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On the coupling of 1D and 3D diffusion-reaction equations. Applications to tissue perfusion problems

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On the coupling of 1D and 3D diffusion-reaction equations. Applications to tissue perfusion problems

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Abstract

In this paper we consider the coupling between two diffusion-reaction problems, one taking place in a three-dimensional domain Ω, the other in a one-dimensional subdomain Λ. This coupled problem is the simplest model of fluid flow in a three-dimensional porous medium featuring fractures that can be described by one-dimensional manifolds. In particular this model can provide the basis for a multiscale analysis of blood flow through tissues, in which the capillary network is represented as a porous matrix, while the major blood vessels are described by thin tubular structures embedded into it: in this case, the model allows the computation of the 3D and 1D blood pressures respectively in the tissue and in the vessels.

The mathematical analysis of the problem requires non-standard tools, since the mass conservation condition at the interface between the porous medium and the one-dimensional manifold has to be taken into account by means of a *measure* term in the 3D equation. In particular, the 3D solution is singular on Λ. In this work, suitable weighted Sobolev spaces are introduced to handle this singularity: the well-posedness of the coupled problem is established in the proposed functional setting. An advantage of such an approach is that it provides a hilbertian framework which may be used for the convergence analysis of finite element approximation schemes. The investigation of the numerical approximation will be the subject of a forthcoming work.

1 Introduction

In this paper we focus on a special class of differential problems, involving a three-dimensional domain $\Omega \subset \mathbb{R}^3$ and a one-dimensional subdomain $\Lambda \subset \Omega$, which is a line parametrised by its curvilinear abscissa s (see fig. 1).

We consider a pair of spaces V_1, V_2 of real functions on Ω, and a space $\hat{V}$ of real functions on Λ: the spaces V_1, V_2, $\hat{V}$ will be specified later. Let $A : V_1 \times V_2 \to \mathbb{R}$, $\hat{A} : \hat{V} \times \hat{V} \to \mathbb{R}$ the following bilinear forms

$$A(u, v) := \int_\Omega A_1 \nabla u \cdot \nabla v \,\mathrm{d}\mathbf{x} + \int_\Omega A_0 uv \,\mathrm{d}\mathbf{x}, \tag{1}$$

$$\hat{A}(\hat{u}, \hat{v}) := \int_\Lambda \hat{A}_1 \frac{\mathrm{d}\hat{u}}{\mathrm{d}s}\frac{\mathrm{d}\hat{v}}{\mathrm{d}s} \,\mathrm{d}s + \int_\Lambda \hat{A}_0 \hat{u}\hat{v} \,\mathrm{d}s, \tag{2}$$

$$\tag{3}$$

where $A_1, A_0 \in L^\infty(\Omega)$ and $\hat{A}_1, \hat{A}_0 \in L^\infty(\Lambda)$. Let[1] $\Pi : V_1 \to L^2(\Lambda)$ be a continuous linear operator, and let be given two continuous functionals $B : V_2 \to \mathbb{R}$, $\hat{B} : \hat{V} \to \mathbb{R}$.

[1] For the sake of simplicity, we will often identify a space of functions defined on a given manifold and the corresponding space of functions defined on the domain of the parametrisation. For instance, if Λ is parametrised by the curvilinear abscissa and has length L, we will identify $L^2(\Lambda)$ and $L^2(0, L)$.

The problem we want to study is: *find* $(u, \hat{u}) \in V_1 \times \hat{V}$ *such that*

$$\begin{cases} A(u, v) + \displaystyle\int_\Lambda \beta(\Pi u - \hat{u}) v \, \mathrm{d}s = B(v) & \forall v \in V_2, \\ \hat{A}(\hat{u}, \hat{v}) - \displaystyle\int_\Lambda \beta(\Pi u - \hat{u}) \hat{v} \, \mathrm{d}s = \hat{B}(\hat{v}) & \forall \hat{v} \in \hat{V}, \end{cases} \tag{4}$$

where $\beta \in L^\infty(\Lambda)$.

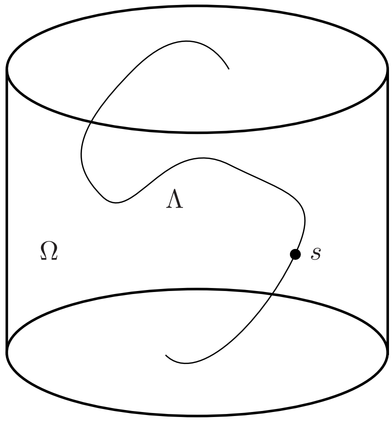

Figure 1: A 3D domain Ω with a 1D sub-domain Λ (the fracture).

Problem (4) is not "standard" because of the special line integral term in the first equation of (4). The continuity of the corresponding bilinear term is ensured only if the space V_2 is equipped with *continuous trace operator* $\gamma_\Lambda : V_2 \to L^2(\Lambda)$ on Λ. This would not be the case if we took $V_2 = H^1(\Omega)$: V_2 has to be "smaller" than $H^1(\Omega)$. As a consequence, V_1 has to be "larger"; in other words, the solution u is not $H^1(\Omega)$.

In this paper we introduce suitable functional spaces V_1, V_2 and $\hat{V}$, carry out the mathematical analysis of the decoupled 3D equation, and prove the well-posedness of the fully coupled 1D-3D problem.

Our main motivation for studying the abstract formulation (4) is that it provides a general paradigm for multiscale flow problems in 3D domains having 1D "fissures", including blood flow through biological tissues. In such cases, u and $\hat{u}$ represent fluid *pressures*, Π is typically an "averaging" operator, having the meaning of the local average of u around a point on Λ, and β plays the role of a hydraulic conductance, so that $\beta(\Pi u - \hat{u})$ represents the linear density of flow rate entering Λ from Ω. Besides the simulation of flow in fractured porous media has gained importance in scientific literature mainly because of its applications in geomechanics, similar models were recently applied in biomechanics for describing physiological flows. For instance, Huyghe, Ooomens and co-workers[8, 9] have developed a theory capable of accounting for the huge range of geometrical scales of the vascular structures: they look at tissue perfusion as blood flow through a hierarchical porous medium, in which each hierarchy of pores describes a network of vessels having comparable sizes. In certain cases, modelling some arteries and arterioles of the considered vascular tree as "pores" of a homogeneous medium may not be satisfactory, and one has to take into account the geometry of the major blood vessels. However, the network of these vessels can still be quite complex (for an example, see fig. 2), which prevents (or at least makes rather difficult) to consider the full three-dimensional Navier-Stokes equations to model blood flow in it. Since blood vessels are thin tubular structures, one possible strategy that allows the reduction of the complexity of the problem at hand consists in representing the latter network as a *one-dimensional fracture in the tissue*. In this case, the problems that are found fall in the family of models (4). We point out that Quarteroni *et alt.*[14, 5] have already studied 1D-3D models for blood flow in arteries, considering a "sequential" coupling. The point of view of this paper is more on the "geometrical" coupling between blood flow in a 1D vessel and through the surrounding tissue.

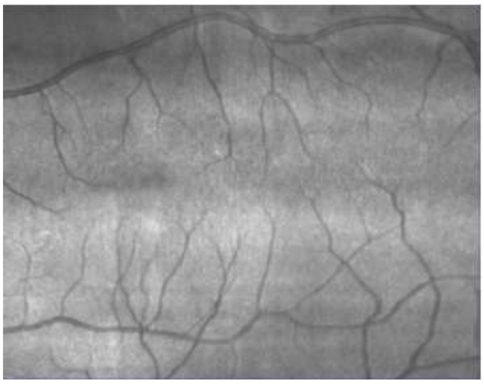

Figure 2: Retinal imaging. Shown is the retinal tissue, with a network of major blood vessels. In the steady case, blood flow can be modelled by 3D Darcy flow in the former and 1D Poiseuille flow in the latter, with suitable coupling conditions.

As an example, let us consider the steady perfusion of a biological tissue, represented by the domain Ω. Suppose that, at the macroscale, tissue's vasculature is given by a single artery; in spite of this artery having a certain radius $R > 0$, we consider the frequent case in which $R \ll \operatorname{diam}(\Omega)$. In this case, the artery can be represented by a one-dimensional subdomain Λ, parametrised by the curvilinear abscissa $s \in [0, L]$, L being the artery length. The averaged blood flow in the capillary matrix of the tissue is governed by Darcy's equation[9]; and if R is small, blood flow in the one-dimensional vessel satisfies a similar elliptic equation (see for instance Fung[6], ch. 5, on steady flow in arteries). To introduce these equations, denote by u, $k : \Omega \to \mathbb{R}$ the blood pressure and Darcy conductivity in the tissue, respectively, and by $\hat{u}$, $\hat{k} : \Lambda \to \mathbb{R}$ the blood pressure and conductivity in the vessel. Let $q : \Omega \to \mathbb{R}$ be the flow rate per unit volume of blood leaving the tissue capillaries and collected by the venous system. Finally, let $\hat{f} : \Lambda \to \mathbb{R}$ be the fluid loss from the vessel to the tissue, i.e. the volume of blood transferred from the vessel to the tissue per unit time and per unit vessel length. Then the tissue and vessel blood pressures satisfy the equations

$$\begin{cases} -\nabla \cdot (k \nabla u) + q - \hat{f}\delta_\Lambda = 0 & \text{in } \Omega, \\ -\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\hat{k}\dfrac{\mathrm{d}\hat{u}}{\mathrm{d}s}\right) + \hat{f} = 0 & \text{in } \Lambda, \end{cases} \tag{5}$$

where we denote by $\hat{f}\delta_\Lambda$ a Dirac measure with mass on Λ and having the density $\hat{f}$. This measure term expresses the conservation of the blood flow rate (the blood that is lost in the vessel goes into the tissue).

