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Stochastic Spectral Galerkin and collocation methods for PDEs with random coefficients: a numerical comparison *

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Abstract

Much attention has recently been devoted to the development of Stochastic Galerkin (SG) and Stochastic Collocation (SC) methods for uncertainty quantification. An open and relevant research topic is the comparison of these two methods. By introducing a suitable generalization of the classical sparse grid SC method, we are able to compare SG and SC on the

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same underlying multivariate polynomial space in terms of accuracy versus computational work. The approximation spaces considered here include isotropic and anisotropic versions of Tensor Product (TP), Total Degree (TD), Hyperbolic Cross (HC) and Smolyak (SM) polynomials. Numerical results for linear elliptic SPDEs indicate a slight computational work advantage of isotropic SC over SG, with SC-SM and SG-TD being the best choices of approximation spaces for each method. Finally, numerical results corroborate the optimality of the theoretical estimate of anisotropy ratios introduced by the authors in a previous work for the construction of anisotropic approximation spaces.

1 Introduction

Nowadays, we observe a widespread need for including uncertainty in mathematical models and quantify its effect on given outputs of interest used in decision making. Such uncertainty may reflect, on one side, our ignorance or inability to properly characterize all input parameters of the mathematical model; on the other side, it may describe intrinsic variability of the event we model. Probability theory offers a natural framework to describe uncertainty, where all uncertain inputs are treated as random variables or more generally as random fields.

Monte Carlo Sampling (MCS) is probably the most natural and widely used technique to forward propagate the input randomness onto the system response or specific quantities of interest. While being very flexible and easy to implement, MCS features a very slow convergence and does not exploit the possible regularity that the solution might have with respect to the input variables.

Much attention has been recently devoted towards alternative methods which exploit such regularity and achieve sometimes a better convergence rate. Stochastic Galerkin (SG) and Stochastic Collocation (SC) are examples of such methods for uncertainty quantification. An open and relevant research topic is the comparison of these two approaches. This work provides, on a couple of numerical examples, a fair comparison between the performances of SG and SC methods *with the same underlying approximation space.*

Traditionally, the SG method approximates the solution in a multivariate polynomial space of given total degree (see e.g. [11, 27, 13] and references therein), or in anisotropic tensor product polynomial spaces [2, 8, 14]. Other global polynomial spaces has been considered recently, see for instance [24, 5], as well as different approximation spaces such as piecewise polynomials [2, 12, 25].

On the other hand the SC method adopted so far for SPDEs follows the classical Smolyak construction, see e.g. [26, 16, 9] and the references therein. It is very relevant to this work the fact that the sparse collocation method considered in [26, 16] leads to an approximate solution in a polynomial space, which we call hereafter Smolyak space, that differs from the total degree polynomial space most commonly used in SG approximation.

In this work we will consider several choices of multivariate polynomial

spaces, namely: tensor product (TP), total degree (TD), hyperbolic cross (HC) and Smolyak (SM) spaces. We consider on the one hand, SG approximations in either of these spaces. On the other hand, we propose a generalization of the classical sparse collocation method that allows us to achieve approximations in these same spaces. By following this path, we are able to compare the two alternative approaches (SG versus SC) given the same underlying multivariate polynomial space.

Once both SG and SC are posed on the same approximation space the second ingredient in a fair comparison is the computational work associated to each of them for the same level of accuracy. Since SC entails the solution of a number of *uncoupled* deterministic problems, its corresponding computational work is directly proportional to the number of collocation points. On the other hand, SG entails the solution of a large system of *coupled* deterministic problems whose size corresponds to the number of stochastic degrees of freedom (sdof). This can be achieved by an iterative strategy, here chosen to be a Preconditioned Conjugate Gradient solver following [18]. Therefore, a natural approximation of its computational work is given by the product of the number of sdof times the number of iterations performed.

This work assesses, on a numerical example having 8 input random variables, the performances of the SG and SC methods in terms of accuracy versus (estimated) computational cost. The numerical study shows that the two approaches have comparable performances. Actually, SC seems to be more efficient for relative errors larger than 10^{-4}, whereas SG is better for smaller errors.

The second numerical example that we propose contains 4 input random variables that have largely different influence on the solution. It is thus suited for anisotropic approximations, where higher polynomials degrees are used to discretize the dependence on the random variables that have a greater influence on the solution. We introduce anisotropic versions of both the SG and SC methods and compare their performances for different choices of anisotropy ratios. The results show that theoretically derived anisotropy ratios following [15] have the best performance and that our formula for the optimal anisotropy ratios is sharp.

2 Problem setting

Let D be a convex bounded polygonal domain in $\mathbb{R}^d$ and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Here Ω is the set of outcomes, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events and $P : \mathcal{F} \to [0,1]$ is a probability measure. Consider the stochastic linear elliptic boundary value problem: find a random function, $u : \Omega \times \overline{D} \to \mathbb{R}$, such that P-almost everywhere in Ω, or in other words almost surely (a.s.), the following equation holds:

$$\begin{cases} -\operatorname{div}(a(\omega,\mathbf{x})\nabla u(\omega,\mathbf{x})) = f(\mathbf{x}) & \mathbf{x} \in D, \\ u(\omega,\mathbf{x}) = 0 & \mathbf{x} \in \partial D. \end{cases} \tag{1}$$

where the operators div and ∇ imply differentiation with respect to the physical coordinate only.

The theory presented in this work extends straightforwardly to the case of a random forcing term $f = f(\omega, \mathbf{x})$ as well as to a non homogeneous, possibly random, Dirichlet datum on the boundary. For easiness of presentation, we will consider the case where the randomness appears only in the diffusion coefficient, which is, however, the most difficult case, since the solution u depends non-linearly on it, whereas it depends linearly on the forcing term and boundary data.

We will make the following assumptions on the random diffusion coefficient:

A1. *$a(\omega, \mathbf{x})$ is strictly positive and bounded with probability 1, i.e. there exist $a_{min} > 0$ and $a_{max} < \infty$ such that*

$$P(a_{min} \leq a(\omega, \mathbf{x}) \leq a_{max},\ \forall \mathbf{x} \in \overline{D}) = 1$$

A2. *$a(\omega, \mathbf{x})$ has the form*

$$a(\omega, \mathbf{x}) = b_0(\mathbf{x}) + \sum_{n=1}^{N} y_n(\omega) b_n(\mathbf{x}) \tag{2}$$

where $\mathbf{y} = [y_1, \ldots, y_N]^T : \Omega \to \mathbb{R}^N$, is a vector of independent random variables.

We denote by $\Gamma_n = y_n(\Omega)$ the image set of the random variable y_n, $\Gamma = \Gamma_1 \times \ldots \times \Gamma_N$, and we assume that the random vector $\mathbf{y}$ has a joint probability density function $\rho : \Gamma \to \mathbb{R}_+$ that factorizes as $\rho(\mathbf{y}) = \prod_{n=1}^{N} \rho_n(y_n),\ \forall \mathbf{y} \in \Gamma$. Observe that for assumption (A1) to hold, the image set Γ has to be a bounded set in $\mathbb{R}^N$.

