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A new mixed method for the Stokes equations based on stress-velocity-vorticity formulation

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Abstract

In this paper, we develop and analyze a mixed finite element method for the Stokes flow. This method is based on a stress-velocity-vorticity formulation. A new discretization is proposed: the stress is approximated using the Raviart-Thomas elements, the velocity and the vorticity by piecewise discontinuous polynomials. It is shown that if the orders of these spaces are properly chosen then the advocated method is stable. We derive error estimates for the Stokes problem, showing optimal accuracy for both the velocity and vorticity.

Keywords: Mixed finite element; Stokes equations; Raviart-Thomas finite element; incompressible fluids

1 Introduction

It is hard to give a precise definition of mixed methods, see also [13], generally the definition refers to a class of methods based on the simultaneous approximations of a primal and a dual quantity: for example displacement and stress for the elastic problem, temperature and heat flux for the heat equation. To clarify the concept let us consider the simplest form of the steady heat equation:

$$-\Delta u = f,$$

where u is the temperature (primal quantity) and f is the source term. In order to obtain the mixed formulation (or dual formulation) we have a second quantity $\boldsymbol{q} = -\operatorname{grad} u$, the heat flux, that is identified as dual variable. So that the heat equation becomes:

$$\operatorname{div} \boldsymbol{q} = f.$$

The dual formulation is written as a system of two equations, the first one is the definition of heat flux, the second one is the energy conservation law. In the same way we can identity a pair of primal and dual equation for the linear elasticity problem. The primal formulation is given by

$$-\operatorname{div}(\mathbb{C} \operatorname{grad} \boldsymbol{u}) = \boldsymbol{f},$$

where $\boldsymbol{u}$ is the displacement and with $\mathbb{C}$ we denote the elasticity tensor. In the dual equation we have to introduce the stress tensor defined as $\sigma = \mathbb{C} \operatorname{grad} \boldsymbol{u}$; the problem can be written as

$$-\operatorname{div}\sigma = \boldsymbol{f},$$

which is the linear momentum conservation law. This is equivalet to the Hellinger-Reissner principle. To be more precise, we have also to enforce the conservation of angular momentum, that is given by the symmetry of σ, this is what makes the construction of a finite element method really complicated.

The usual mixed formulation of Stokes problem is given in terms of velocity and pressure as:

$$\begin{cases} -\Delta \boldsymbol{u} + \operatorname{grad} p = \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} = 0. \end{cases}$$

Even though it is algebraically equivalet to the linear elasticity problem, only a few authors [18, 17, 19] contributed to the development of a proper dual formulation, or a *real* mixed formulation of Stokes equation.

Taking as reference the examples of heat equation and linear elasticity problem, we can say that the usual velocity-pressure formulation is the primal system of equations. The dual one should include at least two equations: the rheological model and the conservation of linear momentum; the unknowns should be the stress tensor and the velocity fields.

The Stokes equations can be seen as a particular case of the linear elasticity problem, in which the bulk modulus tends to infinity (incompressible case). The theory can be deduced using the fact that all the theoretical results about the Hellinger-Reissner formulation are indipendent of the bulk modulus, in particular there are no extra hypotheses for treating the incompressible case, see [4, Section 2].

In this paper we give some novel theoretical results about the Stokes equations in their dual form without using the equivalence with the primal one: this is, to our knowledge, a new contribution that can help in giving a more complete picture of the mixed methods for fluid dynamics problems.

In the dual formulation the stress tensor is one of the unknowns of the problem and its symmetry is fundamental to ensure the conservation of angular momentum, nevertheless the construction of a finite element method honoring this constraint is really hard, see [11] for a list of references related to this issue. Recently the introduction of the finite element exterior calculus [5] gives a new abstract framework for the construction of elements with weakly imposed symmetry [6]. Such construction requires rather sophisticated mathematical tools, here we present a different approach based on more elementary and standard techniques. The results in this paper differ from previous ones since construction relies on Raviart-Thomas finite element space. In the existing literature Brezzi-Douglas-Marini finite element [6] and finite elment spaces augmented with proper bubble functions [3] have been used.

The results in this paper can be compared with the ones in [6]: the Brezzi-Douglas-Marini ($\mathcal{BDM}$) finite element gives a discretization with fewer degrees of freedom with respect to Raviart-Thomas ($\mathcal{RT}$) finite element, but a worse asymptotic order of convergence. We can state also that $\mathcal{BDM}$ finite element ensures the same order of convergence for all the unknowns, on the other side the $\mathcal{RT}$ finite element does not share such property, but it is more accurate. Finally, as pointed out by Suri [30] the constans in the errors estimates for the $\mathcal{RT}$ finite element are truly indipendent on the polynomial degree, on the other side for $\mathcal{BDM}$ finite element there is a small dependence.

The paper is structured as follows, Section 2 contains the basic notations and definitions, in Section 3 the mixed formulation for the Stokes equations is introduced and the well posedness of such problem is proved, then in Section 4 a finite element approximation based on Raviart-Thomas space is given and in Section 5 some numerical tests confirm the theoretical results.

2 Notation and preliminaries

We begin with some basic notations that will be used throughout this paper. We denote by $\mathbb{V}$ the usual finite dimensional Euclidean space $\mathbb{R}^n$ provided with the inner product

$$\boldsymbol{u} \cdot \boldsymbol{v} := \sum_{i=1}^{n} u_i v_i, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}.$$

$\mathbb{M}$ denotes the algebra of $n \times n$ matrices. For any $\sigma \in \mathbb{M}$, σ^T denote the transpose defined as

$$(\sigma \boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (\sigma^T \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}.$$

The space $\mathbb{M}$ is provided with the inner product

$$\sigma : \tau := \operatorname{tr}(\sigma^T \tau) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \tau_{ij}, \quad \sigma, \tau \in \mathbb{M},$$

where tr denote the usual trace linear functional. Throughout the paper we will make an extensive use of the following projectors defined on $\mathbb{M}$

$$\operatorname{sym} \sigma := \frac{\sigma + \sigma^T}{2}, \quad \operatorname{skew} \sigma := \frac{\sigma - \sigma^T}{2}, \quad \operatorname{dev} \sigma := \sigma - \frac{1}{n} \operatorname{tr} \sigma I,$$

where $I \in \mathbb{M}$ is the identity matrix. The ranges of sym and skew are the subspaces of symmetric and skew-symmetric matrices and they are denoted by $\mathbb{S}$ and $\mathbb{K}$ respectively so we have the orthogonal decomposition $\mathbb{M} \sim \mathbb{S} \oplus \mathbb{K}$.

