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Stabilization Strategies for High Order Methods for Transport Dominated Problems

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STABILIZATION STRATEGIES FOR HIGH ORDER METHODS FOR TRANSPORT DOMINATED PROBLEMS

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ABSTRACT. Standard high order Galerkin methods, such as pure spectral or high order finite element methods, have insufficient stability properties when applied to transport dominated problems. In this paper we review some stabilization strategies for pure spectral methods and spectral multidomain approaches.

1. INTRODUCTION

1.1. **General overview.** Finite element upwind approximation techniques have been developed enormously since the pioneering work by Mitchell and Griffiths [30] and the generalization and analysis by Baba and Tabata [1]. The strongly consistent SUPG–method introduced in 1982 by Brooks and Hughes [8] and analyzed by Johnson, Nävert and Pitkäranta [24] opened the door to high order upwind approximations in a finite element framework. Since then, a broad variety of strategies for determining stabilization parameters, generalizations and other approaches have been proposed, see e.g. the book by Roos, Stynes and Tobiska [35].

1.2. **Model problem.** We will concentrate on the following model problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \mathcal{L}u \equiv -\nu\Delta u + \boldsymbol{\beta}\cdot\nabla u + \sigma u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^d$ with boundary $\partial\Omega$. For the problem with non-homogenous boundary conditions standard lifting techniques can be used. The problem is defined by the diffusion coefficient $\nu > 0$, the reaction coefficient $\sigma \geq 0$ and the velocity field $\boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^d$ and we will assume that the following standard coercivity condition holds

$$\sigma - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \geq \sigma_0 > 0.$$

We focus on the case where the advection is dominating the diffusion process; in particular we will later assume that $\sigma = 0$. Nevertheless the case $\sigma \neq 0$ is meaningful too since the reaction term can be interpreted as a time-derivation for time dependent problems. We will also consider the reduced, pure transport problem obtained by setting $\nu = 0$ in (1) and adapting the boundary condition:

$$\begin{aligned} \boldsymbol{\beta}\cdot\nabla u + \sigma u &= f \quad \text{in } \Omega, \\ \boldsymbol{\beta}\cdot\boldsymbol{n} u &= g \quad \text{on } \partial\Omega^-, \end{aligned} \tag{2}$$

where $\partial\Omega^{\pm} = \{\boldsymbol{x} \in \partial\Omega \,:\, \pm\boldsymbol{\beta}\cdot\boldsymbol{n}(\boldsymbol{x}) > 0\}$. The reduced problem has no smoothing properties. Indeed discontinuities in the solution provoked by the data f and g will propagate with the flow field $\boldsymbol{\beta}$ giving rise to internal layers. In the presence of such layers Galerkin methods, such as pure spectral methods or finite element methods, have insufficient stability properties. Indeed due to the conservation properties of standard Galerkin schemes, the high frequency content of the solution will be represented on the (lower) frequencies present in the discrete space leading to spurious oscillations. The cure of such instabilities by controlling the onset of spurious oscillations for high order methods will be the main topic of this article.

Problem (1) provides some smoothing of internal layers, however, in case the layers are unresolved by the finite element method, either by the spacial discretization or by the polynomial degree, the effect will remain the same as for the problem (2), see Figure 1.

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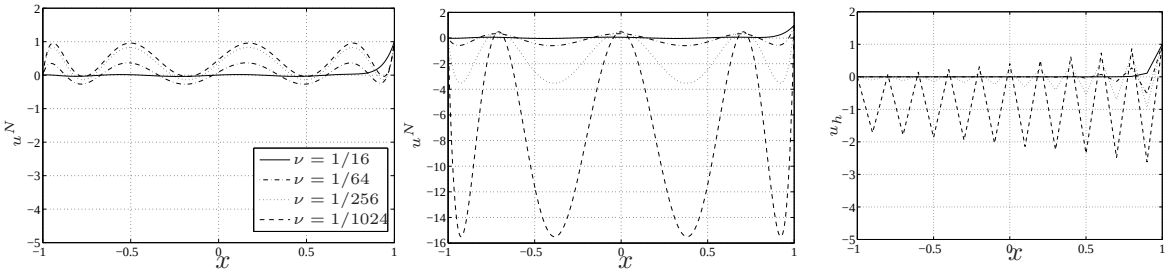

FIGURE 1. Spectral solutions u^N with $N = 9$ (left), $N = 8$ (middle) and piecewise linear finite element solutions u_h with $h = 0.1$ (right) of the advection-diffusion problem $-\nu u_{xx} + u_x = 0,\ -1 < x < 1$, for several values of ν.

1.3. **Outline.** As previously anticipated, in this article we review some basic results about the stabilization of model problem (1) with dominant advection for high order polynomial approximations.

In section 2 we will recall some a-priori estimates for the pure spectral Legendre-Galerkin approximation and discuss the accuracy of transport–dominated problems. An argument based on a maximum principle in the transform space shows that, for odd polynomial degrees, the maximum norm of the approximation is uniformly bounded. Consequently, although oscillations do exist, their amplitude is controlled. For even polynomial degrees, however, the situation is different and the maximum norm of the approximation can not be uniformly bounded.

In section 3 we will revisit the techniques of stabilization for pure spectral methods on single domains. Different strategies to control the oscillations can be invoked. A simple approach consists of applying a filtering procedure, which damps the highest components of the spectrum of the discrete solution. Other stabilization techniques are inspired by procedures that originated in the framework of low-order schemes, such as the h-version of finite elements. A very common strategy for stabilization is the SUPG-stabilization (where SUPG stands for Streamline Upwind Petrov-Galerkin) after Brooks and Hughes [8] and Johnson, Nävert and Pitkäranta [24], or–lately–as the bubble stabilization after Brezzi, Bristeau, Franca, Mallet and Rogé [5] and Brezzi and Russo [7]. An interpretation of the SUPG-method in a spectral framework is given by Canuto [14], Pasquarelli and Quarteroni [31] and Canuto and Puppo [16]. For a general review see [15]. The use of Petrov-Galerkin approach–in which test functions are different from trial functions, the former being biased by the advection (or stream) direction–dates back to the mid 1970's (Christie et al. [17]) and corresponds to the finite element interpretation of upwind collocation or finite difference methods. More recent developements are done by Melenk and Schwab [29] and Gerdes, Melenk, Schwab and Schötzau [19]. A third approach consists of injecting the right amount of artificial viscosity. This idea has been introduced by Maday and Tadmor [27, 37] in the context of high order methods, leading to the concept of spectral viscosity.

