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A functional equation whose unknown is $\mathcal{P}([0,1])$ valued

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Abstract

We study a functional equation whose unknown maps a euclidean space into the space of probability distributions on $[0,1]$. We prove existence and uniqueness of its solution under suitable regularity and boundary conditions and we characterize solutions that are diffuse on $[0,1]$. A canonical solution is obtained by means of a Randomly Reinforced Urn with different reinforcement distributions having equal means.

1 Introduction

The present work treats a particular functional equation whose unknown maps a euclidean space into the space $\mathcal{P}([0,1])$ of probability distributions on $[0,1]$.

Consider two probability distributions μ and ν on the interval $[0,\beta]$, with $\beta > 0$, and assume that μ and ν have the same mean and this is greater than 0. Then, for (x,y) ranging over the subspace $\mathbb{S} = [0,\infty)\times[0,\infty)\setminus(0,0)$ of $\mathbb{R}^2$, define the following equation with *parameters* μ and ν:

$$x\int_0^\beta(\mathcal{G}(x,y)-\mathcal{G}(x+k,y))\mu(dk)+y\int_0^\beta(\mathcal{G}(x,y)-\mathcal{G}(x,y+k))\nu(dk)=0;\ (1)$$

the unknown is the function

$$\mathcal{G}:\mathbb{S}\to\mathcal{P}([0,1]).$$

Without additional constraints or requirements, equation (1) in its complete generality admits infinitely many solutions. For instance, any constant function $\mathcal{G}$ satisfies (1). We will show that under suitable regularity and boundary conditions, the problem described by (1) is well-posed in the sense that its solution exists, it is unique and depends continuously on the boundary datum. Moreover, we will also prove that the solution depends continuously on the parameters μ and ν and we will characterize a class of solutions $\mathcal{G}$ mapping the interior of $\mathbb{S}$ into the subspace of probability distributions diffuse on $[0,1]$.

A particular case of the problem considered in this paper has been studied in [1] where it is proved that, when the two parameters μ and ν are equal, there exists one and only one continuous solution to (1) that maps the x-axis and y-axis borders of $\mathbb{S}$ in the point mass at 1 or at 0, respectively, and that approaches the point mass at $x/(x+y)$ as $x+y$ tends to infinity. We here extend this result to the case of different parameters μ and ν with the same mean, and to more general boundary conditions. These will be described by means of a continuous function $\varphi:[0,1]\to\mathcal{P}([0,1])$ that will represent the *boundary datum* of the problem.

In the next section we set notation and terminology, we formally describe the functional equation problem and we state three results concerning its solution; they will be proved in the rest of the paper. Section 3 deals with the construction of the *canonical solution* to (1) for the special case when the boundary datum $\varphi(t)$ is the point mass at t, for all $t\in[0,1]$; indeed, the canonical solution is obtained by means of a Randomly Reinforced Urn (see [1, 2, 6, 7] and references therein), which represents the

original stimulus to study the functional equation considered in the paper. Canonical solutions are the building block for proving existence, uniqueness and regularity properties of the solution to the functional equation problem with a general boundary datum; this will be shown in Section 4. Section 5 describes functional equation problems whose solution maps the interior of $\mathbb{S}$ into the subspace of $\mathcal{P}([0,1])$ consisting of probability distributions with no point masses. The final Section 6 presents some examples. Auxiliary technical results have been postponed to the Appendix.

2 Problem and main results

In this section we set notation and terminology and we describe the functional equation problem in detail. We also state three main results concerning its solution; they will be proved in the rest of the paper.

2.1 The Wasserstein metric for spaces of probability distributions

For any $\beta \in (0,\infty)$, we endow the set $\mathcal{P}([0,\beta])$ of probability distributions on the real interval $[0,\beta]$ with the 1–Wasserstein metric d_W which metrizes weak convergence. Recall that, for $\xi_1, \xi_2 \in \mathcal{P}([0,\beta])$,

$$d_W(\xi_1,\xi_2) = \int_0^\beta |F_{\xi_1}(t) - F_{\xi_2}(t)|dt = \int_0^1 |q_{\xi_1}(t) - q_{\xi_2}(t)|dt, \qquad (2)$$

where F_ξ and q_ξ are the cumulative distribution function and the quantile function of $\xi \in \mathcal{P}([0,\beta])$, respectively (see [3] for more details). Moreover, by Kantorovich-Rubinstein Theorem,

$$d_W(\xi_1,\xi_2) = \inf\{E(|X_1 - X_2| : X_1 \sim \xi_1, X_2 \sim \xi_2)\} \qquad (3)$$

where the infimum is taken over all joint distributions for the vector of random variables (X_1, X_2) with marginal distributions equal to ξ_1 and ξ_2, respectively. The metric space $(\mathcal{P}([0,\beta]), d_W)$ is complete and compact.

