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Abstract

A section-averaged shallow water model for application to river hydraulics is derived asymptotically, starting from the three-dimensional Reynolds-averaged Navier-Stokes equations for incompressible free surface flows. The resulting section-averaged equations take into account the effects of eddy viscosity, friction and of the three-dimensional geometry of the domain, up to the second order in the ratio between vertical and longitudinal scales. This novel derivation yields a friction term that is similar to that of the classical section-averaged shallow water model, but includes a correction that is dependent on the turbulent vertical viscosity model. Steady state analytic solutions for open channel flow have been computed for the derived model, obtaining solutions that are much closer to those of the three-dimensional model than the solutions computed by the classical one-dimensional shallow-water models.

1 Introduction

The numerical approximation of three-dimensional free surface fluid flow in the context of environmental modelling applications can be computationally very costly. Therefore, whenever the ratio between the vertical and longitudinal scales is small enough, the so-called "Shallow Water" approximation is usually considered. Models based on this approximation are extensively used to simulate various geophysical phenomena, such as rivers and coastal flows [5, 7], oceans and even avalanches [1], and they have been used in hydraulics for a very long time. When the viscosity is neglected and a rectangular channel section is assumed, the derivation of the one-dimensional Shallow Water system is classical [12]. However, this derivation is unsatisfactory since viscosity effects are added *a posteriori* and the three-dimensional geometry is not arbitrary.

In [6], Gerbeau and Perthame derive rigourously, by asymptotic analysis, a one-dimensional viscous Saint-Venant system from the two-dimensional Navier-Stokes equations with molecular viscosity and flat bottom. The effect of the viscosity is recovered in a one-dimensional friction term and in a one-dimensional diffusion term, both obtained through the derivation. The final system is a second order approximation – with respect to the ratio between the vertical and longitudinal scales – of the two-dimensional departing model. Other systems have been derived in the same spirit. In [11], the asymptotic analysis is made through a variable change in a reference domain, independent of the ratio parameter and time. Marche proposes in [9] the derivation of a two-dimensional viscous shallow water system taking into account capillary effects, varying topography, and a molecular viscosity. However, in order to simulate realistic river flows, three-dimensional geometries and turbulence phenomena must be taken into account. Thus, the Reynolds-averaged Navier-Stokes equations (RANS) on an arbitrary three-dimensional domain are a more appropriate starting point for the derivation of simpler systems. In [4], Saleri *et al.* derived a two-dimensional viscous shallow water system from the three-dimensional RANS equations, taking into account a non-flat bottom, atmospheric pressure effects and considering a constant vertical eddy viscosity.

In this paper, we have chosen to proceed as in [6], extending the analysis to the three-dimensional RANS equations with anisotropic Reynolds tensor for free surface flows in arbitrary geometries. We propose a rigourous derivation of a section-averaged system, including the effects of eddy viscosity and friction. This derivation is also aimed at providing an adequate framework for the rigorous derivation of coupling between three- and one-dimensional free surface models. The equation system obtained allows to compute the free surface level of the flow as well as a section-averaged velocity. If applied to flows with rectangular cross-section, this system is similar to the classical section-averaged shallow water equations [10], except for the friction term. Indeed, our derivation shows that, in order to take into account effects up to the second order in the asymptotic parameter, the classical friction term should be corrected by a term which depends on the turbulent vertical viscosity. This conclusion is in good agreement with the one achieved by Gerbeau *et al.* in [6] for two-dimensional flows with constant viscosity over a flat bottom. Indeed, if the vertical viscosity is taken constant and the flow is homogeneous in the transversal direction, we retrieve the same friction correction as in [6]. However, our derivation provides the expression of the friction correction term in a more general case, which includes turbulent flows and three-dimensional arbitrary geometries. In particular, we compute the correction term associated to specific model for the vertical profile of turbulent velocity. Furthermore, for steady state open channel flows admitting analytic solutions of the three-dimensional as well as the simplified models, we show that the solutions computed including our correction term are much closer to those of three dimensional model than those of the standard shallow water model. The friction correction term can be easily included in section averaged models such as the one proposed by Deponti *ea* in [3]. Its use is also expected to ease the coupling of three- and one-dimensional free surface models in the framework of an integrated hydrological basin model.

In the first section of this paper we recall the three-dimensional RANS equations and the boundary conditions closing the problem. Then, we derive the section-averaged shallow water model in section 3 and in section 4 we give the expression of the friction correction term in the laminar and turbulent cases. Finally, in section 5, we compare the analytical solutions of the three-dimensional and the section-averaged models in the particular case of steady state open channel flows with rectangular cross-section, in order to show the accuracy gain achieved by adding the friction correction.

2 The Reynolds-averaged Navier-Stokes equations

2.1 The three-dimensional equations with boundary conditions

We consider the motion of an incompressible fluid with constant density $\rho > 0$, in a three-dimensional domain $\Omega_t = \Omega(t)$ with general transversal section $\omega(t) = \left\{(x, y) \in \mathbb{R}^2 \,/\, 0 \leq x \leq L,\, l_1(x,t) \leq y \leq l_2(x,t)\right\}$, where l_1 and l_2 are the time and space dependent transversal limits of the flow, and L its length. We assume the bottom to be fixed and impervious. We call η and b the functions describing the free surface and the bottom. The water height will be denoted by h, that is $h\,(x, y, t) = \eta\,(x, y, t) - b\,(x, y)$ respectively. The three-dimensional domain is then defined by $\Omega_t = \left\{(x, y, z) \in \mathbb{R}^3 \,/\, (x, y) \in \omega(t),\, b\,(x, y) \leq z \leq \eta\,(x, y, t)\right\}$ as illustrated in Figure 1.

The boundary of the domain Ω_t is denoted by $\partial\Omega_t$ and can be decomposed into four separate parts: the free surface $\Gamma_s(t)$, the bottom surface $\Gamma_b(t)$, the *inflow* boundary $\Gamma_{in}(t)$ and the *outflow* boundary $\Gamma_{out}(t)$.

The governing equations for the motion of the fluid are the incompressible Reynolds-Averaged Navier-Stokes (RANS) equations in Ω_t, valid for

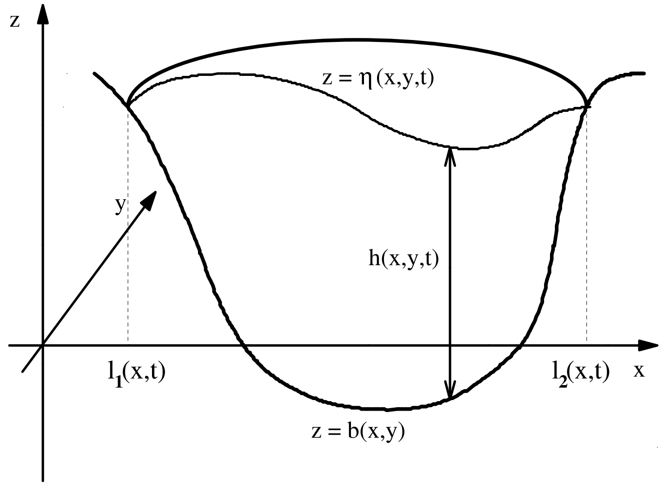

Figure 1: Three-dimensional domain

any $t \in (0, T]$, which can be written as follows:

$$\left\{ \begin{array}{rcl} \dfrac{d\boldsymbol{U}}{dt} - \operatorname{div}\left(\dfrac{1}{\rho}\boldsymbol{\sigma_T}\right) & = & \boldsymbol{f} + \boldsymbol{g}, \\ \operatorname{div}(\boldsymbol{U}) & = & 0, \end{array} \right. \tag{1}$$

where $\boldsymbol{U} = (u, v, w)^T$ is the total velocity of the fluid, $\boldsymbol{\sigma_T}$ is the physical tensor, $\boldsymbol{f} = (f_x, f_y, f_z)^T$ the sum of the external forces applied on the fluid, and $\boldsymbol{g} = (0, 0, -g)^T$ the gravity acceleration. We only consider Newtonian fluids, for which the tensor $\boldsymbol{\sigma_T}$ is written in the following way:

$$\boldsymbol{\sigma_T} \;=\; -p\mathbf{I} \;+\; \boldsymbol{\sigma}, \tag{2}$$

where p is the pressure and $\boldsymbol{\sigma}$ the stress tensor. We consider a turbulence model which is given through an anisotropic relationship between the stress tensor $\boldsymbol{\sigma}$ and the strain-rate tensor

$$\mathbb{D} = \boldsymbol{\nabla U} + (\boldsymbol{\nabla U})^T.$$

Following Levermore and Sammartino in [8], we take:

$$\boldsymbol{\sigma} \;=\; \begin{bmatrix} \boldsymbol{\sigma}_{11} & \mu_h \mathbb{D}_{12} & \mu_v \mathbb{D}_{13} \\ \mu_h \mathbb{D}_{21} & \boldsymbol{\sigma}_{22} & \mu_v \mathbb{D}_{23} \\ \mu_v \mathbb{D}_{31} & \mu_v \mathbb{D}_{32} & \mu_e \mathbb{D}_{33} \end{bmatrix}, \tag{3}$$

where

$$\boldsymbol{\sigma}_{11} = \mu_h(\mathbb{D}_{11} - \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22})) + \mu_e \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22}),$$

and

$$\boldsymbol{\sigma}_{22} = \mu_h(\mathbb{D}_{22} - \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22})) + \mu_e \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22}).$$

The positive coefficients μ_h, μ_v and μ_e are the eddy viscosities. They can be interpreted as the eddy viscosity relative to the horizontal shear motion, the eddy viscosity relative to the vertical shear motion, and the bulk viscosity relative to the expansion rate in the horizontal direction respectively.