In section 4 we review the developement towards spectral multidomain approaches, also known as hp- or SE (standing for Spectral Element) -methods, and in section 5 stabilization techniques for these methods are studied. The methods can be split into two classes. The former consists of element-based stabilizations techniques. Some of them share their origin and motivation with spectral methods. Using the analysis for single domain methods the above introduced techniques can be applied element-wise and results similar to those for spectral methods can be obtained with the only difference that now a parameter for the spacial discretization h is introduced. Other high order methods that are not originated from spectral methods will be discussed as well. The second class consists of face-based stabilization procedures, known as interior penalty methods such as CIP (Continuous Interior Penalty) or discontinuous Galerkin (DG) methods. CIP methods were analysed in the hp-framework by Burman and Ern [9] and DG methods by Houston, Schwab and Süli [22]. Both CIP and DG methods use the solution's fluctuation between elements to stabilize the numerical scheme.

Finally we focus on some recent results on interior penalty methods. Some of the questions that will be considered are how to construct a minimal stabilization approach, i.e. a method that affects the smallest possible portion of the finite element spectrum or vanishes at optimal rate as the polynomial degree increases. This leads to a natural interpretation of local mass conservation in the case of discontinuous approximation of the pure transport problem and to a method with increased robustness with respect to the choice of stabilization parameters for continuous approximation of the advection-diffusion equation.

2. SINGLE DOMAIN METHODS

2.1. **Galerkin formulation of model problem.** In this section we consider Galerkin methods, i.e. methods where the test and trial spaces in the variational formulation are the same. The Galerkin weak formulation of the above problem (1) can be defined as follows. Let $\mathbb{P}_N(\Omega) \subset H^1(\Omega)$ be the space of polynomials, either trigonometric (Fourier) or algebraic (Legendre and Chebychev), of degree N on Ω. The approximation space is defined by

$$V_N = \{v \in \mathbb{P}_N(\Omega) \,:\, v|_{\partial\Omega} = 0\}$$

and the Galerkin method then writes: Find $u^N \in V_N$ such that

$$a(u^N, v^N) = (f, v^N) \quad \forall v^N \in V_N, \tag{3}$$

where $(\cdot,\cdot)$, $(\cdot,\cdot)_\Lambda$ with $\Lambda \subset \Omega$ denotes an inner product of $L^2(\Omega)$, resp. $L^2(\Lambda)$, and might be weighted (as in the case of Chebychev methods). Further, the bilinear form is given by

$$a(u^N, v^N) = (\nu\nabla u^N, \nabla v^N) + (\boldsymbol{\beta}\cdot\nabla u^N, v^N). \tag{4}$$

As previously mentioned, internal layers may appear due to vanishing diffusion. Indeed, the exact solution of problem (1) in one dimension with constant coefficients and non-homogenous Dirichlet boundary conditions, that is

$$\begin{aligned} -\nu u'' + u' &= 0 \quad \text{in } \Omega = (-1,1), \\ u(-1) &= 0, \\ u(1) &= 1, \end{aligned} \tag{5}$$

is given by $u(x) = (e^{(x-1)/\nu} - e^{-2/\nu})/(1 - e^{-2/\nu})$, which has a boundary layer near $x = 1$ of width $O(\nu)$. Obviously, if we fix ν and let N tend to infinity, any spectral approximation u^N to (1) will eventually exhibit exponential convergence to the exact solution. For instance, in the Legendre case, applying Cea's Lemma, one gets the bound

$$\|u - u^N\|_{H^1(0,1)} \leq \frac{C}{\nu} \inf_{v^N \in V_N} \|u - v^N\|_{H^1(0,1)}.$$

Further estimations for the best approximation gives the following bound

$$\|u - u^N\|_{H^1(0,1)} \leq \frac{C'}{\nu} N^{-s} |u|_{H^{s+1}(-1,1)} \leq \frac{C''}{\nu} \left(\frac{1}{\nu^{1/2} N}\right)^s$$

for all $s \geq 1$. This proves the claimed result and, in particular, that spectral convergence is achieved as soon as the boundary layer can be fully resolved by the polynomial degree N.

2.2. **Maximum principle.** Still assuming problem (5), the spectral approximation can be investigated by the so called error equation technique (see e.g. [15] chapter 6) allowing an exact study of the error. For Chebychev approximations we seek for a function

$$u^N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x),$$

where T_k denotes the Chebychev polynomial of degree k. However, results similar to the forthcoming ones hold for Legendre discretizations as well. The resulting error equation writes

$$-\nu u^N_{xx} + u^N_x = \eta T'_N.$$

In this context Canuto [13] proved that for all $\nu > 0$ and $N > 0$,

$$\hat{u}_k > 0 \quad \text{for } k = 1, \ldots, N,$$

which implies the bound $u^N(x) \leq 1$ on $\bar{\Omega}$. In the unresolved regime, the asymptotic behavior of u^N depends on the parity of N. If N is odd, the first coefficient $\hat{u}_0$ is strictly positive, too. An important implicaton of the positivity of all coefficients is that u^N is uniformly bounded in $\bar{\Omega}$, independently of N and ν. In fact,

$$|u^N(x)| \leq \sum_{k=0}^{N} \hat{u}_k |T_k(x)| \leq \sum_{k=0}^{N} \hat{u}_k |T_k(1)| \leq u^N(1) = 1.$$

More precisely, the analysis yields

$$u^N \simeq \frac{1}{2} + \frac{1}{2} T_N \quad \text{in } \Omega.$$

On the other hand, if N is even $\hat{u}_0$ is negative and one has

$$u^N \simeq \hat{u}_0 + \hat{u}_N T_N \quad \text{in } \Omega,$$

with $|\hat{u}_0| \simeq \hat{u}_N \simeq C(\nu N^2)^{-1}$. Hence, in this case u^N is not bounded from below independently of ν

$$-\frac{C}{\nu N} \leq u^N(x) \leq 1.$$

Such a rigorous analysis allows to understand the behavior of spectral solutions to problems with internal layers. These theoretical limits are reflected in numerical examples, see Figure 1.