2.2 The set $\mathcal{P}$ of parameters

For $0 < m_0 \le \beta < \infty$, endow the cartesian product $\mathcal{P}([0,\beta]) \times \mathcal{P}([0,\beta])$ with the Manhattan-distance

$$d_M((\mu_1,\nu_1),(\mu_2,\nu_2)) = d_W(\mu_1,\mu_2) + d_W(\nu_1,\nu_2)$$

and consider the subset $\boldsymbol{\mathcal{P}}$ of couples (μ, ν) of probability distributions with support in $[0, \beta]$ having means that are equal and that are both greater than or equal to m_0, i.e. such that

$$\int_0^\beta k\mu(dk) = \int_0^\beta k\nu(dk) \geq m_0. \tag{4}$$

The elements of $\boldsymbol{\mathcal{P}}$ will act as the parameters for the functional equation (1); note that $\boldsymbol{\mathcal{P}}$ is a closed subset of the metric space $\mathcal{P}([0,\beta]) \times \mathcal{P}([0,\beta])$ and therefore it is compact.

2.3 The set $C([0,1], \mathcal{P}([0,1]))$ of boundary data

A boundary datum φ is defined as a continuous map from $[0,1]$ to $\mathcal{P}([0,1])$. We endow the set of boundary data $C([0,1], \mathcal{P}([0,1]))$ with the sup-distance

$$d_\infty(\varphi_1, \varphi_2) = \sup_{t\in[0,1]} d_W(\varphi_1(t), \varphi_2(t));$$

then $(C([0,1], \mathcal{P}([0,1])), d_\infty)$ is a complete metric space.

From now on, δ will indicate the element of $C([0,1], \mathcal{P}([0,1]))$ defined by setting $\delta(t) = \delta_t$ for $t \in [0,1]$, where δ_t denotes the point mass at t.

2.4 The set $C(\mathbb{S}, \mathcal{P}([0,1]))$ where solutions are to be found

Let $\mathbb{S} = [0,\infty) \times [0,\infty) \setminus (0,0)$ and $C(\mathbb{S}, \mathcal{P}([0,1]))$ be the set of the continuous maps $\mathcal{G} : \mathbb{S} \to \mathcal{P}([0,1])$.

For $n = 1, 2, ...$ let $\mathbb{S}_n = \mathbb{S} \bigcap \{(x,y) \in \mathbb{S} : x + y \geq 1/n\}$ and consider the distance between elements $\mathcal{G}_1, \mathcal{G}_2 \in C(\mathbb{S}_n, \mathcal{P}([0,1]))$ defined by

$$d_n(\mathcal{G}_1, \mathcal{G}_2) = \sup_{(x,y)\in\mathbb{S}_n} d_W(\mathcal{G}_1(x,y), \mathcal{G}_2(x,y)).$$

We then define a new distance d by setting, for all $\mathcal{G}_1, \mathcal{G}_2 \in C(\mathbb{S}, \mathcal{P}([0,1]))$,

$$d(\mathcal{G}_1, \mathcal{G}_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(\mathcal{G}_1 \mid_{\mathbb{S}_n}, \mathcal{G}_2 \mid_{\mathbb{S}_n})}{1 + d_n(\mathcal{G}_1 \mid_{\mathbb{S}_n}, \mathcal{G}_2 \mid_{\mathbb{S}_n})}$$

where $\mathcal{G} \mid_{\mathbb{S}_n}$ indicates the restriction to $\mathbb{S}_n$ of a $\mathcal{G} \in C(\mathbb{S}, \mathcal{P}([0,1]))$.

The distance d metrizes the uniform weak convergence in any closed subset of $\mathbb{S} \bigcup \{(0,0)\}$ which does not contain the origin. Note that convergence with respect to d is equivalent to convergence with respect to all d_n of the corresponding restrictions. The set $(C(\mathbb{S}, \mathcal{P}([0,1])), d)$ is a complete metric space; we will look for elements of this space that are solutions of the functional equation (1).

2.5 The functional equation problem

The Problem object of this paper is now easily stated:
given

$$(\mu,\nu)\in\boldsymbol{\mathcal{P}} \text{ and the boundary datum } \varphi\in C([0,1],\mathcal{P}([0,1])),$$

find

$$\mathcal{G}\in C(\mathbb{S},\mathcal{P}([0,1])) \tag{5a}$$

such that, for all $(x,y)\in\mathbb{S}$,

$$x\int_0^\beta(\mathcal{G}(x,y)-\mathcal{G}(x+k,y))\mu(dk)+y\int_0^\beta(\mathcal{G}(x,y)-\mathcal{G}(x,y+k))\nu(dk)=0, \tag{5b}$$

$$\mathcal{G}(0,y)=\varphi(0), \tag{5c}$$

$$\mathcal{G}(x,0)=\varphi(1), \tag{5d}$$

$$d_W\Big(\mathcal{G}(x,y),\varphi\big(\frac{x}{x+y}\big)\Big)\underset{x+y\to\infty}{\longrightarrow}0. \tag{5e}$$

2.6 Main results

Our first result states that Problem (5) is well-posed in the sense of Hadamard.