The system is closed by suitable initial and boundary conditions. We denote by $\boldsymbol{n}_s$ the outward normal to the surface, wich depends on time:

$$\boldsymbol{n}_s = \frac{1}{\sqrt{1+|\boldsymbol{\nabla}\eta|^2}}(-\frac{\partial\eta}{\partial x}, -\frac{\partial\eta}{\partial y}, 1)^T,$$

and by $\boldsymbol{n}_b$ the outward normal to the bottom:

$$\boldsymbol{n}_b = \frac{1}{\sqrt{1+|\boldsymbol{\nabla}b|^2}}(\frac{\partial b}{\partial x}, \frac{\partial b}{\partial y}, -1)^T.$$

We choose $\boldsymbol{t}_b = (t_{b,1}, t_{b,2})^T$ a basis of the tangential surface to the bottom:

$$t_{b,1} = \frac{1}{\sqrt{1+|\frac{\partial b}{\partial x}|^2}}(1, 0, \frac{\partial b}{\partial x})^T,$$

and

$$t_{b,2} = \frac{1}{\sqrt{1+|\frac{\partial b}{\partial y}|^2}}(0, 1, \frac{\partial b}{\partial y})^T.$$

On the bottom we prescribe the kinematic condition traducing imperviousness,

$$\boldsymbol{U}\cdot\boldsymbol{n}_b \;=\; 0 \qquad \text{on} \quad \Gamma_b(t), \tag{4}$$

as well as a dynamic condition which accounts for friction,

$$(\frac{1}{\rho}\boldsymbol{\sigma_T}\cdot\boldsymbol{n}_b)\cdot\boldsymbol{t}_b \;=\; -\alpha|\boldsymbol{U}|\boldsymbol{U}\cdot\boldsymbol{t}_b \qquad \text{on} \quad \Gamma_b(t), \tag{5}$$

where $\alpha > 0$ is a dimensionless friction coefficient.

At the free surface, the velocity of the fluid is equal to the velocity of the free surface itself. This is expressed by the following kinematic condition:

$$\frac{\partial\eta}{\partial t} \;-\; \boldsymbol{U}\cdot\boldsymbol{n}_s \;=\; 0 \qquad\qquad \text{on} \quad \Gamma_s(t). \tag{6}$$

The dynamical condition at the free surface takes into account the atmospheric stress,

$$\frac{1}{\rho}\boldsymbol{\sigma_T}\cdot\boldsymbol{n}_s \;=\; -\frac{1}{\rho}p_a\boldsymbol{n}_s \quad \text{on} \quad \Gamma_s(t), \tag{7}$$

where p_a is the atmospheric pressure.

2.2 Adimensionalization of the system

Let us consider the following absolute scales: L for the total length, H for the height and U for the x-component of the velocity. We denote by ϵ the ratio between the vertical and the longitudinal scales:

$$\epsilon = \frac{H}{L}.$$

In addition we introduce the following dimensionless quantities:

$$\nu_h = \frac{\mu_h}{\rho\, U\, L}, \qquad \nu_v = \frac{\mu_v}{\rho\, U\, L}, \qquad \nu_e = \frac{\mu_e}{\rho\, U\, L}, \qquad G = \frac{H}{U^2} g, \qquad p_a = \frac{p_a}{U^2}.$$

We then have that the scale for time is L/U, for the vertical velocity $W = \epsilon\, U$, and for the pressure $P = U^2/\rho$. For the sake of simplicity we indicate again by u, v, w, p, η and b, respectively, velocity components, pressure, free surface and bottom elevation, after rescaling. Using these notations in (1) we obtain the following adimensionalized system, written as a function of the primitive unknowns u, v, w and p:

$$\left\{\begin{array}{rcl}
\dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uv}{\partial y} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} & = & \dfrac{\partial}{\partial x}\left((\nu_h + \nu_e)\dfrac{\partial u}{\partial x} - (\nu_h - \nu_e)\dfrac{\partial v}{\partial y}\right) \\[2ex]
& + & \dfrac{\partial}{\partial y}\left(\nu_h(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x})\right) + \dfrac{1}{\epsilon^2}\dfrac{\partial}{\partial z}\left(\nu_v\dfrac{\partial u}{\partial z}\right) + \dfrac{\partial}{\partial z}\left(\nu_v\dfrac{\partial w}{\partial x}\right), \\[2ex]
\dfrac{\partial v}{\partial t} + \dfrac{\partial uv}{\partial x} + \dfrac{\partial v^2}{\partial y} + \dfrac{\partial vw}{\partial z} + \dfrac{\partial p}{\partial y} & = & \dfrac{\partial}{\partial x}\left(\nu_h(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x})\right) \\[2ex]
& & + \dfrac{\partial}{\partial y}\left((\nu_h + \nu_e)\dfrac{\partial v}{\partial y} - (\nu_h - \nu_e)\dfrac{\partial u}{\partial x}\right) + \dfrac{1}{\epsilon^2}\dfrac{\partial}{\partial z}\left(\nu_v\dfrac{\partial v}{\partial z}\right) \\[2ex]
& & + \dfrac{\partial}{\partial z}\left(\nu_v\dfrac{\partial w}{\partial y}\right), \\[2ex]
\epsilon^2\left(\dfrac{\partial w}{\partial t} + \dfrac{\partial uw}{\partial x} + \dfrac{\partial vw}{\partial y} + \dfrac{\partial w^2}{\partial z}\right) + \dfrac{\partial p}{\partial z} & = & -G + \dfrac{\partial}{\partial x}\left(\nu_v\dfrac{\partial u}{\partial z} + \epsilon^2\nu_v\dfrac{\partial w}{\partial x}\right) \\[2ex]
& & + \dfrac{\partial}{\partial y}\left(\nu_v\dfrac{\partial v}{\partial z}\right) + \epsilon^2\dfrac{\partial}{\partial y}\left(\nu_v\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(2\nu_e\dfrac{\partial w}{\partial z}\right), \\[2ex]
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} & = & 0.
\end{array}\right. \tag{8}$$

Coherently, the rescaled boundary conditions are, on the free surface $\Gamma_s(t)$,

$$
\left\{
\begin{aligned}
&\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \ = \ w,\\
&\frac{\partial \eta}{\partial x}\left(p - (\nu_h+\nu_e)\frac{\partial u}{\partial x} + (\nu_h-\nu_e)\frac{\partial v}{\partial y}\right) - \frac{\partial \eta}{\partial y}\left(\nu_h(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\\
&\qquad\qquad\qquad +\nu_v\left(\frac{1}{\epsilon^2}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \ = \ p_a\frac{\partial \eta}{\partial x},\\
&-\frac{\partial \eta}{\partial x}\left(\nu_h\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial \eta}{\partial y}\left(p - (\nu_h+\nu_e)\frac{\partial v}{\partial y} + (\nu_h-\nu_e)\frac{\partial u}{\partial x}\right)\\
&\qquad\qquad\qquad +\nu_v\left(\frac{1}{\epsilon^2}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \ = \ p_a\frac{\partial \eta}{\partial y},\\
&-\frac{\partial \eta}{\partial x}\left(\nu_v\left(\frac{\partial u}{\partial z} + \epsilon^2\frac{\partial w}{\partial x}\right)\right) - \frac{\partial \eta}{\partial y}\left(\nu_v\left(\frac{\partial v}{\partial z} + \epsilon^2\frac{\partial w}{\partial y}\right)\right) - p\\
&\qquad\qquad\qquad\qquad + 2\nu_e\frac{\partial w}{\partial z} \ = \ -p_a,
\end{aligned}
\right.
\tag{9}
$$

and on the bottom $\Gamma_b(t)$,

$$
\left\{
\begin{aligned}
&u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} = w,\\
&\frac{\partial b}{\partial x}\left((\nu_h+\nu_e)\frac{\partial u}{\partial x} - (\nu_h-\nu_e)\frac{\partial v}{\partial y} - 2\nu_e\frac{\partial w}{\partial z}\right) + \frac{\partial b}{\partial y}\left(\nu_h\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right)\\
&\qquad\qquad + \left(\left(\frac{\partial b}{\partial x}\right)^2 - \frac{1}{\epsilon^2}\right)\left(\nu_v\left(\frac{\partial u}{\partial z} + \epsilon^2\frac{\partial w}{\partial x}\right)\right)\\
&\qquad\qquad + \ \frac{\partial b}{\partial x}\frac{\partial b}{\partial y}\left(\nu_v\left(\frac{\partial v}{\partial z} + \epsilon^2\frac{\partial w}{\partial y}\right)\right)\\
&\qquad\qquad = \ -\alpha\sqrt{u^2+v^2+\epsilon^2 w^2}\left(\frac{1}{\epsilon}u + \epsilon\frac{\partial b}{\partial x}w\right)N(b,\epsilon),\\
&\frac{\partial b}{\partial x}\left(\nu_h\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial b}{\partial y}\left((\nu_h+\nu_e)\frac{\partial v}{\partial y} - (\nu_h-\nu_e)\frac{\partial u}{\partial x} - 2\nu_e\frac{\partial w}{\partial z}\right)\\
&\qquad\qquad + \left(\left(\frac{\partial b}{\partial y}\right)^2 - \frac{1}{\epsilon^2}\right)\left(\nu_v\left(\frac{\partial v}{\partial z} + \epsilon^2\frac{\partial w}{\partial y}\right)\right)\\
&\qquad\qquad + \ \frac{\partial b}{\partial x}\frac{\partial b}{\partial y}\left(\nu_v\left(\frac{\partial u}{\partial z} + \epsilon^2\frac{\partial w}{\partial x}\right)\right)\\
&\qquad\qquad = \ -\alpha\sqrt{u^2+v^2+\epsilon^2 w^2}\left(\frac{1}{\epsilon}v + \epsilon\frac{\partial b}{\partial y}w\right)N(b,\epsilon).
\end{aligned}
\right.
\tag{10}
$$

where

$$N(b,\epsilon) = \sqrt{1+\epsilon^2(\frac{\partial b}{\partial x})^2+\epsilon^2(\frac{\partial b}{\partial y})^2}.$$