As regards the boundary conditions for problem (5), we assume that a given flow rate Q is imposed at the vessel inlet $s = 0$, that the reference value of the pressure at the outlet $s = L$ is zero (this is not restrictive, since the pressure is defined up to an arbitrary additive constant), and that no blood crosses the tissue boundaries.

This reads

$$\begin{cases} -k\dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \\[2ex] -\hat{k}\dfrac{\mathrm{d}\hat{u}}{\mathrm{d}s} = Q \quad \text{at } s = 0, \qquad\qquad \hat{u} = 0 \quad \text{at } s = L. \end{cases} \tag{6}$$

Constitutive laws for the fluxes q, $\hat{f}$ have to be chosen. We consider the following linear filtration law:

$$\hat{f}(u, \hat{u})(s) = \beta\left(\hat{u}(s) - \bar{u}(s)\right), \tag{7}$$

where β is a permeability coefficient for the vessel-tissue blood transfer, and $\bar{u}(s)$ is the mean value of u on the circle having the actual vessel radius R, centered on Λ at the point having curvilinear abscissa s, and lying in a normal plane to Λ. In other words, we are assuming that the blood flow rate per unit length from the vessel to the tissue is proportional to the gap between the vessel blood pressure and the local average tissue pressure (computed at the actual vessel-tissue interface). We point out that similar linear constitutive laws expressing the blood flow rate leaving a certain level of the vascular hierarchy and entering into the neighboring levels have been already adopted in biomechanics[9, 8].

As regards the venous out-flux, we consider again a first order conductive law, in which q is proportional to the gap between tissue and venous blood pressures. Moreover, assuming that the venous blood pressure is zero (this reference value is only adopted for the sake of simplicity), we have

$$q = \gamma u, \tag{8}$$

where $\gamma > 0$ is a given conductivity term.

The weak formulation of problem (5) with boundary conditions (6) is actually given by (4), where $\Pi u = \bar{u}$, $B(v) = 0$, $\hat{B}(\hat{v}) = Q\hat{v}(0)$, and $A_1 = k$, $A_0 = \gamma$, $\hat{A}_1 = \hat{k}$, $\hat{A}_0 = 0$.

At our knowledge, a mathematical analysis of such flow problems in 3D porous media with 1D fractures is not available in literature. For 3D media, usually a network of 2D fractures, that are in turn intersecting at a sub-network consisting of 1D fractures, has to be considered (see the work[1] by Alboin, Jaffré, Roberts and Serres): in this way, the problem is traced back to the "standard" coupling of 3D-2D and 2D-1D problems. This is not the case if we consider tissue perfusion problems. The "high dimensional gap" encountered in the related 3D-1D coupled model of blood flow makes the picture rather different, both regarding boundary/interface conditions and the analysis of the resulting coupled problem, and constitutes the subject of this work.

2 Geometry and notations

Let us introduce more precisely the geometry and the notations we will consider in this work (see also fig. 3). Since our model is intended to be applied to blood flow problems, we will refer to Ω as the *tissue* and Λ as the *vessel*.

(i) For the sake of simplicity, we assume that the vessel Λ is a single line:

$$\Lambda = \{\mathbf{x} \in \Omega : \; \mathbf{x} = \mathbf{x}(s), \; s \in [s_1, s_2]\}, \tag{9}$$

where s is the curvilinear abscissa, and $\mathbf{x} : [s_1, s_2] \to \mathbb{R}^3$ is a smooth parametrisation. This assumption can easily be extended to consider branching geometries too.

(ii) We assume that the actual vessel radius is a positive constant $R > 0$. Then, we introduce the actual volume occupied by the vessel as the set of points closer than R to Λ:

$$\Omega^R := \{\mathbf{x} \in \mathbb{R}^3 : \mathrm{dist}(\mathbf{x}, \Lambda) < R\}.$$

This is an auxiliary domain, that we will use only for the analysis of our model. We assume that R is small enough so that $\overline{\Omega^R} \subset \Omega$.

We will equip Ω^R with an atlas consisting of three local maps. To this end, we define

$$\begin{aligned}
\Omega_0^R &= \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{x} = \mathbf{x}_0(s, r, \theta), \\
&\qquad (s, r, \theta) \in (s_1, s_2) \times [0, R) \times [0, 2\pi)\}, \\
\Omega_1^R &= \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{x} = \mathbf{x}_1(r, \theta, \phi), \\
&\qquad (r, \theta, \phi) \in [0, R) \times [0, 2\pi) \times [0, \pi)\}, \\
\Omega_2^R &= \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{x} = \mathbf{x}_2(r, \theta, \phi), \\
&\qquad (r, \theta, \phi) \in [0, R) \times [0, 2\pi) \times [0, \pi)\},
\end{aligned} \tag{10}$$

being

$$\begin{aligned}
\mathbf{x}_0(s, r, \theta) &= \mathbf{x}(s) + \mathbf{n}(s) r \cos\theta + \mathbf{b}(s) r \sin\theta, \\
\mathbf{x}_1(r, \theta, \phi) &= \mathbf{x}(s_1) + \mathbf{n}(s_1) r \cos\theta \sin\phi + \\
&\quad \mathbf{b}(s_1) r \sin\theta \sin\phi + \mathbf{t}(s_1) r \cos\phi, \\
\mathbf{x}_2(r, \theta, \phi) &= \mathbf{x}(s_2) + \mathbf{n}(s_2) r \cos\theta \sin\phi + \\
&\quad \mathbf{b}(s_2) r \sin\theta \sin\phi + \mathbf{t}(s_2) r \cos\phi,
\end{aligned} \tag{11}$$

where $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are the tangent, normal and binormal versors on Λ (see fig. 3). Roughly speaking, Ω^R can be parametrised by an overlapping union of one local map in cylindrical coordinates on Ω_0^R and two mappings in spherical coordinates on Ω_1^R, Ω_2^R.

(iii) We denote by $\Gamma^R = \partial\Omega^R$ the "actual" interface between vessel and tissue. The "cylindrical" part of Γ^R that belongs to the boundary of Ω_0^R will be denoted by

$$\Gamma_0^R = \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{x} = \mathbf{x}_0(s, R, \theta), \quad (s, \theta) \in (s_1, s_2) \times [0, 2\pi)\}.$$

Our basic assumption on the vessel geometry is that the projection from Ω^R to Λ is unique:

$$\forall \mathbf{x} \in \Omega^R : \ \exists! \mathbf{x}_0 \in \Lambda : \mathrm{dist}(\mathbf{x}, \Lambda) = \|\mathbf{x} - \mathbf{x}_0\|. \tag{12}$$

Notice that the projection $\mathbf{x}_0$ exists because Λ is compact. One can show that eq. (12) is satisfied if Λ is smooth enough and R is small. A consequence of (12) is that

$$\mathrm{dist}(\mathbf{x}_0(s, r, \theta), \Lambda) = r \quad \forall (r, s, \theta) \in [s_1, s_2] \times [0, R) \times [0, \pi). \tag{13}$$

(iv) For the perfusion model described in section 1 we have in particular $\Pi u = \bar{u}$, where $\bar{u}$ is the mean value of the function u on circles of radius R laying on the cylindrical surface Γ_0^R and normal to the line Λ. With the notations introduced above, $\bar{u}$ is defined by

$$\bar{u}(s) := \frac{1}{2\pi} \int_0^{2\pi} u(\mathbf{x}_0(s, R, \theta)) \, \mathrm{d}\theta, \tag{14}$$

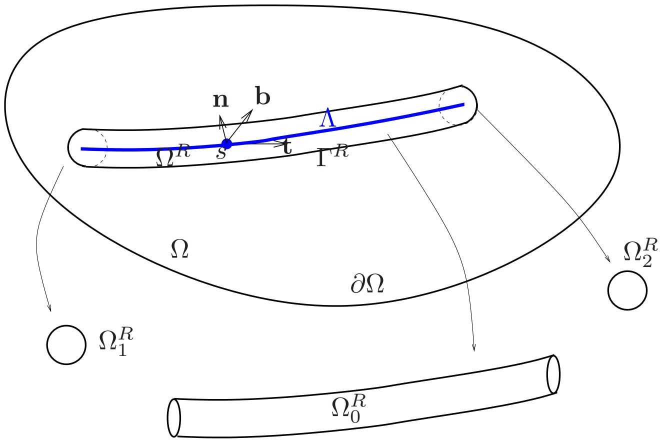

Figure 3: Subdomains in Ω: the 1D domain Λ and the auxiliary 3D domain Ω^R. Shown is the covering of Ω^R by overlapping subsets Ω_0^R, Ω_1^R and Ω_2^R where local cylindrical/spherical coordinates $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ are defined.

at least for a smooth u. As we will see (lemma 4.3), this operator can be extended to a the functional space V_1 in such a way that it is continuous from V_1 to $\hat{V}$. In every case, for the sake of generality, in this paper we always assume Π a generic bounded linear operator from V_1 to $\hat{V}$, so that other possible averaging operators can be considered in our abstract framework.

