t^2}{2(\lambda \Delta t^2/4+1)} \end{bmatrix},$$

and that $G_{\text{GLM}} = W^{-1}DW$ with

$$D = \operatorname{diag}(0, 0, \gamma_1, \gamma_2),$$

where $\gamma_1 = \frac{(2+i\Delta t\sqrt{\lambda})^2}{\Delta t^2\lambda + 4}$ and $\gamma_2 = -\frac{(2i+\Delta t\sqrt{\lambda})^2}{\Delta t^2\lambda + 4}$, and

$$W = \begin{bmatrix} -2 & 1 & \frac{4(\Delta t\sqrt{\lambda}+2i)}{\Delta t^2\lambda(\Delta t\sqrt{\lambda}-6i)} & \frac{4(\Delta t\sqrt{\lambda}-2i)}{\Delta t^2\lambda(\Delta t\sqrt{\lambda}+6i)} \\ -1 & 0 & -\frac{4(\Delta t\sqrt{\lambda}-2i)}{\Delta t^2\lambda(\Delta t\sqrt{\lambda}-6i)} & -\frac{4(\Delta t\sqrt{\lambda}+2i)}{\Delta t^2\lambda(\Delta t\sqrt{\lambda}+6i)} \\ 1 & 0 & -\frac{\Delta t^2\lambda+12+\Delta t\sqrt{\lambda}4i}{\Delta t^2\lambda+36} & -\frac{\Delta t\sqrt{\lambda}+2i}{\Delta t\sqrt{\lambda}+6i} \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

We see that $W = O(\Delta t^{-2})$ and $\rho(G_{\text{GLM}}) = 1$. Moreover, it is possible to compute that $W^{-1} = W_0 + \Delta t W_1 + O(\Delta t^2)$, where

$$W_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad W_1 = \frac{3i}{8} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Now, direct calculations reveal that the vectors

$$\mathbf{q}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 1/2 \\ 0 \\ -1/4 \\ 3/4 \end{bmatrix},$$

satisfy the order conditions (68) and (70) for a = p = 2, thus $\boldsymbol{\theta}^{[n]} = O(\Delta t^3)$ and $T_n = O(\Delta t^3)$. However, it is not possible to find a vector \mathbf{q}_3 satisfying (68) and (70) for a = p = 3. We thus consider the effect of W^{-1} . Since \mathbf{q}_0 , \mathbf{q}_1 , and \mathbf{q}_2 satisfy (70) for a = p = 2, they will clearly satisfy (71) for all $\ell = 0, 1, 2, \ldots$ and $p_{\ell} = 2$, implying that $W^{-1}\boldsymbol{\theta}^{[n]}$ has at least order $O(\Delta t^2)$. Now, one can verify that the vectors

$$\mathbf{q}_{3} = \begin{bmatrix} x \\ y \\ 0 \\ -1/4 \end{bmatrix}, \quad \mathbf{q}_{4} = \begin{bmatrix} x_{0} \\ y_{0} \\ 1/8 \\ -1/4 \end{bmatrix}, \quad \mathbf{q}_{5} = \begin{bmatrix} x_{1} \\ y_{1} \\ \frac{1}{24} + \frac{x}{4} - \frac{y}{2} \\ \frac{5y}{4} - \frac{x}{2} \end{bmatrix},$$

together with \mathbf{q}_0 , \mathbf{q}_1 , and \mathbf{q}_2 , satisfy the conditions (71) for $\ell = 0, 1$ with $p_0 = 5$ and $p_1 = 3$ (for any values of x, y, x_0, y_0, x_1, y_1). Thus, we obtained that $\tilde{\ell} = \arg \min_{\ell} p_{\ell} = 1$, with $p_{\tilde{\ell}} = p_1 = 3$, and hence $W^{-1}\boldsymbol{\theta}^{[n]} = O(\Delta t^5)$ by Theorem 7. Hence, recalling that $T_n = O(\Delta t^3)$ and $W = O(\Delta t^{-2})$, Theorem 5 implies that $\mathbf{e}^{[n]} = O(\Delta t^2)$ and $E^n = O(\Delta t^2)$. Notice that by appropriate choices of x and y, \mathbf{q}_3

can satisfy (68) for a = 3. However, by Theorem 5, this does not improve the order of E^n . We thus conclude that DG2 with \mathbb{P}^1 and $s = \Delta t^2/2$ is second order convergent. Moreover, the reader can verify also in this GLM framework that $s \neq \Delta t^2/2$ does not lead to a second-order convergent method, in agreement with the results of Section 3.1.1.

Finally, we remark that, while the equivalence with GLM is elegant and puts DG2 very clearly into the context of numerical methods for differential equations, it does not lead to a formalism that is simpler to analyze.