Let us now introduce the functional spaces which will be used throughout the paper. Let Ω denotes an open bounded set in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, which, for the sake of simplicity, we suppose to be a n-polytope. Given a vector space $\mathbb{W}$ (for our purposes it can be $\mathbb{R}$, $\mathbb{V}$, $\mathbb{M}$ or one of the previously introduced subspaces) with $C^\infty(\Omega; \mathbb{W})$ we denote the space of infinitely differentiable functions defined on Ω with values in $\mathbb{W}$. Given a real number $p \in [1, \infty)$ with $L^p(\Omega; \mathbb{W})$ we denote the space of p-integrable functions defined on Ω with values in $\mathbb{W}$. For $p = \infty$, the space $L^\infty(\Omega; \mathbb{W})$ is the set of essentially bounded function provided with the norm

$$\|f\|_{\infty,\Omega} = \inf \{\alpha \geq 0 \,|\, |f(x)| \leq \alpha \text{ for almost every } x \in \Omega\}.$$

Given an integer $m \geq 0$ we define the following family of spaces

$$W^{m,p}(\Omega; \mathbb{W}) := \{f \in L^p(\Omega; \mathbb{W}) \,|\, \partial^\alpha f \in L^p(\Omega; \mathbb{W}),\ |\alpha| \leq m\}.$$

When $p = 2$ then $W^{m,2}(\Omega;\mathbb{W})$ is the usual Sobolev space $H^m(\Omega;\mathbb{W})$ provided with the norm

$$\|f\|_{m,\Omega} \coloneqq \sqrt{\sum_{|\alpha|\le m} \|\partial^\alpha f\|^2_{L^2(\Omega;\mathbb{W})}}.$$

We introduce also the space $H(\mathrm{div},\Omega;\mathbb{V})$ of vector-valued functions in $L^2(\Omega;\mathbb{V})$ whose distributional divergence belongs to $L^2(\Omega;\mathbb{R})$, provided with the norm

$$\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega;\mathbb{V})} \coloneqq \sqrt{\|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{0,\Omega}}.$$

In similar way, the space $H(\mathrm{div},\Omega;\mathbb{M})$ denotes the subspace of matrix-valued functions in $L^2(\Omega;\mathbb{M})$ whose distributional divergence (by rows) belongs to $L^2(\Omega;\mathbb{V})$ with the norm

$$\|\sigma\|_{H(\mathrm{div},\Omega;\mathbb{M})} \coloneqq \sqrt{\|\sigma\|^2_{0,\Omega} + \|\mathrm{div}\,\sigma\|^2_{0,\Omega}}.$$

Fixed $\Gamma \subseteq \partial\Omega$ we may define the following pair of trace operators

$$\boldsymbol{v} \in C^\infty(\Omega;\mathbb{V}) \mapsto \boldsymbol{v}|_\Gamma, \qquad \sigma \in C^\infty(\Omega;\mathbb{M}) \mapsto \sigma\boldsymbol{n}|_\Gamma,$$

where $\boldsymbol{n}$ denotes the unit outward normal along Γ. These operators have a unique continuous linear extension on $H^1(\Omega;\mathbb{V})$ and $H(\mathrm{div},\Omega;\mathbb{M})$ respectively [24, Section 1.5] and the images of such trace operators are denoted by $H^{1/2}(\Gamma;\mathbb{V})$ and $H^{-1/2}(\Gamma;\mathbb{V})$ respectively. Moreover, we have the Green's formula

$$\int_\Omega (\mathrm{grad}\,\boldsymbol{v} : \sigma + \boldsymbol{v}\cdot\mathrm{div}\,\sigma)\,d\boldsymbol{x} = \\ = \int_{\partial\Omega} \boldsymbol{v}\cdot(\sigma\boldsymbol{n})\,d\gamma, \quad \forall \boldsymbol{v} \in H^1(\Omega;\mathbb{V}), \forall\sigma \in H(\mathrm{div},\Omega;\mathbb{M}),$$

For sake of simplicity, the following notation is introduced

$$\Sigma_\Gamma \coloneqq \{\sigma \in H(\mathrm{div},\Omega;\mathbb{M}) \mid \sigma\boldsymbol{n}|_\Gamma = \boldsymbol{0}\},$$
$$V \coloneqq L^2(\Omega;\mathbb{V}), \qquad Q \coloneqq L^2(\Omega;\mathbb{M}),$$

moreover

$$\Sigma \coloneqq \Sigma_\varnothing, \qquad \Sigma_0 \coloneqq \Sigma_{\partial\Omega}.$$

The projectors sym and skew induce naturally the definition of the following subspaces

$$\Sigma^{\mathrm{sym}}_\Gamma \coloneqq \left\{\sigma \in \Sigma_\Gamma \,\middle|\, \sigma = \sigma^T\right\}, \quad \Sigma^{\mathrm{skew}}_\Gamma \coloneqq \left\{\sigma \in \Sigma_\Gamma \,\middle|\, \sigma = -\sigma^T\right\},$$

then the projectors can be extended in a natural way to the space Σ_Γ

$$\mathrm{sym} : \Sigma_\Gamma \mapsto \Sigma^{\mathrm{sym}}_\Gamma, \qquad \mathrm{skew} : \Sigma_\Gamma \mapsto \Sigma^{\mathrm{skew}}_\Gamma.$$

The spaces Q^{sym} and Q^{skew} are defined in a similar way. It is then straightforward to prove that the projectors sym and skew induce the following L^2-orthogonal decompositions

$$\Sigma_\Gamma \sim \Sigma^{\mathrm{sym}}_\Gamma \oplus \Sigma^{\mathrm{skew}}_\Gamma, \qquad Q \sim Q^{\mathrm{sym}} \oplus Q^{\mathrm{skew}}.$$

We shall use the expression $A \lesssim B$ to say that there exists a constant C which does not depend on the discretization parameter h such that $A \le CB$.

3 The stress-velocity formulation of Stokes problem

The motion of an incompressible isotropic Newtonian viscous fluid at low Reynolds number is described by the Stokes equations

$$\begin{cases} -\operatorname{div}(2\mu \operatorname{grad}_s \boldsymbol{u}) + \operatorname{grad} p = \boldsymbol{f}, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in } \Omega. \end{cases} \tag{1}$$

Where $\mu \in L^\infty(\Omega)$ is the viscosity of the fluid and it is assumed to be positive; $\boldsymbol{f} \in L^2(\Omega; \mathbb{V})$ is a given external body force, the unknowns are the velocity $\boldsymbol{u} \in H^1(\Omega; \mathbb{V})$ and the pressure $p \in L^2(\Omega)$ and grad_s denotes the symmetric part of the gradient.

Problem (1) must be supplemented with proper boundary conditions: we can assign the velocity on the boundary: given $\boldsymbol{g} \in H^{1/2}(\partial\Omega; \mathbb{V})$ then

$$\boldsymbol{u} = \boldsymbol{g}, \quad \text{on } \partial\Omega,$$

If the function $\boldsymbol{g}$ satisfies the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \, d\gamma = 0,$$

then Problem (1) is well posed in the sense that it admits an unique solution up an additive constant value of the pressure. Alternatively, we can consider the case when a traction condition is enforced; in such case, given $\boldsymbol{h} \in H^{-1/2}(\partial\Omega; \mathbb{V})$, the following condition is used

$$(2\mu \operatorname{grad}_s \boldsymbol{u} - pI)\boldsymbol{n} = \boldsymbol{h}, \quad \text{on } \partial\Omega.$$

Again, if the function $\boldsymbol{h}$ satisfies the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{h} \, d\gamma + \int_{\Omega} \boldsymbol{f} \, d\boldsymbol{x} = \boldsymbol{0}$$

then the problem is well posed, but now the velocity $\boldsymbol{u}$ is uniquely determined up an infinitesimal rigid displacement $\omega\boldsymbol{x} + \boldsymbol{b}$, where $\omega \in \mathbb{K}$ and $\boldsymbol{b} \in \mathbb{V}$. In the mixed case no compatibility condition is needed and the solution is uniquely determined.

For the ease of presentation from now on we consider the full homogeneous velocity problem, where the velocity is assumed to be null along the boundary, for this reason the boundary condition will not be written in the problem formulation any more. Without difficulties all the analysis can be carried out in the inhomogeneous case and in all the other described configurations.