3 The coupled problem and the functional spaces

Problem (5) is a system of two elliptic equations, with the 3D equation featuring a *measure term* in the form of a line integral. Existence and uniqueness results for Dirichlet problems with *measure data* were proved by Stampacchia[17], whereas the semilinear case has been treated by Brezis[3]: in these papers the authors have proved the existence of a solution in $W^{1,q}(\Omega)$, with $1 \leq q < 3/2$ when Ω is in $\mathbb{R}^3$ (in the linear case uniqueness is obtained as well).

In order to provide analytical tools suited also for numerical approximation schemes, in this work we rather consider a hilbertian framework. We point out that Babuška[2] and Scott[16] have already studied the convergence of finite element schemes for Dirichlet problems with Dirac measure data. Nevertheless, their approach is essentially based on spaces $H^s(\Omega)$ with $s \in [0, \frac{1}{2})$, and is not appropriate for problems featuring terms like (7), where one needs a continuous mapping $u \to \bar{u}$ on $L^2(\Lambda)$. Well posedness in spaces $W^{1,q}$ and L^2 finite element scheme convergence was obtained by Casas[4]; however the extension to our problem, where the measure appears not only as datum but in the differential operator as well, is not straightforward at all. Moreover, the measures involved by the problem considered in this paper are not arbitrary: the only measure we deal with (see the strong formulation (5)) is a Dirac measure concentrated on Λ. For all these reasons, we introduce an "*ad hoc*" functional setting, based on weighted Sobolev spaces.

Let $\alpha \in (-1, 1)$; we denote by $L^2_\alpha(\Omega)$ the space of measurable functions u such that

$$\int_\Omega u(\mathbf{x})^2 d^{2\alpha}(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty,$$

where d is the distance from Λ, $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \Lambda)$. This means that $u \in L^2_\alpha(\Omega)$ if and only if $d^\alpha u$ belongs to $L^2(\Omega)$. $L^2_\alpha(\Omega)$ is a Hilbert space, equipped with the scalar product

$$(u,v)_{L^2_\alpha(\Omega)} = \int_\Omega u(\mathbf{x})v(\mathbf{x})d^{2\alpha}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

We also define the following weighted Sobolev space:

$$H^1_\alpha(\Omega) = \left\{u \in L^2_\alpha(\Omega) : \ \nabla u \in L^2_\alpha(\Omega)^3\right\},$$

and its scalar product

$$(u,v)_{H^1_\alpha(\Omega)} = (u,v)_{L^2_\alpha(\Omega)} + (\nabla u, \nabla v)_{L^2_\alpha(\Omega)^3}.$$

It is easily found that, for $\alpha \in (-1,1)$, the weight function d^α belongs to the *Muckenhoupt class* A_2 of functions $w : \mathbb{R}^3 \to \mathbb{R}_+$ such that

$$\sup_{\substack{B=B(\mathbf{x},r)\\ \mathbf{x}\in\mathbb{R}^3, r>0}} \left(\frac{1}{|B|}\int_B w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\left(\frac{1}{|B|}\int_B w(\mathbf{x})^{-1}\,\mathrm{d}\mathbf{x}\right) < +\infty$$

where $B(\mathbf{x},r)$ is the ball centered at $\mathbf{x}$ with radius r, and $|B|$ is its measure. Hence[7, 10], the density of smooth functions, Rellich-Kondriatev theorem and Poincaré inequalities hold true in H^1_α (for the theory of weighted Sobolev spaces, we also refer to Kufner[11]).

Before stating the well-posedness result for the coupled problem (4) (which is the abstract formulation of perfusion models such as (5) with exchange term (7) and venous flux (8)), we consider a simpler 3D decoupled problem. Finally, we extend the previous results to the full coupled problem (4).

4 Analysis of the decoupled problem

Consider the coupled problem (4), and assume that the blood pressure in the vessel Λ is known, $\hat{u} = u_0$. Then we are left with one variational equation only. We introduce the bilinear form

$$\begin{aligned} a(u,v) &= A(u,v) + \int_\Lambda \beta\Pi u(s)v(s)\,\mathrm{d}s \\ &= \int_\Omega A_1\nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\Omega A_0 uv\,\mathrm{d}\mathbf{x} + \int_\Lambda \beta\Pi u(s)v(s)\,\mathrm{d}s, \end{aligned} \tag{15}$$

and the linear functional

$$F(v) = \int_\Lambda \beta u_0 v(s)\,\mathrm{d}s + B(u). \tag{16}$$

The decoupled problem reads: *find $u \in V_1$ such that*

$$a(u,v) = F(v) \qquad \forall v \in V_2. \tag{17}$$

The idea here is to consider $V_1 = H^1_\alpha$ and $V_2 = H^1_{-\alpha}$. To get the existence and uniqueness of this problem, we will make use of a generalised Lax-Milgram theorem[12, 2, 15]:

Theorem 4.1 (Nečas) *Let V_1 and V_2 be two Hilbert spaces, $F \in V_2'$ be a bounded linear functional on V_2 and $a(\cdot,\cdot)$ be a bilinear form on $V_1 \times V_2$ such that*

$$|a(u,v)| \le C_1\|u\|_{V_1}\|v\|_{V_2} \qquad \forall (u,v) \in V_1\times V_2, \tag{18}$$

$$\sup_{u\in V_1} a(u,v) > 0 \qquad \forall v \in V_2, v \neq 0, \tag{19}$$

$$\sup_{\|v\|_{V_2}\le 1} a(u,v) \ge C_2\|u\|_{V_1} \qquad \forall u \in V_1, \tag{20}$$

where C_1 and C_2 are positive constants. Then there exists a unique $u \in V_1$ such that

$$a(u, v) = F(v) \qquad \forall v \in V_2,$$

which depends linearly and continuously on F:

$$\|u\|_{V_1} \leq \frac{1}{C_2} \|F\|_{V_2'}.$$

In order to use this theorem for problem (17), we have to show that:

i) the bilinear form a defined in (15) is continuous on $H^1_\alpha \times H^1_{-\alpha}$ and satisfies the inf-sup inequalities (19), (20);

ii) the functional F defined in (16) is continuous on $H^1_{-\alpha}$.

Let us start with point *ii)*: we can show that if $0 < \alpha < 1$, functions of $H^1_{-\alpha}$ admit a continuous trace operator on the 1D manifold Λ. This implies that F is a continuous functional on $H^1_{-\alpha}$. We will make use of the following weighted Hardy's inequality[13]:

Property 4.1 (Weighted Hardy's inequality) *Let $0 < p \leq q < \infty$, $0 < R \leq \infty$ and let w_1 and w_2 be weight functions defined on $(0, \infty)$. Assume that, for every $r > 0$,*

$$\int_0^r w_2(t)^{\frac{1}{1-p}} \, \mathrm{d}t < \infty.$$

Then, the inequality

$$\left(\int_0^R \left(\int_0^r f(t) \, \mathrm{d}t \right)^q w_1(r) \, \mathrm{d}r \right)^{\frac{1}{q}} \mathrm{d}r \leq C \left(\int_0^R f(r)^p w_2(r) \, \mathrm{d}r \right)^{\frac{1}{p}} \tag{21}$$

holds for all positive functions f on $(0, \infty)$ if and only if

$$D = \sup_{r \in (0,R)} \left(\int_r^R w_1(t) \, \mathrm{d}t \right)^{\frac{1}{q}} \left(\int_0^r w_2(t)^{\frac{1}{1-p}} \, \mathrm{d}t \right)^{\frac{p-1}{p}} < \infty.$$

Moreover, the best constant in (21) satisfies the estimate

$$D \leq C \leq k(p, q) D$$

where

$$k(p, q) = \left(\frac{p + qp - q}{p} \right)^{\frac{1}{q}} \left(\frac{p + qp - q}{(p-1)q} \right)^{\frac{p-1}{p}}.$$

Thanks to weighted Poincaré's inequality (21), we can prove the following trace theorem, which guarantees the continuity of the functional F.