3. Stabilization techniques for spectral methods

We focus on stabilized Galerkin methods that can be written in general as follows: find $u^N \in V_N$ such that

$$a(u^N, v^N) + s(u^N, v^N) = (f, v^N) + s^r(v^N) \quad \forall v^N \in V_N, \tag{6}$$

where $s(\cdot,\cdot)$ is a stabilization bilinear form, $s^r(\cdot)$ the corresponding linear form, being either strongly or weakly consistent. Different approaches have been proposed:

- *SUPG-method*. A tighter control on the variation of the discrete solution is obtained by modifying (3) in a strongly consistent way, i.e. the stabilization operator is defined as:

$$s_{supg}(u^N, v^N) = (\mathcal{L}u^N, \boldsymbol{\beta}\cdot\nabla v^N)_\tau, \tag{7}$$

$$s^r_{supg}(v^N) = (f, \boldsymbol{\beta}\cdot\nabla v^N)_\tau, \tag{8}$$

  where $(\cdot,\cdot)_\tau$ denotes a weighted L^2-inner product, with nonnegative weight τ depending on the discretization but virtually independent of ν in the singular perturbation limit. The added terms gives control on the SUPG-norm of the streamline derivative, i.e. $\|\boldsymbol{\beta}\cdot\nabla u^N\|_\tau$, using a standard coercivity argument.

  The acronym SUPG stands for Streamline Upwind/Petrov-Galerkin. Actually, as pointed out by Hughes [23], this name is motivated by the fact that the method can be formulated as a Petrov-Galerkin with trial space V_N and test space

$$W_N = \{w \in L^2(\Omega) \,:\, \exists v^N \in V_N \text{ s.t. } w = v^N + \delta\boldsymbol{\beta}\cdot\nabla v^N\}.$$

  On the other hand the SUPG method formulated as done in (6) can be regarded as a stabilized Galerkin method.

  The above mentioned weight function yields some freedom. Indeed, there are two strategies for its choice. The first is a constructive recipe and its realization of the weight function has been proposed by Canuto [14] and Pasquarelli and Quarteroni [31], which marries the accuracy of global polynomial expansions with the flexibility of local low-order finite elements. In one dimension let $\{x_j\}_{i=0}^N$ be the $N+1$ Gauss-Lobatto nodes of the computational domain Ω and let $W_{h,0}$ and $V_{h,1}$ be the discontinuous piecewise constant, resp. continuous linear finite element space, on the elements (x_{j-1}, x_j) for $j = 1,\ldots,N$. Further define $J_h : L^2(\Omega) \to W_{h,0}$ the $L^2(\Omega)$-local orthogonal projection and $I_h : C^0(\bar{\Omega}) \to V_{h,1}$ the piecewise linear interpolation. The interpolation I_h has some interesting uniform low-order/high-order interpolation properties, i.e. there exists constants C_i, $i = 1,\ldots,4$, such that for all $v^N \in \mathbb{P}_N(\Omega)$,

$$C_1 \|v^N\|_{L^2(\Omega)} \leq \|I_h v^N\|_{L^2(\Omega)} \leq C_2 \|v^N\|_{L^2(\Omega)},$$
$$C_3 \|v^N_x\|_{L^2(\Omega)} \leq \|(I_h v^N)_x\|_{L^2(\Omega)} \leq C_4 \|v^N_x\|_{L^2(\Omega)}.$$

  Then, following Canuto [14], introduce the SUPG-stabilization term by

$$s_{supg}(u^N, v^N) = \sum_{j=1}^N \tau_j \int_{x_{j-1}}^{x_j} J_h(\mathcal{L}u^N)\beta_h I_h(v^N_x)\,\mathrm{d}x,$$

$$s^r_{supg}(v^N) = \sum_{j=1}^N \tau_j \int_{x_{j-1}}^{x_j} J_h(f)\beta_h I_h(v^N_x)\,\mathrm{d}x,$$

  where $\beta_h = J_h(\beta)$. The choice

$$\tau_j = \min\left(\frac{h_j}{2\|\beta\|_{L^\infty(x_{j-1},x_j)}}, \frac{h_j^2}{12\nu}\right)$$

yields a uniformly stable scheme:

$$\nu\|u_x^N\|^2_{L^2(\Omega)} + \sum_{j=1}^{N} \tau_j \|\beta_h I_h(u_x^N)\|^2_{L^2(x_{j-1},x_j)} + \sigma_0 \|u^N\|^2_{L^2(\Omega)} \leq c\,\|f\|^2_{L^2(\Omega)}.$$

The second strategy is an interpretation by bubble functions on the same subgrid as introduced above. Let us enrich the test and trial space of polynomials by some bubble functions on the subgrid, i.e. set $W_N = \mathbb{P}_N(\Omega) \oplus B_h$, with B_h being such that $\forall v^b \in B_h$, $v^b(x_j) = 0\ \forall j = 0,\ldots,N$. Then, seek $u^N \in \mathbb{P}_N(\Omega)$ and $u^b \in B_h$ such that

$$\begin{aligned} a(u^N, v^N) + a_h(u^b, v^N) &= (f, v^N) \qquad && \forall v^N \in \mathbb{P}_N(\Omega), \\ (J_h(\mathcal{L}u^N), v^b) + a_h(u^b, v^b) &= (J_h f, v^b) \qquad && \forall v^b \in B_h, \end{aligned}$$

with $a_h(u,v) = \nu(u_x, v_x) + (\beta_h u_x, v_x)$. The equation associated to the bubble test functions is then eliminated from the system. This scheme is proven (Canuto and Puppo [16]) to be equivalent to the SUPG spectral scheme (6) with specific choice of (7) and (8), provided

$$\tau_j = \frac{\left(\int_{x_{j-1}}^{x_j} b_j\,dx\right)^2}{\nu h_j \int_{x_{j-1}}^{x_j} (b_{j,x})^2 dx}.$$

The local bubble functions $b_j \in B_{h,j}$ are defined through

$$a_h(b_j, v_j^b) = (1, v_j^b) \quad \forall v_j^b \in B_{h,j}, \tag{9}$$

with $B_{h,j}$ being the restriction of B_h to the subgrid element (x_{j-1}, x_j). There are two variants for the choice of B_h, resp. $B_{h,j}$. Canuto and Puppo [16] proposed a discrete finite element space and consequently equation (9) defines local finite element problems on the subgrid. Brezzi and Russo [7] proposed (in a general finite element Galerkin context) an infinite-dimensional space $B_{h,j} = H_0^1(x_{j-1}, x_j)$ known as residual-free bubble strategy. Then τ_j simplifies to $\tau_j = \int_{x_{j-1}}^{x_j} b_j$ with analytic expression for

$$b_j = \frac{1}{\beta_h}(x - x_{j-1}) - \frac{h_j}{\beta_h}\frac{e^{\beta_h(x-x_j)/\nu} - e^{-\beta_h h_j/\nu}}{1 - e^{-\beta_h h_j/\nu}}.$$

As a result of the bubble enrichment strategy, the variation of u^N at the Gauss-Lobatto points can be controlled and the boundary layer is captured in one subgrid-element. A further post-processing $u_h = I_h u^N$ would filter out spurious oscillations which leads to spectrally accurate solutions. This strategy can be extended straightforwardly to multidimensional problems with tensorized domains.