Theorem 2.1 *A solution to Problem* (5) *exists, it is unique, and it depends continuously on the boundary datum.*

In the rest of the paper, we denote with $\mathcal{G}^{\varphi}_{(\mu,\nu)}$ the unique solution to Problem (5). Theorem 2.1 will be proved first in the special case when the boundary datum is the map δ. Indeed $\mathcal{G}^{\delta}_{(\mu,\nu)}$ is a *canonical solution* for the problem since, for any other boundary datum $\varphi\in C([0,1],\mathcal{P}([0,1]))$, we will show that

$$\mathcal{G}^{\varphi}_{(\mu,\nu)}=\Psi_\varphi(\mathcal{G}^{\delta}_{(\mu,\nu)}), \tag{6}$$

where

$$\Psi_\varphi : C(\mathbb{S},\mathcal{P}([0,1]))\to C(\mathbb{S},\mathcal{P}([0,1]))$$

is the linear map defined by setting, for all $\mathcal{G}\in C(\mathbb{S},\mathcal{P}([0,1]))$,

$$\Psi_\varphi(\mathcal{G})(x,y)=\int_0^1\varphi(t)\mathcal{G}(x,y)(dt)$$

with (x,y) ranging over $\mathbb{S}$.

The second theorem concerns the continuity of the solution to Problem (5) when the parameters of the equation are let to change. Indicate with $\mathbb{G}^{\varphi}$

the set of solutions to Problem (5) obtained by holding fixed the boundary datum $\varphi \in C([0,1], \mathcal{P}([0,1]))$ and by letting the parameters (μ, ν) range over $\boldsymbol{\mathcal{P}}$:

$$\mathbb{G}^{\varphi} = \left\{ \mathcal{G}^{\varphi}_{(\mu,\nu)} \colon (\mu,\nu) \in \boldsymbol{\mathcal{P}} \right\}.$$

Theorem 2.2 *For any given boundary datum $\varphi \in C([0,1], \mathcal{P}([0,1]))$, the map*

$$(\mu,\nu) \mapsto \mathcal{G}^{\varphi}_{(\mu,\nu)},$$

from $\boldsymbol{\mathcal{P}}$ to $\mathbb{G}^{\varphi}$, is uniformly continuous and $\mathbb{G}^{\varphi}$ is compact.

To prove Theorem 2.2 we will first show that it holds for canonical solutions, i.e. for $\mathbb{G}^{\delta}$, and then we will prove that the map Ψ_{φ} is continuous.

The third result regards a different regularity property of the solution to Problem (5), which depends on the boundary datum φ but not on the parameters (μ, ν). Indeed we characterize solutions $\mathcal{G}^{\varphi}_{(\mu,\nu)}$ mapping the interior of $\mathbb{S}$ into the class of probability distributions on $[0,1]$ having no point masses; such solutions will be called *diffuse*.

A boundary datum $\varphi \in C([0,1], \mathcal{P}([0,1]))$ is said to be *monotonic* if, for all $s, t \in [0,1]$, $s \leq t$,

$$\varphi(s) \leq_{st} \varphi(t).$$

For a given $\varphi \in C([0,1], \mathcal{P}([0,1]))$, indicate with Φ the probability distribution on $[0,1]$ obtained as the convex combination with uniform weights of the members of the family $\{\varphi(t) : t \in [0,1]\}$; i.e. $\Phi = \int_0^1 \varphi(t)dt$.

Theorem 2.3 *Assume that the boundary datum φ is monotonic and let $\mathcal{G}^{\varphi}_{(\mu,\nu)}$ be the unique solution to Problem (5). Then:*

1. *If there is (x_0, y_0) in the interior of $\mathbb{S}$ such that $\mathcal{G}^{\varphi}_{(\mu,\nu)}(x_0, y_0)$ has no point masses in $[0,1]$, then $\Phi = \int_0^1 \varphi(t)dt$ has no point masses in $[0,1]$.*

2. *If $\Phi = \int_0^1 \varphi(t)dt$ has no point masses in $[0,1]$, then $\mathcal{G}^{\varphi}_{(\mu,\nu)}(x,y)$ has no point masses in $[0,1]$ for all (x,y) in the interior of $\mathbb{S}$.*

Once again, in Section 5, we will first prove Theorem 2.3 for canonical solutions and then for the general solution $\mathcal{G}^{\varphi}_{(\mu,\nu)}$.

3 Existence of canonical solutions: a Randomly Reinforced Urn

In this section we assume that the boundary datum of Problem (5) is the map $\delta \in C([0,1], \mathcal{P}([0,1]))$; we prove the existence of a solution for this special instance of the problem, by constructing it through a generalized urn scheme. This solution will be called *canonical* since the solution to Problem (5) for a general boundary datum will be obtained by transforming the canonical solution through a suitable map. The generalized urn scheme that we are going to consider is the Randomly Reinforced Urn (RRU) introduced in [6, 7] and further studied in [1, 2, 5, 8]. This is an urn containing balls of different colors, say black and white. The urn is sequentially sampled; whenever a black ball is extracted from the urn, it is replaced in it together with a random number of black balls having distribution μ, whereas if the color of the extracted ball is white, the ball is replaced in the urn together with a random number of white balls having distribution ν. The extra balls added every time the urn is sampled are called reinforcements.