Theorem 4.2 (Λ-trace operator) *Let $0 < \alpha < 1$. There exists a unique linear continuous map*

$$\gamma_\Lambda : H^1_{-\alpha}(\Omega) \to L^2(\Lambda)$$

such that $\gamma_\Lambda \phi = \phi_{|\Lambda}$ for each smooth function $\phi \in C^\infty(\Omega)$. In particular, there exists a positive number $C_\Lambda = C_\Lambda(\alpha)$ such that

$$\|\phi\|_{L^2(\Lambda)} \leq C_\Lambda(\alpha) \|\phi\|_{H^1_{-\alpha}(\Omega)} \qquad \forall \phi \in H^1_{-\alpha}(\Omega).$$

Proof. Let $\phi \in C^\infty(\Omega)$. By using the local cylindrical coordinates $\mathbf{x}_0$ and integrating in Ω_0^R along the radial direction, we have for every $\theta \in [0, 2\pi)$:

$$\phi(s,0,0) = \phi(s,r,\theta) - \int_0^r \frac{\partial \phi}{\partial r}(s,t,\theta)\,\mathrm{d}t,$$

so that, using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, and integrating on Ω_0^R we get

$$\pi R^2 \int_\Lambda \phi(s)^2\,\mathrm{d}s \le 2\int_{\Omega_0^R} \phi(s,r,\theta)^2 r\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}\theta + 2\int_{\Omega_0^R} \left(\int_0^r \frac{\partial\phi}{\partial r}(s,t,\theta)\,\mathrm{d}t\right)^2 r\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}\theta, \tag{22}$$

where $\phi(s) = \phi(s,0,0)$. Now we can use theorem 4.1 and inequality (21) with $p = q = 2$, the weight functions being

$$w_1(t) = t, \quad w_2(t) = t^{1-2\alpha},$$

and $f(t) = |\partial\phi/\partial r(s,t,\theta)|$; in fact, being $\alpha > 0$ we have

$$\int_0^r w_2(t)^{\frac{1}{1-p}}\,\mathrm{d}t = \int_0^r t^{2\alpha-1}\,\mathrm{d}t = \frac{r^{2\alpha}}{2\alpha} < \infty \quad \forall r > 0.$$

In particular,

$$\begin{aligned} D(\alpha) \quad &:= \quad \sup_{r\in(0,R)} \left(\int_r^R t\,\mathrm{d}t\right)^{\frac12} \left(\int_0^r t^{2\alpha-1}\,\mathrm{d}t\right)^{\frac12} \\ &= \quad \sup_{r\in(0,R)} \left[\frac{1}{4\alpha}(R^2 - r^2) r^{2\alpha}\right]^{\frac12} = R^{1+\alpha}\frac{\alpha^{(-1+\alpha)/2}}{2(\alpha+1)^{(1+\alpha)/2}}. \end{aligned}$$

Since $k(2,2) = 2$, we have

$$\int_0^R \left(\int_0^r \left|\frac{\partial\phi}{\partial r}(s,t,\theta)\right| \mathrm{d}t\right)^2 r\,\mathrm{d}r \le C(\alpha)^2 \int_0^R \left|\frac{\partial\phi}{\partial r}(s,r,\theta)\right|^2 r^{1-2\alpha}\,\mathrm{d}r, \tag{23}$$

where $C(\alpha)$ is any number such that

$$D(\alpha) \le C(\alpha) \le 2D(\alpha). \tag{24}$$

Moreover, using the identity $r = \mathrm{dist}(\mathbf{x},\Lambda)$ on Ω_0, and estimates (23) and $1 \le \mathrm{dist}(\mathbf{x},\Lambda)^{-2\alpha} R^{2\alpha}$ $\forall \mathbf{x} \in \Omega_0^R$ in (22), we obtain

$$\begin{aligned} \pi R^2 \int_\Lambda \phi(s)^2\,\mathrm{d}s \quad &\le \quad 2R^\alpha \int_{\Omega_0^R} \phi(s,r,\theta)^2 \mathrm{dist}(\mathbf{x},\Lambda)^{-2\alpha} r\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}\theta + \\ &\qquad 2C(\alpha)^2 \int_{\Omega_0^R} \left(\frac{\partial\phi}{\partial r}(s,r,\theta)\right)^2 \mathrm{dist}(\mathbf{x},\Lambda)^{-2\alpha} r\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}\theta \\ &\le \quad 2\max\{R^\alpha, C(\alpha)^2\} \|\phi\|^2_{H^1_{-\alpha}(\Omega)}, \end{aligned} \tag{25}$$

Hence the following continuity estimate holds:

$$\|\phi\|_{L^2(\Lambda)} \le C_\Lambda(\alpha) \|\phi\|_{H^1_{-\alpha}(\Omega)}$$

where $C_\Lambda(\alpha) = \sqrt{\max\{R^\alpha, C(\alpha)^2\}/(\pi R^2)}$, and ϕ is a smooth function. The extension to $\phi \in H^1_{-\alpha}(\Omega)$ follows by a density argument. From (24) we have that the dependence of C_Λ on α for $\alpha \to 0$ is $C_\Lambda = \mathcal{O}(\alpha^{-1/2})$, and $\lim_{\alpha\to 0} C_\Lambda = \infty$; this confirms that the result is not true anymore if $\alpha = 0$ (non-weighted case).

We are left with the proof of *i)*: the non-trivial step is here the inf-sup inequality (20). To prove this inequality, we will consider an auxiliary problem and the following technical lemma, that we adapted to our case from Voldřich[18].

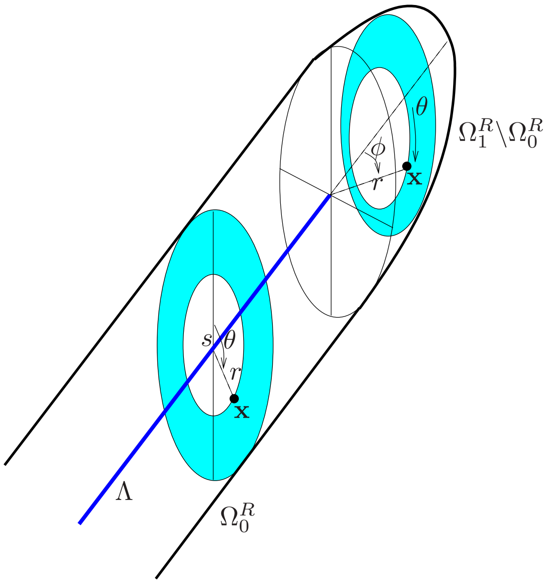

Figure 4: Local coordinates on subdomains of Ω^R.

Lemma 4.1 *Let $\alpha^* \in (0,1)$ and $u \in H^1_\alpha(\Omega)$ be given, with $0 < \alpha \leq \alpha^*$. Consider the θ-Fourier expansions in local coordinates given by*

$$u(s,r,\theta) = \sum_{k\in\mathbb{Z}} A_0^k(r,s)e^{ik\theta} \quad \text{in } \Omega_0^R, \tag{26}$$

$$u(r,\theta,\phi) = \sum_{k\in\mathbb{Z}} A_1^k(r,\phi)e^{ik\theta} \quad \text{in } \Omega_1^R\backslash\Omega_0^R, \tag{27}$$

$$u(r,\theta,\phi) = \sum_{k\in\mathbb{Z}} A_2^k(r,\phi)e^{ik\theta} \quad \text{in } \Omega_2^R\backslash\Omega_0^R. \tag{28}$$

Consider the real function

$$A^0(\mathbf{x}) = \begin{cases} A_0^0(r,s) & \text{in } \Omega_0^R \\ A_1^0(r,\phi) & \text{in } \Omega_1^R\backslash\Omega_0^R \\ A_2^0(r,\phi) & \text{in } \Omega_2^R\backslash\Omega_0^R \end{cases}$$

defined on the whole Ω^R. Furthermore, define

$$\Psi(\mathbf{x}) = \Psi(\mathbf{x};u) = \begin{cases} \Psi(r,y;u) = \displaystyle\int_r^R t^{2\alpha-1}A^0(t,y)\,\mathrm{d}t & \text{in } \Omega^R, \\ 0 & \text{elsewhere,} \end{cases} \tag{29}$$

where y can be either the s or the ϕ local variable, depending on the subdomain of Ω^R $\mathbf{x}$ belongs to (in particular, Ψ is independent of θ).

There are positive constants C_1, C_2, C_3, dependent only on α^*, such that the following estimates hold $\forall\alpha \in (0,\alpha^*]$:

$$\|u - A^0\|_{L^2_{\alpha-1}(\Omega^R)} \leq C_1\|\nabla u\|_{L^2_\alpha(\Omega)}, \tag{30}$$

$$\|\Psi\|_{L^2_{-\alpha}(\Omega)} \leq C_2\|u\|_{L^2_\alpha(\Omega)}, \tag{31}$$

$$\left\|d^{2\alpha-1}u\nabla d + \nabla\Psi\right\|_{L^2_{-\alpha}(\Omega^R)} \leq C_3\|\nabla u\|_{L^2_\alpha(\Omega^R)}, \tag{32}$$

where as usual $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x},\Lambda)$.

The proof is given in the appendix. Thanks to this lemma we can prove the following auxiliary result.

Lemma 4.2 *Let $A_0, A_1 \in L^\infty(\Omega)$; assume that there exists a constant $A_{\min} > 0$ such that $A_0, A_1 \geq A_{\min}$ in Ω. Then, there is a constant $\delta \in (0,1)$ such that for each $\alpha \in (0,\delta)$ there exists a unique $w \in H^1_\alpha(\Omega)$ satisfying*

$$A(w,\psi) = \widetilde{F}(\psi) \qquad \forall \psi \in H^1_{-\alpha}(\Omega),$$

where A is defined by (1) and $\widetilde{F}$ is a given continuous linear functional on $H^1_{-\alpha}(\Omega)$.