4 Time DG methods for first-order systems: DG1

For T > 0, we rewrite (27) as a system of first-order differential equations,

$$\begin{cases} \dot{u}(t) - w(t) = 0 & \forall t \in (0, T], \\ \dot{w}(t) + \lambda u(t) = 0 & \forall t \in (0, T], \\ u(0) = \hat{u}_0, \\ w(0) = \hat{u}_1, \end{cases}$$
(72)

or in compact form as

$$\begin{cases} \dot{\boldsymbol{z}}(t) = L\boldsymbol{z}(t), & \forall t \in (0,T], \\ \boldsymbol{z}(0) = \hat{\boldsymbol{z}}_0, \end{cases}$$
(73)

where $\boldsymbol{z}(t) := [\boldsymbol{u}(t), \boldsymbol{w}(t)]^{\top}$, $L := \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix}$, and $\hat{\boldsymbol{z}}_0 := [\hat{\boldsymbol{u}}_0, \hat{\boldsymbol{u}}_1]^{\top}$. In order to construct a DG method for (73), we focus as for the second-order case in Section 3 on one time slab I_n . Multiplying (73) by a (regular enough) test function $\boldsymbol{v}(t)$ and integrating in time over I_n we obtain

$$(\dot{\boldsymbol{z}},\boldsymbol{v})_{I_n} - (L\boldsymbol{z},\boldsymbol{v})_{I_n} = 0.$$
(74)

Noting that $z \in H^1(0,T)$, since $u \in H^2(0,T)$ and $w = \dot{u}$, we add to (74) the vanishing term $[z]_{n-1} \cdot v(t_{n-1}^+)$ and get

$$(\dot{\boldsymbol{z}}, \boldsymbol{v})_{I_n} - (L\boldsymbol{z}, \boldsymbol{v})_{I_n} + [\boldsymbol{z}]_{n-1} \cdot \boldsymbol{v}_{n-1}^+ = 0.$$
 (75)

Summing over all time slabs we obtain the problem: find $z_{DG} \in [\mathbb{V}_{DG}]^2$ such that

$$\mathcal{B}(\boldsymbol{z}_{DG}, \boldsymbol{v}) = \mathcal{G}(\boldsymbol{v}) \quad \forall \, \boldsymbol{v} \in [\mathbb{V}_{DG}]^2, \tag{76}$$

where $\mathcal{B}(\cdot, \cdot) : [\mathbb{V}_{DG}]^2 \times [\mathbb{V}_{DG}]^2 \to \mathbb{R}$ is defined by

$$\mathcal{B}(\boldsymbol{z}, \boldsymbol{v}) := \sum_{n=1}^{N} \left((\dot{\boldsymbol{z}}, \boldsymbol{v})_{I_n} - (L\boldsymbol{z}, \boldsymbol{v})_{I_n} \right) + \sum_{n=1}^{N-1} [\boldsymbol{z}]_n \cdot \boldsymbol{v}(t_n^+) + \boldsymbol{z}(0^+) \cdot \boldsymbol{v}_0^+,$$
(77)

and the linear functional $\mathcal{G}(\cdot) : [\mathbb{V}_{DG}]^2 \to \mathbb{R}$ is

$$\mathcal{G}(\boldsymbol{v}) := \widehat{\boldsymbol{z}}_0 \cdot \boldsymbol{z}_0^+. \tag{78}$$

Note that the bilinear form $\mathcal{B}(\cdot, \cdot)$ is well defined whenever its arguments are, at least, $H^1(I_n)$ functions for any $n = 1, \ldots, N$. The existence and uniqueness of the discrete solution and stability bounds in a suitable mesh-dependent norm can be found in [5].

4.1 Algebraic formulation for DG discretization in first-order form: DG1

As in Section 3.1 we focus on a generic time slab I_n , where a local polynomial degree r is used, and we fix a basis $\{[\psi_{n,j}(t), 0]^T, [0, \psi_{n,j}(t)]^T\}_{j=1,...,r}$ for the polynomial space $[\mathbb{P}^r(I_n)]^2$, 2(r+1) being the dimension of the local finite-dimensional space $[\mathbb{V}_n^r]^2$. We write the trial function z as a linear combination of the basis functions, i.e.,

$$\boldsymbol{z}(t) = \sum_{j=1}^{r+1} u_{n,j} \begin{bmatrix} \psi_{n,j}(t) \\ 0 \end{bmatrix} + \sum_{j=1}^{r+1} w_{n,j} \begin{bmatrix} 0 \\ \psi_{n,j}(t) \end{bmatrix}, \quad \text{for } t \in I_n.$$