In order to derive the mixed formulation of Stokes equations the definition of stress tensor σ is used, then the problem can be written as a system of three equations

$$\begin{cases} \sigma = 2\mu \operatorname{grad}_s \boldsymbol{u} - pI, & \text{in } \Omega, \\ \operatorname{div} \sigma + \boldsymbol{f} = 0, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in } \Omega, \end{cases}$$

which are the constitutive equation, the equation of conservation of linear momentum and the incompressibility constraint respectively. Generally the quantity σ is dropped by substituting the first equation in the second one, obtaining the usual velocity-pressure formulation. In this work we leave untouched the second equation and the pressure p is dropped in the same way as in [17, 18, 19]. Noticing that $\operatorname{tr}\sigma = np$ and using the incompressibility constraint the constitutive equation can be rewritten as

$$\frac{1}{2\mu}\operatorname{dev}\sigma = \operatorname{grad}_s \boldsymbol{u}.$$

Then the problem can be recasted in a form that involves only the stress and velocity variables

$$\begin{cases} \dfrac{1}{2\mu}\operatorname{dev}\sigma = \operatorname{grad}_s \boldsymbol{u}, & \text{in } \Omega, \\ \operatorname{div}\sigma + \boldsymbol{f} = 0, & \text{in } \Omega. \end{cases}$$

Note that if this problem admits a solution then the velocity satisfies the incompressibility condition without enforcing it:

$$\operatorname{div}\boldsymbol{u} = \operatorname{tr}(\operatorname{grad}_s \boldsymbol{u}) = \frac{1}{2\mu}\operatorname{tr}(\operatorname{dev}\sigma) = 0.$$

The weak formulation of this problem reads: find $(\sigma, \boldsymbol{u}) \in \Sigma^{\text{sym}} \times V$ such that

$$\begin{cases} \displaystyle\int_\Omega \frac{1}{2\mu}\operatorname{dev}\sigma : \operatorname{dev}\tau + \boldsymbol{u}\cdot\operatorname{div}\tau \, d\boldsymbol{x} = 0, \\ \displaystyle\int_\Omega (\operatorname{div}\sigma + \boldsymbol{f})\cdot\boldsymbol{v}\, d\boldsymbol{x} = 0, \end{cases} \tag{2}$$

for all $\tau \in \Sigma^{\text{sym}}$ and $\boldsymbol{v} \in V$. Clearly if $(\boldsymbol{u}, p) \in H^1(\Omega;\mathbb{V}) \times L^2(\Omega)$ is a weak solution of Problem (1) than the pair $(\hat{\sigma}, \boldsymbol{u})$, where $\hat{\sigma} = 2\mu \operatorname{grad}_s \boldsymbol{u} - pI$, is a solution of Problem (2). This is possible only because $\boldsymbol{f} \in L^2(\Omega;\mathbb{V})$, in such case from the Green's formula it follows that $\operatorname{div}\hat{\sigma} \in L^2(\Omega;\mathbb{V})$; if $\boldsymbol{f}$ is less regular, for example $\boldsymbol{f} \in H^{-1}(\Omega;\mathbb{V})$, then Problem (1) could admit a weak solution but it would not be possible to prove that $\hat{\sigma}$ has the required regularity for obtaining the equivalence of the problems.

Problem (2) is a saddle point problem, then a proper finite element approximation of it must satisfy the well known $\inf - \sup$ condition [8, 12] in order to prevent spurius modes. The definition of a pair of finite element spaces that honors both the $\inf - \sup$ condition and the symmetry constraint of the stress tensor is extremely difficulty and generally leads to a definition of spaces with a huge number of degrees of freedom (24 per triangle [7] and 162 per tetrahedral [1, 2]), that makes them unsuitable for real applications. Two different approaches are possible to bypass the problem: using a stabilization technique like least squares formulations to avoid the $\inf - \sup$ condition [10, 16], or to weaken the symmetry requirement enforcing it only weakly with a Lagrange multiplier [3, 6]. Here we adopt the second technique; for this reason a third equation equivalent to the conservation of angular momentum and a new unknown $\omega \in Q^{\text{skew}}$, the *vorticity*, are introduced. The weak formulation of the problem becomes: find $(\sigma, \boldsymbol{u}, \omega) \in \Sigma \times V \times Q^{\text{skew}}$ such that

$$\begin{cases} \displaystyle\int_\Omega \frac{1}{2\mu} \operatorname{dev} \sigma : \operatorname{dev} \tau + \boldsymbol{u} \cdot \operatorname{div} \tau + \omega : \tau \, d\boldsymbol{x} = 0, \\ \displaystyle\int_\Omega (\operatorname{div} \sigma + \boldsymbol{f}) \cdot \boldsymbol{v} \, d\boldsymbol{x} = 0, \\ \displaystyle\int_\Omega \sigma : \phi \, d\boldsymbol{x} = 0, \end{cases} \tag{3}$$

for all $\tau \in \Sigma$, $\boldsymbol{v} \in V$ and $\phi \in Q^{\text{skew}}$. Again, if $(\sigma, \boldsymbol{u})$ is a solution of Problem (2) than the triple $(\sigma, \boldsymbol{u}, \hat{\omega})$, where $\hat{\omega} = \operatorname{skew}(\operatorname{grad} \boldsymbol{u})$, is a solution of Problem (3).

As previously said if a solution of this problem exists then it can not be unique: the stress tensor σ is unique up to an additive constant value of pressure. For this reason the proper functional space for the stress is given by the quotient space $\Sigma^0 := \Sigma/\mathbb{R}$, where it is considered the identification

$$\alpha \in \mathbb{R} \mapsto \alpha I \in \Sigma,$$

or equivalently the projector

$$\sigma \in \Sigma \mapsto \sigma - \left(\frac{1}{|\Omega|} \int_\Omega \operatorname{tr} \sigma \, d\boldsymbol{x} \right) \frac{1}{n} I \in \Sigma^0,$$

where $|\Omega|$ denotes the measure of the domain Ω

The existence and uniqueness of solution for Problem (3) can be prooved using the equivalence between Problem (1) and Problem (3), like in [4, Theorem 2.1]. Here we given a new proof based of the abstract theory of saddle point problems (for the details see [15, Chapter II]). The bilinear forms $a : \Sigma^0 \times \Sigma^0 \to \mathbb{R}$ and $b : \Sigma^0 \times V \oplus Q^{\text{skew}} \to \mathbb{R}$ are defined as

$$a(\sigma, \tau) := \int_\Omega \frac{1}{2\mu} \operatorname{dev} \sigma : \operatorname{dev} \tau \, d\boldsymbol{x},$$

$$b(\sigma, (\boldsymbol{u}, \omega)) := \int_\Omega \boldsymbol{u} \cdot \operatorname{div} \sigma + \omega : \sigma \, d\boldsymbol{x}.$$

Clearly both of them are bounded, hence continuous, so that to prove the well posedness of Problem (3) it is necessary to verify that:

- the bilinear form a is strictly coercive on the subspace

$$\begin{aligned} \operatorname{Ker} B &:= \{\sigma \in \Sigma^0 \,|\, b(\sigma, (\boldsymbol{u}, \omega)) = 0 \quad \forall (\boldsymbol{u}, \omega) \in V \oplus Q^{\text{skew}}\} = \\ &= \{\sigma \in \Sigma^0 \cap \Sigma^{\text{sym}} \mid \operatorname{div} \sigma = 0\}; \end{aligned}$$

- the $\inf - \sup$ condition is satisfied, *i.e.* there exists $\beta > 0$ such that for all $(\boldsymbol{u}, \omega) \in V \oplus Q^{\text{skew}}$

$$\sup_{\sigma \in \Sigma^0 \setminus \{0\}} \frac{b(\sigma, (\boldsymbol{u}, \omega))}{\|\sigma\|_{\Sigma^0}} \geq \beta \|(\boldsymbol{u}, \omega)\|_{V \oplus Q^{\text{skew}}}.$$

Lemma 1 *Let be $\hat{\Sigma}$ the Hilbert space obtained from $\mathcal{C}^\infty(\Omega; \mathbb{M})$ by completion with respect to the seminorm induced by the bilinear form*

$$(\sigma, \tau) \mapsto \int_\Omega \operatorname{dev} \sigma : \operatorname{dev} \tau + \operatorname{div} \sigma \cdot \operatorname{div} \tau \, d\boldsymbol{x}.$$

Then $\hat{\Sigma}$ is isomorphic to Σ^0.