3 Derivation of the section-averaged shallow water model

3.1 Second order approximation in ϵ

In order to derive our section-averaged shallow water model, a number of approximations have to be performed. Firstly, we assume that the vertical eddy viscosity is first order with respect to the ratio between the vertical and longitudinal scales, that is,

$$\nu_v \;=\; \epsilon\,\nu_{v,0}, \tag{11}$$

where $\nu_{v,0}$ is a given positive quantity. This assumption can be justified by a simple dimensional analysis. Indeed, the eddy viscosity is homogeneous to a length times a velocity, and more precisely

$$\mu \;\sim\; l_m^2\,||\mathbb{D}||, \tag{12}$$

where l_m is the mixing length of the turbulent flow and $||\mathbb{D}||$ is the norm of the strain-rate tensor. When considering the vertical eddy viscosity, l_m is homogeneous to a height and the strain-rate tensor reduces to the vertical acceleration, then we conclude that

$$\mu_v \;\sim\; l_m^2\,||\frac{\partial \boldsymbol{U}}{\partial z}||. \tag{13}$$

Note that Prandtl's mixing length model – see for instance [10] – is based on this assumption. Adimensionalizing this expression of μ_v gives:

$$\hat{\mu_v} \;\sim\; H^2\,\hat{l_m}^2\sqrt{\frac{U^2}{H^2}\left((\frac{\partial \hat{u}}{\partial z})^2+(\frac{\partial \hat{v}}{\partial z})^2+(\epsilon\frac{\partial \hat{w}}{\partial z})^2\right)} \;\sim\; UH\,\hat{l_m}^2\,|\frac{\partial \hat{u}}{\partial z}|, \tag{14}$$

where the "hat" denotes here the adimensional variables. Thus

$$\nu_v \;=\; \frac{\hat{\mu_v}}{\rho UL} \;\sim\; \epsilon\,\frac{\hat{l_m}^2}{\rho}\,|\frac{\partial \hat{u}}{\partial z}| \;=\; O(\epsilon). \tag{15}$$

Moreover, the horizontal and bulk viscosities are of same order as the vertical eddy viscosity, and therefore we can write:

$$\nu_h \;=\; \epsilon\,\nu_{h,0}, \qquad \nu_e \;=\; \epsilon\,\nu_{e,0}, \tag{16}$$

where $\nu_{h,0}$ and $\nu_{e,0}$ are two given positive quantities. Finally, we assume a slow varying bathymetry in the longitudinal direction, as it has been done often in these derivations – see for instance [4] –, and we consider a constant atmospheric pressure, that is

$$\frac{\partial b}{\partial x} = O(\epsilon) \quad \text{and} \quad \boldsymbol{\nabla} p_a = 0. \tag{17}$$

Since our aim is to obtain a second order approximation with respect to ϵ of the three-dimensional system, we neglect quantities which of order $O(\epsilon^2)$. In this way, under the previous assumptions, (8) becomes:

$$
\left\{
\begin{array}{rcl}
\dfrac{\partial u}{\partial t}+\dfrac{\partial u^2}{\partial x}+\dfrac{\partial uv}{\partial y}+\dfrac{\partial uw}{\partial z}+\dfrac{\partial p}{\partial x} &=& \epsilon\,\dfrac{\partial}{\partial x}\left((\nu_{h,0}+\nu_{e,0})\dfrac{\partial u}{\partial x}-(\nu_{h,0}-\nu_{e,0})\dfrac{\partial v}{\partial y}\right)\\
&+& \epsilon\,\dfrac{\partial}{\partial y}\left(\nu_{h,0}(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x})\right)+\dfrac{1}{\epsilon}\,\dfrac{\partial}{\partial z}\left(\nu_{v,0}\dfrac{\partial u}{\partial z}\right)+\epsilon\,\dfrac{\partial}{\partial z}\left(\nu_{v,0}\dfrac{\partial w}{\partial x}\right),\\
\dfrac{\partial v}{\partial t}+\dfrac{\partial uv}{\partial x}+\dfrac{\partial v^2}{\partial y}+\dfrac{\partial vw}{\partial z}+\dfrac{\partial p}{\partial y} &=& \epsilon\,\dfrac{\partial}{\partial x}\left(\nu_{h,0}(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x})\right)\\
&+& \epsilon\,\dfrac{\partial}{\partial y}\left((\nu_{h,0}+\nu_{e,0})\dfrac{\partial v}{\partial y}-(\nu_{h,0}-\nu_{e,0})\dfrac{\partial u}{\partial x}\right)+\dfrac{1}{\epsilon}\,\dfrac{\partial}{\partial z}\left(\nu_{v,0}\dfrac{\partial v}{\partial z}\right)\\
&+& \epsilon\,\dfrac{\partial}{\partial z}\left(\nu_{v,0}\dfrac{\partial w}{\partial y}\right),\\
\dfrac{\partial p}{\partial z} &=& -G+\epsilon\,\dfrac{\partial}{\partial x}\left(\nu_{v,0}\dfrac{\partial u}{\partial z}\right)+\epsilon\,\dfrac{\partial}{\partial y}\left(\nu_{v,0}\dfrac{\partial v}{\partial z}\right)+\epsilon\,\dfrac{\partial}{\partial z}\left(2\nu_{e,0}\dfrac{\partial w}{\partial z}\right),\\
\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z} &=& 0.
\end{array}
\right.
\tag{18}
$$

together with boundary conditions on the free surface $\Gamma_s(t)$,

$$
\left\{
\begin{array}{l}
\dfrac{\partial \eta}{\partial t}+u\dfrac{\partial \eta}{\partial x}+v\dfrac{\partial \eta}{\partial y} = w,\\
\dfrac{\partial \eta}{\partial x}\left(p-\epsilon\,(\nu_{h,0}+\nu_{e,0})\dfrac{\partial u}{\partial x}+\epsilon\,(\nu_{h,0}-\nu_{e,0})\dfrac{\partial v}{\partial y}\right)-\dfrac{\partial \eta}{\partial y}\left(\epsilon\,\nu_{h,0}(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y})\right)\\
\qquad +\dfrac{1}{\epsilon}\,\nu_{v,0}\dfrac{\partial u}{\partial z}+\epsilon\,\nu_{v,0}\dfrac{\partial w}{\partial x} = p_a\dfrac{\partial \eta}{\partial x},\\
-\dfrac{\partial \eta}{\partial x}\left(\epsilon\,\nu_{h,0}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right)+\dfrac{\partial \eta}{\partial y}\left(p-\epsilon\,(\nu_{h,0}+\nu_{e,0})\dfrac{\partial v}{\partial y}+\epsilon\,(\nu_{h,0}-\nu_{e,0})\dfrac{\partial u}{\partial x}\right)\\
\qquad +\dfrac{1}{\epsilon}\,\nu_{v,0}\dfrac{\partial v}{\partial z}+\epsilon\,\nu_{v,0}\dfrac{\partial w}{\partial y} = p_a\dfrac{\partial \eta}{\partial y},\\
-\dfrac{\partial \eta}{\partial x}(\epsilon\,\nu_{v,0}\dfrac{\partial u}{\partial z})-\dfrac{\partial \eta}{\partial y}(\epsilon\,\nu_{v,0}\dfrac{\partial v}{\partial z})-p+2\epsilon\nu_{e,0}\dfrac{\partial w}{\partial z} = -p_a,
\end{array}
\right.
\tag{19}
$$

and on the bottom $\Gamma_b(t)$,

$$
\left\{
\begin{aligned}
& u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} = w, \\
& \frac{\partial b}{\partial x}\left(\epsilon\,(\nu_{h,0}+\nu_{e,0})\frac{\partial u}{\partial x} - \epsilon\,(\nu_{h,0}-\nu_{e,0})\frac{\partial v}{\partial y} - 2\epsilon\,\nu_{e,0}\frac{\partial w}{\partial z}\right) \\
& \qquad + \frac{\partial b}{\partial y}\left(\epsilon\,\nu_{h,0}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right) - \frac{1}{\epsilon}\,\nu_{v,0}\frac{\partial u}{\partial z} + \epsilon\,\nu_{v,0}\frac{\partial w}{\partial x} \\
& \qquad\qquad = -\frac{\alpha}{\epsilon}\sqrt{u^2+v^2}u, \\
& \frac{\partial b}{\partial x}\left(\epsilon\nu_{h,0}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right) + \frac{\partial b}{\partial y}\left(\epsilon\,(\nu_{h,0}+\nu_{e,0})\frac{\partial v}{\partial y} - \epsilon\,(\nu_{h,0}-\nu_{e,0})\frac{\partial u}{\partial x}\right. \\
& \qquad \left. - 2\epsilon\,\nu_{e,0}\frac{\partial w}{\partial z}\right) + \left((\frac{\partial b}{\partial y})^2 - \frac{1}{\epsilon^2}\right)\left(\epsilon\,\nu_{v,0}\frac{\partial v}{\partial z}\right) \\
& \qquad\qquad = -\alpha|\boldsymbol{u}|\left(\frac{1}{\epsilon}\,v + \epsilon\,\frac{\partial b}{\partial y}w\right).
\end{aligned}
\right.
\tag{20}
$$