By choosing v in (75) as the *i*-th basis function we obtain the system

$$B\widehat{\boldsymbol{z}} = \boldsymbol{c},\tag{79}$$

where

$$B := \begin{bmatrix} I_{2(r+1)} & & & \\ B_{-} & B_{+} & & \\ & B_{-} & B_{+} & \\ & & \ddots & \ddots & \\ & & & B_{-} & B_{+} \end{bmatrix}, \quad \hat{z} := \begin{bmatrix} \hat{z}_{0} \\ \hat{z}_{1} \\ \hat{z}_{2} \\ \vdots \\ \hat{z}_{N} \end{bmatrix}, \quad c := \begin{bmatrix} c_{0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here, $\hat{z}_n \in \mathbb{R}^{2(r+1)(N+1)}$ is the vector containing the coefficients

$$\widehat{\boldsymbol{z}}_n := [u_{n,1}, w_{n,1}, \dots, u_{n,r+1}, w_{n,r+1}]^{\top}, \quad n = 1, \dots, N,$$

and \hat{z}_0 and c_0 contain the initial conditions, and the local matrices B_- and B_+ are given by

$$B_{-} := N^0 \otimes I_2, \tag{80}$$

$$B_{+} := (N^{1} + N^{3}) \otimes I_{2} - N^{2} \otimes L,$$
(81)

where $I_2 \in \mathbb{R}^2$ is the identity matrix and

$$\begin{split} N_{ij}^0 &:= -\langle \psi_{n,j}, \psi_{n+1,i} \rangle_{t_n}, \quad N_{ij}^1 &:= (\dot{\psi}_{n+1,j}, \psi_{n+1,i})_{I_n}, \\ N_{ij}^2 &:= (\psi_{n+1,j}, \psi_{n+1,i})_{I_n}, \quad N_{ij}^3 &:= \langle \psi_{n+1,j}, \psi_{n+1,i} \rangle_{t_n}, \end{split}$$

for i, j = 1, ..., r + 1. Clearly, (79) can be written as

$$B_{+}\hat{z}_{n+1} + B_{-}\hat{z}_{n} = \mathbf{0} \text{ for } n = 0, 1, \dots, N-1,$$
 (82)

however, to derive the analogy presented in the following section, we rewrite (82) as

$$\widehat{B}_{+}\boldsymbol{z}_{n+1} = -\widehat{B}_{-}\boldsymbol{z}_{n} \text{ for } n = 0, 1, \dots, N-1,$$
(83)

or equivalently as

$$(I_{r+1} \otimes I_2 - N^4 \otimes L)\boldsymbol{z}_{n+1} = -(N^5 \otimes I_2)\boldsymbol{z}_n,$$
(84)

where $N^4 = (N^1 + N^3)^{-1}N^2$, $N^5 = (N^1 + N^3)^{-1}N^0$ and I_{r+1} is the $(r+1) \times (r+1)$ identity matrix. Note that the system matrix in (84) is invertible because of the well-posedness of problem (76). To see this directly at the algebraic level one has to prove that $\det(I_{r+1} \otimes I_2 - N^4 \otimes L) = \det(I_{r+1} + \lambda(N^4)^2) \neq 0$. Thus, one needs to study the eigenvalues of $I_{r+1} + \lambda(N^4)^2$, which are equal to $1 + \lambda(\sigma(N^4))^2$, where $\sigma(N^4)$ are the eigenvalues of N^4 . In general, since the eigenvalues of N^4 are complex, the invertibility depends on their specific values. In Sections 4.3, 4.4 and 4.5, we will show that for polynomial degree r = 1, 2, 3 the system matrix (84) is invertible and in particular that $N_4 = \Delta tA$, cf. (9), with A defined by the specific Butcher tableau (6).

4.2 DG1 as Implicit Runge-Kutta scheme

In this section, we start presenting the general analogy between the DG discretization (75) and Implicit Runge Kutta (IRK) schemes by adapting the original proof in [25] to problem (72). Next, we give an equivalent result by considering the matrix formulation (84).

We consider the problem (75), i.e., for any I_n , n = 1, ..., N, find $\boldsymbol{z} \in [\mathbb{P}^r(I_n)]^2$ such that

$$(\dot{\boldsymbol{z}} - L\boldsymbol{z}, \boldsymbol{v})_{I_n} + [\boldsymbol{z}]_{n-1} \cdot \boldsymbol{v}_{n-1}^+ = 0, \quad \forall \boldsymbol{v} \in [\mathbb{P}^r(I_n)]^2,$$
(85)

with $z_0^- = \hat{z}_0$, and replace the first integral in (85) by an interpolatory quadrature formula,

$$\int_{t_{n-1}}^{t_n} (\dot{\boldsymbol{z}} - L\boldsymbol{z}) \, ds = \Delta t \sum_{i=1}^{r+1} b_i (\dot{\boldsymbol{z}} - L\boldsymbol{z}) (t_{n-1}^i) + \mathcal{O}(\Delta t^{p+1}), \quad r+1 \le p \le 2r+1, \tag{86}$$

where $t_{n-1}^i = t_{n-1} + c_i \Delta t$, for $1 \leq i \leq r+1$, $c_1 = 0$, and where b_i and c_i are weights and nodes in [0, 1]. Using (86) we can rewrite (85) for all I_n , n = 1, ..., N: we look for $\boldsymbol{z} \in [\mathbb{P}^r(I_n)]^2$ that satisfies

$$\Delta t \sum_{i=1}^{r+1} b_i (\dot{\boldsymbol{z}} - L \boldsymbol{z}) (t_{n-1}^i) \cdot \boldsymbol{v}(t_{n-1}^i) + [\boldsymbol{z}]_{n-1} \cdot \boldsymbol{v}_{n-1}^+ = 0 \quad \forall \boldsymbol{v} \in [\mathbb{P}^r(I_n)]^2.$$
(87)