Proof This proof follows the same of [21, Theorem 2.1]. Clearly $\Sigma^0 \subseteq \hat{\Sigma}$, then we have to prove the inverse inclusion. Let $\sigma \in \hat{\Sigma}$, clearly $\operatorname{dev} \sigma \in L^2(\Omega;\mathbb{M})$, $\operatorname{div} \sigma \in L^2(\Omega;\mathbb{V})$ and

$$\operatorname{grad}(\operatorname{tr}\sigma) = n(\operatorname{div}\sigma - \operatorname{div}(\operatorname{dev}\sigma)) \in H^{-1}(\Omega;\mathbb{V}),$$

then for each $i, j = 1, \dots, n$

$$\operatorname{grad}\sigma_{ij} = \operatorname{grad}(\operatorname{dev}\sigma)_{ij} + \frac{1}{n}\operatorname{grad}(\operatorname{tr}\sigma)\delta_{ij} \in H^{-1}(\Omega;\mathbb{V}).$$

Hence $\sigma \in \Sigma$ by a lemma of J.-L. Lions [22, Theorem 3.2]. Then the identity map $i : \Sigma^0 \to \hat{\Sigma}$ is continuous and surjective. Injectivity holds since the elements $\tau \in \Sigma^0$ are such that

$$\int_\Omega \operatorname{tr}\tau \, d\boldsymbol{x} = 0.$$

By the Banach open mapping theorem also the inverse map i^{-1} is continuous. Hence i is an isomorphism.

In the case of full stress boundary condition the same result can be proved for the space of functions $\sigma \in \Sigma_0$. In this case it is necessary to consider the space $\mathcal{C}_0^\infty(\Omega;\mathbb{M})$ and the proof proceeds in the same way, but the injectivity is given by the boundary condition. Clearly, the same is true for the mixed boundary condition case: it is enough to put a stress boundary condition on a portion of boundary $\Gamma \subset \partial\Omega$ with not null capacity in order to ensure the injectivity of map i.

Corollary 1 *The bilinear form $a(\cdot,\cdot)$ is strictly coercive on the space* $\operatorname{Ker} B$.

Proof It is enough to observe that the space $\operatorname{Ker} B$ is contained in the subspace of divergence-free functions of Σ^0 and use Lemma 1.

The proof of the validity of $\inf - \sup$ condition is given by a standard argument used in the context of saddle point problems and the same technique will be used for the discrete problem in the next section. Then we are ready to state and prove the following lemma that concludes the proof of the well posedness of Problem (3). This result is equivalent to [6, Theorem 7.1], but our proof is different and we think that this proof gives a clearer image about the structure of Problem (3) and the physical meaning of each term.

Lemma 2 *There exists $\beta > 0$ such that for all $(\boldsymbol{u}, \omega) \in V \oplus Q^{\text{skew}}$*

$$\sup_{\sigma \in \Sigma^0 \setminus \{0\}} \frac{b(\sigma, (\boldsymbol{u}, \omega))}{\|\sigma\|_{\Sigma^0}} \geq \beta \|(\boldsymbol{u}, \omega)\|_{V \oplus Q^{\text{skew}}}.$$

Proof Given a pair of functions $(\boldsymbol{u}, \omega) \in V \oplus Q^{\text{skew}}$ let us now deal with the variational problem: find $\boldsymbol{\phi} \in H_0^1(\Omega;\mathbb{V})$ such that

$$\int_\Omega \operatorname{grad}_s \boldsymbol{\phi} : \operatorname{grad}_s \boldsymbol{\psi} \, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\psi} + \omega : \operatorname{grad} \boldsymbol{\psi} \, d\boldsymbol{x}$$

for all $\boldsymbol{\psi} \in H_0^1(\Omega;\mathbb{V})$. The existence and uniqueness of a solution of this problem is ensured by the Lax-Milgram theorem and it is a standard application of Korn inequality [21, Theorem 2.3]. Moreover, from the Lax-Milgram theorem the following stability inequality holds

$$\|\mathrm{grad}_s\, \boldsymbol{\phi}\|_{0,\Omega} \leq C\|(\boldsymbol{u},\omega)\|_{V\oplus Q^{\mathrm{skew}}},$$

where C is a positive constant that is strictly related to Ω and does not depend on data $(\boldsymbol{u},\omega)$. Now define $\sigma = \omega - \mathrm{grad}_s\, \boldsymbol{\phi}$ then by Green's formula the distributional divergence of σ is equal to $\boldsymbol{u}$, so we obtain

$$\sigma = \omega - \mathrm{grad}_s\, \boldsymbol{\phi}, \qquad \mathrm{div}\, \sigma = \boldsymbol{u},$$

and we get the stability estimate

$$\|\sigma\|_{\Sigma^0} \leq \|\sigma\|_{\Sigma} \leq \sqrt{1+C^2}\|(\boldsymbol{u},\omega)\|_{V\oplus Q^{\mathrm{skew}}}.$$

We can state that, given $(\boldsymbol{u},\omega) \in V \oplus Q^{\mathrm{skew}}$ there exists $\sigma^0 \in \Sigma^0$, that is the projection of σ in Σ^0, such that

$$b(\sigma^0,(\boldsymbol{u},\omega)) = b(\sigma,(\boldsymbol{u},\omega)) = \int_\Omega \|\boldsymbol{u}\|^2 + \|\omega\|^2\, d\boldsymbol{x},$$

then the $\inf-\sup$ condition holds with $\beta = 1/\sqrt{1+C^2}$ by the stability estimate.

This lemma can not be used directly to prove that the $\inf-\sup$ condition is satisfied also for full stress and mixed boundary conditions. Nevertheless in these situation the proof is still valid replacing Σ^0 with Σ_Γ and noticing that by construction $\sigma\boldsymbol{n}|_\Gamma = \boldsymbol{0}$.

Remark 1 The space of infinitesimal rigid displacements plays a crucial role in the Stokes problem. As previously noted if the traction condition is enforced along all the boundary of Ω then the velocity is defined up to an infinitesimal rigid displacement. In such case Σ_0 is the proper space for the stress unknown, then, by the abstract theory of saddle point problems, the components $(\boldsymbol{u},\omega)$ are uniquely defined up to an element of the subspace

$$\mathrm{Ker}\, B^T := \big\{(\boldsymbol{u},\omega) \in V \oplus Q^{\mathrm{skew}} \,\big|\, b(\sigma,(\boldsymbol{u},\omega)) = 0 \quad \forall \sigma \in \Sigma_0\big\}.$$

It coincides with the space of infinitesimal rigid displacements:

$$\mathrm{Ker}\, B^T = \big\{(\omega\boldsymbol{x}+\boldsymbol{b},\omega) \in V \oplus Q^{\mathrm{skew}} \,\big|\, \omega \in \mathbb{K} \text{ and } \boldsymbol{b} \in \mathbb{V}\big\}.$$

Moreover the same technique can be applied in even more complicated cases: for example only the normal or the tangential component of the velocity can be constrained, and then it is necessary to enforce the complementary part of stress. Using these boundary conditions it is not hard to define problems with a solution which is unique up to a translation.