After assumption (A2), the solution u of (1) depends on the single realization $\omega \in \Omega$ only through the value taken by the random vector $\mathbf{y}$. We can therefore replace the probability space $(\Omega, \mathcal{F}, P)$ with $(\Gamma, B(\Gamma), \rho(\mathbf{y})d\mathbf{y})$, where $B(\Gamma)$ denotes the Borel σ-algebra on Γ and $\rho(\mathbf{y})d\mathbf{y}$ is the distribution measure of the vector $\mathbf{y}$.

Finally, we introduce the functional space $H^1(D)$ of square integrable functions in D with square integrable distributional derivatives; its subspace $H^1_0(D)$ of functions with zero trace on the boundary, and the space $L^2_\rho(\Gamma)$ of square integrable functions on Γ with respect to the measure $\rho(\mathbf{y})d\mathbf{y}$.

We are now in the position to write a weak formulation of problem (1):
find $u \in H^1_0(D) \otimes L^2_\rho(\Gamma)$ such that $\forall v \in H^1_0(D) \otimes L^2_\rho(\Gamma)$

$$\int_\Gamma \int_D \left(b_0(\mathbf{x}) + \sum_{n=1}^{N} y_n b_n(\mathbf{x}) \right) \nabla u(\mathbf{x}, \mathbf{y}) \cdot \nabla v(\mathbf{x}, \mathbf{y})\, \rho(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \int_\Gamma \int_D f(\mathbf{x}) v(\mathbf{x}, \mathbf{y})\, \rho(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}. \tag{3}$$

Under assumption (A1), a straightforward application of the Lax-Milgram lemma allows to prove that there exists a unique solution to problem (3) for any $f \in L^2(D)$. Moreover, the following estimate holds:

$$\|\nabla u\|_{L^2(D)\otimes L^2_\rho(\Gamma)} \leq \frac{C_p}{a_{min}} \|f\|_{L^2(D)}$$

where C_p is the Poincaré constant such that $\|u\|_{L^2(D)} \leq C_p \|\nabla u\|_{L^2(D)}$ for any $u \in H^1_0(D)$.

It is well known (see e.g. [3, 14]) that the solution depends analytically on each parameter $y_n \in \Gamma_n$. In particular, denoting $\Gamma^*_n = \prod_{j\neq n} \Gamma_j$ and $\mathbf{y}^*_n$ an arbitrary element of Γ^*_n, there exists a constant M and regions $\Sigma_n \subset \mathbb{C}$ in the complex plane for $n = 1, \ldots, N$, with $\Sigma_n \supset \Gamma_n$, in which the solution $u(\mathbf{x}, y_n, \mathbf{y}^*_n)$ admits an analytic continuation $u(\mathbf{x}, z, \mathbf{y}^*_n),\ z \in \Sigma_n$. Moreover

$$\max_{z\in\Sigma_n} \max_{\mathbf{y}^*_n \in \Gamma^*_n} \|\nabla u(\cdot, z, \mathbf{y}^*_n)\|_{H^1(D)} \leq M, \qquad \text{for } n = 1, \ldots, N.$$

2.1 Finite element approximation in the physical space

Let $\mathcal{T}_h$ be a triangulation of the physical domain D and $V_h(D) \subset H^1_0(D)$ a finite element space of piecewise continuous polynomials on $\mathcal{T}_h$, with dimension $N_h = dim(V_h(D))$. We introduce the *semi-discrete* problem: *find* $u_h \in V_h(D) \otimes L^2_\rho(\Gamma)$ *such that* $\forall v_h \in V_h(D)$

$$\int_D \left(b_0(\mathbf{x}) + \sum_{n=1}^{N} y_n b_n(\mathbf{x}) \right) \nabla u_h(\mathbf{x}, \mathbf{y}) \cdot \nabla v_h(\mathbf{x})\, d\mathbf{x} = \int_D f(\mathbf{x}) v_h(\mathbf{x})\, d\mathbf{x}, \quad \rho\text{-a.e. in } \Gamma. \tag{4}$$

Problem (4) admits a unique solution for almost every $\mathbf{y} \in \Gamma$. Moreover, u_h satisfies the same analyticity result as the continuous solution u.

Let $\{\phi_i\}_{i=1}^{N_h}$ be a Lagrangian basis of $V_h(D)$ and consider the expansion of the semi-discrete solution as $u_h(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N_h} u_i(\mathbf{y}) \phi_i(\mathbf{x})$. Denoting by $\mathbf{U}(\mathbf{y}) = [u_1(\mathbf{y}), \ldots, u_{N_h}(\mathbf{y})]^T$ the vector of nodal values as functions of the random variables $\mathbf{y}$, problem (4) can be written in algebraic form as

$$\left(K_0 + \sum_{n=1}^{N} y_n K_n \right) \mathbf{U}(\mathbf{y}) = \mathbf{F}, \qquad \rho\text{-a.e. in } \Gamma \tag{5}$$

where $(K_n)_{ij} = \int_D b_n(\mathbf{x}) \nabla \phi_j(\mathbf{x}) \cdot \nabla \phi_i(\mathbf{x})$, for $n = 0, \ldots, N$, are deterministic stiffness matrices and $\mathbf{F}_i = \int_D f(\mathbf{x}) \phi_i(\mathbf{x})$ is a deterministic right hand side.

In writing (5) we have heavily exploited the fact that the random diffusion coefficient is an affine function of the random variables y_n. This allows of an efficient evaluation of the stochastic stiffness matrix $A(\mathbf{y}) = K_0 + \sum_{n=1}^{N} y_n K_n$ in any point $\mathbf{y} \in \Gamma$ and greatly simplifies the implementation of the SG method that will be presented in the next section.

3 Polynomial approximation in the stochastic dimension

We seek a further approximation of $u_h(\cdot, \mathbf{y})$ with respect to $\mathbf{y}$ by global polynomials, which is sound because of the analyticity of the semi-discrete solution with respect to the input random variables $\mathbf{y}$.

In this work we aim at comparing numerically several choices of multivariate polynomials spaces. We remark that the choice of the polynomial space is critical when the number of input random variables, N, is large, since the number of stochastic degrees of freedom might grow very fast with N, even exponentially, for instance when isotropic tensor product polynomials are used, cf. (6). This effect is known as the *curse of dimensionality*.

Let $w \in \mathbb{N}$ be an integer index denoting the level of approximation and $\mathbf{p} = (p_1, \ldots, p_N)$ a multi-index. We introduce a sequence of increasing index sets $\Lambda(w)$ such that $\Lambda(0) = \{(0, \ldots, 0)\}$ and $\Lambda(w) \subseteq \Lambda(w+1)$, for $w \geq 0$. Finally, we denote by $\mathbb{P}_{\Lambda(w)}(\Gamma)$ the multivariate polynomial space

$$\mathbb{P}_{\Lambda(w)}(\Gamma) = span \left\{ \prod_{n=1}^{N} y_n^{p_n}, \text{ with } \mathbf{p} \in \Lambda(w) \right\}$$

and seek a *fully discrete* approximation $u_{hw} \in V_h(D) \otimes \mathbb{P}_{\Lambda(w)}(\Gamma)$.