- *Filtering*. A technique in the spirit of post-processing is to apply a filter on the coefficients of the approximation, in which case no further stabilization is needed, i.e. we can set $s \equiv 0$ and $s^r \equiv 0$. In the one dimensional case let us write the approximation as

$$u^N(x) = \sum_{|k|\leq N} \hat{u}_k \varphi_k(x),$$

with $\{\varphi_k\}$ being the family of orthogonal polynomials, either trigonometric $\{e^{ikx}\}$ (Fourier) or algebraic with $\{L_k(x)\}$ (Legendre) or $\{T_k(x)\}$ (Chebychev). Then, the filtered solution writes as

$$u_f^N(x) = \sum_{|k|\leq N} \sigma_k \hat{u}_k \varphi_k(x),$$

where the smoothing factors σ_k satisfy

$$\begin{aligned} \sigma_k &= \sigma(k/N) \quad k = 0,\ldots,N, \quad \text{in the Legendre/Chebychev case, or} \\ \sigma_k &= \sigma(\pi k/N) \quad k = -N,\ldots,N, \quad \text{in the Fourier case.} \end{aligned}$$

The function $\sigma = \sigma(\theta)$ is real, even, and satisfies the following three conditions:

(i) σ is $(p-1)$-times continuously differentiable in $\mathbb{R}$, for some $p \geq 1$,
(ii) $\sigma(\theta) = 0$ if $|\theta| \geq 1$,
(iii) $\sigma(0) = 1$, $\sigma^{(j)}(0) = 0$ for $1 \leq j \leq p-1$.

Such a function is termed a filter of order p (Vandeven [38]). Applying filters does not allow to recover spectral convergence near discontinuities or steep gradients. However filters reduce oscillations and allow to obtain spectral convergence away from internal layers which would not be possible otherwise since the oscillations occur on a global scale.

- *Spectral viscosity method*. The spectral viscosity method was introduced by Tadmor and Maday [27, 37]. The basic idea is to introduce numerical dissipation only on the high order modes of the approximation, those responsible for the Gibbs oscillations. Thus, the scheme is only weakly consistent. In the one dimensional case, let us write the approximation u^N as

$$u^N(x) = \sum_{|k|\leq N} \hat{u}_k \varphi_k(x).$$

In a Galerkin framework, the spectral viscosity method takes the form

$$s_{spv}(u^N, v^N) = \epsilon_N(Q\partial_x u^N, \partial_x v^N) \quad \text{and} \quad s^r_{spv} \equiv 0,$$

with $Q\partial_x u^N = \sum_{|k|>m_N} \hat{Q}_k \hat{u}_k \varphi_k'(x)$. The method is characterized by the three following ingedients:

  - the viscosity amplitude ϵ_N, which vanishes in the limit $N \to \infty$;
  - the spectral viscosity smoothing factor $\hat{Q}_k$, that behaves as

$$1 - \left(\frac{m_N}{|k|}\right)^{\rho} \leq \hat{Q}_k \leq 1,$$

    with a method dependent factor ρ;
  - the inviscid spectrum $m_N \approx N^{\theta}$, for some $\theta\ < 1$.

Spectral superviscosity can be introduced if the spectral viscosity filtering is applied on higher order derivatives of the solution, say

$$s_{spv}(u^N, v^N) = \epsilon_N(Q\partial_x^s u^N, \partial_x^s v^N), \quad \text{for } s > 1.$$

4. Multidomain spectral methods

The application of classical spectral methods is limited to simple geometries. Multidomain spectral methods (or spectral element methods) merge geometrical flexibility of FEM with high-order (spectral) accuracy of classical spectral methods.

The multidomain approach consists of introducing a splitting of the domain Ω into elements $\kappa \in \mathcal{K}$ of maximal diameter h_κ such that $\bar{\Omega} = \cup_{\kappa\in\mathcal{K}}\bar{\kappa}$. Let us denote the set of algebraic polynomials of degree N_κ on an element κ as $\mathbb{P}_{N_\kappa}(\kappa)$. Then the approximation space V_δ, δ representing the discretization factors $\{h_\kappa\}_{\kappa\in\mathcal{K}}$ and $\{N_\kappa\}_{\kappa\in\mathcal{K}}$, is either made of continuous functions

$$V_\delta = \{v \in C^0(\bar{\Omega}) \ : \ v|_\kappa \in \mathbb{P}_{N_\kappa}(\kappa),\ \forall\kappa \in \mathcal{K},\ v|_{\partial\Omega} = 0\},$$

or by discontinuous functions. Discontinuous approximations will be discussed in section 5.2. The Galerkin method of problem (1) then writes: Find $u_\delta \in V_\delta$ such that

$$a(u_\delta, v_\delta) = (f, v_\delta) \quad \forall v_\delta \in V_\delta, \tag{10}$$

where $(\cdot,\cdot)$, resp. $(\cdot,\cdot)_\Lambda$ denotes the $L^2(\Omega)$- resp. $L^2(\Lambda)$-scalar product. Further, the bilinear form $a(\cdot,\cdot)$ is given by

$$a(u_\delta, v_\delta) = (\nu\nabla u_\delta, \nabla v_\delta) + (\boldsymbol{\beta}\cdot\nabla u_\delta, v_\delta).$$

Remark that hp or hN discretizations can be cast in this form and depend entirely on the finite element space V_δ. It is well known that the solution of (10) may exhibit interior or outflow layers in case the advection dominates. Unless the local Peclet number ($\mathrm{Pe}_\kappa = \frac{|\beta|h_\kappa}{\nu}$) is smaller than 1, i.e. these layers are fully resolved by the finite element space, they give rise to spurious oscillations that may propagate throughout the computational domain. The reason for this loss of stability can be explained by Cea's lemma:

$$\|u - u_\delta\|_{H^1(\Omega)} \leq C\frac{|\boldsymbol{\beta}|}{\nu} \inf_{v_\delta\in V_\delta} \|u - v_\delta\|_{H^1(\Omega)}.$$

If $\nu \ll 1$, we observe that the constant $\frac{|\boldsymbol{\beta}|}{\nu}$ becomes large and the estimate essentially useless. The loss of stability comes from the loss of coercivity for small ν combined with the nonsymmetric term that is continuous on the spaces L^2 and H^1.