Next, we define

$$\begin{cases} \boldsymbol{z}_{n-1} := \boldsymbol{z}(t_{n-1}^{-}) \\ \boldsymbol{z}_{n-1}^{1} := \boldsymbol{z}(t_{n-1}^{+}) = \boldsymbol{z}(t_{n-1}^{1}), \\ \boldsymbol{z}_{n-1}^{i} := \boldsymbol{z}(t_{n-1}^{i}), \qquad 2 \le i \le r+1, \end{cases}$$

and introduce the Lagrange polynomials

$$\ell_i(t) := \prod_{j=2, j \neq i}^{r+1} \frac{t - c_j}{c_i - c_j}, \quad 2 \le i \le r+1.$$

Following [25] one can prove that scheme (87) is equivalent to the implicit Runge-Kutta method

$$\begin{cases} \boldsymbol{z}_{n-1}^{i} = \boldsymbol{z}_{n-1} + \Delta t \sum_{j=1}^{r+1} a_{ij} L \boldsymbol{z}_{n-1}^{j} & 1 \le i \le r+1, \\ \boldsymbol{z}_{n} = \boldsymbol{z}_{n-1} + \Delta t \sum_{j=1}^{r+1} b_{j} L \boldsymbol{z}_{n-1}^{j}, \end{cases}$$
(88)

where $a_{i1} = b_1$ and $a_{ij} = \int_0^{c_i} \ell_j(s) ds - b_1 \ell_j(c_1)$, for $1 \le i \le r+1$, $2 \le j \le r+1$. To do so, we consider the basis functions $\{\psi_i\}_{i=1}^{r+1} = \{[\psi_i(t), 0]^T, [0, \psi_i(t)]^T\}$ for i = 1, ..., r+1 for $[\mathbb{P}^r(I_n)]^2$ such that $\psi_i(t_{n-1}^j) = \delta_{ij}$ for $1 \le i, j \le r+1$. Next, we replace v by ψ_i in (87) to get

$$\begin{cases} \boldsymbol{z}(t_{n-1}^{+}) - \boldsymbol{z}(t_{n-1}^{-}) + \Delta t b_1 \left(\dot{\boldsymbol{z}}(t_{n-1}^{1}) - L \boldsymbol{z}(t_{n-1}^{1}) \right) = \boldsymbol{0}, \\ \dot{\boldsymbol{z}}(t_{n-1}^{i}) - L \boldsymbol{z}(t_{n-1}^{i}) = \boldsymbol{0}, \end{cases}$$
(89)

and use the second equation above to obtain

$$\dot{\boldsymbol{z}}_{h}(t) = \sum_{j=2}^{r+1} \ell_{j} \left(\frac{t - t_{n-1}}{\Delta t} \right) \dot{\boldsymbol{z}}_{h}(t_{n-1}^{i}) = \sum_{j=2}^{r+1} \ell_{j} \left(\frac{t - t_{n-1}}{\Delta t} \right) L \boldsymbol{z}(t_{n-1}^{j}).$$

Taking $t = t_{n-1} = t_{n-1}^1$ and using the first equation in (89) yields

$$\boldsymbol{z}_{n-1}^{1} = \boldsymbol{z}_{n-1} + \Delta t b_1 \Big(L \boldsymbol{z}_{n-1}^{1} - \sum_{j=2}^{r+1} \ell_j(c_1) L \boldsymbol{z}_{n-1}^{j} \Big).$$
(90)

On the other hand, for $2 \leq i \leq r+1$, we have

$$\boldsymbol{z}_{n-1}^{i} = \boldsymbol{z}_{n-1}^{1} + \int_{t_{n-1}^{1}}^{t_{n-1}^{i}} \dot{\boldsymbol{z}}_{h}(s) \, ds,$$

and then

$$\boldsymbol{z}_{n-1}^{i} = \boldsymbol{z}_{n-1}^{1} + \Delta t \left(b_{1} L \boldsymbol{z}_{n-1}^{1} + \sum_{j=2}^{r+1} \left(\int_{0}^{c_{i}} \ell_{j}(s) \, ds - b_{1} \ell_{j}(c_{1}) \right) L \boldsymbol{z}_{n-1}^{j} \right)$$

Similarly for t_n^- we get

$$\boldsymbol{z}_{n} = \boldsymbol{z}_{n-1} + \Delta t \left(b_{1} L \boldsymbol{z}_{n-1}^{1} + \sum_{j=2}^{r+1} \left(\int_{0}^{1} \ell_{j}(s) \, ds - b_{1} \ell_{j}(c_{1}) \right) L \boldsymbol{z}_{n-1}^{j} \right),$$

and noticing that $\int_0^1 \ell_j(s) \, ds = b_1 \ell(\xi_1) + b_j$ we obtain

$$\mathbf{z}_n = \mathbf{z}_{n-1} + \Delta t \sum_{j=1}^{r+1} b_j L \mathbf{z}_{n-1}^j.$$
 (91)

Equations (90)-(91) are identical to the equations in (88), which means that the discontinuous Galerkin method leads to the one-step method (88). On the other hand, system (88) can be viewed as a discontinuous Galerkin method.