On a rich enough probability space $(\Omega, \mathcal{A}, \mathcal{P})$, define two independent infinite sequences of random elements, $\{U_n\}$ and $\{(V_n, W_n)\}$; $\{U_n\}$ is a sequence of i.i.d. random variables uniformly distributed on $[0,1]$, while $\{(V_n, W_n)\}$ is a sequence of i.i.d. bivariate random vectors with components uniformly distributed on $[0,1]$. Then, define an infinite sequence $\{(R_X(n), R_Y(n))\}$ of bivariate random vectors by setting, for all n,

$$R_X(n) = q_\mu(V_n) \text{ and } R_Y(n) = q_\nu(W_n),$$

where q_μ and q_ν are the quantile functions of two distributions μ and ν having support in $[0, \beta]$, with $\beta > 0$. Let x and y be two non-negative real numbers such that $x + y > 0$. Set $X_0 = x$, $Y_0 = y$, and, for $n = 0, 1, 2, ...$, let

$$\begin{cases} X_{n+1} &= X_n + R_X(n+1)\mathbb{I}(n+1), \\ Y_{n+1} &= Y_n + R_Y(n+1)(1 - \mathbb{I}(n+1)), \end{cases} \tag{7}$$

where the variable $\mathbb{I}(n+1)$ is the indicator of the event $\{U_{n+1} \leq X_n(X_n + Y_n)^{-1}\}$. The law of $\{(X_n, Y_n)\}$ is that of the stochastic process counting, along the sampling sequence, the number of black and white balls present in a RRU with initial composition (x, y) and reinforcement distributions equal to μ and ν, respectively.

For $n = 0, 1, 2, \ldots$ let $D_n = X_n + Y_n$ be the total number of balls present in the urn at time n and set

$$Z_n(x, y) = X_n / D_n;$$

$Z_n(x,y)$ represents the proportion of black balls in a *RRU* with initial composition (x,y), before the $(n+1)$-th ball is sampled from it. In [7] it is proved that $\{Z_n(x,y)\}$ is eventually a bounded sub- or super-martingale, and it thus converges almost surely, and in L^p, for $1 \leq p \leq \infty$, to a random variable $Z_\infty(x,y) \in [0,1]$; moreover, when μ and ν have different means, $Z_\infty(x,y)$ is the point mass concentrated in 1 or 0, according to whether the mean of μ is greater or smaller than that of ν. However, when the means of μ and ν are the same, the distribution of $Z_\infty(x,y)$ is unknown, apart from a few special cases, see [1] and [5].

For a given couple $(\mu,\nu) \in \boldsymbol{\mathcal{P}}$, let

$$\mathcal{F}_{(\mu,\nu)} : \mathbb{S} \to \mathcal{P}([0,1])$$

be the map which assigns to every $(x,y) \in \mathbb{S}$ the distribution of the limit proportion $Z_\infty(x,y)$ of a *RRU* with initial composition (x,y) and reinforcement distributions μ and ν. In the special case where $\mu = \nu$, the map $\mathcal{F}_{(\mu,\mu)}$ has been characterized in [1] as the unique solution to Problem (5) when the boundary datum is δ. We now extend this result to the general case $(\mu,\nu) \in \boldsymbol{\mathcal{P}}$.

Proposition 3.1 *$\mathcal{F}_{(\mu,\nu)}$ is a solution to Problem (5) when its boundary datum is equal to δ.*

In order to prove Proposition 3.1 we need some auxiliary results; when they do not depend on the parameters $(\mu,\nu) \in \boldsymbol{\mathcal{P}}$, and there is no place for misunderstanding, we write $\mathcal{F}$ for $\mathcal{F}_{(\mu,\nu)}$. Some technicalities connected with the Doob's decomposition of the process $\{Z_n\}$ have been postponed to the Appendix.

The distance between $\mathcal{F}$, evaluated at (x,y), and the boundary datum, evaluated at $x/(x+y)$, is controlled in the following lemma; this distance is uniformly bounded, provided that the size of the urn initial composition is sufficiently large.

Lemma 3.1 *If $x+y \geq 2\beta$,*

$$d_W\big(\mathcal{F}(x,y),\ \delta_{\frac{x}{x+y}}\big) < 2\sqrt{\frac{\beta}{x+y}}.$$

Proof Note that, by (3), $d_W\big(\mathcal{F}(x,y), \delta_{\frac{x}{x+y}}\big) = \mathrm{E}(|Z_\infty(x,y) - \frac{x}{x+y}|)$. Set

$$m = \int_0^\beta k\mu(dk) = \int_0^\beta k\nu(dk);$$

then $m \geq m_0$. From Lemma A.2 and Lemma A.3 with $n = 0$, and because $x + y \geq 2\beta$ implies $\sqrt{\frac{\beta}{x+y}} < 1$, we have

$$d_W\big(\mathcal{F}(x,y),\ \delta_{\frac{x}{x+y}}\big) \leq \sqrt{\frac{\beta}{x+y}} + \frac{\beta}{x+y} < \sqrt{\frac{\beta}{x+y}} + \sqrt{\frac{\beta}{x+y}}.$$

□

The Markov inequality together with Lemma 3.1 imply the following corollary.

Corollary 3.1 *If* $x + y \geq 2\beta$,

$$\mathbb{P}\Big(\big|Z_\infty(x,y) - \frac{x}{x+y}\big| > h_0\Big) \leq \frac{2}{h_0}\sqrt{\frac{\beta}{x+y}}$$

for every $h_0 > 0$.