In the following we consider four possible choices of index sets:

Tensor product polynomial space (TP)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \max_{n=1\ldots,N} p_n \leq w\} \tag{6}$$

Total degree polynomial space (TD)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \sum_{n=1}^{N} p_n \leq w\} \tag{7}$$

Hyperbolic cross space (HC)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \prod_{n=1}^{N} (p_n + 1) \leq w + 1\} \tag{8}$$

Smolyak polynomial space (SM)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \sum_{n=1}^{n} f(p_n) \leq f(w)\}, \quad \text{with } f(p) = \begin{cases} 0, & p = 0 \\ 1, & p = 1 \\ \lceil \log_2(p) \rceil, & p \geq 2 \end{cases} \tag{9}$$

TP and TD spaces are the most common choices. The first suffers greatly from the curse of dimensionality and is impractical for a large dimension N. The second has a reduced curse of dimensionality and has been widely used in SG approximations (see e.g. [11, 27, 13, 17, 23]). HC spaces have been introduced in [1] in the context of approximation of periodic functions by trigonometric polynomials. Recently they have been used to solve elliptic PDEs in high dimension in [21]. Finally, the SM space is an unusual choice in the context of SG approximations. The reason for introducing it will be made clear later, as this space appears naturally when performing interpolation on a sparse grid following the Smolyak construction (see Section 3.2). Observe that the Smolyak space is similar to the hyperbolic cross space; indeed, the HC index set can be equivalently written as $\Lambda^{HC}(w) = \{\mathbf{p} \in \mathbb{N}^N : \quad \sum_{n=1}^N \log_2(p_n + 1) \leq \log_2(w + 1)\}$. Other polynomial spaces have been introduced e.g. in [24].

It is also useful to introduce *anisotropic* versions of these spaces. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}_+^N$ be a vector of positive weights, and $\alpha_{min} = \min_n \boldsymbol{\alpha}$. The anisotropic version of the spaces previously defined reads:

Anisotropic tensor product polynomial space (ATP)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \quad \max_{n=1\ldots,N} \alpha_n p_n \leq \alpha_{min} w\} \tag{10}$$

Anisotropic total degree polynomial space (ATD)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \quad \sum_{n=1}^N \alpha_n p_n \leq \alpha_{min} w\} \tag{11}$$

Anisotropic hyperbolic cross space (AHC)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \quad \prod_{n=1}^N (p_n + 1)^{\frac{\alpha_n}{\alpha_{min}}} \leq w + 1\} \tag{12}$$

Anisotropic Smolyak polynomial space (ASM)

$$\Lambda(w) = \{\mathbf{p} \in \mathbb{N}^N : \quad \sum_{n=1}^N \alpha_n f(p_n) \leq \alpha_{min} f(w)\} \tag{13}$$

In all cases introduced except for the Smolyak space, the maximum polynomial degree used in each direction y_n does not exceed the index w and there is at least one direction (corresponding to the minimum weight α_{min}) for which the monomial y_n^w is in the polynomial space. For the Smolyak space this property holds only if $\log_2(w)$ is integer.

In the next sections we introduce and compare two possible ways of obtaining a *fully-discrete* approximation $u_{hw} \in V_h(D) \otimes \mathbb{P}_{\Lambda(w)}(\Gamma)$, namely Galerkin projection and collocation on a suitable sparse grid.

3.1 Stochastic Galerkin approximation

The Stochastic Galerkin (SG) - Finite Element approximation consists in restricting the weak formulation (3) to the subspace $V_h(D) \otimes \mathbb{P}_{\Lambda(w)}(\Gamma)$ and reads: *find* $u_{hw}^{SG} \in V_h(D) \otimes \mathbb{P}_{\Lambda(w)}(\Gamma)$ *such that* $\forall v_{hw} \in V_h(D) \otimes \mathbb{P}_{\Lambda(w)}(\Gamma)$

$$\int_\Gamma \int_D \left(b_0(\mathbf{x}) + \sum_{n=1}^{N} y_n b_n(\mathbf{x}) \right) \nabla u_{hw}^{SG}(\mathbf{x},\mathbf{y}) \cdot \nabla v_{hw}(\mathbf{x},\mathbf{y})\, \rho(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \int_\Gamma \int_D f(\mathbf{x}) v_{hw}(\mathbf{x},\mathbf{y})\, \rho(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}. \quad (14)$$

Let $\{\psi_{n,p}\}_{p=0}^{\infty}$ be the sequence of orthonormal polynomials in Γ_n with respect to the weight ρ_n, i.e. for any $n = 1, \dots, N$ and $p \geq 0$

$$\int_{\Gamma_n} \psi_{n,p}(t) v(t) \rho_n(t)\, dt = 0 \quad \forall v \in \mathbb{P}_{p-1}(\Gamma_n).$$

Given a multi-index $\mathbf{p} = (p_1, \dots, p_N)$, let $\psi_{\mathbf{p}}(\mathbf{y}) = \prod_{n=1}^{N} \psi_{n,p_n}(y_n)$ be the product of one dimensional orthonormal polynomials. Then a basis for the space $\mathbb{P}_{\Lambda(w)}(\Gamma)$ is given by $\{\psi_{\mathbf{p}},\ \ \mathbf{p} \in \Lambda(w)\}$ and the SG solution can be expanded as

$$u_{hw}^{SG}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{p}\in\Lambda(w)} u_{\mathbf{p}}(\mathbf{x}) \psi_{\mathbf{p}}(\mathbf{y}) = \sum_{\mathbf{p}\in\Lambda(w)} \sum_{i=1}^{N_h} u_{\mathbf{p},i} \phi_i(\mathbf{x}) \psi_{\mathbf{p}}(\mathbf{y}). \quad (15)$$

Given this expansion and exploiting the orthonormality of the basis $\{\psi_{\mathbf{p}}(\mathbf{y})\}$, one can easily compute mean and variance of u_{hw}^{SG} as $\mathbb{E}\left[u_{hw}^{SG}\right](\mathbf{x}) = u_{\mathbf{0}}(\mathbf{x})$ and $\mathbb{V}\mathrm{ar}\left[u_{hw}^{SG}\right](\mathbf{x}) = \sum_{\mathbf{p}\in\Lambda(w)} u_{\mathbf{p}}^2(\mathbf{x}) - \mathbb{E}\left[u_{hw}^{SG}\right]^2(\mathbf{x})$.

Let $\mathbf{U_p} = [u_{\mathbf{p},1}, \dots, u_{\mathbf{p},N_h}]^T$ be the vector of nodal values of the finite element solution corresponding to the $\mathbf{p}$ multi-index. Then inserting expression (15) into (14) and recalling the definition of the deterministic stiffness matrices K_n, we obtain the *system of* $N_w = dim(\mathbb{P}_{\Lambda(w)}(\Gamma))$ *coupled finite element problems*

$$K_0 \mathbf{U_p} + \sum_{n=1}^{N} \sum_{\mathbf{q}\in\Lambda(w)} G_{\mathbf{p},\mathbf{q}}^n K_n \mathbf{U_q} = \mathbf{F}\delta_{\mathbf{0p}}, \qquad \forall \mathbf{p} \in \Lambda(w). \quad (16)$$

where $G_{\mathbf{p},\mathbf{q}}^n = \int_\Gamma y_n \psi_{\mathbf{p}}(\mathbf{y}) \psi_{\mathbf{q}}(\mathbf{y}) \rho(\mathbf{y})\, d\mathbf{y}$ and δ_{ij} is the usual Kroneker symbol. $G_{\mathbf{p},\mathbf{q}}^n$ can be explicitly calculated via the well known three terms relation for orthogonal polynomials, see e.g. [10, 20].