Moreover, there is a positive number $\widetilde{C} = \widetilde{C}(\alpha, A_{\min}, \|A_0\|_\infty, \|A_1\|_\infty) > 0$ such that

$$\|w\|_{H^1_\alpha(\Omega)} \leq \widetilde{C}\|\widetilde{F}\|_{H^1_{-\alpha}(\Omega)'} \qquad \forall \widetilde{F} \in H^1_{-\alpha}(\Omega)'. \tag{33}$$

Proof. The idea of the proof is to apply the Nečas' theorem 4.1, with $V_1 = H^1_\alpha(\Omega)$, $V_2 = H^1_{-\alpha}(\Omega)$, for $\alpha > 0$ small enough. The bilinear form A is continuous on $H^1_\alpha(\Omega) \times H^1_{-\alpha}(\Omega)$, since

$$\begin{aligned}
|A(w,\psi)| &\leq \left|\int_\Omega A_1 \nabla w \cdot \nabla\psi \,\mathrm{d}\mathbf{x}\right| + \left|\int_\Omega A_0 w\psi \,\mathrm{d}\mathbf{x}\right| \\
&= \left|\int_\Omega A_1 d^\alpha \nabla w \cdot d^{-\alpha}\nabla\psi \,\mathrm{d}\mathbf{x}\right| + \left|\int_\Omega A_0 d^\alpha w d^{-\alpha}\psi \,\mathrm{d}\mathbf{x}\right| \\
&\leq \|A_1\|_{L^\infty}\|\nabla w\|_{L^2_\alpha}\|\nabla\psi\|_{L^2_{-\alpha}} + \|A_0\|_{L^\infty}\|w\|_{L^2_\alpha}\|\psi\|_{L^2_{-\alpha}} \\
&\leq \max\{\|A_1\|_{L^\infty}, \|A_0\|_{L^\infty}\}\|w\|_{H^1_\alpha}\|\psi\|_{H^1_{-\alpha}}.
\end{aligned}$$

Now, let $\psi \in H^1_{-\alpha}(\Omega)$, $\psi \neq 0$; since $\alpha > 0$, we have $H^1_{-\alpha}(\Omega) \subset H^1_\alpha(\Omega)$, so that

$$\sup_{u \in H^1_\alpha} A(u,\psi) \geq A(\psi,\psi) \geq A_{\min}\|\nabla\psi\|^2_{L^2(\Omega)} + A_{\min}\|\psi\|^2_{L^2(\Omega)} > 0.$$

Hence, A is non-degenerate and hypothesis (19) is satisfied.

To prove that (20) holds, it is sufficient to show that there are positive constants m, M, such that for every $w \in H^1_\alpha$ there is $\psi \in H^1_{-\alpha}$ satisfying

$$\|\psi\|_{H^1_{-\alpha}} \leq m\|w\|_{H^1_\alpha}, \tag{34}$$

$$A(w,\psi) \geq M\|w\|^2_{H^1_\alpha}. \tag{35}$$

Then, (20) holds with $C_2 = M/m$.

Set

$$\psi(\mathbf{x}) = \widetilde{d}(\mathbf{x})^{2\alpha} w(\mathbf{x}) + 2\alpha\Psi(\mathbf{x}), \tag{36}$$

where $\widetilde{d}$ is the following Lipschitz continuous function

$$\widetilde{d}(\mathbf{x}) = \max\{\mathrm{dist}(\mathbf{x},\Lambda), R\} = \begin{cases} \mathrm{dist}(\mathbf{x},\Lambda) & \text{in } \Omega^R, \\ R & \text{elsewhere,} \end{cases} \tag{37}$$

and $\Psi = \Psi(\mathbf{x}; w)$ is the auxiliary function introduced in lemma 4.1, associated to $u = w$. Notice that $\widetilde{d}$ is equivalent to the distance function d in the sense that

$$\left(\frac{R}{\mathrm{diam}(\Omega)}\right) d \leq \widetilde{d} \leq d \quad \text{on } \Omega. \tag{38}$$

Thanks to (38) and (31) we have

$$\|\psi\|_{L^2_{-\alpha}} \leq \|w\|_{L^2_\alpha} + 2\|\Psi\|_{L^2_{-\alpha}} \leq m_1\|w\|_{H^1_\alpha}.$$

Moreover, since

$$\nabla\psi = \widetilde{d}^{2\alpha}\nabla w + 2\alpha(\widetilde{d}^{2\alpha-1} w\nabla\widetilde{d} + \nabla\Psi)$$

observing that $\widetilde{d} = d$ on Ω^R, $\Psi = 0$ on $\Omega\backslash\Omega^R$, $\nabla\widetilde{d} = \nabla\Psi = 0$ on $\Omega\backslash\Omega^R$, and using estimate (32) of lemma 4.1, we have

$$\|\nabla\psi\|_{L^2_{-\alpha}(\Omega)} \le \|\nabla w\|_{L^2_\alpha(\Omega)} + 2\|d^{2\alpha-1}w\nabla d + \nabla\Psi\|_{L^2_{-\alpha}(\Omega^R)} \le m_2\|w\|_{L^2(\Omega)}.$$

Hence, (34) is satisfied with $m^2 = m_1^2 + m_2^2$. Let $\alpha^* \in (0,1)$: by lemma 4.1, for any $\alpha \in (0,\alpha^*]$, constants m_1, m_2 and m only depend on α^*. Now, since

$$\begin{aligned} A(w,\psi) \ge & A_{\min}\int_\Omega \widetilde{d}^{2\alpha}|\nabla w|^2\,\mathrm{d}\mathbf{x} + A_{\min}\int_\Omega \widetilde{d}^{2\alpha} w^2\,\mathrm{d}\mathbf{x} \\ & + 2\alpha\int_{\Omega^R} A_1\nabla w\cdot(\widetilde{d}^{2\alpha-1}w\nabla\widetilde{d} + \nabla\Psi)\,\mathrm{d}\mathbf{x} + 2\alpha\int_{\Omega^R} A_0 w\Psi\,\mathrm{d}\mathbf{x}, \end{aligned}$$

we can use estimate (38) to obtain

$$\begin{aligned} A(w,\psi) \quad \ge \quad & A_{\min}\|\nabla w\|^2_{\widetilde{L}^2_\alpha(\Omega)} + A_{\min}\|w\|^2_{\widetilde{L}^2_\alpha(\Omega)} \\ & -2\alpha\|A_1\|_{L^\infty}\|\nabla w\|_{\widetilde{L}^2_\alpha(\Omega^R)}\|d^{2\alpha-1}u\nabla d + \nabla\Psi\|_{\widetilde{L}^2_{-\alpha}(\Omega^R)} \\ & -2\alpha\|A_0\|_{L^\infty}\|w\|_{\widetilde{L}^2_\alpha(\Omega^R)}\|\Psi\|_{\widetilde{L}^2_{-\alpha}(\Omega^R)} \\ \ge \quad & A_{\min}\|\nabla w\|^2_{\widetilde{L}^2_\alpha(\Omega)} + A_{\min}\|w\|^2_{\widetilde{L}^2_\alpha(\Omega)} \\ & -2\alpha\left(\|A_1\|_{L^\infty}C_3\|\nabla w\|^2_{\widetilde{L}^2_\alpha(\Omega^R)} + \|A_0\|_{L^\infty}C_2\|w\|^2_{\widetilde{L}^2_\alpha(\Omega^R)}\right) \end{aligned} \tag{39}$$

where $C_2 = C_2(\alpha^*)$, $C_3 = C_3(\alpha^*)$ are the constants in estimates (31), (32), and, for any subset $A \subset \Omega$ and function f, we define

$$\|f\|^2_{\widetilde{L}^2_\alpha(A)} := \int_A \widetilde{d}^{2\alpha}|f|^2\,\mathrm{d}\mathbf{x}.$$

Of course $\|f\|_{L^2_\alpha(\Omega^R)} = \|f\|_{\widetilde{L}^2_\alpha(\Omega^R)}$, and $\|\cdot\|_{\widetilde{L}^2_\alpha(\Omega)}$, $\|\cdot\|_{L^2_\alpha(\Omega)}$ are equivalent norms, since thanks to (38) we have

$$\frac{R^\alpha}{\mathrm{diam}(\Omega)^\alpha}\|f\|_{L^2_\alpha(\Omega)} \le \|f\|_{\widetilde{L}^2_\alpha(\Omega)} \le \|f\|_{L^2_\alpha(\Omega)}.$$

From (39) we get

$$\begin{aligned} A(w,\psi) \quad \ge \quad & (A_{\min} - 2\alpha\max\{C_3\|A_1\|_{L^\infty}, C_2\|A_0\|_{L^\infty}\})\left(\|w\|^2_{\widetilde{L}^2_\alpha(\Omega)} + \|\nabla w\|^2_{\widetilde{L}^2_\alpha(\Omega)^3}\right) \\ \ge \quad & (A_{\min} - 2\alpha\max\{C_3\|A_1\|_{L^\infty}, C_2\|A_0\|_{L^\infty}\})\frac{R^{2\alpha}}{\mathrm{diam}(\Omega)^{2\alpha}}\|w\|^2_{H^1_\alpha(\Omega)}. \end{aligned}$$

Defining the following α-independent quantity

$$\delta = \min\left\{\alpha^*, \frac{A_{\min}}{2\max\{C_2\|A_1\|_{L^\infty}, C_3\|A_0\|_{L^\infty}\}}\right\}, \tag{40}$$

for $0 < \alpha < \delta$ we have

$$A(w,\psi) \ge M\|w\|^2_{H^1_\alpha(\Omega)}, \tag{41}$$

where

$$M = A_{\min}(1 - \alpha/\delta)\frac{R^{2\alpha}}{\mathrm{diam}(\Omega)^{2\alpha}},$$

so that (35) holds.

In this case, theorem 4.1 applies. This proves the theorem and the estimate (33), with $\widetilde{C} = m/M$.

Now, let us consider problem (17). The next theorem establishes that this problem is well-posed, at least for $\|\beta\|_\infty$ small.