Lemma 3.2 *For all* $n_0 \geq 1$ *and* $\epsilon > 0$, *there is* $N = N(\epsilon, n_0)$ *such that,*

$$\mathrm{E}\left(|Z_n(x,y) - Z_\infty(x,y)|\right) \leq \epsilon,$$

if $n \geq N$ *and* $x + y \geq 1/n_0$.

Proof Equation (A.36) yields

$$\mathbb{P}(D_n < t) = \mathbb{P}\Big(\frac{1}{D_n} > \frac{1}{t}\Big) \leq t\,\mathrm{E}\,\frac{1}{D_n} < t\frac{1 + n_0(\beta - m_0)}{m_0(n-1) + \beta} \tag{8}$$

for all $t > 0$. Set

$$t = \max\{16\beta/\epsilon^2, 2\beta\} \tag{9}$$

and

$$N \geq \frac{\frac{2t}{\epsilon}(1 + n_0(\beta - m_0)) - \beta}{m_0} + 1. \tag{10}$$

From (8) and (10), we get

$$\mathbb{P}(D_N < t) < \frac{\epsilon}{2}. \tag{11}$$

Moreover, since the process $\{(X_n, Y_n)\}$ is Markov, it follows from Lemma 3.1 and (9) that, for $n \geq N$ and $\omega \in \{D_N \geq t\}$,

$$\mathrm{E}\left(|Z_\infty - Z_n|\Big|(X_n, Y_n)\right)(\omega) = d_W\Big(\mathcal{F}(X_n(\omega), Y_n(\omega)),\ \delta_{\frac{X_n(\omega)}{X_n(\omega)+Y_n(\omega)}}\Big) \leq \frac{\epsilon}{2}. \tag{12}$$

Since $\{D_{n+1} < t\} \subseteq \{D_n < t\}$ for all n, (11) and (12) imply that

$$\begin{aligned}\mathrm{E}[|Z_\infty - Z_n|] &= \mathrm{E}[|Z_\infty - Z_n|; \{D_n \geq t\}] + \mathrm{E}[|Z_\infty - Z_n|; \{D_n < t\}] \\ &\leq \mathrm{E}\left(\mathrm{E}(|Z_\infty - Z_n| 1_{\{D_n \geq t\}} \big| (X_n, Y_n))\right) + \mathbb{P}(D_N < t) \\ &\leq \epsilon\end{aligned}$$

for $n \geq N$. □

The next result can be read as a bound on the modulus of continuity of $\mathcal{F}$ when evaluated at the inner points of $\mathbb{S}$.

Lemma 3.3 *For all $n_0 \geq 1$ and $\epsilon > 0$, there is $\eta = \eta(\epsilon, n_0)$, increasing with ϵ and $1/n_0$, such that*

$$d_W(\mathcal{F}(x,y), \mathcal{F}(\bar{x}, \bar{y})) < \epsilon,$$

if $|x - \bar{x}| + |y - \bar{y}| < \eta$ and $\min\{x + y, \bar{x} + \bar{y}\} \geq 1/n_0$.

Proof Let $N = N(\epsilon/4, n_0)$ be given by Lemma 3.2. Then:

$$\begin{aligned}&d_W(\mathcal{F}(x,y), \mathcal{F}(\bar{x}, \bar{y})) \qquad (13)\\ &\quad \leq E[|Z_\infty(x,y) - Z_N(x,y)|] + E[|Z_\infty(\bar{x}, \bar{y}) - Z_N(\bar{x}, \bar{y})|] + E[|Z_N(x,y) - Z_N(\bar{x}, \bar{y})|] \\ &\quad \leq \frac{\epsilon}{2} + E[|Z_N(x,y) - Z_N(\bar{x}, \bar{y})|].\end{aligned}$$

For controlling the last term, we adopt a coupling argument as in [1]. Consider two different randomly reinforced urns, the first one with initial composition (x, y) and second one with $(\bar{x}, \bar{y})$. The two urns are coupled in the sense that the same processes $\{U_n\}$ and $\{(V_n, W_n)\}$ generate both $\{(X_n(x,y), Y_n(x,y))\}$ and $\{(X_n(\bar{x}, \bar{y}), Y_n(\bar{x}, \bar{y}))$ according to the dynamics described in (7). With the same arguments as in [1, pages 701-702], one may show that

$$\mathrm{E}[|Z_N(x,y) - Z_N(\bar{x}, \bar{y})|] \leq (1 + N) \frac{|x - \bar{x}| + |y - \bar{y}|}{\min\{x + y, \bar{x} + \bar{y}\}};$$

therefore, if $\eta \leq \frac{\epsilon}{2(1+N)n_0}$,

$$\mathrm{E}[|Z_N(x,y) - Z_N(\bar{x}, \bar{y})|] \leq \frac{\epsilon}{2}.$$

□

Proof of Proposition 3.1 By considering the conditional distribution of $Z_\infty(x,y)$, given $\mathbb{I}(1), R_X(1)$ and $R_Y(1)$, and taking the expected values, one immediately verifies that $\mathcal{F}_{(\mu,\nu)}$ satisfies equation (5b) for all $(x,y) \in \mathbb{S}$. Conditions

(5c) and (5d) are also easily verified when $\varphi = \delta$. Finally, (5a) and (5e) are consequences of Lemma 3.3 and of Lemma 3.1, respectively. □

The next result proves a further regularity property of $\mathcal{F}_{(\mu,\nu)}$.