Notice that, neglecting terms in $O(\epsilon)$ in $(20)_2$ and $(20)_3$, we obtain the classical boundary condition on the bottom (see e.g. [2]) :

$$\nu_{v,0}\frac{\partial u}{\partial z} \;=\; -\,\alpha\sqrt{u^2+v^2}\;u\,.$$

3.2 Vertical integration of the equations

Let us now vertically-integrate the momentum equation $(18)_1$ for u between the bottom and the free surface. For any three-dimensional variable f, we denote with $\bar{f}$ the average along the vertical direction,

$$\bar{f}(x,y,t) \;=\; \frac{1}{h(x,y,t)}\int_b^\eta f(x,y,z,t)dz.$$

Making use of the Leibnitz rule, $(18)_1$ becomes:

$$
\begin{aligned}
&\frac{\partial h\bar{u}}{\partial t} + \frac{\partial h\bar{u^2}}{\partial x} + \frac{\partial h\bar{uv}}{\partial y} + \frac{\partial h\bar{p}}{\partial x} + \; uw(\eta) - uw(b) - \frac{\partial \eta}{\partial t} u(\eta) - \frac{\partial \eta}{\partial x} u^2(\eta) \\
&\quad + \quad \frac{\partial b}{\partial x} u^2(b) - \frac{\partial \eta}{\partial y} uv(\eta) + \frac{\partial b}{\partial y} uv(b) - \frac{\partial \eta}{\partial x} p(\eta) + \frac{\partial b}{\partial x} p(b) \\
&\quad = \quad \frac{\partial}{\partial x} \int_b^\eta \left(\epsilon\,(\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - \epsilon\,(\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) dz \\
&\quad + \quad \frac{\partial}{\partial y} \int_b^\eta \left(\epsilon\, \nu_{h,0} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \right) dz \\
&\quad - \quad \frac{\partial \eta}{\partial x} \left(\epsilon\,(\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - \epsilon\,(\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) \Big|_{z=\eta} \\
&\quad - \quad \frac{\partial \eta}{\partial y} \left(\epsilon\, \nu_{h,0} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \right) \Big|_{z=\eta} \\
&\quad + \quad \frac{\partial b}{\partial x} \left(\epsilon\,(\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - \epsilon\,(\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) \Big|_{z=b} \\
&\quad + \quad \frac{\partial b}{\partial y} \left(\epsilon\, \nu_{h,0} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \right) \Big|_{z=b} \\
&\quad + \quad \frac{1}{\epsilon}\; \nu_{v,0} \frac{\partial u}{\partial z}\Big|_{\eta - b} \; + \; \epsilon\; \nu_{v,0} \frac{\partial w}{\partial x}\Big|_{\eta - b}.
\end{aligned}
$$

Using now the kinematic boundary conditions $(19)_1$ and $(20)_1$, as well as $(19)_2$ and $(20)_2$, the equation reduces to:

$$
\begin{aligned}
&\frac{\partial h\bar{u}}{\partial t} + \frac{\partial h\bar{u^2}}{\partial x} + \frac{\partial h\bar{uv}}{\partial y} + \frac{\partial h\bar{p}}{\partial x} = \; p_a \frac{\partial \eta}{\partial x} \; - \; \frac{\alpha |\boldsymbol{u}(b)|}{\epsilon} u(b) \; - \; \frac{\partial b}{\partial x} p(b) \\
&\quad + \quad 2\,\epsilon\; \frac{\partial b}{\partial x} (\nu_{e,0} \frac{\partial w}{\partial z})\Big|_{z=b} \; + \; \epsilon\; \frac{\partial}{\partial x} \int_b^\eta \left((\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - (\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) dz \\
&\quad + \quad \epsilon\; \frac{\partial}{\partial y} \int_b^\eta \left(\nu_{h,0} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \right) dz \,.
\end{aligned}
\tag{21}
$$

On the other hand, by vertically-integrating equation $(18)_3$ between z and the free surface, we obtain:

$$
\begin{aligned}
p \;&=\; p(\eta) \;+\; G(\eta - z) \;-\; \epsilon \int_z^\eta \left(\frac{\partial}{\partial x}(\nu_{v,0} \frac{\partial u}{\partial \delta}\,) + \frac{\partial}{\partial y}(\nu_{v,0} \frac{\partial v}{\partial \delta}) \right) d\delta \\
&- \;\; 2\,\epsilon\;(\nu_{e,0} \frac{\partial w}{\partial z})\Big|_{z=\eta} \;+\; 2\,\epsilon\,\nu_{e,0} \frac{\partial w}{\partial z} \\
&=\; p(\eta) \;+\; G(\eta - z) \;-\; \epsilon\; \frac{\partial}{\partial x} \int_z^\eta \left(\nu_{v,0} \frac{\partial u}{\partial \delta} \right) d\delta \;+\; \epsilon\; \frac{\partial \eta}{\partial x} \left(\nu_{v,0} \frac{\partial u}{\partial z} \right) \Big|_{z=\eta} \\
&- \epsilon\; \frac{\partial}{\partial y} \int_z^\eta \left(\nu_{v,0} \frac{\partial v}{\partial \delta} \right) d\delta \;+\; \epsilon\; \frac{\partial \eta}{\partial y} \left(\nu_{v,0} \frac{\partial v}{\partial z} \right) \Big|_{z=\eta} \\
&- 2\,\epsilon\;(\nu_{e,0} \frac{\partial w}{\partial z})\Big|_{z=\eta} \;+\; 2\,\epsilon\,\nu_{e,0} \frac{\partial w}{\partial z}\,.
\end{aligned}
$$

Applying now the dynamic condition $(19)_4$ at the free surface, we deduce the following expression for the pressure:

$$
\begin{aligned}
p \;&=\; p_a \;+\; G(\eta - z) \;+\; 2\,\epsilon\,\nu_{e,0} \frac{\partial w}{\partial z} \;-\; \epsilon\; \frac{\partial}{\partial x} \int_z^\eta \left(\nu_{v,0} \frac{\partial u}{\partial \delta} \right) d\delta \\
&-\; \epsilon\; \frac{\partial}{\partial y} \int_z^\eta \left(\nu_{v,0} \frac{\partial v}{\partial \delta} \right) d\delta \;+\; O(\epsilon^2).
\end{aligned}
\tag{22}
$$

Note that the pressure at the bottom is given by

$$
\begin{aligned}
p(b) \;&=\; p_a \;+\; Gh \;+\; 2\,\epsilon\,(\nu_{e,0} \frac{\partial w}{\partial z})\Big|_{z=b} \;-\; \epsilon\; \frac{\partial}{\partial x} \int_b^\eta \left(\nu_{v,0} \frac{\partial u}{\partial z} \right) dz \\
&-\; \epsilon\; \frac{\partial}{\partial y} \int_z^\eta \left(\nu_{v,0} \frac{\partial v}{\partial \delta} \right) d\delta \;+\; O(\epsilon^2),
\end{aligned}
\tag{23}
$$

and therefore, recalling that $\dfrac{\partial b}{\partial x} = 0(\epsilon)$, we can conclude

$$
\frac{\partial b}{\partial x} p(b) \;=\; p_a \frac{\partial b}{\partial x} \;+\; Gh\, \frac{\partial b}{\partial x} + O(\epsilon^2). \tag{24}
$$

Let us now vertically-integrate this expression of the pressure from the bottom to the surface:

$$
\begin{aligned}
h\bar{p} \;&=\; hp_a \;+\; G\frac{h^2}{2} \;+\; 2\,\epsilon\int_b^\eta (\nu_{e,0}\frac{\partial w}{\partial z})dz \\
&\quad -\,\epsilon\int_b^\eta \left(\frac{\partial}{\partial x}\int_z^\eta (\nu_{v,0}\frac{\partial u}{\partial \delta})d\delta + \frac{\partial}{\partial y}\int_z^\eta (\nu_{v,0}\frac{\partial v}{\partial \delta})d\delta\right) dz \\
&=\; hp_a \;+\; G\frac{h^2}{2} \;+\; 2\,\epsilon\int_b^\eta (\nu_{e,0}\frac{\partial w}{\partial z})dz \\
&\quad -\;\; \epsilon\;\left(\frac{\partial}{\partial x}\int_b^\eta \int_z^\eta (\nu_{v,0}\frac{\partial u}{\partial \delta})d\delta dz + \frac{\partial}{\partial y}\int_b^\eta \int_z^\eta (\nu_{v,0}\frac{\partial v}{\partial \delta})d\delta dz\right) \\
&\quad +\;\; \epsilon\;\left(\frac{\partial b}{\partial x}\int_b^\eta (\nu_{v,0}\frac{\partial u}{\partial z})dz + \frac{\partial b}{\partial y}\int_b^\eta (\nu_{v,0}\frac{\partial v}{\partial z})dz\right) \\
&=\;\; hp_a \;+\; G\frac{h^2}{2} \;+\; 2\,\epsilon\int_b^\eta (\nu_{e,0}\frac{\partial w}{\partial z})dz \\
&\qquad -\;\; \epsilon\;\left(\frac{\partial}{\partial x}\int_b^\eta \int_z^\eta (\nu_{v,0}\frac{\partial u}{\partial \delta})d\delta dz + \frac{\partial}{\partial y}\int_b^\eta \int_z^\eta (\nu_{v,0}\frac{\partial v}{\partial \delta})d\delta dz\right)\;.
\end{aligned}
\tag{25}
$$

We thus have

$$
\begin{aligned}
\frac{\partial h\bar{u}}{\partial t} \;+\; \frac{\partial h\bar{u^2}}{\partial x} \;+\; \frac{\partial h\bar{uv}}{\partial y} \;+\; G\frac{\partial}{\partial x}\frac{h^2}{2} \;&=\; -\;\frac{\alpha|\boldsymbol{u}(b)|}{\epsilon}u(b) \;-\; Gh\frac{\partial b}{\partial x} \\
&\quad -\;\; 2\,\epsilon\,\frac{\partial}{\partial x}\left(\int_b^\eta (\nu_{e,0}\frac{\partial w}{\partial z})dz\right) \;+\; \epsilon\;\frac{\partial}{\partial y}\int_b^\eta \left(\nu_{h,0}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right) dz \\
&\quad +\;\; \epsilon\;\frac{\partial}{\partial x}\int_b^\eta \left((\nu_{h,0}+\nu_{e,0})\frac{\partial u}{\partial x} - (\nu_{h,0}-\nu_{e,0})\frac{\partial v}{\partial y}\right) dz \\
&\quad +\;\; \epsilon\;\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\int_b^\eta \int_z^\eta (\nu_{v,0}\frac{\partial u}{\partial \delta})d\delta dz + \frac{\partial}{\partial y}\int_b^\eta \int_z^\eta (\nu_{v,0}\frac{\partial v}{\partial \delta})d\delta dz\right)\;.
\end{aligned}
\tag{26}
$$