The resulting matrix of the algebraic system (16) is highly sparse, symmetric and positive definite. See e.g. [18] for sparsity plots. For its solution we consider a Preconditioned Conjugate Gradient (PCG) method with block diagonal preconditioner $P_{\mathbf{q},\mathbf{q}} = K_0 + \sum_{n=1}^{N} G_{\mathbf{q},\mathbf{q}}^n K^n$ as suggested in [18]. It follows easily from assumption A1 that the condition number of the preconditioned matrix is

independent of the discretization parameters both in the physical and stochastic spaces, See [19, 7] for a detailed analysis of the condition number of the SG matrix.

Each PCG iteration implies the solution of N_w deterministic problems with matrix $P_{\mathbf{q},\mathbf{q}}$. If the finite element discretization is relatively coarse and the dimension of the probability space is moderate, a Cholesky factorization of all matrices $P_{\mathbf{q},\mathbf{q}}$ could be computed once and for all. In general, this strategy could lead to excessive memory requirements and an iterative method should be preferred. Observe that in certain cases (e.g. for uniform random variables) all blocks are equal and this reduces considerably the computational burden.

Let us now denote by W_{FE} the cost for solving one deterministic problem and by N_{iter} the number of PCG iterations. In this work we focus on the computational cost for solving the linear system (16) and neglect the time for assembling the full stochastic matrix, which highly depends on how much the computer code has been optimized. Therefore, we can estimate the total cost W_{SGFE} for SG - finite element as

$$W_{SGFE} \approx N_w * W_{FE} * N_{iter}. \tag{17}$$

This estimate will be used to compare the SG method with the SC method in the numerical tests presented in Section 4.

3.2 Stochastic collocation approximation on sparse grids

The Stochastic Collocation (SC) - Finite Element method consists in collocating the semi-discrete problem (4) in a set of points $\{\boldsymbol{\theta}_j \in \Gamma, \quad j = 1, \ldots, M_w\}$, i.e. computing the solutions $u_h(\cdot, \boldsymbol{\theta}_j)$ and building a global polynomial approximation u_{hw}^{SC} (not necessarily interpolatory) upon those evaluations: $u_{hw}^{SC}(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{M_w} u_h(\mathbf{x}, \boldsymbol{\theta}_j)\tilde{\psi}_j(\mathbf{y})$ for suitable multivariate polynomials $\{\tilde{\psi}_j\}_{j=1}^{M_w}$.

We consider here a generalization of the classical Smolyak construction (see e.g. [22, 4]) to build a multivariate polynomial approximation on a sparse grid. For each direction y_n we introduce a sequence of one dimensional polynomial interpolant operators of increasing order: $\mathcal{U}_n^{m(i)} : C^0(\Gamma_n) \to \mathbb{P}_{m(i)-1}(\Gamma_n)$. Here $i \geq 1$ denotes the level of approximation and $m(i)$ the number of collocation points used to build the interpolation at level i, with the requirement that $m(1) = 1$ and $m(i) < m(i+1)$ for $i \geq 1$. In addition, let $m(0) = 0$ and $\mathcal{U}_n^{m(0)} = 0$. In this work the collocation points $\{\theta_{n,j}^{(i)}, \ j = 1, \ldots, m(i)\}$ for the one dimensional interpolation formula $\mathcal{U}_n^{m(i)}$ will be taken as the Gauss points with respect to the weight ρ_n, that is the zeros of the orthogonal polynomial $\psi_{n,m(i)}$. To simplify the presentation of the sparse grid approximation (18), we now introduce the difference operators

$$\Delta_n^{m(i)} = \mathcal{U}_n^{m(i)} - \mathcal{U}_n^{m(i-1)}.$$

Given an integer $w \geq 0$ and a multi-index $\mathbf{i} = (i_1, \ldots, i_N) \in \mathbb{N}_+^N$, $\mathbf{i} \geq \mathbf{1}$, we introduce a function $g : \mathbb{N}_+^N \to \mathbb{N}$ strictly increasing in each argument and define a sparse grid approximation of u_h as

$$u_{hw}^{SC} = \mathcal{S}_w^{m,g}[u_h] = \sum_{\mathbf{i} \in \mathbb{N}_+^N : g(\mathbf{i}) \leq w} \bigotimes_{n=1}^{N} \Delta_n^{m(i_n)}(u_h). \tag{18}$$

The previous formula implies evaluation of the function u_h in a finite set of points $\mathcal{H}_w^{m,g} \subset \Gamma$ (*sparse grid*). From the construction (18) one can easily build the corresponding quadrature formula, and evaluate e.g. $\mathbb{E}\left[u_{hw}^{SC}\right](\mathbf{x}) = \sum_{j=1}^{M_\omega} \omega_j u_h(\mathbf{x}, \boldsymbol{\theta}_j)$ and $\mathbb{V}\mathrm{ar}\left[u_{hw}^{SC}\right] = \omega_j u_h^2(\mathbf{x}, \boldsymbol{\theta}_j) - \mathbb{E}\left[u_{hw}^{SC}\right]^2(\mathbf{x})$. To fully characterize the sparse approximation operator $\mathcal{S}_w^{m,g}$ one has to provide the two strictly increasing functions $m : \mathbb{N}_+ \to \mathbb{N}_+$ and $g : \mathbb{N}_+^N \to \mathbb{N}$. The first defines the relation between the level i and the number of points $m(i)$ in the corresponding one dimensional polynomial interpolation formula $\mathcal{U}^{m(i)}$, while the second characterizes the set of multi-indices used to construct the sparse approximation. Since m is not surjective in $\mathbb{N}^+$ (unless it is affine) we introduce a *left inverse* $m^{-1}(k) = \min\{i \in \mathbb{N}_+ : \ m(i) \geq k\}$. Observe that with this choice m^{-1} is a (non-strictly) increasing function satisfying $m^{-1}(m(i)) = i$, and $m(m^{-1}(k)) \geq k$.

Let $\mathbf{m}(\mathbf{i}) = (m(i_1), \ldots, m(i_N))$ and consider the polynomial order set

$$\Lambda^{m,g}(w) = \{\mathbf{p} \in \mathbb{N}^N, \ \ g(\mathbf{m}^{-1}(\mathbf{p} + \mathbf{1})) \leq w\}.$$

The following result characterizes the polynomial space underlying the sparse approximation $\mathcal{S}_w^{m,g}[u_h]$:

Proposition 1

a) For any $f \in C^0(\Gamma)$, we have $\mathcal{S}_w^{m,g}[f] \in \mathbb{P}_{\Lambda^{m,g}(w)}$.

b) Moreover, $\mathcal{S}_w^{m,g}[v] = v, \ \ \forall v \in \mathbb{P}_{\Lambda^{m,g}(w)}$.