5. MULTIDOMAIN METHODS: STABILIZATION TECHNIQUES

We focus on stabilization techniques of the form: find $u_\delta \in V_\delta$ such that

$$a(u_\delta, v_\delta) + s(u_\delta, v_\delta) = (f, v_\delta) + s^r(v_\delta) \quad \forall v_\delta \in V_\delta,$$

with some method dependent stabilization bilinear form $s(\cdot,\cdot)$ and its linear counterpart $s^r(\cdot)$ to guarantee strong consistency. The bilinear form $a(\cdot,\cdot)$ is defined by (4). We address two cases:

- *Element-based stabilization* operators:

$$s(u_\delta, v_\delta) = \sum_{\kappa\in\mathcal{K}} \left(L_1(u_\delta), L_2(v_\delta)\right)_\kappa, \quad s^r(v_\delta) = \sum_{\kappa\in\mathcal{K}} \int_\kappa L_3(v_\delta)\mathrm{d}x.$$

- *Face-based stabilization* operators:

$$s(u_\delta, v_\delta) = \sum_{F\in\mathcal{F}_i} \left(L_1(u_\delta), L_2(v_\delta)\right)_F, \quad s^r(v_\delta) = 0,$$

  where $\mathcal{F}_i$ denotes the set of interior faces of the mesh $\mathcal{K}$.

The operators L_i are method specific operators that act either on the elements or on their faces.

5.1. **Element-based stabilization techniques.** This family of methods uses stabilization operators that act on the elements κ. The investigations of a stability- and convergence-analysis for spectral methods can be reused applying affine transformations, which will introduce the spatial discretization parameter h, to the reference element. Let us further introduce the method dependent function $H(h_\kappa, N_\kappa)$ that is chosen in a way to guarantee the best possible approximation properties of the scheme.

- The *SUPG-method* was introduced by Brooks and Hughes [8], Johnson, Nävert and Pitkäranta [24] and analyzed in a hp-framework by Houston, Schwab and Süli [21]. The stabilization operator in this case is essentially the same as for the spectral approach, with the only difference that for reasons of well-posedness of the integrals the bilinear form has to be written elementwise as

$$\begin{aligned} s_{supg}(u_\delta, v_\delta) &= \sum_{\kappa\in\mathcal{K}} \left(H(h_\kappa, N_\kappa)|\boldsymbol{\beta}|^{-1}\mathcal{L}u_\delta, \boldsymbol{\beta}\cdot\nabla v_\delta\right)_\kappa, \\ s^r_{supg}(v_\delta) &= \left(H(h_\kappa, N_\kappa)|\boldsymbol{\beta}|^{-1} f, \boldsymbol{\beta}\cdot\nabla v_\delta\right). \end{aligned}$$

  Observe that the stencil and the data-structure of the SUPG-method is still the same compared with standard FEM. On the other hand, this method has several disadvantages. Firstly, stiff source terms must be included in the stabilization. Secondly, when applied to time-dependent problems the time derivation has to be included in the stabilization which prompts a space-time finite element scheme for high order time steps. And thirdly, in the analysis the stabilization parameter γ turns out to be dependent on the diffusion parameter ν which is unconvenient for non-linear problems where the diffusion depends on the solution itself or for linear problems with anisotropic diffusion coupling.

Nevertheless there are several approaches not originated from spectral methods. All the methods presented below share the property that $s^r \equiv 0$. Consequently, the drawback of the SUPG-method, that is the fact that the source terms and time derivatives has to be included in the stabilization, is eliminated, allowing more flexibility with respect to time-stepping and stiff source terms.

- The *orthogonal subscale method* (OS) introduced by Blasco and Codina [3] consists of using the stabilization operator

$$s_{osm}(u_\delta, v_\delta) = \left(H(h_\kappa, N_\kappa)|\boldsymbol{\beta}|^{-1}(\mathrm{I}_d - \Pi_\delta)(\boldsymbol{\beta}\cdot\nabla u_\delta), (\mathrm{I}_d - \Pi_\delta)(\boldsymbol{\beta}\cdot\nabla u_\delta)\right),$$

  where $\Pi_\delta : L^2(\Omega) \to V_\delta$ denotes a suitable weighted global L^2-projection defined by

$$\left(H(h_\kappa, N_\kappa)|\boldsymbol{\beta}|^{-1}\Pi_\delta(w), v_\delta\right) = \left(H(h_\kappa, N_\kappa)|\boldsymbol{\beta}|^{-1} w, v_\delta\right) \quad \forall v_\delta \in V_\delta, w \in L^2(\Omega).$$

  The stabilization parameter is now independent of the diffusion parameter. However, since the global projection has to be computed, the number of unknowns is doubled and the matrix have approximately four times more nonzero entries than standard FEM.

- The *subgrid viscosity method* (SV) introduced by Guermond [20] is defined by
$$s_{sv}(u_\delta, v_\delta) = \left(H(h_\kappa, N_\kappa)|\boldsymbol{\beta}|^{-1}\boldsymbol{\beta}\cdot\nabla(\mathrm{I}_d - i_{2\delta})u_\delta, \boldsymbol{\beta}\cdot\nabla(\mathrm{I}_d - i_{2\delta})v_\delta\right),$$
where $i_{2\delta}$ denotes a local interpolation operator onto a continuous finite element space of characteristic mesh size $2h$ and polynomial degree N or of mesh size h and polynomial degree $N-1$. This method shares with the previous one the property that the stabilization parameter is independent of the diffusion coefficient. The introduction of a fine grid introduces an increase of the number of unknowns, the stabilization operator introduces non-standard couplings between basis functions which yields a larger system matrix. In addition, an extra data-structure is needed for the computation of the local interpolation.
- The *local projection stabilization method* (LPS) introduced by Becker and Braack [2], Braack and Burman [4] and analyzed for arbitrary polynomial orders by Matthies, Skrzypacz and Tobiska [28] consists of
$$s_{lps}(u_\delta, v_\delta) = \left(H(h_\kappa, N_\kappa)|\boldsymbol{\beta}|^{-1}(\mathrm{I}_d - \pi_{2\delta})\boldsymbol{\beta}\cdot\nabla u_\delta, (\mathrm{I}_d - \pi_{2\delta})\boldsymbol{\beta}\cdot\nabla v_\delta\right),$$
where $\pi_{2\delta}$ denotes a local projection operator onto a discontinuous finite element space of characteristic mesh size $2h$ and polynomial degree $N-1$. In certain cases this method coincides with the subgrid viscosity method. The stabilization parameter is independent of the diffusion coefficient. This method shares the same disadvantages of the subgrid viscosity method.