Next, with a different point of view, we prove that schemes of the form (83) are IRK methods with (r + 1)-stages, cf. [32, Remark 1]. Moreover, for polynomial degree r = 1, 2, 3, we show that the DG scheme is equivalent to an IRK Lobatto-IIIC method, by computing the entries of the time matrices N_4 and N_5 through a Gauss-Legendre-Lobatto (GLL) quadrature formula having r + 1 points and weights, and consider the basis functions $\{\psi_\ell\}_{\ell=1}^{r+1}$ as the characteristic polynomials associated with these points. We show, under specific assumptions, that the Runge-Kutta method (7) (and (9)) is equivalent to the DG scheme (84).

Theorem 8 (equivalence between DG and RK). Assume that s = r + 1, $N^4 = \Delta tA$, $b_i = a_{s,i}$ for $i = 1, \ldots, s$, that the matrix $N^5 \in \mathbb{R}^{(r+1)\times(r+1)}$ is zero up to the last column having all entries equal to -1, and that the matrix $(I_s \otimes I_2 - N^4 \otimes L)$ is invertible, cf. (84). If $[\hat{z}_0]_{2r+1,2r+2} = z_0$, then

$$[\widehat{z}_n]_{2r+1,2r+2} = z_n \quad and \quad \widehat{z}_n = (I_{r+1} \otimes L^{-1})k_{n-1} \quad for \ n = 1, \dots, N.$$
(92)

Proof. The proof is by induction. Since the relation $[\hat{z}_0]_{2r+1,2r+2} = z_0$ holds by assumption, we assume that (92) holds for n and we prove it for n+1. To prove the second equation in (92), a direct calculation using the structure of N^5 and the induction hypothesis $[\hat{z}_n]_{2r+1,2r+2} = z_n$ allow us to obtain that $\mathbf{1} \otimes z_n = -(N^5 \otimes I_2)\hat{z}_n$, where $\mathbf{1} \in \mathbb{R}^s$ is a vector with all components equal to 1. Now, noticing that the matrices $I_s \otimes L^{-1}$ and $(I_s \otimes I_2 - N^4 \otimes L)$ commute², and using (9), we can compute

$$-(N^{5} \otimes I_{2})\widehat{\boldsymbol{z}}_{n} = \mathbf{1} \otimes \boldsymbol{z}_{n} = (I_{s} \otimes L^{-1})(\mathbf{1} \otimes L\boldsymbol{z}_{n}) = (I_{s} \otimes L^{-1})\boldsymbol{f}$$
$$= (I_{s} \otimes L^{-1})(I_{s} \otimes I_{2} - \Delta tA \otimes L)\boldsymbol{k}_{n}$$
$$= (I_{s} \otimes L^{-1})(I_{s} \otimes I_{2} - N^{4} \otimes L)\boldsymbol{k}_{n}$$
$$= (I_{s} \otimes I_{2} - N^{4} \otimes L)(I_{s} \otimes L^{-1})\boldsymbol{k}_{n},$$

which shows that the vector $(I_s \times L^{-1})\mathbf{k}_n$ satisfies (84). Thus, by the invertibility of $(I_s \otimes I_2 - N^4 \otimes L)$ it follows that $\hat{\mathbf{z}}_{n+1} = (I_s \otimes L^{-1})\mathbf{k}_n$. To prove the first equation in (92), since $b_i = a_{s,i}$ for all i, we obtain from (7) that

$$\boldsymbol{k}_{s,n} = L\left(\boldsymbol{z}_n + \Delta t \sum_{j=1}^s a_{s,j} \boldsymbol{k}_{j,n}\right) = L\left(\boldsymbol{z}_n + \Delta t \sum_{j=1}^s b_j \boldsymbol{k}_{j,n}\right) = L \boldsymbol{z}_{n+1}.$$

Using this relation, we can compute

$$\hat{\boldsymbol{z}}_{n+1} = (I_s \otimes L^{-1})\boldsymbol{k}_n = \begin{bmatrix} L^{-1} & & \\ & \ddots & \\ & & L^{-1} & \\ & & & L^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_{1,n} \\ \vdots \\ \boldsymbol{k}_{s-1,n} \\ L\boldsymbol{z}_{n+1} \end{bmatrix} = \begin{bmatrix} L^{-1}\boldsymbol{k}_{1,n} \\ \vdots \\ L^{-1}\boldsymbol{k}_{s-1,n} \\ \boldsymbol{z}_{n+1} \end{bmatrix},$$

which shows that $[\hat{z}_{n+1}]_{2r+1,2r+2} = z_{n+1}$, and the claim follows.