4 Finite element discretization of the mixed formulation

In this section we derive the discrete version of Problem (3) by adapting the theory developed in the previous section. The results of this section are strictly related to the ones developed in [6, Section 6], but there are two main differences:

- we have chosen to adopt the Raviart-Thomas element [29, 26] instead of the Brezzi-Douglas-Marini element [14, 27];
- the framework of finite element exterior calculus is not required: the analysis exploits the Fortin's trick [23, Proposition 4.1] to prove the good properties of the proposed finite element approximation.

As usual, now suppose we are given a regular family $\mathcal{T}_h$ of triangulations of the domain Ω consisting of n-simplices [20, Chapter 2], this means that there exists a constant $C > 0$ such that

$$\sup_{K\in\mathcal{T}_h} h_K \leq h, \quad \text{and} \quad \sup_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \leq C,$$

where h_K and ρ_K are the diameter and the inradius of the n-simplex K respectively. Given an integer $k \geq 0$ with $\mathcal{P}_k(K)$ is denoted the space of real polynomials of degree less or equal to k defined on the n-simplex K and with $\mathcal{RT}_k(K)$ the Raviart-Thomas space of index k on the n-simplex K. Using such spaces then we can introduce the conforming approximation of spaces Σ, V and Q^{skew}: for each integer $k \geq 0$ we define the finite element spaces

$$\begin{aligned}
\Sigma_{h,k}^{\mathcal{RT}} &:= \{\sigma_h \in \Sigma \mid \sigma_h|_K \in \mathbb{V} \otimes \mathcal{RT}_k(K),\, \forall K \in \mathcal{T}_h\},\\
V_{h,k} &:= \{\boldsymbol{v}_h \in V \mid \boldsymbol{v}_h|_K \in \mathbb{V} \otimes \mathcal{P}_k(K),\, \forall K \in \mathcal{T}_h\},\\
Q_{h,k}^{\text{skew}} &:= \big\{\omega_h \in Q^{\text{skew}} \,\big|\, \omega_h|_K \in \mathbb{K} \otimes \mathcal{P}_k(K),\, \forall K \in \mathcal{T}_h\big\}.
\end{aligned}$$

The space $\Sigma_{h,k}^{\mathcal{RT}}$ is defined by approximating each row of the tensor $\sigma \in \Sigma$ with a vector in the Raviart-Thomas space. Because $\Sigma_{h,k}^{\mathcal{RT}} \subset \Sigma$ then for each $\sigma_h \in \Sigma_{h,k}^{\mathcal{RT}}$ the normal component $\sigma_h \boldsymbol{n}$ is continuous across the elements, from the physical point of view this is equivalent to the action-reaction law and the Cauchy's fundamental theorem [31, Chapter III]. The spaces $V_{h,k}$ and $Q_{h,k}$ are the space of piecewise polynomials of degree at most k with values in $\mathbb{V}$ and $\mathbb{K}$ respectively.

Before stating the main result let us briefly sum up the properties of the chosen finite element spaces. The interpolation operator $\Pi_{h,k}^{\mathcal{RT}}$ associated with the Raviart-Thomas space of index k is defined as

$$\begin{aligned}
&\int_{K'} (\boldsymbol{q} - \Pi_{h,k}^{\mathcal{RT}}\boldsymbol{q}) \cdot \boldsymbol{n}\, \phi\, d\gamma = 0, \quad \forall \phi \in \mathcal{P}_k(K'),\, \forall K' \in \mathcal{E}_h,\\
&\int_{K} (\boldsymbol{q} - \Pi_{h,k}^{\mathcal{RT}}\boldsymbol{q}) \cdot \boldsymbol{\phi}\, d\boldsymbol{x} = 0, \quad \forall \boldsymbol{\phi} \in \mathbb{V} \otimes \mathcal{P}_{k-1}(K),\, \forall K \in \mathcal{T}_h,
\end{aligned} \tag{4}$$

where $\mathcal{E}_h$ is the set of $(n-1)$-simplices in $\mathcal{T}_h$. From the Green's identity it follows that

$$\int_K \operatorname{div}(\boldsymbol{q} - \Pi_{h,k}^{\mathcal{RT}}\boldsymbol{q})\, \phi\, d\boldsymbol{x} = 0, \quad \forall \phi \in \mathcal{P}_k(K),\, \forall K \in \mathcal{T}_h \tag{5}$$

or equivalently $\operatorname{div} \mathcal{RT}_k(K) = \mathcal{P}_k(K)$. With $P_{h,k}$ are denoted both the interpolation operators from V into $V_{h,k}$ and from Q^{skew} into $Q_{h,k}^{\text{skew}}$.

Theorem 1 *For each integer $k \geq 1$ there exists an unique*

$$(\sigma_h, \boldsymbol{u}_h, \omega_h) \in \Sigma_{h,k}^{\mathcal{RT},0} \times V_{h,k} \times Q_{h,k-1}^{\text{skew}}$$

such that

$$\begin{cases} \displaystyle\int_\Omega \frac{1}{2\mu} \operatorname{dev} \sigma_h : \operatorname{dev} \tau_h + \boldsymbol{u}_h \cdot \operatorname{div} \tau_h + \omega_h : \tau_h \, d\boldsymbol{x} = 0, \\ \displaystyle\int_\Omega (\operatorname{div} \sigma_h + \boldsymbol{f}) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} = 0, \\ \displaystyle\int_\Omega \sigma_h : \phi_h \, d\boldsymbol{x} = 0, \end{cases} \tag{6}$$

for all $\tau_h \in \Sigma_{h,k}^{\mathcal{RT}}$, $\boldsymbol{v}_h \in V_{h,k}$ and $\phi_h \in Q_{h,k-1}^{\text{skew}}$. Moreover there exists $\beta > 0$ indipendent from h such that for all $(\boldsymbol{u}_h, \omega_h) \in V_{h,k} \oplus Q_{h,k-1}^{\text{skew}}$

$$\sup_{\sigma_h \in \Sigma_{h,k}^0 \setminus \{0\}} \frac{b(\sigma_h, (\boldsymbol{u}_h, \omega_h))}{\|\sigma_h\|_{\Sigma^0}} \geq \beta \|(\boldsymbol{u}_h, \omega_h)\|_{V \oplus Q^{\text{skew}}}.$$

PROOF Again we have to prove:

- the bilinear form a is strictly coercive on the space $\operatorname{Ker} B_h$

$$\operatorname{Ker} B_h \coloneqq \{\sigma \in \Sigma_{h,k}^0 \,|\, b(\sigma_h, (\boldsymbol{u}_h, \omega_h)) = 0 \;\forall (\boldsymbol{u}_h, \omega_h) \in V_{h,k} \oplus Q_{h,k-1}^{\text{skew}}\};$$

- since all the functional spaces are finite dimension then the $\inf - \sup$ condition is satisfied with a constant $\beta_h \geq 0$, then we have to prove that β_h is bounded away from zero in order to ensure the convergence of the finite element method, *i.e.* there exists $\beta_0 > 0$ such that $\beta_h \geq \beta$ for all $h > 0$.

The first point is simple to prove, by (5) we have that

$$\operatorname{Ker} B_h \subset \{\sigma \in \Sigma^0 \mid \operatorname{div} \sigma = \boldsymbol{0}\},$$

then the coercivity follows directly from Lemma 1.