In all the above described methods an h-analysis leads to the following a priori estimate. Let u be the exact solution of problem (1) satisfying $u \in H^k(\Omega)$ for some integer $k \geq 1$. Then, for any integer s, $1 \leq s \leq \min(k, N+1)$, there holds

$$\|u - u_\delta\|_{L^2(\Omega)} + \|h^{\frac{1}{2}}\boldsymbol{\beta}\cdot\nabla(u - u_\delta)\|_{L^2(\Omega)} + |\nu^{\frac{1}{2}}(u - u_\delta)|_{H^1(\Omega)} \leq c(\nu^{\frac{1}{2}} + |\boldsymbol{\beta}|^{\frac{1}{2}}h^{\frac{1}{2}})h^{s-1}|u|_{H^s(\Omega)},$$

with $c > 0$ being independent of the mesh size h, but possibly dependent on the polynomial degree N.

The previous three methods use artificial viscosity that acts only on the finest scales, thus a scale separation is inevitable. Let us either write

$$u_h = Pu_h + (I_d - P)u_h \quad \text{(subgrid viscosity with } P = i_{2\delta}\text{), or}$$
$$\boldsymbol{\beta}\cdot\nabla u_h = P(\boldsymbol{\beta}\cdot\nabla u_h) + (I_d - P)(\boldsymbol{\beta}\cdot\nabla u_h) \quad \text{(OS with } P = \Pi_\delta \text{ and LPS with } P = \pi_{2\delta}\text{)}$$

for the method dependent interpolation/projection operator P and the identity operator I_d. Then, artificial viscosity is added only on the fine scales defined by $\boldsymbol{\beta}\cdot\nabla(I_d - P)u_\delta$, resp. $(I_d - P)(\boldsymbol{\beta}\cdot\nabla u_\delta)$. Figure 2 illustrates the different scales of the above mentioned quantities for low order approximations where the SV- and LPS-method coincides.

5.2. **Face-based stabilization techniques.** Face-based stabilization methods are known as interior penalty methods. This name is originated by the fact that a consistent stabilization operator is added that acts only on the interior faces of the mesh. This stabilization- or penalization-term gives sufficient control on the oscillations. From a theoretical point of view it is advantageous to consider weakly imposed boundary conditions for such methods. We present two kind of methods, one using continuous and the other discontinuous finite element spaces. Let us first introduce some additional notations in order to handle the supplementary terms on the faces. Denote by $\mathcal{F} = \mathcal{F}_i \cup \mathcal{F}_e$ the set of all faces of the mesh where $\mathcal{F}_i$ denotes the interior and $\mathcal{F}_e \subset \partial\Omega$ the exterior faces. Further let us split $\mathcal{F}_e = \mathcal{F}_- \cup \mathcal{F}_+$ with $\mathcal{F}_\pm$ the faces contained in $\partial\Omega^\pm$. The operator $[u_\delta]|_F$ denotes the jump of u_δ over the face $F = \partial\kappa_1 \cap \partial\kappa_2$, i.e.

$$[u_\delta]|_F = u_\delta|_{\kappa_1}\boldsymbol{n}_1 + u_\delta|_{\kappa_2}\boldsymbol{n}_2,$$

with $\boldsymbol{n}_i$ being the outer normal of κ_i. Similarly we define the normal resp. tangential jump of the gradient by

$$[\nabla u_\delta]_n|_F = \nabla u_\delta|_{\kappa_1}\cdot\boldsymbol{n}_1 + \nabla u_\delta|_{\kappa_2}\cdot\boldsymbol{n}_2 \quad \text{resp.} \quad [\nabla u_\delta]_t|_F = \nabla u_\delta|_{\kappa_1}\times\boldsymbol{n}_1 + \nabla u_\delta|_{\kappa_2}\times\boldsymbol{n}_2$$

and the average operators by

$$\{u_\delta\}|_F = \tfrac{1}{2}(u_\delta|_{\kappa_1} + u_\delta|_{\kappa_2}) \quad \text{resp.} \quad \{\nabla u_\delta\}|_F = \tfrac{1}{2}(\nabla u_\delta|_{\kappa_1} + \nabla u_\delta|_{\kappa_2}).$$

- The *continuous interior penalty method* (CIP) introduced by Douglas and Dupont [18] and analyzed by Burman and Hansbo [10] uses a stabilization operator that penalizes the normal jump of the gradient of the solution, i.e.
$$s_{cip}(u_\delta, v_\delta) = (\gamma H(h, N)|\boldsymbol{\beta}\cdot\boldsymbol{n}|[\nabla u_\delta]_n, [\nabla v_\delta]_n)_{\mathcal{F}_i},$$