An analogous equation can be obtained by vertically integrating the continuity equation $(18)_4$:

$$
\frac{\partial h}{\partial t} + \frac{\partial h\bar{u}}{\partial x} + \frac{\partial h\bar{v}}{\partial y} \;=\; 0. \tag{27}
$$

3.3 Section averaged equations

Since the equations are now vertically-integrated, they are defined on $\omega(x,t)\times I$, where

$$\omega(x,t) = \{\ (x,y)\ /\ x \in [x_1,x_2] \text{ and } y \in [l_1(x,t), l_2(x,t)\ \}.$$

We can therefore integrate them along the y-axis between $l_1(x,t)$ and $l_2(x,t)$. In addition we point out that, for any scalar quantity f,

$$\left(\int_b^\eta f dz\right)\Big|_{y=l_1} \;=\; \left(\int_b^\eta f dz\right)\Big|_{y=l_2} \;=\; 0. \tag{28}$$

This assumption is justified in the case of a natural river, whose depth tends to zero as the banks are approached. Note however that we can retrieve the same section averaged model under the hypothesis that

$$\frac{\partial l_1}{\partial x} \;\equiv\; O(\epsilon^2) \quad \text{and} \quad \frac{\partial l_2}{\partial x} \;\equiv\; O(\epsilon^2), \tag{29}$$

as happens for instance in straight or mildly curved channels. For the sake of clarity we do only report the derivation in the first case, that is with hypothesis (28).

We denote

$$\begin{aligned} A(x,t) \;&=\; \int_{l_1}^{l_2} h(x,y,t)dy, \\ \bar{\bar{f}}(x,y,t) \;&=\; \frac{1}{A(x,t)}\int_{l_1}^{l_2}\int_b^\eta f(x,y,z,t)\,dzdy, \end{aligned}$$

$$Q(x,t) \;=\; \int_{l_1}^{l_2}\int_b^\eta u(x,y,z,t)\,dzdy \;=\; A(x,t)\bar{\bar{u}}(x,y,t).$$

Let us first integrate the momentum equation (26) on u. Using the Leibnitz rule and (28), we obtain:

$$\begin{aligned} \frac{\partial(A\bar{\bar{u}})}{\partial t} \;+\; \frac{\partial(A\,\overline{\overline{u^2}})}{\partial x} \;+\; G\int_{l_1}^{l_2}\frac{\partial}{\partial x}\left(\frac{h^2}{2}\right)dy \;&=\; -\,\frac{1}{\epsilon}\int_{l_1}^{l_2}\alpha|\boldsymbol{u}(b)|u(b)dy \\ -\quad & G\int_{l_1}^{l_2} h\frac{\partial b}{\partial x}dy \;-\; 2\,\epsilon\,\frac{\partial}{\partial x}\left(\int_{l_1}^{l_2}\int_b^\eta(\nu_{e,0}\frac{\partial w}{\partial z})dzdy\right) \\ +\quad & \epsilon\,\frac{\partial}{\partial x}\int_{l_1}^{l_2}\int_b^\eta\left((\nu_{h,0}+\nu_{e,0})\frac{\partial u}{\partial x} - (\nu_{h,0}-\nu_{e,0})\frac{\partial v}{\partial y}\right)dzdy \\ +\quad & \epsilon\,\frac{\partial^2}{\partial x^2}\left(\int_{l_1}^{l_2}\int_b^\eta\int_z^\eta(\nu_{v,0}\frac{\partial u}{\partial \delta})d\delta\,dz\,dy\right). \end{aligned} \tag{30}$$

Note that

$$G\int_{l_1}^{l_2} \frac{\partial}{\partial x}\left(\frac{h^2}{2}\right) dy \;=\; G\int_{l_1}^{l_2} \frac{\partial h}{\partial x} h dy \;=\; G\int_{l_1}^{l_2} \left(\frac{\partial \eta}{\partial x} h - \frac{\partial b}{\partial x} h\right) dy \,.$$

Note also that, from the continuity equation,

$$-\,2\,\epsilon\,\frac{\partial}{\partial x}\left(\int_{l_1}^{l_2}\int_{b}^{\eta} (\nu_{e,0}\frac{\partial w}{\partial z}) dz dy\right) \;=\; 2\,\epsilon\,\frac{\partial}{\partial x}\left(\int_{l_1}^{l_2}\int_{b}^{\eta} \nu_{e,0}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y})\, dz dy\right)\;,$$

and then, using this last expression in the right-hand side of (30), we obtain the following section averaged momentum equation:

$$\begin{aligned}
\frac{\partial Q}{\partial t} \;+\; \frac{\partial \tilde{Q}^2}{\partial x} \;+\; G\int_{l_1}^{l_2} h\frac{\partial \eta}{\partial x} dy \;&=\; -\,\frac{1}{\epsilon}\int_{l_1}^{l_2} \alpha|\boldsymbol{u}(b)|u(b)dy \\
&+\;\; \epsilon\,\frac{\partial}{\partial x}\int_{l_1}^{l_2}\int_{b}^{\eta}\left(\nu_{h,0}(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y})\right) dz dy \\
&+\;\; 3\,\epsilon\,\frac{\partial}{\partial x}\int_{l_1}^{l_2}\int_{b}^{\eta}\left(\nu_{e,0}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y})\right) dz dy \\
&+\;\; \epsilon\,\frac{\partial^2}{\partial x^2}(\int_{l_1}^{l_2}\int_{b}^{\eta}\int_{z}^{\eta}(\nu_{v,0}\frac{\partial u}{\partial \delta})\, d\delta\, dz\, dy)\,,
\end{aligned} \tag{31}$$

where

$$\tilde{Q} \;=\; \left(\int_{l_1}^{l_2}\int_{b}^{\eta} u^2(x,y,z,t)\, dz dy\right)^{1/2} .$$

Denoting by β the momentum correction coefficient (or Boussinesq coefficient)

$$\beta \;=\; \frac{1}{A}\int_{l_1}^{l_2}\int_{b}^{\eta} \frac{u^2}{\bar{\bar{u}}^2}\, dz\, dy,$$

we have that

$$\frac{\partial \tilde{Q}^2}{\partial x} \;=\; \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right).$$

The integration of the continuity equation $(27)_4$ gives

$$\frac{\partial A}{\partial t} \;+\; \frac{\partial Q}{\partial x} \;=\; 0, \tag{32}$$

that is the classical continuity equation of the one-dimensional open channel equations.

3.4 Asymptotic analysis of the section-averaged equations

We now go back to the three-dimensional equations in order to model the friction term and show that we can neglect the last viscous term in the right-hand side of the momentum equation (31).

From the three-dimensional momentum equation $(18)_1$ we deduce that

$$\frac{\partial}{\partial z}\left(\nu_{v,0}\frac{\partial u}{\partial z}\right) \;=\; O(\epsilon).$$

In addition, boundary condition $(19)_2$ indicates that $\dfrac{\partial}{\partial z}\left(\nu_{v,0}\dfrac{\partial u}{\partial z}\right) = O(\epsilon)$ at the free surface, from which we conclude that

$$\nu_{v,0}\frac{\partial u}{\partial z} = O(\epsilon) \quad \text{on } \Omega_t, \tag{33}$$

and thus

$$u(x,y,z,t) \;=\; \bar{u}(x,y,t) + O(\epsilon). \tag{34}$$

Equation (33) has two important consequences. First, it shows that the friction term $\alpha|\boldsymbol{u}|$ is necessarily also of the first order in ϵ. Indeed, from boundary condition $(20)_2$, we have that $\nu_{v,0}\dfrac{\partial u}{\partial z} \;=\; \alpha|\boldsymbol{u}|u \;+\; O(\epsilon^2)$. Thus, since $\nu_{v,0}\dfrac{\partial u}{\partial z} = O(\epsilon)$ on Ω_t and u is independent of ϵ, we have that $\alpha|\boldsymbol{u}| = O(\epsilon)$. In the following we will thus assume that

$$\alpha|\boldsymbol{u}| = \epsilon\alpha_0. \tag{35}$$

On the other hand equation (33) shows that the third viscous term in the momentum equation (31) is second order in ϵ.