Proof. Let us denote by $\mathbb{P}_{\mathbf{m(i)-1}}$ the tensor product polynomial space

$$\mathbb{P}_{\mathbf{m(i)-1}} = span\{\prod_{n=1}^{N} y_n^{p_n}, \ \ p_n \leq m(i_n) - 1\}.$$

Clearly we have that $\bigotimes_{n=1}^{N} \Delta_n^{m(i_n)}(f) \in \mathbb{P}_{\mathbf{m(i)-1}}(\Gamma)$ and

$$\begin{aligned}
\mathcal{S}_w^{m,g}[f] &\in span\Big\{ \bigcup_{\mathbf{i}\in\mathbb{N}_+^N:\, g(\mathbf{i})\le w} \mathbb{P}_{\mathbf{m(i)-1}}(\Gamma)\Big\} \\
&\equiv span\Big\{ \bigcup_{\mathbf{i}\in\mathbb{N}_+^N:\, g(\mathbf{i})\le w} span\{\prod_{n=1}^{N} y_n^{p_n},\ \ \mathbf{p}\le \mathbf{m(i)}-\mathbf{1}\}\Big\} \\
&\equiv span\Big\{ \bigcup_{\mathbf{i}\in\mathbb{N}_+^N:\, g(\mathbf{i})\le w} span\{\prod_{n=1}^{N} y_n^{p_n},\ \ \mathbf{m}^{-1}(\mathbf{p}+\mathbf{1})\le \mathbf{i}\}\Big\} \\
&\equiv span\{\prod_{n=1}^{N} y_n^{p_n},\ \ g(\mathbf{m}^{-1}(\mathbf{p}+\mathbf{1}))\le w\} =: \mathbb{P}_{\Lambda^{m,g}(w)}(\Gamma).
\end{aligned}$$

This proves a). Due to linearity in (18), to prove point b) we only need to show that the approximation formula $\mathcal{S}_w^{m,g}$ is exact for all monomials $\prod_{n=1}^{N} y_n^{p_n}$ with $\mathbf{p}\in\Lambda^{m,g}(w)$. We have

$$\begin{aligned}
\mathcal{S}_w^{m,g}\left[\prod_{n=1}^{N} y_n^{p_n}\right] &= \sum_{\mathbf{i}\in\mathbb{N}_+^N:\, g(\mathbf{i})\le w} \bigotimes_{n=1}^{N} \Delta_n^{m(i_n)}\mathbf{y^p} \\
&= \sum_{\mathbf{i}\in\mathbb{N}_+^N:\, g(\mathbf{i})\le w} \prod_{n=1}^{N} \left((\mathcal{U}^{m(i_n)} - \mathcal{U}^{m(i_n-1)})y_n^{p_n}\right).
\end{aligned}$$

Observe that $\mathcal{U}^{m(i_n)}y_n^{p_n}$ will be an exact interpolation whenever $m(i_n)\ge p_n+1$ and therefore the term $\prod_{n=1}^{N}(\mathcal{U}^{m(i_n)}-\mathcal{U}^{m(i_n-1)})y_n^{p_n}$ will vanish if any of the $m(i_n-1)\ge p_n+1$ or equivalently if there exists at least one n such that $i_n\ge m^{-1}(p_n+1)+1$. Let $\bar{i}_n = m^{-1}(p_n+1)$ for $n=1,\dots,N$. The multi-index $\bar{\mathbf{i}}=(\bar{i}_1,\dots,\bar{i}_N)$ satisfies the constraint $g(\bar{\mathbf{i}})\le p$.

Then, the previous formula reduces to

$$\begin{aligned}
\mathcal{S}_w^{m,g}\left[\prod_{n=1}^{N} y_n^{p_n}\right] &= \sum_{\mathbf{i}\le\bar{\mathbf{i}}} \prod_{n=1}^{N} \left((\mathcal{U}^{m(i_n)} - \mathcal{U}^{m(i_n-1)})y_n^{p_n}\right) \\
&= \prod_{n=1}^{N} \sum_{i_n=0}^{\bar{i}_n} \left((\mathcal{U}^{m(i_n)} - \mathcal{U}^{m(i_n-1)})y_n^{p_n}\right) = \prod_{n=1}^{N} \mathcal{U}^{m(\bar{i}_n)}y_n^{p_n}.
\end{aligned}$$

The final result follows from the fact that $m(\bar{i}_n) = m(m^{-1}(p_n+1))\ge p_n+1$ and therefore the interpolant $\mathcal{U}^{m(\bar{i}_n)}$ is exact for $y_n^{p_n}$.

□

Remark 1 *Observe that in the previous Lemma we have never used the assumption that the one dimensional interpolants are based on Gauss points. Hence,*

the previous result still holds for interpolants based on arbitrary (distinct) knots and for an arbitrary strictly increasing function $m(i)$.

We recall that the most typical choice of m and g is given by (see [22, 4])

$$m(i) = \begin{cases} 1, & \text{for } i = 1 \\ 2^{i-1} + 1, & \text{for } i > 1 \end{cases} \quad \text{and } g(\mathbf{i}) = \sum_{n=1}^{N} (i_n - 1)$$

This choice of m, combined with the choice of Clenshaw-Curtis interpolation points (extrema of Chebyshev polynomials) leads to nested sequences of one dimensional interpolation formulas and a reduced sparse grid. In the same vein, it is possible to show that the underlying polynomial space associated to the operator $S_w^{m,g}$ is the Smolyak space $\mathbb{P}_{\Lambda(w)}$ defined in (9).

On the other hand, if we choose $m(i) = i$, it is easy to find functions g for the construction of sparse collocation approximations in the polynomial spaces introduced in Section 3, namely tensor product (6), total degree (7) and hyperbolic cross (8) spaces. Table 1 summarizes several available. It is also straightforward

Approx. space	SC: m, g	SG: $\Lambda(w)$
Tensor Product (TP)	$m(i) = i$ $g(\mathbf{i}) = \max_n(i_n - 1) \leq w$	$\{\mathbf{p} \in \mathbb{N}^N : \ \max_n p_n \leq w\}$
Total Degree (TD)	$m(i) = i$ $g(\mathbf{i}) = \sum_n(i_n - 1) \leq w$	$\{\mathbf{p} \in \mathbb{N}^N : \ \sum_n p_n \leq w\}$
Hyperbolic Cross (HC)	$m(i) = i$ $g(\mathbf{i}) = \prod_n(i_n) \leq w + 1$	$\{\mathbf{p} \in \mathbb{N}^N : \ \prod_n(p_n + 1) \leq w + 1\}$
Smolyak (SM)	$m(i) = \begin{cases} 2^{i-1} + 1, \, i > 1 \\ 1, \, i = 1 \end{cases}$ $g(\mathbf{i}) = \sum_n(i_n - 1) \leq w$	$\{\mathbf{p} \in \mathbb{N}^N : \ \sum_n f(p_n) \leq f(w)\}$ $f(p) = \begin{cases} 0, \, p = 0 \\ 1, \, p = 1 \\ \lceil \log_2(p) \rceil \, p \geq 2 \end{cases}$

Table 1: Sparse approximation formulas and corresponding underlying polynomial space

to build the corresponding anisotropic sparse approximation formulas.