Proposition 3.2 *The map*

$$(\mu, \nu) \mapsto \mathcal{F}_{(\mu,\nu)},$$

from $(\boldsymbol{\mathcal{P}}, d_M)$ *to* $(C(\mathbb{S}, \mathcal{P}([0,1])), d)$*, is uniformly continuous.*

Proof Let $A : C(\mathbb{S}, \mathcal{P}([0,1])) \times \boldsymbol{\mathcal{P}} \to C(\mathbb{S}, \mathcal{P}([0,1]))$ be the operator defined by setting, for every $\mathcal{H} \in C(\mathbb{S}, \mathcal{P}([0,1]))$ and $(\mu, \nu) \in \boldsymbol{\mathcal{P}}$,

$$A(\mathcal{H}, (\mu,\nu))(x,y) = \tfrac{x}{x+y} \int_0^\beta \mathcal{H}(x+k, y)\mu(dk) + \tfrac{y}{x+y} \int_0^\beta \mathcal{H}(x, y+k)\nu(dk) \quad (14)$$

$$= \tfrac{x}{x+y} \int_0^1 \mathcal{H}(x + q_\mu(t), y)dt + \tfrac{y}{x+y} \int_0^1 \mathcal{H}(x, y + q_\nu(t))dt, \quad (15)$$

where (x, y) ranges over $\mathbb{S}$.

Let $n \geq 1$. Then

$$d_n(A(\mathcal{H}_1, (\mu,\nu))|_{\mathbb{S}_n}, A(\mathcal{H}_2, (\mu,\nu))|_{\mathbb{S}_n}) \leq d_n(\mathcal{H}_1|_{\mathbb{S}_n}, \mathcal{H}_2|_{\mathbb{S}_n}), \quad (16)$$

for every $\mathcal{H}_1, \mathcal{H}_2 \in C(\mathbb{S}, \mathcal{P}([0,1]))$ and $(\mu, \nu) \in \boldsymbol{\mathcal{P}}$. Indeed, for every $(x, y) \in \mathbb{S}_n$,

$$\begin{aligned} &d_W(A(\mathcal{H}_1, (\mu,\nu))(x,y), A(\mathcal{H}_2, (\mu,\nu))(x,y)) \\ &\quad \leq \frac{x}{x+y} \int_0^\beta d_W(\mathcal{H}_1(x+k, y), \mathcal{H}_2(x+k, y))\mu(dk) \\ &\qquad + \frac{y}{x+y} \int_0^\beta d_W(\mathcal{H}_1(x, y+k), \mathcal{H}_2(x, y+k))\nu(dk) \\ &\quad \leq \sup_{(x',y') \in \mathbb{S}_n} d_W(\mathcal{H}_1(x', y'), \mathcal{H}_2(x', y')). \end{aligned}$$

Moreover, if $\mathcal{H} \in C(\mathbb{S}, \mathcal{P}([0,1]))$ is Lipschitz on $\mathbb{S}_n$ with Lipschitz constant K_n, then, for every $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \boldsymbol{\mathcal{P}}$,

$$d_n(A(\mathcal{H}, (\mu_1,\nu_1))|_{\mathbb{S}_n}, A(\mathcal{H}, (\mu_2,\nu_2))|_{\mathbb{S}_n}) \leq K_n d_M((\mu_1,\nu_1), (\mu_2,\nu_2)), \quad (17)$$

since, for every $(x, y) \in \mathbb{S}_n$,

$$\begin{aligned} &d_W(A(\mathcal{H}, (\mu_1,\nu_1))(x,y), A(\mathcal{H}, (\mu_2,\nu_2))(x,y)) \\ &\quad \leq \frac{x}{x+y} \int_0^1 d_W(\mathcal{H}(x + q_{\mu_1}(t), y), \mathcal{H}(x + q_{\mu_2}(t), y))dt \\ &\qquad + \frac{y}{x+y} \int_0^1 d_W(\mathcal{H}(x, y + q_{\nu_1}(t)), \mathcal{H}(x, y + q_{\nu_2}(t)))dt \\ &\quad \leq K_n(\tfrac{x}{x+y} d_W(\mu_1, \mu_2) + \tfrac{y}{x+y} d_W(\nu_1, \nu_2)) \\ &\quad \leq K_n d_M((\mu_1,\nu_1), (\mu_2,\nu_2)). \end{aligned} \quad (18)$$