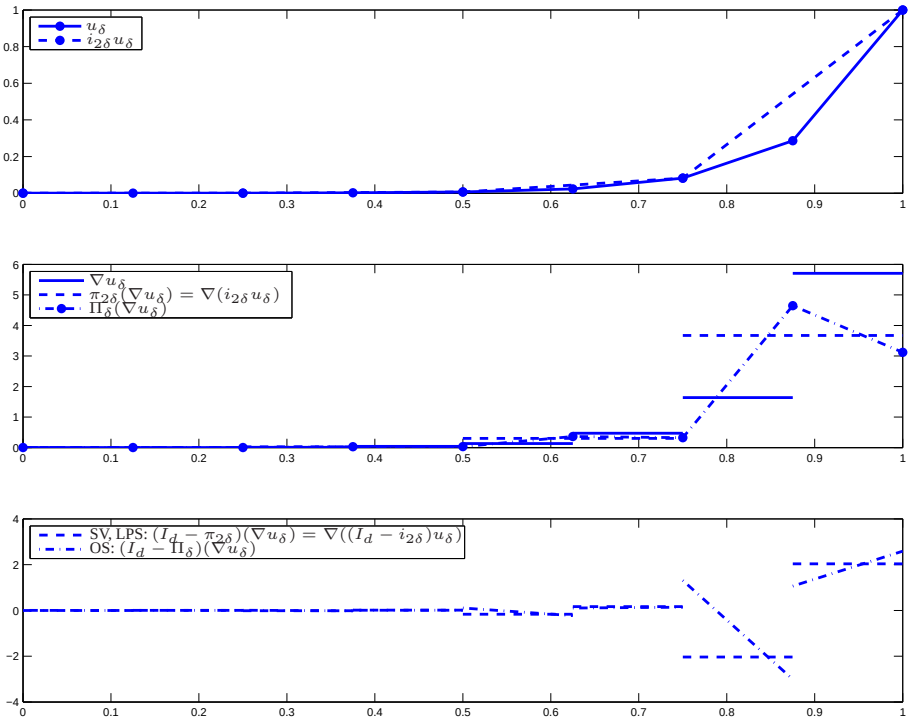

FIGURE 2. Different scales related to the projection/interpolation operators of the OS-, SV- and LPS-method for an example where the SV- and LPS-method coincide and for low order approximations.

where $\gamma > 0$ and $H(h, N) = h^2 N^{-3.5}$. Recently, an hp-analysis (for the finite element community, p is the local polynomial degree, which is denoted by N in our case) for optimal convergence in the case of interior penalty stabilization leading to the following quasi optimal convergence for (2) and in the transport dominated regime for (1) was carried out [9]: Let u be the exact solution of problem (1) and satisfy $u \in H^k(\Omega)$ for some integer $k \geq 1$. Then, for any integer s, $1 \leq s \leq \min(k, N+1)$, there holds

$$\|u - u_\delta\|_{L^2(\Omega)} + s_{cip}(u - u_\delta, u - u_\delta)^{\frac{1}{2}} \leq c N^{\frac{1}{4}} \left(\frac{h}{N}\right)^{s-\frac{1}{2}} |u|_{H^s(\Omega)}, \tag{11}$$

with $c > 0$ being a constant independent of h and N. We make two remarks. Firstly, the graph-norm could be included into (11), but the estimate is not expected to be optimal with respect to N due to the inverse inequality involved in the proof. Secondly, the slight sub-optimality yielded by the presence of the factor $N^{\frac{1}{4}}$ in (11) vanishes in the context of dominating diffusion where the standard optimal estimate is recovered. Figure 3 shows the convergence behavior measured in the L^2-norm of this method under h- and N-refinement for problem (2). Observe the optimal convergence under h-refinement with expected super-convergence (order $N + 1$ instead of $N + 1/2$) and the exponential convergence under N-refinement (N-refinement in fact means increasing N).

This method is flexible with time-stepping, source terms are not included in stabilization and that the stabilization parameter is robust with respect to the diffusion coefficient for approximation of the advection-diffusion equation. On the other hand, the stencil is increased leading to an increase in the number of non-zero elements in the system matrix. Additionally, an extra mesh data-structure is needed for the computations of the gradients jumps.

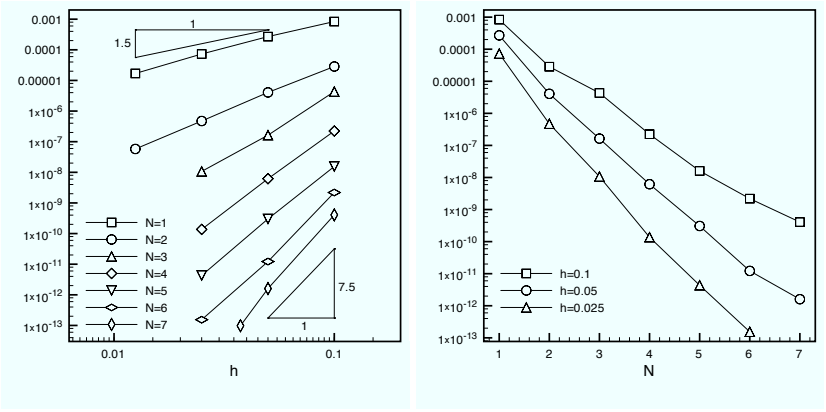

FIGURE 3. Convergence behavior of the CIP-method for the advection-reaction equation under h-refinement (left) and N-refinement (right) measured in the L^2-norm.

- The *discontinuous Galerkin method* (DG) introduced by Reed and Hill [34] and analyzed by Lesaint and Raviart [26], and Johnson and Pitkäranta [25] uses approximation functions that are discontinuous on mesh faces, but restricted to an element being still a polynomial approximation of degree N. Thus, let us introduce the discontinuous finite element space by

$$W_\delta = \{v \in L^2(\Omega) \ : \ v|_\kappa \in \mathbb{P}_{N_\kappa}(\kappa), \ \forall \kappa \in \mathcal{K}\}.$$

Since this is a non-conforming finite element space, i.e. $W_\delta \not\subset H^1(\Omega)$, integration by parts have to be computed elementwise which leads to supplementary terms on the faces

$$a_{dg}(u_\delta, v_\delta) = (\{\nu \nabla u_\delta\}, [v_\delta])_{\mathcal{F}} + ([u_\delta], \{\nu \nabla v_\delta\})_{\mathcal{F}} + ([u_\delta], \{\boldsymbol{\beta} v_\delta\})_{\mathcal{F}_i \cup \mathcal{F}_+}.$$

The discontinuities give sufficient liberty to capture oscillations providing that the jumps of the solution are controlled. This leads to the following stabilization operator

$$s_{dg}(u_\delta, v_\delta) = \big((\gamma_d \nu h^{-1} + \gamma_a |\boldsymbol{\beta} \cdot \boldsymbol{n}|)[u_\delta], [v_\delta]\big)_{\mathcal{F}_i} + (\gamma_d \nu h^{-1} u_\delta, v_\delta)_{\mathcal{F}_e},$$

with $\gamma_d, \gamma_a > 0$. The problem reads: find $u_\delta \in W_\delta$ such that

$$a(u_\delta, v_\delta) + a_{dg}(u_\delta, v_\delta) + s_{dg}(u_\delta, v_\delta) = (f, v_\delta) \quad \forall v_\delta \in W_\delta.$$

The hp-analysis carried out in [22] gives us the following optimal estimates: Let u be the exact solution of problem (1) satisfying $u \in H^1(\Omega)$ and $u \in H^k(\kappa)$ for all $\kappa \in \mathcal{K}$ and for some integer $k \geq 1$. Then, for any integer s, $1 \leq s \leq \min(k, N+1)$, there holds

$$\|u - u_\delta\|_{L^2(\Omega)} + s_{dg}(u - u_\delta, u - u_\delta)^{\frac{1}{2}} \leq c \left(\frac{h}{N}\right)^{s-\frac{1}{2}} |u|_{H^s(\Omega)}, \tag{12}$$

with $c > 0$ being a constant independent of h and N. In the diffusion dominated case, the estimates are suboptimal with respect to N due to the presence of the factor $N^{\frac{1}{2}}$.