Let now $\mathcal{H}_w^{m,g}$ be the sparse grid associated to the formula $\mathcal{S}_w^{m,g}$ and $M_w = \#\mathcal{H}_w^{m,g}$ the number of distinct collocation points in $\mathcal{H}_w^{m,g}$. To form the sparse collocation solution $u_{h,w}$ we only have to solve M_w *independent* deterministic problems. Observe, however, that in general the number of points M_w is much larger than the dimension N_w of the corresponding polynomial space $\mathbb{P}_{\Lambda^{m,g}(w)}$. The computational cost of the SC - Finite Element method can therefore be estimated as

$$W_{SCFE} \approx M_w * W_{FE}, \tag{19}$$

to be compared with the cost of the SG - Finite Element method in the same polynomial space, given by (17).

4 Numerical results

4.1 Test case 1: isotropic problem

In this first test case we consider a thermal diffusion problem in the form of (1) defined in the unit square $[0,1]^2$, with homogeneous Dirichlet boundary conditions and stochastic conductivity coefficient that depends on a finite, small, number of random variables. The coefficient is chosen in such a way that each random input has more or less the same influence on the solution (isotropic problem).

Fig. 1-left shows the geometry of the test case. The forcing term is deterministic, $f(\mathbf{x}) = 100\chi_F(\mathbf{x})$, where $\chi_F(\mathbf{x})$ is the indicator function of F, a square subdomain with side length equal to 0.2, centered in the domain. The material features 8 circular inclusions with radius $r = 0.13$ and symmetrically distributed with respect to the center of the square, each with a uniformly distributed random conductivity. Let $\chi_n(\mathbf{x}), n = 1, .., 8$ be the indicator function for each circle. The expression of the stochastic conductivity coefficient is then in the form of (2), with $b_n(\mathbf{x}) = \chi_n(\mathbf{x})$:

$$a(\omega, \mathbf{x}) = b_0(\mathbf{x}) + \sum_{n=1}^{8} y_n(\omega)\chi_n(\mathbf{x}), \quad \text{with } b_0 = 1 \text{ and } y_n(\omega) \sim \mathcal{U}(-0.99, -0.2)$$

As a consequence, the basis functions $\psi_{n,p}$ for SG methods will be Legendre polynomials orthonormal with respect to the uniform probability measure in $[-0.99, -0.2]$, and the collocation points for SC will be the corresponding Gauss points.

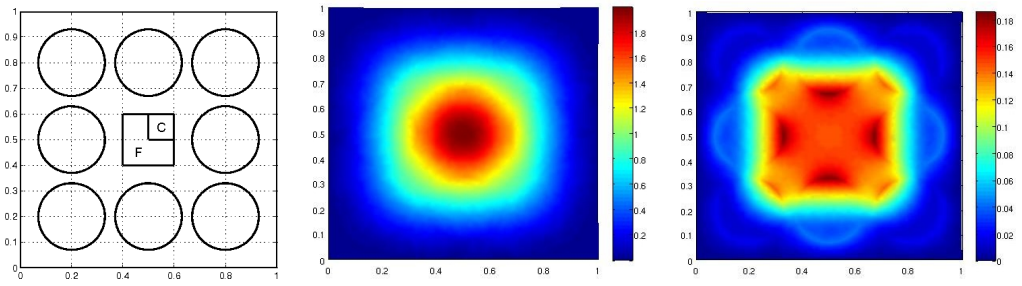

Figure 1: Left: geometry for test case 1. Middle: expected value of the solution. Right: standard deviation of the solution.

We will compare the accuracy of the Stochastic Galerkin (SG) and Stochastic Collocation (SC) methods by looking at statistical indicators of two quantities of interest:

- $\psi_1(u) = \int_F u(\mathbf{x})d\mathbf{x}$;
- $\psi_2(u) = \int_C \partial_x u(\mathbf{x})d\mathbf{x}$.

The quantity $\psi_2(u)$ is defined only on C, the upper right part of F, since by symmetry its expected value on F is 0 whatever (isotropic) Galerkin or Collocation approximation is considered.

Let u_p be an approximate solution (computed either with SG or SC) and u_{ex} the exact solution. For both quantities ψ_1 and ψ_2 we will check the convergence of the following errors:

- error in the mean: $\varepsilon_{\text{mean}}[\psi_j] = |\mathbb{E}[\psi_j(u_p)] - \mathbb{E}[\psi_j(u_{ex})]|$;
- error in the variance: $\varepsilon_{\text{var}}[\psi_j] = |\mathbb{V}\text{ar}[\psi_j(u_p)] - \mathbb{V}\text{ar}[\psi_j(u_{ex})]|$;
- error in L^2 norm: $\varepsilon_{\text{norm}}[\psi_j] = \sqrt{\mathbb{E}[(\psi_i(u_p) - \psi_i(u_{ex}))^2]}$.

Since we do not know the exact solution for this problem, we will check the convergence of the statistical indicators with respect to an overkill solution, which we consider close enough to the exact one. To this end we take the solution computed with SG-TD at level 9, which has approximately 24000 stochastic degrees of freedom (sdof). The L^2 error will be calculated via a MCS approximation, i.e. $\varepsilon_{\text{norm}}[\psi_j] \simeq \frac{1}{M}\left(\sum_{l=1}^{M}[\psi_j(u_p(\mathbf{y}_l)) - \psi_j(u_{ex}(\mathbf{y}_l))]^2\right)^{1/2}$, where $\mathbf{y}_l$, $l = 1, .., M$, are M randomly chosen points in Γ. To this end we have used $M = 1000$ points.

We remark that here and in the following test all the computations are performed on the same physical mesh, which is supposed to be refined enough to solve adequately the elliptic problem for every value $\mathbf{y}$ of the random variables. Moreover notice that, as stated in section 2.1, the FEM solution and the exact solution have the same regularity with respect to the stochastic variables. Therefore we expect the convergence in the stochastic dimension not to be affected by space discretization.

We have compared the performances of the SG and Collocation methods with the four choices of polynomial spaces presented in Table 1. In our convergence plots we have also added the performance of the classical MCS method.

Fig. 2 shows the error $\varepsilon_{\text{mean}}[\psi_1]$ versus the estimated computational cost (normalized to the cost W_{FE} of a deterministic solve) given by formula (17) for SG methods and (19) for SC methods. For the MCS method the cost is simply $M * W_{FE}$, where M is the number of samples used. The MCS has been repeated 20 times and only the average error over the 20 repetitions is shown.

As one can see, MCS has the worst performance, followed by tensor product polynomial spaces both in the SG and SC version, as expected. All other choices lead to similar, however much more accurate, results, with TD being the best space for Galerkin method and SM the best for Collocation.

We notice that different choices of collocation points for SC-SM (Gauss versus Clenshaw Curtis) lead to similar results (see Fig. 2-right). Therefore from now on we will only use SC-SM with Gauss points.

From Fig. 2-right we conclude that the SC method is the best method with respect to the computational cost, at least for "practical" tolerances, while, for

very small tolerances ($\leq 10^{-10}$), SG is a better choice. The same happens also for the other error indicators $\varepsilon_{\text{var}}[\psi_1]$ and $\varepsilon_{\text{norm}}[\psi_1]$, (see Fig. 3), as well as for the quantity ψ_2 (see Fig. 4).