4.3 Polynomial degree r = 1

For \mathbb{V}_n^1 we consider the same basis functions as in Section 3.1.1, and use the GLL nodes and weights, i.e.,

$${x_1, x_2} = {-1, 1}, {w_1, w_2} = {1, 1}.$$

The matrices N_4 and N_5 are then given by

$$N_4 = \Delta t \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad N_5 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix},$$
(93)

²This follows using the property of the Kronecker product $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

which corresponds to the Butcher tableau

$$\begin{array}{c|cccc} 0 & 1/2 & -1/2 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

for (9), i.e., the 2-stage Lobatto IIIC method, see [21].

Corollary 9 (DG is a RK Lobatto IIIC - \mathbb{P}^1). Let r = 1. Denote by $\{(\mathbf{k}_n, \mathbf{z}_n)\}_n$, with $\mathbf{k}_n = [\mathbf{k}_{1,n}, \mathbf{k}_{2,n}]^{\top}$, the sequence produced by the Lobatto IIIC scheme (7)-(8) starting from \mathbf{z}_0 , and by $\{\hat{\mathbf{z}}_n\}_n$ the sequence generated by the DG scheme (84) starting from $\hat{\mathbf{z}}_0$. If $[\hat{\mathbf{z}}_0]_{3,4} = \mathbf{z}_0$, then

$$[\widehat{\boldsymbol{z}}_n]_{3,4} = \boldsymbol{z}_n \quad and \quad \widehat{\boldsymbol{z}}_n = (I_2 \otimes L^{-1})\boldsymbol{k}_{n-1} \quad for \ n = 1, \dots, N.$$
 (94)

Proof. The result follows by Theorem 8. In fact, the coefficients b_j and $a_{j,i}$ and the matrix N^5 (given in (93)) clearly satisfy the hypotheses of Theorem 8. Moreover, a direct calculation reveals that $K := (I_s \otimes I_2 - \Delta t A \otimes L)$ is

$$K = \begin{bmatrix} 1 & -\frac{\Delta t}{2} & 0 & \frac{\Delta t}{2} \\ \frac{\Delta t\lambda}{2} & 1 & -\frac{\Delta t\lambda}{2} & 0 \\ 0 & -\frac{\Delta t}{2} & 1 & -\frac{\Delta t}{2} \\ \frac{\Delta t\lambda}{2} & 0 & \frac{\Delta t\lambda}{2} & 1 \end{bmatrix}$$

and det $K = \frac{4+\lambda^2 \Delta t^4}{4} > 0.$

4.4 Polynomial degree r = 2

For \mathbb{V}_n^2 we consider the same basis functions as in Section 3.1.2, and use again the GLL nodes and weights, i.e.,

$${x_1, x_2, x_3} = {-1, 0, 1}, {w_1, w_2, w_3} = {1/3, 4/3, 1/3}$$

The matrices N_4 and N_5 then become

$$N_4 = \Delta t \begin{bmatrix} 1/6 & -1/3 & 1/6\\ 1/6 & 5/12 & -1/12\\ 1/6 & 2/3 & 1/6 \end{bmatrix}, \quad N_5 = \begin{bmatrix} 0 & 0 & -1\\ 0 & 0 & -1\\ 0 & 0 & -1 \end{bmatrix},$$
(95)

which corresponds to the Butcher tableau

$$\begin{array}{c|ccccc} 0 & 1/6 & -1/3 & 1/6 \\ 1/2 & 1/6 & 5/12 & -1/12 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

for (9), i.e., the 3-stage Lobatto IIIC method, see [21].

Corollary 10 (DG is a RK Lobatto IIIC - \mathbb{P}^2). Let r = 2. Denote by $\{(\mathbf{k}_n, \mathbf{z}_n)\}_n$, with $\mathbf{k}_n = [\mathbf{k}_{1,n}, \mathbf{k}_{2,n}]^{\top}$, the sequence produced by the Lobatto IIIC scheme (7)-(8) starting from \mathbf{z}_0 , and by $\{\hat{\mathbf{z}}_n\}_n$ the sequence generated by the DG scheme (84) starting from $\hat{\mathbf{z}}_0$. If $[\hat{\mathbf{z}}_0]_{5,6} = \mathbf{z}_0$, then

$$[\widehat{\boldsymbol{z}}_n]_{5,6} = \boldsymbol{z}_n \quad and \quad \widehat{\boldsymbol{z}}_n = (I_3 \otimes L^{-1})\boldsymbol{k}_{n-1} \quad for \ n = 1, \dots, N.$$
 (96)