Theorem 4.3 *Let $A_0, A_1 \in L^\infty(\Omega)$, and assume that there exist a constant $A_{\min} > 0$ such that $A_0, A_1 \geq A_{\min}$ in Ω.*

Let $\Pi : H^1_\alpha(\Omega) \to L^2(\Lambda)$, a and F be respectively a bounded linear operator, the bilinear form (15) and the linear functional (16), where $B \in H^1_{-\alpha}(\Omega)'$.

Then, there is a constant $\delta \in (0,1)$ and a positive function $\beta_{\max}(\alpha)$, such that if $\alpha \in (0,\delta)$ and $\|\beta\|_\infty < \beta_{\max}(\alpha)$ problem

$$a(u,v) = F(v) \qquad \forall v \in H^1_{-\alpha}(\Omega),$$

admits a unique solution $u \in H^1_\alpha(\Omega)$.

Moreover, there exists a positive number $C = C(\alpha, A_{\min}, \|A_0\|_\infty, \|A_1\|_\infty, \|\beta\|_\infty)$ such that:

$$\|u\|_{H^1_\alpha(\Omega)} \leq C\left(\|u_0\|_{L^2(\Lambda)} + \|B\|_{H^1_{-\alpha}(\Omega)'}\right). \tag{42}$$

Proof. For any given $\alpha^* \in (0,1)$, let $0 < \alpha \leq \alpha^*$. The first two terms of a in equation (15) are obviously continuous on $H^1_\alpha(\Omega) \times H^1_{-\alpha}(\Omega)$. The third term is also continuous: indeed,

$$\begin{aligned}\left|\int_\Lambda \beta\Pi u(s)v(s)\,\mathrm{d}s\right| &\leq \|\beta\|_\infty \|\Pi u\|_{L^2(\Lambda)} \|\gamma_\Lambda v\|_{L^2(\Lambda)} \\ &\leq \|\beta\|_\infty K_\Lambda(\alpha) C_\Lambda(\alpha) \|u\|_{H^1_\alpha(\Omega)} \|v\|_{H^1_{-\alpha}(\Omega)},\end{aligned} \tag{43}$$

where we denote by $K_\Lambda(\alpha)$ the norm of the bounded operator $\Pi : H^1_\alpha(\Omega) \to L^2(\Lambda)$, and by $C_\Lambda(\alpha)$ the norm of the trace operator $\gamma_\Lambda : H^1_{-\alpha}(\Omega) \to L^2(\Lambda)$ (given in theorem 4.2).

Similarly, thanks to theorem 4.2, F is a continuous linear functional on $H^1_{-\alpha}(\Omega)$, and

$$\|F\| \leq C_\Lambda(\alpha) \|u_0\|_{L^2(\Lambda)} + \|B\|_{H^1_{-\alpha}(\Omega)'}. \tag{44}$$

Let $v \in H^1_{-\alpha}(\Omega)$, $v \neq 0$; to show that bilinear form a is non-degenerate, we take advantage of lemma 4.2, and choose u as the solution of

$$A(u,\psi) = \widetilde{F}(\psi) \qquad \forall \psi \in H^1_{-\alpha}(\Omega),$$

with

$$\widetilde{F}(\psi) = (v,\psi)_{H^1_{-\alpha}(\Omega)}.$$

Obviously $\widetilde{F}(\psi)$ is a continuous linear functional on $H^1_{-\alpha}(\Omega)$, and

$$\|\widetilde{F}\|_{H^1_{-\alpha}(\Omega)'} = \|v\|_{H^1_{-\alpha}(\Omega)}.$$

Moreover,

$$\begin{aligned} a(u,v) &= A(u,v) + \int_\Lambda \beta\Pi u(s)v(s)\,\mathrm{d}s \\ &= \|v\|^2_{H^1_{-\alpha}(\Omega)} + \int_\Lambda \beta\Pi u(s)v(s)\,\mathrm{d}s. \end{aligned} \tag{45}$$

Using the estimation (33) of theorem 4.2, we have

$$\|u\|_{H^1_\alpha(\Omega)} \leq \widetilde{C}\|\widetilde{F}\|_{H^1_{-\alpha}(\Omega)'} = \widetilde{C}\|v\|_{H^1_{-\alpha}(\Omega)} \tag{46}$$

with $\widetilde{C} = \widetilde{C}(\alpha, A_{\min}, \|A_0\|_\infty, \|A_1\|_\infty)$. Thanks to (44), (45) and (46), we get

$$a(u,v) \geq (1 - \|\beta\|_\infty K_\Lambda C_\Lambda \widetilde{C})\|v\|^2_{H^1_{-\alpha}(\Omega)},$$

so that, if $0 \le \|\beta\|_\infty \le \dfrac{1}{K_\Lambda C_\Lambda \widetilde{C}}$, the bilinear form a is non-degenerate.

Now, let $u \in H^1_\alpha(\Omega)$, and set

$$v = \widetilde{d}^{2\alpha} u + 2\alpha\Psi,$$

as done in the proof of lemma 4.2, with $\Psi(\mathbf{x}) = \Psi(\mathbf{x}; u)$. It has already been shown in the proof of lemma 4.2 (eq. (34), (40), (41)), that constants $m, \delta > 0$ exist, independent of α, such that

$$\|v\|_{H^1_{-\alpha}} \le m\|u\|_{H^1_\alpha},$$

and

$$A(u, v) \ge M\|u\|^2_{H^1_\alpha(\Omega)},$$

where $M = M(\alpha) = A_{\min}(1 - \alpha/\delta)R^{2\alpha}\mathrm{diam}(\Omega)^{-2\alpha}$.

Hence, we have:

$$a(u, v) \ge M(\alpha)\|u\|^2_{H^1_\alpha(\Omega)} + \int_\Lambda \beta\Pi u(s)v(s)\,\mathrm{d}s. \tag{47}$$

By (44) we can estimate the line integral as follows

$$a(u, v) \ge (M(\alpha) - m\|\beta\|_\infty K_\Lambda(\alpha)C_\Lambda(\alpha))\,\|u\|^2_{H^1_\alpha(\Omega)}.$$

Defining

$$\beta_{\max} = \beta_{\max}(\alpha) = \min\left\{\frac{1}{K_\Lambda C_\Lambda \widetilde{C}}, \frac{M}{mK_\Lambda C_\Lambda}\right\},$$

for $\alpha \in (0, \delta)$ and $\|\beta\|_\infty \le \beta_{\max}$ Nečas' theorem applies. In particular, (44) implies estimate (42) with

$$C = \frac{m \max\{C_\Lambda(\alpha), 1\}}{M(\alpha) - m\|\beta\|_\infty K_\Lambda(\alpha)C_\Lambda(\alpha)}$$

and the proof is complete.

We point out that theorem 4.3 applies for $\Pi u = \bar{u}$, which is a bounded linear operator, as the next lemma states.

Lemma 4.3 *Let $\alpha \in (-1, 1)$: the linear mapping $u \to \bar{u}$ from $H^1_\alpha(\Omega)$ to $L^2(\Lambda)$ is bounded.*

Proof. For any $u \in C^\infty(\Omega)$,

$$\int_\Lambda \bar{u}(s)^2\,\mathrm{d}s = \int_\Lambda \left(\frac{1}{2\pi}\int_0^{2\pi} u(s, R, \theta)\,\mathrm{d}\theta\right)^2 \mathrm{d}s$$
$$\le \int_\Lambda \frac{1}{2\pi}\int_0^{2\pi} u(s, R, \theta)^2\,\mathrm{d}\theta\,\mathrm{d}s = \frac{1}{2\pi R}\|u\|^2_{L^2(\Gamma_0^R)},$$

where Γ_0^R is the “cylindrical” part of the actual vessel surface (see sec. 2). Since $\mathrm{dist}(\Gamma_0^R, \Lambda) = R > 0$, the trace operator from $H^1_\alpha(\Omega)$ to $L^2(\Gamma_0^R)$ is continuous. Thanks to the density of smooth functions, $\bar{u}$ is thus extended to a bounded linear operator from $H^1_\alpha(\Omega)$ to $L^2(\Lambda)$.

Remark. The “coercivity” of a (in the sense of Nečas' theorem) in theorem 4.3 is obtained also thanks to the term $\int_\Omega A_0 uv\,\mathrm{d}\mathbf{x}$ and to the hypothesis $A_0 \ge A_{\min} > 0$. In the case $\Pi u = \bar{u}$, an alternative analysis is possible, in which A_0 can be zero. Assuming instead $\beta \ge \beta_{\min} > 0$, one might investigate if the term $\int_\Lambda \beta\bar{u}v\,\mathrm{d}s$ can supply to the loss of coercivity, at least for R *small enough*; but this approach is by far more complex.