We should point out that the plots may not represent a completely fair comparison. Actually, the solution of the global linear system for SG method is performed through preconditioned conjugate gradient iterations, with a fixed tolerance ($\epsilon = 10^{-12}$); this clearly over-resolves the system when the error in the stochastic dimension is much larger than ϵ. The performance of SG may be therefore improved by tuning the tolerance of the PCG method to an *a posteriori* estimation of the stochastic error. However, we have observed that running the same SG simulations with tolerance $\epsilon = 10^{-8}$ changes only slightly the results, so we can say that the choice of the tolerance for the PCG method is not deeply affecting our performance/cost analysis.

It is also instructive to look at the convergence plots of the error versus the dimension of the stochastic space (Fig. 5). As expected from L^2 optimality, for a given polynomial space the Galerkin solution is more accurate than the collocation solution. We remind once more, however, that the computational cost in the two cases is quite different and the convergence plots in Fig. 2 give a more complete picture of the performances of the two methods.

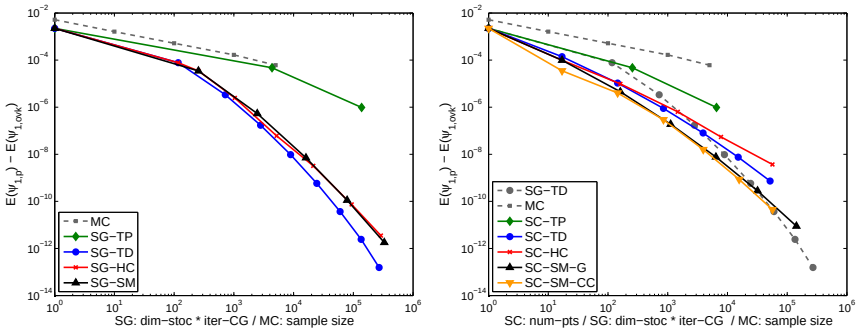

Figure 2: Error $\varepsilon_{\text{mean}}[\psi_1]$ versus estimated computational cost. Left: comparison between SG methods and Monte Carlo. Right: comparison between SC methods and SG-TD.

4.2 Test case 2: anisotropic problem

In this test we consider an anisotropic problem in which different random variables contribute differently to the total variability of the solution, in order to study the advantages of the anisotropic version of the SC and SG methods. We take the geometry and problem definition similar to test case 1; however, since our focus is on anisotropy, we consider only 4 inclusions (the ones in the corners, cf. Fig.6-left) so that we can test many different choices of the weights that define the anisotropic spaces (10)-(13). Nonetheless, the anisotropic setting is

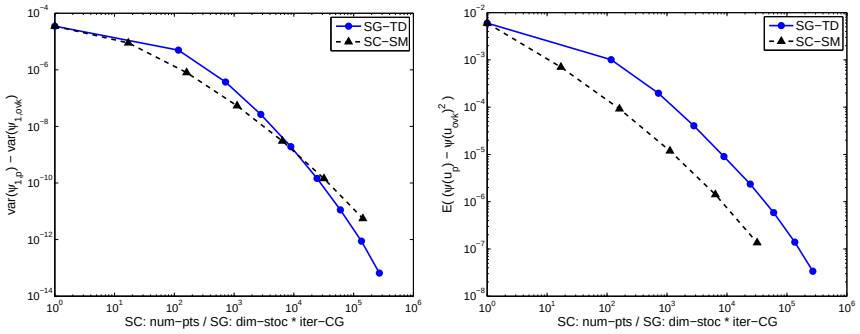

Figure 3: Convergence curves for $\varepsilon_{\text{var}}[\psi_1]$ (left) and $\varepsilon_{\text{norm}}[\psi_1]$ (right) with respect to the computational cost. Comparison between SG-TD and SC-SM methods.

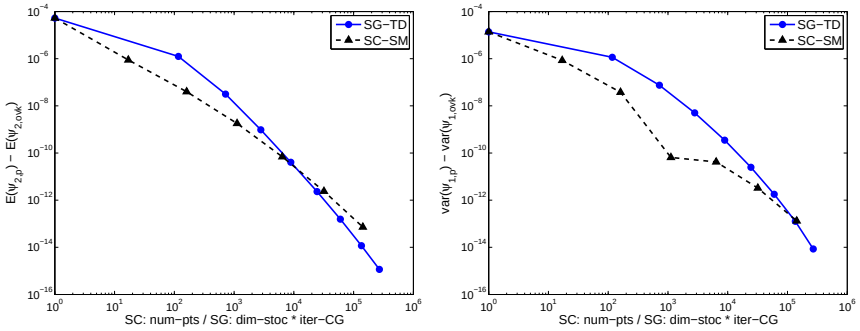

Figure 4: Convergence curves for $\varepsilon_{\text{mean}}[\psi_2]$ (left) and $\varepsilon_{\text{var}}[\psi_2]$ (right) with respect to the computational cost. Comparison between SG-TD and SC-SM methods.

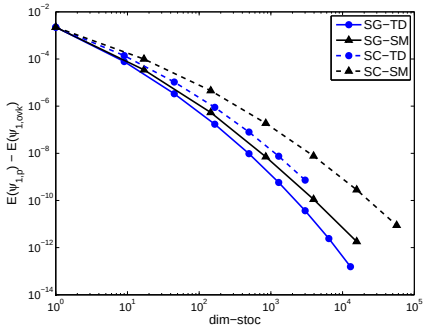

Figure 5: Convergence curves for $\varepsilon_{\text{mean}}[\psi_1]$ with respect to the dimension of the stochastic space. Comparison between SG and SC methods with TD and SM polynomial spaces.

particularly meant to be used in high dimensional spaces (see e.g. [15]). For convenience we consider a forcing term uniformly distributed on the whole domain

and we look just at $\varepsilon_{\text{mean}}[\psi_1]$.

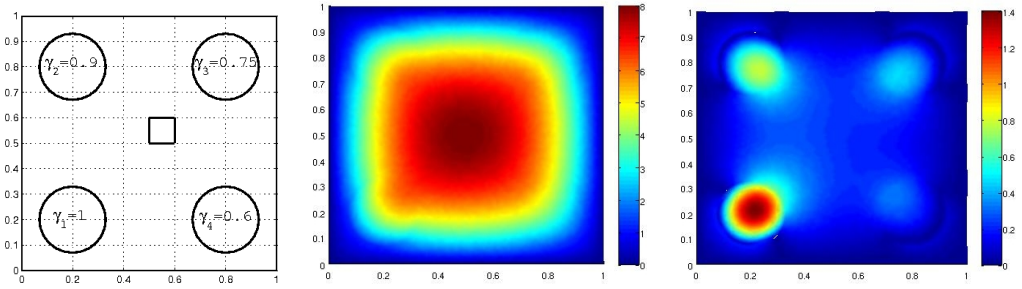

Figure 6: Left: geometry for test case 2. Middle: expected value of the solution. Right: standard deviation of the solution.