This method is flexible with time-stepping schemes and source terms are not included in the stabilization. The local support of the basis functions makes the method suitable for hp-strategies with hanging nodes. On the other hand in the case of convection-diffusion problems, the stabilization parameter depends on the diffusion coefficient since also the Laplace operator has to be stabilized and the a priori estimates for diffusion dominated problems are subpoptimal with a factor $N^{\frac{1}{2}}$. However, this suboptimality is not observed numerically and can be circumvented by adding another stabilization term, see [36]. Further the number of unknowns is increased compared to continuous methods which yields a larger matrix and some extra data-structure is needed for the computation of the jumps.

The argument to obtain control of the jumps relies on coercivity. The interpretation of the upwind method as a penalty method was pointed out by Brezzi and coworkers [6]. Recently Burman and Stamm [11, 12] showed that for quadratic or higher approximations the previous choice of stabilization can be

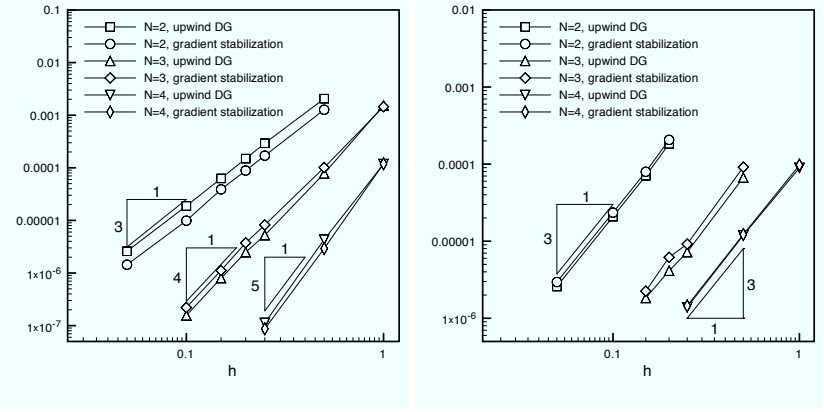

FIGURE 4. Convergence behavior of the upwind DG-method and the DG-method using the jump of the gradient for the advection-reaction equation under h-refinement for a smooth solution (left) and a irregular solution (right) measured in the L^2-norm.

reduced, still controlling the whole jump of the solution, but this time using an inf-sup argument. They only stabilize the tangential jump of the gradient, i.e.

$$s_{dg,min}(u_\delta, v_\delta) = \big((\gamma_d \nu h^{-1} + \gamma_a |\boldsymbol{\beta}\cdot\boldsymbol{n}|)[\nabla u_\delta]_t, [\nabla v_\delta]_t\big)_{\mathcal{F}_i} + (\gamma_d \nu h^{-1} \nabla u_\delta \times \boldsymbol{n}, \nabla v_\delta \times \boldsymbol{n})_{\mathcal{F}_e}$$

and use a mixed formulation. In this approach the stabilization affects a smaller portion of the finite element spectrum. This leads to a very natural interpretation of local mass conservation of the pure transport problem, i.e.

$$\int_{\partial\kappa} \boldsymbol{\beta}\cdot\boldsymbol{n}_\kappa \{u_\delta\}\, ds = \int_\kappa f\, dx. \tag{13}$$

In spite of the fact that only the tangential jump of the gradient is stabilized, quasi optimal convergence can be recovered for (2) and for (1) in the transport dominated regime. More precisely the convergence result reads as follows: Let u be the exact solution of problem (1) satisfying $u \in H^1(\Omega)$ and $u \in H^k(\kappa)$ for all $\kappa \in \mathcal{K}$ and for some integer $k \geq 1$. Then, for any integer s, $1 \leq s \leq \min(k, N+1)$, there holds

$$\|u - u_\delta\|_{L^2(\Omega)} + \|h^{\frac{1}{2}}\boldsymbol{\beta}\cdot\nabla(u-u_\delta)\|_{L^2(\Omega)} + s_{dg}(u-u_\delta, u-u_\delta)^{\frac{1}{2}} \leq c h^{s-\frac{1}{2}} |u|_{H^s(\Omega)},$$

$c > 0$ being a constant independent on h. Note that this provides a better error control than the L^2-norm, similarly to what we have obtained in (12) for the standard DG-method.

Figure 4 illustrates the h-convergence of the standard upwind DG-method and the above introduced DG-method for problem (2) with smooth solution (left) and restricted Sobolev regularity (right). The errors are measured in the L^2-norm. Observe the optimal convergence with respect to h which is $N+1$ resp. 3 (with the expected super-convergence of $h^{\frac{1}{2}}$) and in general the similar behavior of the two methods.

6. CONCLUSION

Comparing the DG- with the continuous versions of the finite element method is not easy. A discontinuous finite element space allows easier implementation of hp-strategies with hanging nodes. On the other hand the disadvantage is the increased number of degrees of freedom. But this argument gets less and less relevant while increasing the local polynomial degree N since basis-functions having its support on the edges increase proportionally to N while basis-functions having their support only on the interior of the element increase as N^2. In general we conclude that DG-methods are successful for pure transport problems and stabilization parameter independent local mass conservation can be achieved, see (13). For convection-diffusion problems continuous approximations are attractive since the stabilization can be made independent of the diffusion parameter. A comparison with respect to different aspects of the continuous methods can be found in Table 6.

	SUPG	OS	SV	LPS	CIP
Time-stepping, source terms	-	+	+	+	+
Stabilization par. indep. on diffusion coefficient	-	+	+	+	+
Same degree of freedom as standard FEM	+	-	+	+	+
Same stencil as standard FEM	+	+	-	-	-
Data-structure*	+	+	-	-	-

TABLE 1. Comparison of some relevant aspects for different schemes. *: Need use of hierarchical meshes or to know solution on heighbor elements, in contrast to standard FEM.

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