Furthermore, from (22) we know that

$$p(x,y,z,t) \;=\; p_a + G(\eta - z) + O(\epsilon). \tag{36}$$

Using now (34) and (36) in the three-dimensional momentum equation $(18)_1$ we can write:

$$\begin{aligned}
\frac{1}{\epsilon}\frac{\partial}{\partial z}\left(\nu_{v,0}\frac{\partial u}{\partial z}\right) &= \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} + O(\epsilon) \\
&= \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}w}{\partial z} + G\frac{\partial \eta}{\partial x} + O(\epsilon) \\
&= \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + G\frac{\partial \eta}{\partial x} + O(\epsilon).
\end{aligned} \tag{37}$$

Note that for the last step we use the fact that, from continuity,

$$\bar{u}\,\frac{\partial w}{\partial z} \;=\; -\bar{u}\,(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}) \;=\; -\bar{u}\,(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y})+O(\epsilon).$$

On the other hand the vertically-integrated momentum equation (26) gives

$$\begin{aligned} & h\left\{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+G\frac{\partial h}{\partial x}\right\}+\bar{u}\left\{\frac{\partial h}{\partial t}+\frac{\partial h\bar{u}}{\partial x}+\frac{\partial h\bar{v}}{\partial y}\right\} \\ & \qquad = \;-\;\alpha_0 u(b)\;+\;O(\epsilon), \end{aligned}$$

and using the vertically-averaged continuity equation (27),

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+G\frac{\partial h}{\partial x} \;=\; -\;\frac{\alpha_0 u(b)}{h}\;+\;O(\epsilon).$$

Replacing this expression in (37) we have that

$$\frac{1}{\epsilon}\frac{\partial}{\partial z}\left(\nu_{v,0}\frac{\partial u}{\partial z}\right) \;=\; -\;\frac{\alpha_0 u(b)}{h}\;+\;O(\epsilon).$$

Let us now integrate this expression from the bottom b to z:

$$\frac{1}{\epsilon}\nu_{v,0}\frac{\partial u}{\partial z} \;=\; \frac{1}{\epsilon}(\nu_{v,0}\frac{\partial u}{\partial z})_{|_{z=b}} \;-\; \frac{\alpha_0(z-b)}{h}u(b)\;+\;O(\epsilon).$$

Using boundary condition $(20)_2$ we get:

$$\frac{1}{\epsilon}\nu_{v,0}\frac{\partial u}{\partial z} \;=\; \alpha_0 u(b)\left(1-\frac{z-b}{h}\right)\;+\;O(\epsilon),$$

so that

$$\frac{\partial u}{\partial z} \;=\; \epsilon\alpha_0 u(b)\left(\frac{\eta-z}{h\,\nu_{v,0}}\right)\;+\;O(\epsilon^2).$$

We vertically-integrate again this expression from the bottom b to z, yielding

$$\begin{aligned} u \;&=\; u(b)+\epsilon\alpha_0 u(b)\int_b^z \frac{\eta-\delta}{h\,\nu_{v,0}}\,d\delta\;+\;O(\epsilon^2) \\ &=\; u(b)\left(1+\frac{\epsilon\alpha_0}{h}\int_b^z\frac{\eta-\delta}{\nu_{v,0}}\,d\delta\right)\;+\;O(\epsilon^2). \end{aligned} \tag{38}$$

Integrating now on the vertical and dividing by the water height h we obtain:

$$\bar{u} \;=\; u(b)\,(1+\epsilon\,\alpha_0\,\bar{\nu}_{v,0})\;+\;O(\epsilon^2)\,, \tag{39}$$

where for the sake of simplicity we have denoted

$$\bar{\nu}_{v,0}=\frac{1}{h^2}\int_b^\eta\int_b^z\frac{\eta-\delta}{\nu_{v,0}}\,d\delta\,dz.$$

Equation (39) leads us to two important results. On one hand, it gives us some information about the Boussinesq coefficient β. Indeed, from (38) we deduce that

$$u^2 \;=\; u^2(b)\left(1 + \frac{2\epsilon\alpha_0}{h}\int_b^z \frac{\eta-\delta}{\nu_{v,0}}\,d\delta\right) \;+\; O(\epsilon^2),$$

and therefore that

$$\begin{aligned}\frac{1}{h}\int_b^\eta u^2 dz \;&=\; u^2(b)\,(1+2\epsilon\alpha_0\bar{\nu}_{v,0}) \;+\; O(\epsilon^2)\\ &=\; \bar{u}^2\,(1-2\epsilon\alpha_0\bar{\nu}_{v,0})(1+2\epsilon\alpha_0\bar{\nu}_{v,0}) \;+\; O(\epsilon^2)\\ &=\; \bar{u}^2 \;+\; O(\epsilon^2).\end{aligned}$$

Thus

$$\bar{u^2} \;=\; \bar{u}^2 \;+\; O(\epsilon^2), \tag{40}$$

which means that, up to the second order in ϵ, the Boussinesq coefficient β only depends on the transversal variations of the velocity u (and not on its vertical variations). Indeed:

$$\beta \;=\; \frac{1}{A}\int_{l_1}^{l_2}\int_b^\eta \frac{u^2}{\bar{\bar{u}}^2}\,dzdy \;=\; A\frac{\displaystyle\int_{l_1}^{l_2} h\bar{u}^2 dy}{\left(\displaystyle\int_{l_1}^{l_2} h\bar{u}\,dy\right)^2} + O(\epsilon^2).$$

On the other hand equation (39) allows to model the friction term. Indeed, we have now the following expression of the velocity on the bottom with respect to the vertically-averaged velocity $\bar{u}$:

$$u(b) \;=\; \frac{\bar{u}}{1+\epsilon\alpha_0\bar{\nu}_{v,0}} \;+\; O(\epsilon^2) \;=\; \bar{u}\;(1-\epsilon\alpha_0\bar{\nu}_{v,0}) \;+\; O(\epsilon^2). \tag{41}$$

However, we recall that

$$\alpha_0 = \frac{\alpha|\boldsymbol{u}(b)|}{\epsilon}.$$

The friction coefficient α_0 depends itself on the value of the velocity at the bottom, therefore expression (41) is unsatisfactory for the purpose of expressing $u(b)$ in terms of $\bar{u}$. To overcome this difficulty we use the following approximation of $u(b)$ to the second order in ϵ given by (41):

$$u(b) \;=\; \bar{u}\;(1-\epsilon\alpha_0\bar{\nu}_{v,0}) + O(\epsilon^2).$$

Since ϵ is very small we can assume that $|1-\epsilon\alpha_0\bar{\nu}_{v,0}|$ is positive and therefore:

$$|u(b)| \;=\; |\bar{u}|\;(1-\epsilon\alpha_0\bar{\nu}_{v,0}) + O(\epsilon^2).$$

We then approximate $|\boldsymbol{u}(b)|$ with $|u(b)|$ in α_0, so that we can write:

$$\alpha_0 \approx \frac{\alpha}{\epsilon}|u(b)| \approx \frac{\alpha}{\epsilon}|\bar{u}|(1-\alpha|\bar{u}|\bar{\nu}_{v,0}).$$

Thus

$$\alpha_0(1+\alpha\bar{\nu}_{v,0}|\bar{u}|) \approx \frac{\alpha}{\epsilon}|\bar{u}| + O(\epsilon^2).$$

Neglecting the $O(\epsilon^2)$ term, we finally obtain an expression of α_0 which is independent of the velocity at the bottom:

$$\alpha_0 \;\approx\; \frac{\alpha|\bar{u}|}{\epsilon\,(1+\alpha\bar{\nu}_{v,0}|\bar{u}|)}. \tag{42}$$

Using this approximation of α_0, together with expression (41), we can approximate the friction term in (31) as follows:

$$-\frac{1}{\epsilon}\int_{l_1}^{l_2} \alpha\,|u(b)|\,u(b)dy \;\approx\; -\frac{1}{\epsilon}\int_{l_1}^{l_2} \frac{\alpha\,|\bar{u}|}{1+2\,\alpha\,\bar{\nu}_{v,0}\,|\bar{u}|}\;\bar{u}\;dy\;. \tag{43}$$

In this way we have overcome the initial difficulty and we use expression (43) to model the friction term in the momentum equation (31).

3.5 The section averaged shallow water model

We have derived a section-averaged shallow water model which is an approximation of the second order in ϵ of the initial three-dimensional free surface flow problem (1) with boundary conditions (4)-(7). Switching to the dimensional variables, this model writes:

$$\left\{\begin{aligned}
&\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + g\int_{l_1}^{l_2} h\frac{\partial \eta}{\partial x}dy \;=\; -\int_{l_1}^{l_2}\frac{\alpha\,|\bar{u}|}{1+c_\alpha}\;\bar{u}\;dy\\
&\qquad\qquad + \;3\,\frac{\partial}{\partial x}\left(\int_{l_1}^{l_2}\int_{b}^{\eta}\frac{\mu_e}{\rho}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dzdy\right)\\
&\qquad\qquad + \;\frac{\partial}{\partial x}\left(\int_{l_1}^{l_2}\int_{b}^{\eta}\frac{\mu_h}{\rho}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dzdy\right)\\
&\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} \;=\; 0,
\end{aligned}\right. \tag{44}$$

where

$$c_\alpha \;=\; \frac{2\,\alpha\,|\bar{u}|}{h^2}\int_b^\eta\int_b^z \frac{\rho\,(\eta-\delta)}{\mu_v}d\delta\,dz \tag{45}$$

acts as a correction to the classical one-dimensional friction term, and β is the momentum correction coefficient

$$\beta \;=\; A\frac{\displaystyle\int_{l_1}^{l_2} h\bar{u}^2 dy}{(\displaystyle\int_{l_1}^{l_2} h\bar{u}\,dy\,)^2}. \tag{46}$$

This model results of a direct, rigorous asymptotic derivation from the three-dimensional free surface flow equations. In addition, this derivation is very general since it is valid for flows with arbitrary cross-section and non-constant, turbulent viscosity. Thus, we expect that the coupling of such a reduced model to a three-dimensional model to be easier and yield better results.

4 Computation of the corrected friction term

In this section we give an explicit expression of the friction correction to use in the laminar case and with a parabolic turbulence model for the vertical eddy viscosity.

4.1 The laminar case

We first consider the case where a constant vertical viscosity μ_v is used. Note that in order to be consistent with our analysis, its adimensional value $\nu_v = \dfrac{\mu_v}{\rho U L}$ must be $O(\epsilon)$. In that case we have that

$$\int_b^\eta \int_b^z \frac{\rho\,(\eta-\delta)}{\mu_v} d\delta\, dz \;=\; \frac{\rho}{\mu_v}\frac{h^3}{3},$$

and therefore the correction (45) of the friction term is:

$$c_\alpha = \frac{2}{3}\frac{\rho}{\mu_v}\alpha\, h\, |\bar{u}|. \tag{47}$$

Note that in this case we retrieve a friction correction term which is very similar to the one presented by Gerbeau *et al.* in [6], which is :

$$c_\kappa = \frac{1}{3}\frac{\rho}{\mu_v}\kappa\, h, \tag{48}$$

where κ is the friction coefficient taken in their model. In our analysis, we have taken $\kappa = \alpha|\boldsymbol{u}(b)|$, where α is a dimensionless friction coefficient and $|\boldsymbol{u}(b)|$ is the module of the horizontal velocity on the bottom. That explains the difference between both friction correction terms.