The random coefficient is $a(\omega, \mathbf{x}) = 1 + \sum_{n=1}^{4} \gamma_n y_n(\omega)\chi_n(\mathbf{x})$, with $y_n(\omega) \sim \mathcal{U}(-0.99, 0)$ and $\gamma_n \leq 1$. The values of the coefficients γ_n are shown in Fig. 6-left. Notice that these values give different importance to the four random variables. In particular, the inclusion in the bottom-left corner has the largest variance and we expect it to contribute the most to the total variance of the solution. It is therefore intuitively justified to use polynomial degrees higher in the corresponding direction of the stochastic multidimensional space rather than in the other ones. Fig. 6 also shows the mean value (middle) and the standard deviation (right) of the solution.

Our goal is to assess the performances of anisotropic polynomial spaces in comparison with their isotropic counterpart. For this we need to estimate the weights to be used in the construction of the anisotropic polynomial space.

We follow closely the argument in [15]. The overall random conductivity coefficient in the n-th inclusion Ω_n is a uniform random variable $\mathcal{U}(a_n, b_n)$ with $a_n = 1 - 0.99\gamma_n$ and $b_n = 1$. This can be rewritten as

$$a(\omega, \mathbf{x})_{|\Omega_n} = \frac{a_n + b_n}{2} + \frac{b_n - a_n}{2}\hat{y}_n, \qquad \text{with } \hat{y}_n \sim \mathcal{U}(-1, 1).$$

It is easy to show that the solution $u = u(\cdot, \hat{y}_n)$ admits an analytic continuation in the complex region $\Sigma_n = \{z \in \mathbb{C} : \mathfrak{Re}(z) > -w_n\}$ with $w_n = \frac{a_n+b_n}{b_n-a_n} = \frac{2-0.99\gamma_n}{0.99\gamma_n}$, which contains, in particular, the interior of the ellipse

$$\mathcal{E}_{\rho_n} = \left\{ z \in \mathbb{C} : \mathfrak{Re}(z) = \frac{\rho_n + \rho_n^{-1}}{2}\cos\phi,\ \mathfrak{Im}(z) = \frac{\rho_n - \rho_n^{-1}}{2}\sin\phi,\ \phi \in [0, 2\pi) \right\}$$

with $\rho_n = w_n + \sqrt{w_n^2 - 1}$.

Standard spectral approximation analysis (see e.g. [6]) allows us to say that interpolation of $u(\cdot, \hat{y}_n)$ in $p_n + 1$ Gauss-Legendre points converges exponentially fast with rate $e^{-g_n p_n}$, with $g_n = \log \rho_n = \log(w_n + \sqrt{w_n^2 - 1})$.

Therefore the *theoretical estimate* (*a priori* choice) of the weight to be used for the n-th variable is $\alpha_n = g_n$. The larger γ_n, the smaller the corresponding weight α_n. In practice, we have renormalized the weights by dividing them by the smallest one. Notice that the spaces (10)-(13) remain unchanged by this normalization. The corresponding theoretical weights are in this case $\alpha^{th} = [1,\ 3.5,\ 5.5,\ 7.5]$. To assess the effectiveness of the proposed theoretical estimate, we also consider the weights $\alpha = [1, 2, 3, 4]$ (nearly half the theoretical estimate) and $\alpha = [1, 7, 11, 15]$ (twice the theoretical estimate). Finally, we have also considered an experimental (*a posteriori*) estimate of the coefficients (as suggested in [15]), where the exponential decay $e^{-g_n p_n}$ is estimated numerically by increasing the approximation level in only one direction at a time; the resulting weights are $\alpha^{exp} = [1,\ 2.5,\ 4,\ 5.5]$.

In this example we consider only SG methods in anisotropic TD spaces as they seem to be the most appropriate for this type of problem. Similarly, we restrict our study only to SC methods in the same ATD spaces, so they are directly comparable with the corresponding Galerkin version. The use of SC-ASM methods is expected to give even better results.

We have computed the SG-ATD and SC-ATD with the different choices of weights up to level $w = 21$ and compared them with an overkill solution computed by SG-TD isotropic method at level $w = 22$. This solution has about 14000 sdof . In comparison, the SG-ATD solution has 837 sdof with weights $\alpha = [1, 2, 3, 4]$, 434 sdof with the experimental weights $\alpha^{exp} = [1, 2.5, 4, 5.5]$, 220 sdof with the theoretical weights $\alpha^{th} = [1, 3.5, 5.5, 7.5]$, and 68 sdof with the weights $\alpha = [1, 7, 11, 15]$. We observe that the level $w = 22$ isotropic TD space contains all the ATD spaces with level $w < 22$, therefore our overkill solution is much more accurate than the other ones considered here.

Fig. 7 shows the error in computing $\mathbb{E}[\psi_1]$ versus the estimated computational cost when using the SG-ATD (left) or SC-ATD (right) methods. For reference purposes we have also added the convergence plot for MCS.

First, we observe that SC and SG outperform the standard MCS. Fig. 7 also shows that the theoretical estimate of the weights performs better than all other choices and seems to be very close to optimum for both SC and SG methods, while the *a posteriori* choice gives slightly worse results although the convergence curve is smoother.

In Fig. 8 we compare the performances of the SG-ATD and SC-ATD methods with the theoretical and experimental choices of the weights. In this test, the collocation method seems to be superior to the Galerkin one, even for very small tolerances.

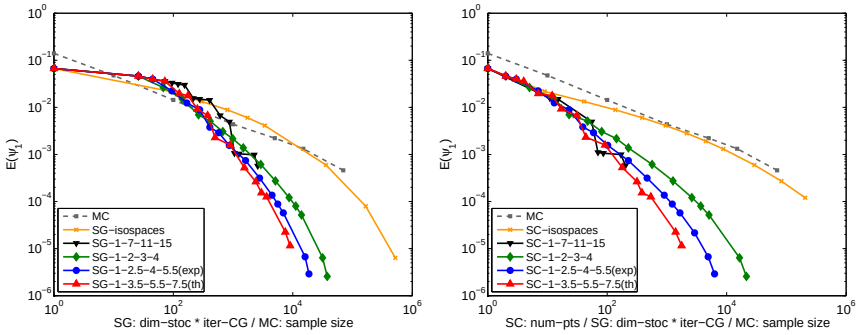

Figure 7: Performance of SG-ATD (left) and SC-ATD (right) methods with different choices of weights, in the computation of $\mathbb{E}[\psi_1]$. Error $\varepsilon_{\text{mean}}[\psi_1]$ versus computational cost.

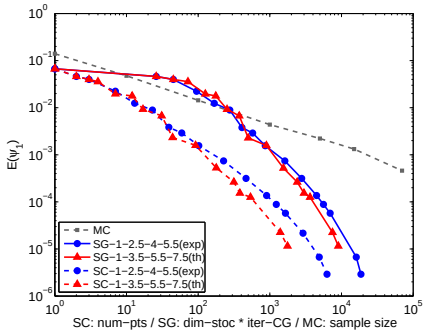

Figure 8: Comparison between SG-ATD and SC-ATD methods with best weights in the computation of $\mathbb{E}[\psi_1]$. Error $\varepsilon_{\text{mean}}[\psi_1]$ versus computational cost.

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