5 The 1D-3D coupled problem

The methods we used in section 4 apply to the abstract coupled problem (4) without any substantial modification. Let us consider the case of mixed Neumann-Dirichlet homogeneous boundary conditions, being the Dirichlet homogeneous condition imposed at the vessel end $s = s_2$ to the 1D variable. Define the subspace

$$\hat{V} = \left\{ \hat{u} \in H^1(\Lambda) : \hat{u}(s_2) = 0 \right\},$$

and consider the spaces $\mathbf{V}_1 = H^1_\alpha(\Omega) \times \hat{V}$, $\mathbf{V}_2 = H^1_{-\alpha}(\Omega) \times \hat{V}$; for every $\mathbf{u} = (u, \hat{u}) \in \mathbf{V}_1$ and $\mathbf{v} = (v, \hat{v}) \in \mathbf{V}_2$, define the following bilinear form and linear functional

$$a(\mathbf{u}, \mathbf{v}) = A(u, v) + \hat{A}(\hat{u}, \hat{v}) + \int_\Lambda \beta(\Pi u - \hat{u})(v - \hat{v}) \, \mathrm{d}s, \tag{48}$$

$$F(\mathbf{v}) = B(v) + \hat{B}(\hat{v}), \tag{49}$$

with $\Pi : H^1_\alpha(\Omega) \to L^2(\Lambda)$ a bounded linear operator, $B \in V_1'$, $\hat{B} \in \hat{V}'$. The next theorem states the well-posedness of the problem (4) for $\|\beta\|_\infty$ small.

Theorem 5.1 *Let $A_i \in L^\infty(\Omega)$, $\hat{A}_i \in L^\infty(\Lambda)$ and assume that $A_i \geq A_{\min}$, $\hat{A}_0 \geq 0$, $\hat{A}_1 \geq A_{\min}$, with $A_{\min} > 0$ a constant and $i = 0, 1$.*

Let $\Pi : H^1_\alpha(\Omega) \to L^2(\Lambda)$, a and F be respectively a bounded linear operator, the bilinear form (48) and the linear functional (49), where $B \in V_1'$, $\hat{B} \in \hat{V}'$.

Then, there is a $\delta \in (0, 1)$ and a positive function $\beta_{\max}(\alpha)$ such that if $\alpha \in (0, \delta)$ and $\|\beta\|_\infty < \beta_{\max}(\alpha)$ there exists a unique $\mathbf{u} \in \mathbf{V}_1$ such that

$$a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_2.$$

Moreover, there is a positive number $C = C(\alpha, A_{\min}, \|A_i\|_\infty, \|\hat{A}_i\|_\infty, \|\beta\|_\infty)$ such that:

$$\|\mathbf{u}\|_{\mathbf{V}_1} \leq C \|F\|_{\mathbf{V}_2'}. \tag{50}$$

Proof. We follow the same steps of the proof of theorem 4.3. First of all, let us prove that a is continuous on $\mathbf{V}_1 \times \mathbf{V}_2$. We already know that the bilinear terms A, $\hat{A}$ in eq. (48) are respectively continuous on $H^1_\alpha(\Omega) \times H^1_{-\alpha}(\Omega)$ and $\hat{V} \times \hat{V}$. To see that the remaining coupling term is continuous on $\mathbf{V}_1 \times \mathbf{V}_2$, notice that

$$\begin{aligned}\left| \int_\Lambda \beta(\Pi u - \hat{u})(v - \hat{v}) \, \mathrm{d}s \right| &\leq \|\beta\|_\infty \|\Pi u - \hat{u}\|_{L^2(\Lambda)} \|\gamma_\Lambda v - \hat{v}\|_{L^2(\Lambda)} \\ &\leq \|\beta\|_\infty (K_\Lambda(\alpha) \|u\|_{H^1_\alpha(\Omega)} + \|\hat{u}\|_{\hat{V}})(C_\Lambda(\alpha) \|v\|_{H^1_{-\alpha}(\Omega)} + \|\hat{v}\|_{\hat{V}}),\end{aligned}$$

where $K_\Lambda(\alpha) = \|\Pi\|$ and $C_\Lambda(\alpha) = \|\gamma_\Lambda\|$ are respectively the norm of the bounded linear operator $\Pi : H^1_\alpha(\Omega) \to L^2(\Lambda)$ and the norm of the trace operator $\gamma_\Lambda : H^1_{-\alpha}(\Omega) \to L^2(\Lambda)$ (see th. 4.2). Thus, we have

$$\left| \int_\Lambda \beta(\Pi u - \hat{u})(v - \hat{v}) \, \mathrm{d}s \right| \leq \|\beta\|_\infty C(\alpha) \|\mathbf{u}\|_{\mathbf{V}_1} \|\mathbf{v}\|_{\mathbf{V}_2}, \tag{51}$$

where $C(\alpha) = \max\{1, K_\Lambda(\alpha), C_\Lambda(\alpha)\}$.

Now, we have to show that a satisfies the inequalities (19) and (20) of the Nečas' theorem 4.1.

a) Given $\mathbf{v} = (v, \hat{v}) \in \mathbf{V}_2$, we choose $\mathbf{u} \in \mathbf{V}_1$ as $\mathbf{u} = (u, \hat{v})$ where u is the solution of $A(u, \psi) = (v, \psi)_{H^1_{-\alpha}} \forall \psi \in H^1_{-\alpha}(\Omega)$, to show that a is non-degenerate. Thanks to lemma 4.2, we have $\|u\|_{H^1_\alpha(\Omega)} \leq \widetilde{C}(\alpha) \|v\|_{H^1_{-\alpha}(\Omega)}$, where $\widetilde{C}$ also depends on $A_{\min}$ and $\|A_i\|_\infty$, $i = 0, 1$. By estimate (51), defining $C'(\alpha) =$

$C(\alpha)\max\{1, \widetilde{C}(\alpha)\}$, and using Poincaré's inequality $\|\hat{v}\|_{L^2(\Lambda)} \leq C_P \|\mathrm{d}\hat{v}/\mathrm{d}s\|_{L^2(\Lambda)}$ in $\hat{V}$, we get

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) &\geq A(u,v) + \hat{A}(\hat{v},\hat{v}) - \|\beta\|_\infty C(\alpha)\|\mathbf{u}\|_{\mathbf{V}_1}\|\mathbf{v}\|_{\mathbf{V}_2} \\
&\geq \|v\|^2_{H^1_{-\alpha}(\Omega)} + A_{\min}\|\mathrm{d}\hat{v}/\mathrm{d}s\|^2_{L^2(\Lambda)} - \|\beta\|_\infty C'(\alpha)\|\mathbf{v}\|^2_{\mathbf{V}_2} \\
&\geq \|v\|^2_{H^1_{-\alpha}(\Omega)} + A_{\min}\frac{1}{2}\left(\|\mathrm{d}\hat{v}/\mathrm{d}s\|^2_{L^2(\Lambda)} + C_P^{-2}\|\hat{v}\|^2_{L^2(\Lambda)}\right) - \|\beta\|_\infty C'(\alpha)\|\mathbf{v}\|^2_{\mathbf{V}_2} \\
&\geq \min\left\{1, A_{\min}\frac{\min\{1, C_P^{-2}\}}{2}\right\}\|\mathbf{v}\|^2_{\mathbf{V}_2} - \|\beta\|_\infty C'(\alpha)\|\mathbf{v}\|^2_{\mathbf{V}_2}.
\end{aligned}
$$

Hence, if $\|\beta\|_\infty < \min\{1, A_{\min}\frac{\min\{1,C_P^{-2}\}}{2}\}/C'(\alpha)$ then a is non-degenerate.

b) Given $\mathbf{u} = (u,\hat{u}) \in \mathbf{V}_1$, consider $\mathbf{v} = (v,\hat{v}) = (\widetilde{d^{2\alpha}}u + 2\alpha\Psi, \hat{u}) \in \mathbf{V}_2$, as in the proof of theorems 4.2 and 4.3. We have that there exist two constants $m, \delta > 0$, both independent of α, such that

$$
\|v\|_{H^1_{-\alpha}(\Omega)} \leq m\|u\|_{H^1_\alpha(\Omega)}, \quad A(u,v) \geq M\|u\|^2_{H^1_\alpha(\Omega)},
$$

with $M = M(\alpha) = A_{\min}(1-\alpha/\delta)R^{2\alpha}\mathrm{diam}(\Omega)^{-2\alpha}$. Using (51) we have

$$
a(\mathbf{u},\mathbf{v}) \geq M(\alpha)\|u\|^2_{H^1_\alpha(\Omega)} + A_{\min}\|\mathrm{d}\hat{u}/\mathrm{d}s\|^2_{L^2(\Lambda)} - \|\beta\|_\infty C''(\alpha)\|\mathbf{u}\|^2_{\mathbf{V}_1}
$$

where $C''(\alpha) = C(\alpha)\max\{1, m\}$. For $\alpha \in (0,\delta)$ $M(\alpha) > 0$, and

$$
a(\mathbf{u},\mathbf{v}) \geq C'''(\alpha)\|\mathbf{u}\|^2_{\mathbf{V}_1} - \|\beta\|_\infty C''(\alpha)\|\mathbf{u}\|^2_{\mathbf{V}_1},
$$

where $C'''(\alpha) = \min\left\{M(\alpha), A_{\min}\frac{\min\{1,C_p^{-2}\}}{2}\right\} > 0$. The inequality (20) of the Nečas' theorem is thus satisfied if $\|\beta\|_\infty < C'''(\alpha)/C''(\alpha)$.

The proof is completed by taking $\beta_{\max}(\alpha)$ as the lowest between the upper bounds for $\|\beta\|_\infty$ found in a) and b).

As an immediate application of th. 5.1 for $\Pi u = \bar{u}$, and thanks to lemma 4.3, the well-posedness of the tissue perfusion problem (5) with boundary conditions (6) follows.