If the flow is homogeneous in the y-direction and has a rectangular-cross-section, $\bar{u} = \dfrac{Q}{A}$ and the friction term in (44) writes

$$-\,\frac{\alpha\,|Q|}{h^2\, l\left(1 + \dfrac{2}{3}\dfrac{\rho}{\mu_v}\alpha\,\dfrac{|Q|}{l}\right)}\; Q\;. \tag{49}$$

4.2 Parabolic model for the vertical eddy viscosity

Let us now consider a turbulence model which assumes a parabolic distribution of the vertical eddy viscosity over the water depth:

$$\frac{\mu_v}{\rho} \;=\; \nu_m \;+\; \kappa\, u^*(z-b)\left(1 - \frac{(z-b)}{h}\right), \tag{50}$$

where κ is the von Karman constant, $u*$ the modulus of the friction velocity and ν_m the molecular kinematic viscosity. Note that a simple dimensional analysis shows that we have $\nu_v = \dfrac{\mu_v}{\rho U L} = 0(\epsilon)$ as expected.

We remark that we have slightly modified the classical parabolic turbulence model (50) in order to simplify the analytical integration in the computation of the correction term (45). Indeed, we have used

$$\frac{\mu_v}{\rho} \;=\; (\nu_m \;+\; \kappa\, u^*(z-b)) \left(1 - \frac{(z-b)}{h}\right). \tag{51}$$

Note that this modification does not change significantly the profile of the vertical viscosity.

By analytical computation we have that:

$$\int_b^\eta \int_b^z \frac{\rho\,(\eta-\delta)}{\mu_v} d\delta\, dz \;=\; \frac{h^2}{\kappa u^*} \left(\left(1 + \frac{\nu_m}{\kappa u^* h}\right) ln \left(1 + \frac{\kappa u^* h}{\nu_m}\right) - 1 \right).$$

The friction correction (45) then writes:

$$c_\alpha \;=\; \frac{2\,\alpha\,|\bar{u}|}{\kappa u^*} \left(\left(1 + \frac{\nu_m}{\kappa u^* h}\right) ln \left(1 + \frac{\kappa u^* h}{\nu_m}\right) - 1 \right).$$

Following the Chézy law we have that

$$\frac{|\bar{u}|}{u^*} \;=\; \frac{1}{\sqrt{\alpha}}\,,$$

therefore the friction correction is

$$c_\alpha \;=\; \frac{2\sqrt{\alpha}}{\kappa} \left(\left(1 + \frac{\nu_m}{\kappa\sqrt{\alpha}|\bar{u}|h}\right) ln \left(1 + \frac{\kappa\sqrt{\alpha}|\bar{u}|h}{\nu_m}\right) - 1 \right). \tag{52}$$

If the flow is homogeneous in the y-direction and has a rectangular-cross-section, $\bar{u} = \dfrac{Q}{A}$ and the friction term in (44) writes

$$- \frac{\alpha\;|Q|}{h^2\, l \left(1 + \dfrac{2\sqrt{\alpha}}{\kappa} \left(\left(1 + \dfrac{\nu_m}{\kappa\sqrt{\alpha}} \dfrac{l}{|Q|}\right) ln \left(1 + \dfrac{\kappa\sqrt{\alpha}}{\nu_m} \dfrac{|Q|}{l}\right) - 1 \right)\right).}\, Q. \tag{53}$$

5 Comparison of the three-dimensional and the section-averaged solutions in the case of flows with rectangular cross-section

Our aim is now to illustrate the accuracy gain achieved by taking into account the correction of the friction term in the section-averaged model. For this purpose, we restrict ourselves to the case of rectangular cross-section open channels, for which steady state solutions are available. Note that these flows are representative of the main physical features of river flows and are commonly used as a first benchmark in many hydraulics applications.

In this case the water depth h is constant along the y-direction and, denoting by l the width of the river, the section area is $A = lh$. In addition we suppose $\mu_e = \mu_h = 0$. The section averaged shallow water model then writes in the more classical form:

$$\left\{\begin{array}{l} \dfrac{\partial Q}{\partial t} \ + \ \dfrac{\partial}{\partial x}\left(\beta \dfrac{Q^2}{A}\right) \ + \ gA\dfrac{\partial \eta}{\partial x} \ = \ - \displaystyle\int_{l_1}^{l_2} \dfrac{\alpha \ |\bar{u}|}{1 + c_\alpha} \ \bar{u} \ dy \\ \dfrac{\partial A}{\partial t} \ + \ \dfrac{\partial Q}{\partial x} \ = \ 0. \end{array}\right. \tag{54}$$

Note that the Boussinesq term (46) then reduces to $\beta \ = \ l \dfrac{\displaystyle\int_{l_1}^{l_2} \bar{u}^2 dy}{(\displaystyle\int_{l_1}^{l_2} \bar{u} \, dy\,)^2}$.

We emphasize the fact that in this particular case we obtain the classical section-averaged equations [3] with a correction of the friction term.

Remark 5.1 *If the flow is homogeneous in the y-direction, we have that $\bar{u} = \dfrac{Q}{A}$ and therefore the friction term writes*

$$- \ \frac{\alpha \, |Q| \, l}{A^2 \ (1 + c_\alpha)} \ Q \ .$$

Without correction the friction term reduces to

$$- \ \frac{\alpha \ |Q| \ l}{A^2} \ Q,$$

which is the expression of the friction in the classical section-averaged shallow water equations.

We choose a three-dimensional test case with an analytic solution, to be compared to the analytic solution of the section-averaged model with and without friction correction. The test case consists of a steady state turbulent flow in a channel with a slight slope i_F, as illustrated in figure 2.

We take the channel as the reference configuration – (x, y, z) in figure 2 – and we suppose that $\nu_h = \nu_e = 0$. The flow is steady and uniform in the x-direction, and the free surface is perfectly parallel to the bottom, that is:

$$\boldsymbol{\nabla} \eta = (\frac{\partial b}{\partial x}, 0)^T, \qquad \boldsymbol{U} = (u, 0, 0)^T. \tag{55}$$

Rewriting the three-dimensional RANS equations (1) in the new reference

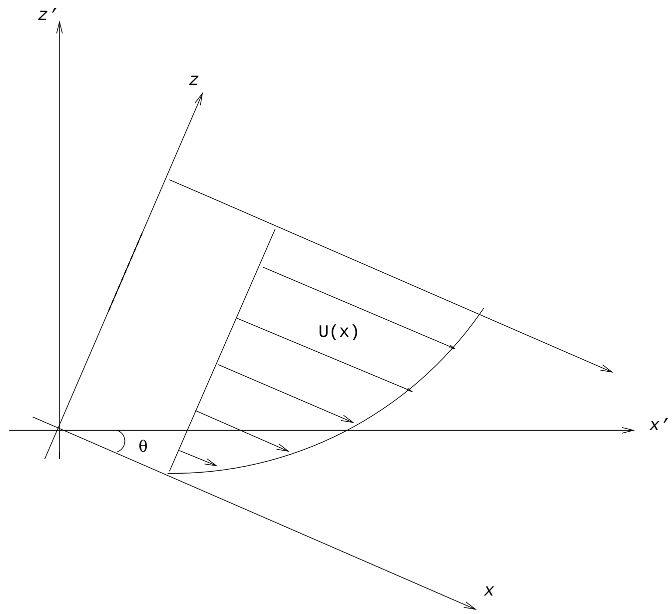

Figure 2: Uniform flow in a chanel with slope

configuration, and considering (55), we retrieve the following system:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\sin\theta + (\cos^2\theta - \sin^2\theta)^2 \frac{\partial}{\partial z}(\frac{\mu_v}{\rho}\frac{\partial u}{\partial z}) \tag{56}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g\cos\theta + 2\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)\frac{\partial}{\partial z}(\frac{\mu_v}{\rho}\frac{\partial u}{\partial z}), \tag{57}$$

where θ is the angle of the slope. The boundary conditions on the free surface are:

$$p = p_a \quad \text{and} \quad \frac{\mu_v}{\rho}\frac{\partial u}{\partial z} = 0, \tag{58}$$

and on the bottom:

$$\frac{\mu_v}{\rho}\frac{\partial u}{\partial z} = \alpha\,\psi(\theta)\,|u|\,u, \tag{59}$$

where $\psi(\theta) = \dfrac{1}{(\cos^2\theta - \sin^2\theta)^2}$. From $(56)_2$ we deduce that the pressure is independent of x, and therefore $\dfrac{\partial p}{\partial x} = 0$. Equation $(56)_1$ reduces to:

$$\frac{\partial}{\partial z}(\frac{\mu_v}{\rho}\frac{\partial u}{\partial z}) = -g\phi(\theta), \tag{60}$$

where $\phi(\theta) = \dfrac{\sin\theta}{(\cos^2\theta - \sin^2\theta)^2}$. Integrating (60) from an arbitrary elevation z to the free surface η, and using boundary condition (58) we obtain:

$$\frac{\partial u}{\partial z} = g\phi(\theta)\frac{\rho(\eta - z)}{\mu_v}. \tag{61}$$

Integrating now (61) from z to the bottom we obtain the following expression of the velocity:

$$u \;=\; u_{|_{(z=b)}} \;+\; g\phi(\theta)\int_b^z \frac{\rho(\eta-\delta)}{\mu_v} d\delta. \tag{62}$$

This expression can be vertically-integrated on the entire water column in order to retrieve an expression of the flow. Indeed,

$$Q \;=\; \int_{l_1}^{l_2}\int_b^{\eta} udzdy \;=\; \int_{l_1}^{l_2}\left(h\,u_{|_{(z=b)}} \;+\; g\phi(\theta)\int_b^{\eta}\int_b^z \frac{\rho(\eta-\delta)}{\mu_v} d\delta dz\right)dy,$$

and since the flow is homogeneous in the y-direction:

$$Q \;=\; A\,u_{|_{(z=b)}} \;+\; g\,l\,\phi(\theta)\int_b^{\eta}\int_b^z \frac{\rho(\eta-\delta)}{\mu_v} d\delta dz\,.$$

Let us now retrieve an expression of the velocity at the bottom. From (61) with $z = b$ we have that

$$\frac{\mu_v}{\rho}\frac{\partial u}{\partial z}_{|_{z=b}} \;=\; g\phi(\theta)h\,.$$

Using boundary condition (59) we obtain:

$$\alpha|u|_{|_{z=b}} u_{|_{z=b}}\psi(\theta) \;=\; gh\phi(\theta),$$

and since in the particular case we are considering the velocity is always positive, we have that:

$$u_{|_{z=b}} \;=\; \sqrt{\frac{g\,h\sin\theta}{\alpha}}\,. \tag{63}$$

Finally we have derived the following expression of the flow:

$$q \;=\; \frac{Q}{l} \;=\; h\sqrt{\frac{g\,h\sin\theta}{\alpha}} \;+\; g\,\phi(\theta)\int_b^{\eta}\int_b^z \frac{\rho(\eta-\delta)}{\mu_v} d\delta dz\,, \tag{64}$$

which is an analytic solution of the three-dimensional problem considered in this section.