The proof of the second point uses the Fortin's trick [23]: we shall prove that the operator $\Pi_{h,k}^{\mathcal{RT}} : \Sigma \to \Sigma_{h,k}^{\mathcal{RT}}$ is a uniformly continuous operator such that

$$b(\sigma - \Pi_{h,k}^{\mathcal{RT}} \sigma, (\boldsymbol{v}_h, \omega_h)) = 0, \quad \forall (\boldsymbol{u}_h, \omega_h) \in V_{h,k} \oplus Q_{h,k-1}^{\text{skew}}. \tag{7}$$

Because the projection $\Sigma \to \Sigma_{h,k}^{\mathcal{RT}}$ is done row-wise its uniform continuity is a direct consequence of the results for the Laplace problem in mixed form [23, Example 5.1]. The identity (7) is equivalent to (4) and (5).

It is clear from the previous proof the reason why it is not possible to define a stable space with the lowest-order Raviart-Thomas elements ($k = 0$). In such case it is necessary to stabilize the method augmenting the space for the stress tensor with suitable bubble functions: the analysis of this method is reported in [11, Example 2].

From the general theory of the saddle point problem it follows the error estimate

$$\begin{aligned} &\|\sigma - \sigma_h\|_\Sigma + \|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|\omega - \omega_h\|_{Q^{\text{skew}}} \\ &\quad \lesssim \inf_{\tau_h \;\; in \Sigma_{h,k}^{\mathcal{RT},0}} \|\sigma - \tau_h\|_\Sigma + \inf_{\boldsymbol{v}_h \in V_{h,k}} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \inf_{\phi_h \in Q_{h,k-1}^{\text{skew}}} \|\omega - \phi_h\|_{Q^{\text{skew}}}. \end{aligned}$$

From this inequality it is easy to see the importance of balancing the quality of the approximation for each component of the solution, as discussed in [11, Section 3.2]. Following this theory our method would give rise to an error estimate of the order of $p-1$ due to the vorticity term. On the other hand a more detailed analysis can show a optimal approximation estimates for all components with the exception of the L^2-error of the stress field.

Before stating the result about the error estimates it is important to remark that even if the space $\operatorname{Ker} B_h$ is not contained in $\operatorname{Ker} B$, because the symmetry is enforced only weakly, the following equivalence is still valid: for all $\sigma_h \in \Sigma_{h,k}^{\mathcal{RT}}$

$$\operatorname{div} \sigma_h = 0 \Leftrightarrow \int_\Omega \boldsymbol{v}_h \cdot \operatorname{div} \sigma_h \, d\boldsymbol{x} = 0 \quad \forall \boldsymbol{v}_h \in V_{h,k}.$$

Theorem 2 *If* $(\sigma, \boldsymbol{u}, \omega)$ *is the solution of Problem* (3) *and* $(\sigma_h, \boldsymbol{u}_h, \omega_h)$ *is solution of* (6)*, then the following error estimates hold*

$$\|\sigma - \sigma_h\|_{\Sigma^0} \lesssim \inf_{\tau_h \in \Sigma_{h,k}^{\mathcal{RT},0}} \|\sigma - \tau_h\|_{\Sigma^0} + \inf_{\phi_h \in Q_{h,k-1}^{\text{skew}}} \|\omega - \phi_h\|_{Q^{\text{skew}}},$$

$$\|\operatorname{div}(\sigma - \sigma_h)\|_{0,\Omega} \leq \inf_{\tau_h \in \Sigma_{h,k}^{\mathcal{RT},0}} \|\operatorname{div}(\sigma - \tau_h)\|_{0,\Omega},$$

$$\begin{aligned}\|(\boldsymbol{u} - \boldsymbol{u}_h, \omega - \omega_h)\|_{V \oplus Q^{\text{skew}}} \lesssim \|\sigma - \sigma_h\|_{\Sigma^0} + \\ + \inf_{(\boldsymbol{v}_h, \phi_h) \in V_{h,k} \oplus Q_{h,k-1}^{\text{skew}}} \|(\boldsymbol{u} - \boldsymbol{v}_h, \omega - \phi_h)\|_{V \oplus Q^{\text{skew}}},\end{aligned}$$

moreover, if the domain Ω *is convex, we have also*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \lesssim \|\boldsymbol{u} - P_{h,k}\boldsymbol{u}\|_V + h\left(\|\sigma - \sigma_h\|_{\Sigma^0} + \|\omega - \omega_h\|_{Q^{\text{skew}}}\right).$$

Proof Fixed $\boldsymbol{f} \in L^2(\Omega; \mathbb{V})$ we define the affine manifold $Z_{h,k}(\boldsymbol{f})$ as the set of $\sigma_h \in \Sigma_{h,k}^{\mathcal{RT},0}$ such that

$$b(\sigma_h, (\boldsymbol{v}_h, \omega_h)) + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h = 0 \quad \forall (\boldsymbol{v}_h, \omega_h) \in V_{h,k} \oplus Q_{h,k-1}^{\text{skew}}.$$

Chosen $\tau_h \in Z_{h,k}(\boldsymbol{f})$ then $z_h = \tau_h - \sigma_h$ belongs to $\operatorname{Ker} B_h$, then by the coercivity proved in the previous theorem there exists $\alpha > 0$ such that $\alpha \|z_h\|_\Sigma^2 \leq a(z_h, z_h)$. Now, by linearity and using that σ and σ_h are respectively solution of the continuous and discrete problem:

$$\alpha \|z_h\|_\Sigma^2 \leq a(\tau_h - \sigma, z_h) - b(z_h, (\boldsymbol{u} - \boldsymbol{u}_h, \omega - \omega_h)).$$

Recall that $\operatorname{div} z_h = \mathbf{0}$ because $z_h \in \operatorname{Ker} B_h$, then this inequality, by continuity and equation of angular momentum conservation, reduces to

$$\alpha \|\tau_h - \sigma_h\|_\Sigma \leq \|a\| \|\sigma - \tau_h\|_\Sigma + \|\omega - \phi_h\|_{Q^{\text{skew}}}.$$

for all $\tau_h \in Z_{h,k}(\boldsymbol{f})$ and $\phi_h \in Q_{h,k-1}^{\text{skew}}$. From here it is a standard argument to obtain the first error estimate, see [15, Section II.2.2].

By linearity, from the equation of linear momentum conservation

$$\int_\Omega \operatorname{div}(\sigma - \sigma_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} = 0 \quad \forall \boldsymbol{v}_h \in V_{h,k}.$$

Taken $\boldsymbol{v}_h = \operatorname{div}(\tau_h - \sigma_h)$ with $\tau_h \in \Sigma_{h,k}^{\mathcal{RT},0}$ it is easy to prove the second error estimate.

Again, by linearity, from the equations of conservation of linear and angular momentum, for all $\tau_h \in \Sigma_{h,k}^{\mathcal{RT},0}$ and $(\boldsymbol{v}_h, \phi_h) \in V \oplus Q^{\text{skew}}$

$$b(\tau_h, (\boldsymbol{u}_h - \boldsymbol{v}_h, \omega_h - \phi_h)) = a(\sigma - \sigma_h, \tau_h) + b(\tau_h, (\boldsymbol{u} - \boldsymbol{v}_h, \omega - \phi_h)),$$

using the continuity of both forms a and b, the $\inf - \sup$ condition the following estimates follows

$$\|(\boldsymbol{u}_h - \boldsymbol{v}_h, \omega_h - \phi_h)\|_{V \oplus Q^{\text{skew}}} \lesssim \|\sigma - \sigma_h\|_{\Sigma^0} + \|(\boldsymbol{u} - \boldsymbol{v}_h, \omega - \phi_h)\|_{V \oplus Q^{\text{skew}}}.$$

The third estimate now follows directly by the triangle inequality.