Now, for every $\mathcal{H} \in C(\mathbb{S}, \mathcal{P}([0,1]))$ and $(\mu,\nu) \in \boldsymbol{\mathcal{P}}$, set $A^0(\mathcal{H},(\mu,\nu)) = \mathcal{H}$ and, for $N = 1, 2, ...$ define iteratively

$$A^N(\mathcal{H},(\mu,\nu)) = A(A^{N-1}(\mathcal{H},(\mu,\nu)),(\mu,\nu)).$$

Consider $\mathcal{H}_0 \in C(\mathbb{S}, \mathcal{P}([0,1]))$ defined by setting $\mathcal{H}_0(x,y) = \delta(\frac{x}{x+y})$ for every $(x,y) \in \mathbb{S}$; then $Z_0^{(\mu,\nu)}(x,y)$ has distribution $\mathcal{H}_0(x,y)$, while, for $N = 1, 2, ...$, $Z_N^{(\mu,\nu)}(x,y)$ has distribution $A^N(\mathcal{H}_0,(\mu,\nu))(x,y)$, where, for clarity of exposition, the exponent of the Z variables is evidence for the reinforcement distributions of the *RRU* under consideration. Note that, for $n \geq 1$, $\mathcal{H}_0$ is a Lipschitz map from $\mathbb{S}_n$ to $\mathcal{P}([0,1])$ with Lipschitz constant n. Moreover, it is not difficult to show, with computations analogous to those appearing in (18), that the operator A preserves the Lipschitz property with the same constant; hence, for $(\mu,\nu) \in \boldsymbol{\mathcal{P}}$ and $N = 1, 2, ...$, $A^N(\mathcal{H}_0,(\mu,\nu))$ is a Lipschitz map from $\mathbb{S}_n$ to $\mathcal{P}([0,1])$ with Lipschitz constant n.

Let $(\mu_1,\nu_1),(\mu_2,\nu_2) \in \boldsymbol{\mathcal{P}}$, $n, N \geq 1$ and, for ease of notation, set $\mathcal{H}_i = A^{N-1}(\mathcal{H}_0,(\mu_i,\nu_i))$, for $i = 1, 2$; then

$$\begin{aligned} d_n(A^N(\mathcal{H}_0,(\mu_1,\nu_1))|_{\mathbb{S}_n}, A^N(\mathcal{H}_0,(\mu_2,\nu_2))|_{\mathbb{S}_n}) &= d_n(A(\mathcal{H}_1,(\mu_1,\nu_1))|_{\mathbb{S}_n}, A(\mathcal{H}_2,(\mu_2,\nu_2))|_{\mathbb{S}_n}) \\ &\leq d_n(A(\mathcal{H}_1,(\mu_1,\nu_1))|_{\mathbb{S}_n}, A(\mathcal{H}_2,(\mu_1,\nu_1)|_{\mathbb{S}_n})) \\ &\quad + d_n(A(\mathcal{H}_2,(\mu_1,\nu_1))|_{\mathbb{S}_n}, A(\mathcal{H}_2,(\mu_2,\nu_2))|_{\mathbb{S}_n}) \\ &\leq d_n(\mathcal{H}_1|_{\mathbb{S}_n}, \mathcal{H}_2|_{\mathbb{S}_n}) + n d_M((\mu_1,\nu_1),(\mu_2,\nu_2)), \end{aligned} \tag{19}$$

the last inequality being a consequence of (16) and (17). By iteratively applying (19), it follows that

$$d_n(A^N(\mathcal{H}_0,(\mu_1,\nu_1)), A^N(\mathcal{H}_0,(\mu_2,\nu_2))) \leq nN d_M((\mu_1,\nu_1),(\mu_2,\nu_2)).$$

Therefore, for every $n \geq 1$ and $\epsilon > 0$, if $N = N(\epsilon, n)$ is chosen according to Lemma 3.2, we obtain

$$\begin{aligned} d_n(\mathcal{F}_{(\mu_1,\nu_1)}|_{\mathbb{S}_n}, \mathcal{F}_{(\mu_2,\nu_2)}|_{\mathbb{S}_n}) &\leq d_n(\mathcal{F}_{(\mu_1,\nu_1)}|_{\mathbb{S}_n}, A^N(\mathcal{H}_0,(\mu_1,\nu_1))|_{\mathbb{S}_n}) \\ &\quad + d_n(A^N(\mathcal{H}_0,(\mu_1,\nu_1))|_{\mathbb{S}_n}, A^N(\mathcal{H}_0,(\mu_2,\nu_2))|_{\mathbb{S}_n}) \\ &\quad + d_n(A^N(\mathcal{H}_0,(\mu_2,\nu_2))|_{\mathbb{S}_n}, \mathcal{F}_{(\mu_2,\nu_2)}|_{\mathbb{S}_n}) \\ &\leq nN d_M((\mu_1,\nu_1),(\mu_2,\nu_2)) \\ &\quad + \sup_{(x,y)\in\mathbb{S}_n} \Big[\mathrm{E}\left(|Z_\infty^{(\mu_1,\nu_1)}(x,y) - Z_N^{(\mu_1,\nu_1)}(x,y)|\right) \\ &\qquad + \mathrm{E}\left(|Z_N^{(\mu_2,\nu_2)}(x,y) - Z_\infty^{(\mu_2,\nu_2)}(x,y)|\right)\Big] \\ &\leq nN d_M((\mu_1,\nu_1),(\mu_2,\nu_2)) + 2\epsilon. \end{aligned}$$