This three-dimensional solution is to be compared with the analytic solution of the section-averaged model (54) with and without friction correction. In the particular case considered here we can easily derive the following analytic solution to the section-averaged equations:

$$q \;=\; h\sqrt{\frac{g\,h\,i_F}{\alpha}(1+c_\alpha)}\;. \tag{65}$$

Note that if the correction of the friction term is not taken into account in the section-averaged model, the analytic solution is:

$$q \;=\; h\sqrt{\frac{g\,h\,i_F}{\alpha}}\;. \tag{66}$$

Since c_α depends on the flow rate q, and μ_v also when using a parabolic turbulence model, equations (64), (65) and (66) yield an implicit relation between q and h. We have solved this relation for different values of the water height h, in order to compare the analytic solutions of the different models. Indeed, we have compared the solutions in the laminar case – with constant vertical vicosity $\nu = 0.01$ – and in the turbulent case – using the parabolic model (50) for the turbulent vertical viscosity. We use the Chézy friction term $\alpha = \dfrac{g}{\chi}$ with $\chi = 30$ and $\chi = 60$, respectively, in the case of a slope $i_F = 10^{-4}$. Figure 3 shows the profile of the analytical water height h as a function of the flow q in the laminar case for $\chi = 66.5$ (left) and $\chi = 30$ (right). The starred line corresponds to the three-dimensional solution given by (64), the dashed line corresponds to the solution to the section-averaged model with friction correction given by (65), wheareas the dotted line corresponds to the solution without friction correction (66).

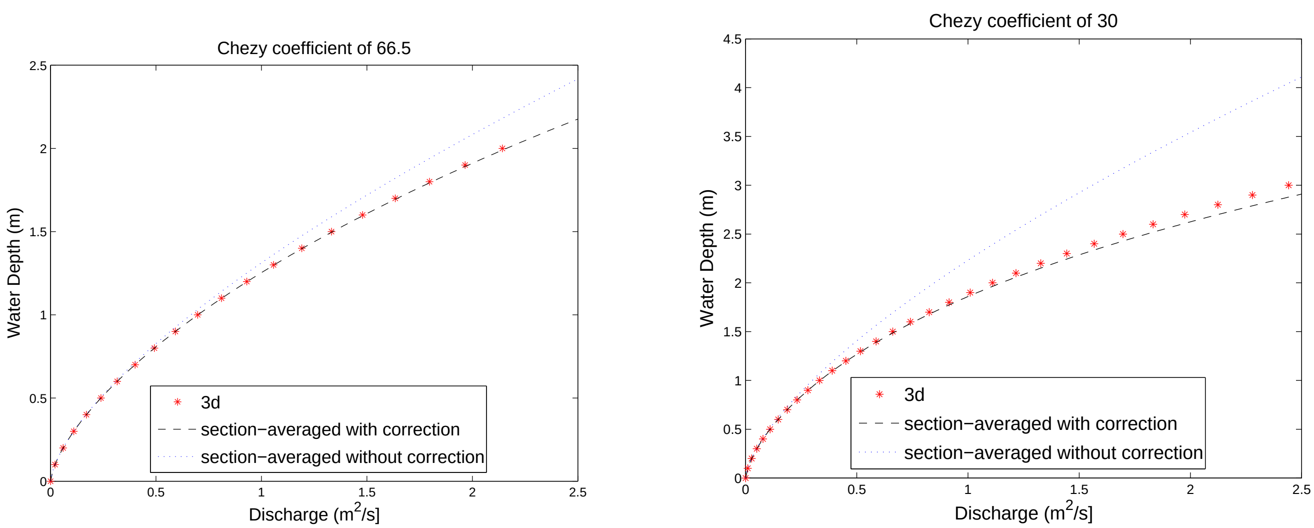

Figure 3: Analytic solutions for the three-dimensional problem (starred line), the section-averaged problem with friction correction (dashed line) and without correction (dotted line). Laminar case with $\nu = 0.01$ and with $\chi = 66.5$ (left) and $\chi = 30$ (right).

Figure 4 shows the same profile in the turbulent case.

As we can see, the analytic solution of the section-averaged model is much closer to the three-dimensional solution when the friction correction is taken into account. This is true when taking a constant vertical viscosity, as well as when using the parabolic turbulence model. The results obtained in this test case, which is a relevant regime for river hydraulics, confirm that classical friction term in the section-averaged shallow water equations should be corrected as defined in (44).

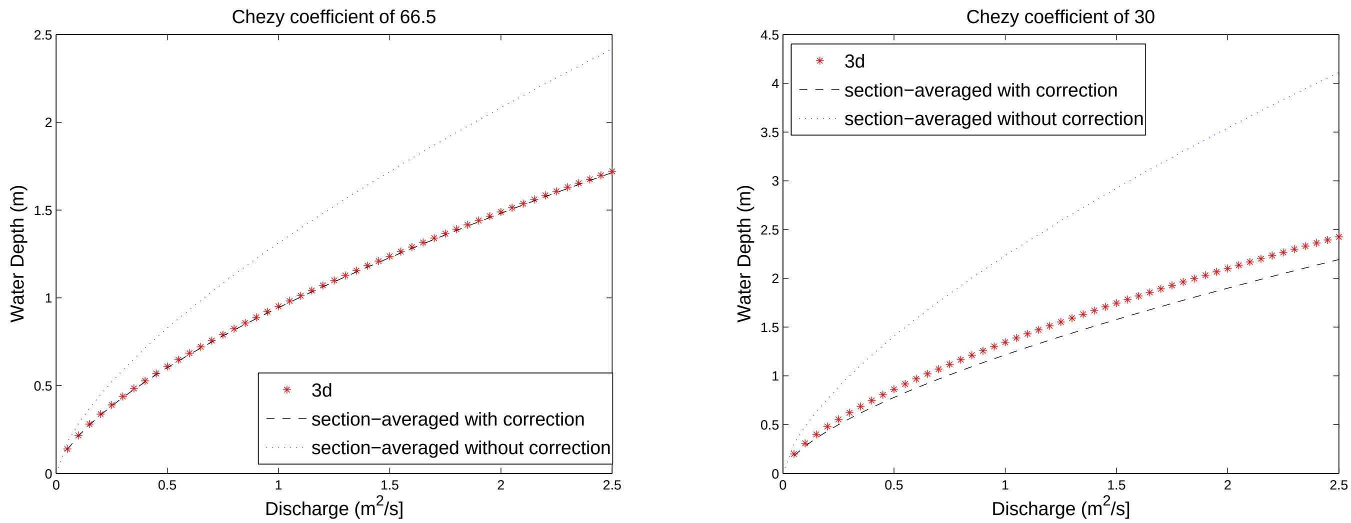

Figure 4: Analytic solutions for the three-dimensional problem (starred line), the section-averaged problem with friction correction (dashed line) and without correction (dotted line). Case with a parabolic turbulent vertical viscosity and with $\chi = 66.5$ (left) and $\chi = 30$ (right).

6 Conclusion

In this paper, we have extended the analysis of [6] to the three-dimensional RANS equations with anisotropic Reynolds tensor for free surface flows in arbitrary geometries. A rigourous derivation of a section-averaged system has been proposed, including the effects of eddy viscosity and friction. When applied to flows with rectangular cross-section, this system is similar to the classical section-averaged shallow water equations [10], except for the friction term. Indeed, our derivation shows that, in order to take into account effects up to the second order in the asymptotic parameter, the classical friction term should be corrected by a term which depends on the turbulent vertical viscosity.

This conclusion is in good agreement with the one achieved by Gerbeau *et al.* in [6] for two-dimensional flows with constant viscosity over a flat bottom. Indeed, if the vertical viscosity is taken constant and the flow is homogeneous in the transversal direction, we retrieve the same friction correction as in [6]. Our derivation provides the expression of the friction correction term in a more general case than those treated by [6], including turbulent flows and three-dimensional arbitrary geometries. In particular, we compute the correction term associated to specific model for the vertical profile of turbulent velocity. Furthermore, for steady state open channel flows admitting analytic solutions of the three-dimensional as well as the simplified models, we have shown that the solutions computed including our correction term are much closer to those of three dimensional model than those of the standard shallow water model. In forthcoming work, we plan to take advantage of the present results by including the friction correction term in section averaged models such as the one proposed by Deponti *ea* in [3]. Its use is also expected to ease the coupling of three- and one-dimensional free surface models in the framework of an integrated hydrological basin model.

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