The last estimate is obtained by duality. Let be $\boldsymbol{U}$ and P the solution of the Dirichlet homogeneous problem

$$\begin{cases} \operatorname{div}(2\mu \operatorname{grad}_s \boldsymbol{U} - PI) = P_{h,k}\boldsymbol{u} - \boldsymbol{u}_h, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{U} = 0, & \text{in } \Omega, \\ \boldsymbol{U} = \boldsymbol{0}, & \text{on } \partial\Omega, \end{cases}$$

then set $T = 2\mu \operatorname{grad}_s \boldsymbol{U} - PI$. Exploiting the commutativity property $\operatorname{div} \Pi_{h,k}^{\mathcal{RT}} = P_{h,k} \operatorname{div}$ of the Raviart-Thomas finite elements we obtain

$$\begin{aligned} \|P_{h,k}\boldsymbol{u} - \boldsymbol{u}_h\|_V^2 &= \int_\Omega \operatorname{div} T \cdot (P_{h,k}\boldsymbol{u} - \boldsymbol{u}_h) \, d\boldsymbol{x} = \\ &= \int_\Omega \operatorname{div} (\Pi_{h,k}^{\mathcal{RT}} T) \cdot (\boldsymbol{u} - \boldsymbol{u}_h) \, d\boldsymbol{x} = \end{aligned}$$

then we use that $\boldsymbol{u}$ and $\boldsymbol{u}_h$ are solution of Problem (3) and (6)

$$\begin{aligned} &= -\int_\Omega \left[\frac{1}{2\mu} \operatorname{dev}(\sigma - \sigma_h) + (\omega - \omega_h) \right] : \Pi_{h,k}^{\mathcal{RT}} T \, d\boldsymbol{x} = \\ &= \int_\Omega \left[\frac{1}{2\mu} \operatorname{dev}(\sigma - \sigma_h) + (\omega - \omega_h) \right] : (T - \Pi_{h,k}^{\mathcal{RT}} T) \, d\boldsymbol{x} + \\ &\quad - \int_\Omega (\sigma - \sigma_h) : \operatorname{grad}_s \boldsymbol{U} \, d\boldsymbol{x} = \end{aligned}$$

using the orthogonal decomposition $\operatorname{grad} \boldsymbol{U} = \operatorname{sym} \operatorname{grad} \boldsymbol{U} + \operatorname{skew} \operatorname{grad} \boldsymbol{U}$ and integrating by parts

$$\begin{aligned} &= \int_\Omega \left[\frac{1}{2\mu} \operatorname{dev}(\sigma - \sigma_h) + (\omega - \omega_h) \right] : (T - \Pi_{h,k}^{\mathcal{RT}} T) \, d\boldsymbol{x} + \\ &\quad + \int_\Omega \operatorname{div}(\sigma - \sigma_h) \cdot (\boldsymbol{U} - P_{h,k}\boldsymbol{U}) \, d\boldsymbol{x} + \\ &\quad + \int_\Omega (\sigma - \sigma_h) : \operatorname{skew}(\operatorname{grad} \boldsymbol{U} - P_{h,k-1} \operatorname{grad} \boldsymbol{U}) \, d\boldsymbol{x}. \end{aligned}$$

Table 1: Summary of asymptotic order of convergence.

	$\mathcal{RT}_k$	$\mathcal{BDM}_k$
$\|\sigma - \sigma_h\|_{\Sigma^0}$	k	k
$\|\operatorname{div}(\sigma - \sigma_h)\|_{0,\Omega}$	$k+1$	k
$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V$	$k+1$	k
$\|\sigma - \sigma_h\|_{Q^{\text{skew}}}$	k	k

Using the elliptic regularity, which holds when the domain Ω is convex, the norms $\|\boldsymbol{U}\|_{2,\Omega}$ and $\|T\|_{1,\Omega}$ are bounded by $\|P_{h,k}\boldsymbol{u} - \boldsymbol{u}_h\|_V$. Moreover, the interpolation operators satisfy the following estimates [28, Section 3.4]

$$\left\|T - \Pi_{h,k}^{\mathcal{RT}} T\right\|_{0,\Omega} \lesssim h\|T\|_{1,\Omega}, \quad \|P_{h,k}\boldsymbol{U} - \boldsymbol{U}\|_{0,\Omega} \lesssim h\|\boldsymbol{U}\|_{1,\Omega},$$

$$\|P_{h,k-1} \operatorname{grad} \boldsymbol{U} - \operatorname{grad} \boldsymbol{U}\|_{0,\Omega} \lesssim h\|\boldsymbol{U}\|_{2,\Omega}.$$

From here the last estimate follows directly by the triangle inequality.

This theorem is still valid if the finite space chosen for discretization is the one introduced in [6]. As previously remarked, in the work of Arnold et al. the finite element space used for the stress unknown is defined as follow

$$\Sigma_{h,k}^{\mathcal{BDM}} := \{\sigma_h \in \Sigma \mid \sigma_h|_K \in \mathbb{V} \otimes \mathcal{BDM}_k(K), \forall K \in \mathcal{T}_h\},$$

where $\mathcal{BDM}_k$ denotes the Brezzi-Douglas-Marini finite element [14, 27] of degree k. In this case the method is stable if the finite element space $\Sigma_{h,k}^{\mathcal{BDM}} \times V_{h,k-1} \times Q_{h,k-1}^{\text{skew}}$ is used and Theorem 2 is still valid since the proof does not rely on the definition of the finite element space, but only on the commutativity property of the interpolation operators, that in the case of Brezzi-Douglas-Marini elements is given by

$$\operatorname{div} \Pi_{h,k}^{\mathcal{BDM}} = P_{h,k-1} \operatorname{div},$$

where in this case $\Pi_{h,k}^{\mathcal{BDM}}$ denotes the extensions to the tensors (row-wisely) of interpolation operator associated to the space $\mathcal{BDM}_k$.

These two different finite element spaces are related by the following inclusions

$$\Sigma_{h,k}^{\mathcal{BDM}} \times V_{h,k-1} \times Q_{h,k-1}^{\text{skew}} \subset \Sigma_{h,k}^{\mathcal{RT}} \times V_{h,k} \times Q_{h,k-1}^{\text{skew}} \subset \Sigma_{h,k+1}^{\mathcal{BDM}} \times V_{h,k} \times Q_{h,k}^{\text{skew}},$$

for $k \geq 1$. So the Brezzi-Douglas-Marini finite element gives a discretization with fewer degrees of freedom, but a worse asymptotic order of convergence. In Table 1 the asymptotic order of convergence are summed up: it is clear that both methods are not optimal in the stress variable, but the $\mathcal{BDM}$ finite element gives a more balanced method, on the other side the $\mathcal{RT}$ finite element are more accurate. It is also worth to mention the paper of Suri [30] where a more detailed analysis for rectangular higher order methods for heat equation is reported. The author shows that the Raviart-Thomas are optimal, in the sense that the constants in the errors estimates do not depend on h and k, on the other side for the Brezzi-Douglas-Marini there is a small dependence on k (see the cited paper for the technical details).

Table 2: Errors and convergence rate for the case $k = 1$.