Proof. The result follows again by Theorem 8. In fact, the coefficients b_j and $a_{j,i}$ and the matrix N^5 (given in (95)) clearly fulfill the hypotheses of Theorem 8, and a direct calculation reveals that $K := (I_s \otimes I_2 - \Delta t A \otimes L)$ is

$$K = \begin{bmatrix} 1 & -\frac{\Delta t}{6} & 0 & \frac{\Delta t}{3} & 0 & -\frac{\Delta t}{6} \\ \frac{\Delta t\lambda}{6} & 1 & -\frac{\Delta t\lambda}{3} & 0 & \frac{\Delta t\lambda}{6} & 0 \\ 0 & -\frac{\Delta t}{6} & 1 & -\frac{5\Delta t}{12} & 0 & \frac{\Delta t}{12} \\ \frac{\Delta t\lambda}{6} & 0 & \frac{5\Delta t\lambda}{12} & 1 & -\frac{\Delta t\lambda}{12} & 0 \\ 0 & -\frac{\Delta t}{6} & 0 & -\frac{2\Delta t}{3} & 1 & -\frac{\Delta t}{6} \\ \frac{\Delta t\lambda}{6} & 0 & \frac{2\Delta t\lambda}{3} & 0 & \frac{\Delta t\lambda}{6} & 1 \end{bmatrix},$$

and det $K = \frac{576 + 36\lambda \Delta t^2 + \lambda^3 \Delta t^6}{576} > 0.$

4.5 Polynomial degree r = 3

For \mathbb{V}_n^3 we consider the same basis functions as in Section 3.1.2, and use the GLL nodes and weights

The matrices N_4 and N_5 become

$$N_4 = \Delta t \begin{bmatrix} 1/12 & -\sqrt{5}/12 & \sqrt{5}/12 & -1/12 \\ 1/12 & 1/4 & (10 - 7\sqrt{5})/60 & \sqrt{5}/60 \\ 1/12 & (10 + 7\sqrt{5})/60 & 1/4 & -\sqrt{5}/60 \\ 1/12 & 5/12 & 5/12 & 1/12 \end{bmatrix}, \quad N_5 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (97)$$

corresponding to the Butcher tableau

for (9), i.e., the 4-stage Lobatto IIIC method, see [21].

Corollary 11 (DG is a RK Lobatto IIIC - \mathbb{P}^3). Let r = 3. Denote by $\{(\mathbf{k}_n, \mathbf{z}_n)\}_n$, with $\mathbf{k}_n = [\mathbf{k}_{1,n}, \mathbf{k}_{2,n}]^{\top}$, the sequence produced by the Lobatto IIIC scheme (7)-(8) starting from \mathbf{z}_0 , and by $\{\hat{\mathbf{z}}_n\}_n$ the sequence generated by the DG scheme (84) starting from $\hat{\mathbf{z}}_0$. If $[\hat{\mathbf{z}}_0]_{7,8} = \mathbf{z}_0$, then

$$[\widehat{\boldsymbol{z}}_n]_{7,8} = \boldsymbol{z}_n \quad and \quad \widehat{\boldsymbol{z}}_n = (I_4 \otimes L^{-1})\boldsymbol{k}_{n-1} \quad for \ n = 1, \dots, N.$$
 (98)

Proof. The result follows again by Theorem 8: the coefficients b_j and $a_{j,i}$ and the matrix N^5 (given in (97)) satisfy the hypotheses of Theorem 8, and a direct calculation reveals that $K := (I_s \otimes I_4 - \Delta t A \otimes L)$ is

$$K = \begin{bmatrix} 1 & -\frac{\Delta t}{12} & 0 & \frac{\sqrt{5}\Delta t}{12} & 0 & -\frac{\sqrt{5}\Delta t}{12} & 0 & \frac{\Delta t}{12} \\ \frac{\Delta t\lambda}{12} & 1 & -\frac{\sqrt{5}\Delta t\lambda}{12} & 0 & \frac{\sqrt{5}\Delta t\lambda}{12} & 0 & -\frac{\Delta t\lambda}{12} & 0 \\ 0 & -\frac{\Delta t}{12} & 1 & -\frac{\Delta t}{4} & 0 & a_{-} & 0 & -\frac{\sqrt{5}\Delta t}{60} \\ \frac{\Delta t\lambda}{12} & 0 & \frac{\Delta t\lambda}{4} & 1 & -a_{-} & 0 & \frac{\sqrt{5}\Delta t\lambda}{60} & 0 \\ 0 & -\frac{\Delta t}{12} & 0 & -a_{+} & 1 & -\frac{\Delta t}{4} & 0 & \frac{\sqrt{5}\Delta t}{60} & 0 \\ \frac{\Delta t\lambda}{12} & 0 & a_{+} & 0 & \frac{\Delta t\lambda}{4} & 1 & -\frac{\Delta t\lambda}{60} & 0 \\ 0 & -\frac{\Delta t}{12} & 0 & -\frac{5\Delta t}{12} & 0 & -\frac{5\Delta t}{12} & 1 & -\frac{\Delta t}{12} \\ \frac{\Delta t\lambda}{12} & 0 & \frac{5\Delta t\lambda}{12} & 0 & \frac{5\Delta t\lambda}{12} & 0 & \frac{\Delta t\lambda}{12} & 1 \end{bmatrix},$$

where $a_{-} = \Delta t \lambda (7\sqrt{5} - 10), a_{+} = \Delta t \lambda (7\sqrt{5} + 10)$ and det $K = \frac{\Delta t^{*} \lambda^{4}}{129600} + \frac{\Delta t^{*} \lambda^{2}}{900} + \frac{2\Delta t^{2} \lambda}{45} + 1 > 0$