6 On the FE approximation of coupled 1D-3D elliptic problems

Since standard finite element functions are continuous and have a trace on any one-dimensional manifold in Ω, the Galerkin discretisation of problem (4) is straightforward. However, the convergence analysis of the resulting numerical scheme is not trivial. The functional setting we have introduced in this work provides a tool for the convergence analysis of finite element schemes, thanks to results based on theorem 4.1 and analogous to Cea's lemma[15, 2].

Figure 5 shows an example of FE numerical approximation for a flow problem with branching 1D geometry inside a porous cylinder. The details about the FE scheme and its convergence will be given in a forthcoming work.

7 Conclusions

In this paper, a coupled 1D-3D diffusion-reaction problem has been considered, for modelling flow in porous media with thin tubular fractures, as for instance in the

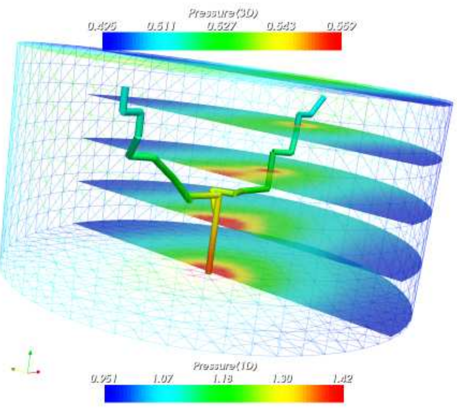

Figure 5: Finite element approximation of a 1D-3D flow problem: Ω is a cylinder and λ is a branching 1D subdomain. Shown are the computed 1D pressure and some slices of the 3D pressure.

case of blood flow through tissues. A mathematical analysis of this problem, based on suitable weighted Sobolev spaces, has been carried out to show that the coupled problem is well-posed. Our investigation is the basis for forthcoming studies on the numerical approximation of the solutions, as well as the application of the model to the simulation of tissue perfusion in relevant cases of physiological interest.

A Proof of lemma 4.1

Proof. The coefficients of Fourier expansions are given by standard formulae, for example in Ω_0^R we have

$$A_0^k(r,s) = \frac{1}{2\pi}\int_0^{2\pi} u(s,r,\theta)e^{-ik\theta}\,\mathrm{d}\theta.$$

Since $A_0^{-k}(r,s) = \overline{A_0^k(r,s)}$, A_0^0 is a real function, and so are functions A_1^0, A_2^0. Actually, $A^0(\mathbf{x})$ is the average of u on the circle described by $\theta \in [0,2\pi]$, keeping the other local variables constant and equal to those of point $\mathbf{x}$. Incidentally, this gives a geometrical interpretation for Ψ too, as the integral of $\frac{1}{2\pi}\mathrm{dist}(\mathbf{x},\Lambda)^{2\alpha-2}u(\mathbf{x})$ on the shaded areas associated with $\mathbf{x}$ in fig. 4. Even if we have two kinds of local variables (spherical and cylindrical), we will consider only the cylindrical subdomain Ω_0^R, since calculations for the remaining hemispherical subdomains are carried on in the same way.

From now on, when the integration intervals are omitted, it is understood that they are $r \in (0,R)$, $\theta \in (0,2\pi)$, $s \in (s_1,s_2)$, and $\phi \in (0,\pi/2)$. Thanks to Parseval's equality and to the orthogonality of the Fourier components, we have

$$\int_0^{2\pi} \big(u(s,r,\theta) - A_0^0(r,s)\big)^2\,\mathrm{d}\theta = 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} |A_0^k(r,s)|^2, \tag{52}$$

so that we can write

$$\begin{aligned}\|u - A^0\|^2_{L^2_{\alpha-1}(\Omega_0^R)} &= \int r^{2\alpha-2}[u(s,r,\theta) - A_0^0(r,s)]^2 r\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\theta \\ &= 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} \int r^{2\alpha-1}|A_0^k(r,s)|^2\,\mathrm{d}r\,\mathrm{d}s. \end{aligned}\tag{53}$$

On the other hand, being $|\nabla u|^2 \geq \frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2$, we have

$$\begin{aligned}\|\nabla u\|^2_{L^2_\alpha(\Omega)} &\geq \int r^{2\alpha}\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2 r\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\sum_{k\in\mathbb{Z}}\int r^{2\alpha-1}k^2|A_0^k(r,s)|^2\,\mathrm{d}r\,\mathrm{d}s\\ &= 2\pi\sum_{k\in\mathbb{Z}\setminus\{0\}}\int r^{2\alpha-1}k^2|A_0^k(r,s)|^2\,\mathrm{d}r\,\mathrm{d}s,\end{aligned}$$

where Parseval's formula for the θ-derivative has been used. Since in the last sum $k^2\geq 1$, comparing with (53) we have

$$\|u-A^0\|_{L^2_{\alpha-1}(\Omega_0^R)} \leq \|\nabla u\|_{L^2_\alpha(\Omega)}.$$

Analogous estimates on $\Omega_1^R\setminus\Omega_0^R$ and $\Omega_2^R\setminus\Omega_0^R$ follow in an similar way and (30) is proved.

The $L^2_{-\alpha}$ norm of Ψ on Ω_0^R is given by

$$\|\Psi\|^2_{L^2_{-\alpha}(\Omega_0^R)} = 2\pi\int r^{-2\alpha}\left(\int_r^R t^{2\alpha-1}A_0^0(t,s)\,\mathrm{d}t\right)^2 r\,\mathrm{d}s\,\mathrm{d}r.$$

Now we use the following weighted Hardy's inequality[18]

$$\int_0^R r^{-\beta}\left(\int_r^R t^{\beta-1}f(t)\,\mathrm{d}t\right)^2\mathrm{d}r \leq \left(\frac{2}{1-\beta}\right)^2\int_0^R r^\beta f(r)^2\,\mathrm{d}r, \qquad \beta<1, \tag{54}$$

with $f(t)=tA_0^0(t,s)$, $\beta=2\alpha-1$ (which is <1 since $0<\alpha\leq\alpha^*<1$). We get

$$\begin{aligned}\|\Psi\|^2_{L^2_{-\alpha}(\Omega_0^R)} &\leq 2\pi\left(\frac{1}{1-\alpha}\right)^2\int A_0^0(r,s)^2r^{2\alpha+1}\,\mathrm{d}r\,\mathrm{d}s\\ &\leq \left(\frac{1}{1-\alpha}\right)^2\int u(r,s,\theta)^2r^{2\alpha+1}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\theta \leq \left(\frac{1}{1-\alpha^*}\right)^2\|u\|^2_{L^2_\alpha(\Omega_0^R)},\end{aligned} \tag{55}$$

where Parseval's formula has been used again. Analogous estimates are found on $\Omega_1^R\setminus\Omega_0^R$ and $\Omega_2^R\setminus\Omega_0^R$, where we make use of (54) with $f(t)=A_i^0(t,\phi)t^2$, $i=1,2$, and $\beta=2\alpha-2$ due to the extra r term coming from the integration formula in spherical coordinates; therefore, since $\Psi=0$ outside Ω^R, (31) is proved.

Now let us show (32). We recall the following formulae in Ω_0^R

$$\frac{\partial u}{\partial s}(s,r,\theta) = \sum_{k\in\mathbb{Z}}\frac{\partial A_0^k}{\partial s}(r,s)e^{ik\theta}, \tag{56}$$

and

$$\nabla\Psi = -\mathbf{e}_r r^{2\alpha-1}A_0^0(r,s) + \mathbf{e}_s\int_r^R t^{2\alpha-1}\frac{\partial A_0^0}{\partial s}(t,s)\,\mathrm{d}t, \quad d^{2\alpha-1}\nabla d = r^{2\alpha-1}\mathbf{e}_r,$$

where $\mathbf{e}_r$ and $\mathbf{e}_s$ are the versors associated to the r and s local coordinates. We have

$$\begin{aligned}\|d^{2\alpha-1}u\nabla d+\nabla\Psi\|^2_{L^2_{-\alpha}(\Omega_0^R)} &\leq \|u-A^0\|^2_{L^2_{\alpha-1}(\Omega_0^R)}\\ &\quad + 2\pi\int r^{-2\alpha}\left(\int_r^R t^{2\alpha-1}\frac{\partial A_0^0}{\partial s}(t,s)\,\mathrm{d}t\right)^2 r\,\mathrm{d}s\,\mathrm{d}r.\end{aligned}$$

The first term can be estimated by means of (30), so that we are left with the second one. We can proceed as for eq. (55), using (54) and Parseval's equality for the Fourier expansion (56); we get

$$\begin{aligned} 2\pi \int r^{-2\alpha} \left(\int_r^R t^{2\alpha-1} \frac{\partial A_0^0}{\partial s}(t,s)\,\mathrm{d}t \right)^2 r\,\mathrm{d}s\,\mathrm{d}r \\ \leq 2\pi \left(\frac{1}{1-\alpha} \right)^2 \int \left(\frac{\partial A_0^0}{\partial s} \right)^2 r^{2\alpha+1}\,\mathrm{d}r\,\mathrm{d}s \\ \leq \left(\frac{1}{1-\alpha} \right)^2 \left\| \frac{\partial u}{\partial s} \right\|^2_{L^2_\alpha(\Omega_0^R)} \leq \left(\frac{1}{1-\alpha^*} \right)^2 \|\nabla u\|^2_{L^2_\alpha(\Omega)}. \end{aligned} \quad (57)$$

Analogous estimates can be derived from integration over the remaining subdomains of Ω^R. This completes the proof.

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