$\Sigma_{h,1}^{\mathcal{RT}} \times V_{h,1} \times Q_{h,0}^{\text{skew}}$

h	$\lVert\sigma - \sigma_h\rVert_{0,\Omega}$		$\lVert\text{div}\,(\sigma - \sigma_h)\rVert_{0,\Omega}$	
6.40×10^{-1}	2.58×10^{-1}	–	3.08×10^{-2}	–
3.44×10^{-1}	1.15×10^{-1}	1.30	7.72×10^{-3}	2.22
1.73×10^{-1}	5.00×10^{-2}	1.21	1.70×10^{-3}	2.21
8.80×10^{-2}	2.53×10^{-2}	1.00	4.30×10^{-4}	2.03
4.41×10^{-2}	1.25×10^{-2}	1.02	1.06×10^{-4}	2.04
2.24×10^{-2}	6.23×10^{-3}	1.03	2.66×10^{-5}	2.04
h	$\lVert\boldsymbol{u} - \boldsymbol{u}_h\rVert_V$		$\lVert\omega - \omega_h\rVert_{Q^{\text{skew}}}$	
6.40×10^{-1}	1.81×10^{-2}	–	1.13×10^{-1}	–
3.44×10^{-1}	4.28×10^{-3}	2.31	5.49×10^{-2}	1.17
1.73×10^{-1}	9.15×10^{-4}	2.25	2.49×10^{-2}	1.16
8.80×10^{-2}	2.32×10^{-4}	2.02	1.26×10^{-2}	1.00
4.41×10^{-2}	5.70×10^{-5}	2.04	6.25×10^{-3}	1.02
2.24×10^{-2}	1.42×10^{-5}	2.05	3.12×10^{-3}	1.03

$\Sigma_{h,1}^{\mathcal{BDM}} \times V_{h,0} \times Q_{h,0}^{\text{skew}}$

h	$\lVert\sigma - \sigma_h\rVert_{0,\Omega}$		$\lVert\text{div}\,(\sigma - \sigma_h)\rVert_{0,\Omega}$	
6.40×10^{-1}	2.55×10^{-1}	–	4.31×10^{-1}	–
3.44×10^{-1}	1.16×10^{-1}	1.26	2.14×10^{-1}	1.12
1.73×10^{-1}	5.01×10^{-2}	1.23	9.98×10^{-2}	1.11
8.80×10^{-2}	2.54×10^{-2}	1.00	5.13×10^{-2}	0.98
4.41×10^{-2}	1.25×10^{-2}	1.03	2.55×10^{-2}	1.01
2.24×10^{-2}	6.23×10^{-3}	1.03	1.28×10^{-2}	1.02
h	$\lVert\boldsymbol{u} - \boldsymbol{u}_h\rVert_V$		$\lVert\omega - \omega_h\rVert_{Q^{\text{skew}}}$	
6.40×10^{-1}	1.94×10^{-1}	–	1.14×10^{-1}	–
3.44×10^{-1}	9.82×10^{-2}	1.09	5.50×10^{-2}	1.17
1.73×10^{-1}	4.58×10^{-2}	1.11	2.49×10^{-2}	1.16
8.80×10^{-2}	2.35×10^{-2}	0.98	1.26×10^{-2}	1.00
4.41×10^{-2}	1.17×10^{-2}	1.01	6.25×10^{-3}	1.02
2.24×10^{-2}	5.88×10^{-3}	1.02	3.12×10^{-3}	1.03

Table 3: Errors and convergence rate for the case $k = 2$.

$\Sigma_{h,2}^{\mathcal{RT}} \times V_{h,2} \times Q_{h,1}^{\text{skew}}$

h	$\|\sigma - \sigma_h\|_{0,\Omega}$		$\|\text{div}\,(\sigma - \sigma_h)\|_{0,\Omega}$	
6.40×10^{-1}	2.28×10^{-2}	–	1.99×10^{-3}	–
3.44×10^{-1}	5.83×10^{-3}	2.19	2.75×10^{-4}	3.18
1.73×10^{-1}	1.26×10^{-3}	2.23	2.84×10^{-5}	3.32
8.80×10^{-2}	3.70×10^{-4}	1.81	3.79×10^{-6}	2.97
4.41×10^{-2}	8.08×10^{-5}	2.21	4.70×10^{-7}	3.03
h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V$		$\|\omega - \omega_h\|_{Q^{\text{skew}}}$	
6.40×10^{-1}	1.00×10^{-3}	–	1.11×10^{-2}	–
3.44×10^{-1}	1.35×10^{-4}	3.22	2.87×10^{-3}	2.17
1.73×10^{-1}	1.38×10^{-5}	3.33	6.26×10^{-4}	2.22
8.80×10^{-2}	1.85×10^{-6}	2.97	1.65×10^{-4}	1.97
4.41×10^{-2}	2.26×10^{-7}	3.05	4.05×10^{-5}	2.04

$\Sigma_{h,2}^{\mathcal{BDM}} \times V_{h,1} \times Q_{h,1}^{\text{skew}}$

h	$\|\sigma - \sigma_h\|_{0,\Omega}$		$\|\text{div}\,(\sigma - \sigma_h)\|_{0,\Omega}$	
6.40×10^{-1}	2.27×10^{-2}	–	3.08×10^{-2}	–
3.44×10^{-1}	5.85×10^{-3}	2.18	7.72×10^{-3}	2.22
1.73×10^{-1}	1.26×10^{-3}	2.24	1.70×10^{-3}	2.21
8.80×10^{-2}	3.70×10^{-4}	1.81	4.30×10^{-4}	2.03
4.41×10^{-2}	1.21×10^{-4}	1.62	6.69×10^{-3}	-3.98[a]
h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V$		$\|\omega - \omega_h\|_{Q^{\text{skew}}}$	
6.40×10^{-1}	1.19×10^{-2}	–	1.11×10^{-2}	–
3.44×10^{-1}	2.95×10^{-3}	2.25	2.87×10^{-3}	2.18
1.73×10^{-1}	6.25×10^{-4}	2.20	6.27×10^{-4}	2.22
8.80×10^{-2}	1.64×10^{-4}	2.04	1.65×10^{-4}	1.97
4.41×10^{-2}	4.04×10^{-5}	2.03	4.49×10^{-5}	1.85

[a] This value is probably due to the particular implementation of the $\mathcal{BDM}$ finite element.

5 Numerical experiments

In this section we report the results of some numerical experiments carried out using the finite element method proposed in this paper and in [6]. In particular we measure the order of convergence for the cases $k = 1$ and $k = 2$. We consider the full Dirichlet Stokes problem in the domain

$$\Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, \|\boldsymbol{x}\| < 1 \right\},$$

whose analytical solution is given by

$$\boldsymbol{u} = \begin{pmatrix} -\cos x \sin y \\ \sin x \cos y \end{pmatrix}, \qquad p = -\frac{1}{4}(\cos(2x) + \cos(2y)).$$

The implementation of this test case is coded in Python with the aid of Fenics [25], for the solution of the linear system the PETSc [9] built in LU solver has been used. The full code is available to the following URL:

`https://gist.github.com/mattiapenati/0acde0f4aac174d98742`

The results are reported in the Table 2 and 3 where h represented the maximum diameter of the triangulation. The theoretical results are confirmed since both methods achieve the predicted order of convergence.

6 Conclusions

In this paper we have proposed a new mixed formulation of Stokes problem and we have given a detailed analysis of a new numerical finite element method based on the Raviart-Thomas element. With respect to existing literature our proposal does not rely on the equivalence between the primal formulation (velocity-pressure) and the dual one (stress-velocity-vorticity); moreover the proposed error analysis is more detailed than the existing ones and it shows the optimality of a new family of finite element space for the Stokes problem.

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