5 Numerical results: DG2 vs DG1

We now compare the DG2 method proposed in Section 3 and the DG1 in Section 4 numerically. For the model problem (27), we consider the regular solution $u_{ex} = \cos(t) + \sin(t)$ in the time interval (0, 20] and set $\lambda = 1$, $\hat{u}_0 = 1$ and $\hat{u}_1 = 1$. For the DG2 scheme, we show in Figure 3 the computed errors $\|\boldsymbol{e}_n\| = \|\boldsymbol{u}_n^{xr} - \boldsymbol{u}_n\|$ as a function of the time step Δt , by varying the polynomial degree r = 1, 2, 3. In Figure 3 we select s = 0 (left) and $s = \frac{\Delta t^2}{2}$ (right) in (29). The numerical results confirm the theoretical ones in Lemma 1. In particular, for s = 0 we can observe an order of convergence of $\mathcal{O}(\Delta t^r)$, cf. Figure 3-left, while for $s = \frac{\Delta t^2}{2}$ the order of convergence is $\mathcal{O}(\Delta t^2)$ for r = 1 and $\mathcal{O}(\Delta t)$ for r = 2, 3. This agrees with the findings in Section 3. Next, to further illustrate these findings, we plot in Figure 4 the consistency error $\|V^{-1}\boldsymbol{\theta}_n\|$ as a function of the time step Δt by varying the polynomial degree r = 1, 2, 3, with s = 0 (left) and with $s = \frac{\Delta t^2}{2}$ (right). We see that for s = 0 the asymptotic trend is $\mathcal{O}(\Delta t^{r+1})$ while for $s = \frac{\Delta t^2}{2}$ we obtain $\mathcal{O}(\Delta t^2)$ for r = 2, 3 and $\mathcal{O}(\Delta t^3)$ for r = 1. Again, this is aligned with the theoretical results of Section 3.



Figure 3: DG2: computed convergence errors $||\boldsymbol{e}_n||$ as a function of the time step Δt for r = 1, 2, 3 with s = 0 (left) and $s = \frac{\Delta t^2}{2}$ (right).

For the DG1 method of Section 4 we show the asymptotic rate of convergence with respect to the time step Δt in Figure 5. As shown is Section 2.2 we obtain the rate of convergence $\mathcal{O}(\Delta t^{r+1})$ for r = 1, 2, 3 when computing $||e_n||$, cf. Figure 5-left, and $\mathcal{O}(\Delta t^{2(r+1)-2})$ when considering the error at the final time T, cf. Figure 5-right. As a general remark, we can say that DG1 outperforms DG2 by retaining a higher order of accuracy for the same polynomial degree r. Finally, we compute the condition number of the system matrix A^+ in (36) and B^+ in (82) and plot the results in Figure 6. The results for Δt tending to zero show that for the DG2 method, the condition number of A^+ behaves like $\mathcal{O}(\Delta t^{-2})$ while for the DG1 scheme the condition number of B^+ is almost constant. This result still favors the DG1 method, which is therefore preferable to the DG2 scheme.

6 Conclusions

We examined discontinuous Galerkin (DG) discretizations as time integrators for second-order differential systems, exploring two formulations: one in the original problem's second-order form, and another one in its first-order formulation. We presented a new convergence analysis framework for the secondorder formulation, studying both consistency and accuracy of the resulting schemes. We also showed the equivalence between the DG formulation and other classical time integrator schemes such as the Newmark scheme and General Linear Methods. We also provided an in-depth review of the first-order formulation, complemented by a new algebraic proof that shows the equivalence of the DG method with the Lobatto IIIC schemes. We compared the two different formulations in terms of accuracy, consistency, and computational cost. From the results obtained, we conclude that the DG formulation for first-order systems is to be preferred to that for the second-order equation, from every point of view.

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Figure 4: DG2: computed consistency errors $||V^{-1}\theta_n||$ as a function of the time step Δt for r = 1, 2, 3 with s = 0 (left) and $s = \frac{\Delta t^2}{2}$ (right).

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Figure 5: DG1: computed convergence errors $||\boldsymbol{e}_n||$ (left) and $|\boldsymbol{e}_{N-1}|$ (right) as a function of the time step Δt for r = 1, 2, 3.

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Figure 6: Computed condition number for the system matrix A^+ in (36) (left) and B^+ in (82) (right).

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