This shows that the map $(\mu,\nu) \mapsto \mathcal{F}_{(\mu,\nu)}|_{\mathbb{S}_n}$, from $(\boldsymbol{\mathcal{P}}, d_M)$ to $(C(\mathbb{S}_n, \mathcal{P}([0,1])), d_n)$, is continuous for every n. Hence the map $(\mu,\nu) \mapsto \mathcal{F}_{(\mu,\nu)}$ from $(\boldsymbol{\mathcal{P}}, d_M)$ to

$(C(\mathbb{S}, \mathcal{P}([0,1])), d)$ is continuous; since $\boldsymbol{\mathcal{P}}$ is compact, it is also uniformly continuous. □

4 The solution to the functional equation problem

In this section we prove Theorem 2.1 and Theorem 2.2. In particular we show existence and uniqueness of the solution to Problem (5) when the boundary datum is a generic element of $C([0,1], \mathcal{P}([0,1]))$. Existence is shown by means of a constructive proof based on the canonical solution described in Section 3. Uniqueness is proved through a fixed point argument.

Given $\varphi \in C([0,1]\mathcal{P}([0,1]))$, define the map $\Gamma_\varphi : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ by setting, for every $\xi \in \mathcal{P}([0,1])$,

$$\Gamma_\varphi(\xi)(B) = \int_0^1 \varphi(t)(B)\xi(dt),$$

where B ranges over the Borel sets in $[0,1]$.

Lemma 4.1 *For any given $\varphi \in C([0,1]\mathcal{P}([0,1]))$, the map Γ_φ is uniformly continuous.*

Proof Since $\varphi \in C([0,1], \mathcal{P}([0,1]))$, φ is uniformly continuous and bounded: i.e. for any $\epsilon > 0$, there is an $\eta = \eta(\epsilon, \varphi)$ such that

$$d_W(\varphi(t_1), \varphi(t_2)) \le \epsilon, \qquad \text{if } |t_1 - t_2| \le \eta, \tag{20}$$

while

$$d_W(\varphi(t_1), \varphi(t_2)) \le 1, \qquad \text{for all } t_1, t_2 \in [0,1]. \tag{21}$$

Now, take $\xi_1, \xi_2 \in \mathcal{P}([0,1])$ such that $d_W(\xi_1, \xi_2) < \epsilon\eta$. We are going to prove that $d_W(\Gamma_\varphi(\xi_1), \Gamma_\varphi(\xi_2)) \le 2\epsilon$.

Because of (3), there is a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$ carrying a couple of random variables X_1, X_2 such that $X_1 \sim \xi_1$, $X_2 \sim \xi_2$, and $d_W(\xi_1, \xi_2) = \mathrm{E}(|X_1 - X_2|) \le \epsilon\eta$; by Markov inequality,

$$\widetilde{\mathbb{P}}(|X_1 - X_2| > \eta) \le \epsilon. \tag{22}$$

On the product probability space $(\widetilde{\Omega} \times [0,1], \widetilde{\mathcal{A}} \otimes \mathcal{B}([0,1]), \widetilde{\mathbb{P}} \otimes Leb)$ define the random variables

$$\zeta_1(\omega, t) = \inf\left\{z\colon \int_{[0,z]} \varphi(\eta_1(\omega))(ds) \ge t\right\} = q_{\varphi(\eta_1(\omega))}(t)$$

and

$$\zeta_2(\omega,t)=\inf\Big\{z\colon \int_{[0,z]}\varphi(\eta_2(\omega))(ds)\geq t\Big\}=q_{\varphi(\eta_2(\omega))}(t),$$

where q_ξ indicates the quantile function relative to the probability distribution $\xi\in\mathcal{P}([0,1])$.

For $i=1,2$, note that $\varphi(X_i)$ is the conditional distribution of ζ_i, given X_i; thus, $\zeta_i\sim\Gamma_\varphi(\xi_i)$. Moreover, for all $\omega\in\widetilde{\Omega}$,

$$\begin{aligned} d_W(\varphi(X_1(\omega)),\varphi(X_2(\omega)))&=\int_0^1|q_{\varphi(X_1(\omega))}(t)-q_{\varphi(X_2(\omega))}(t)|dt\\ &=\int_0^1|\zeta_1(\omega,t)-\zeta_2(\omega,t)|dt. \end{aligned}$$

Hence,

$$\begin{aligned} d_W(\Gamma_\varphi(\xi_1),\Gamma_\varphi(\xi_2)) &\leq \mathrm{E}(|\zeta_1-\zeta_2|)\\ &= \mathrm{E}\left(\mathrm{E}\left(|\zeta_1-\zeta_2|\big|X_1,X_2\right)\right)\\ &= \mathrm{E}\left(d_W(\varphi(X_1),\varphi(X_2))\right). \end{aligned}\tag{23}$$

Now, let $F=\{|X_1-X_2|>\eta\}$. From (20), (21) and (22) one obtains:

$$\begin{aligned} \mathrm{E}\left(d_W(\varphi(X_1),\varphi(X_2))\right) &= \mathrm{E}\left(d_W(\varphi(X_1),\varphi(X_2));F\right)+\mathrm{E}\left(d_W(\varphi(X_1),\varphi(X_2));F^c\right)\\ &\leq \widetilde{\mathbb{P}}(F)+\epsilon\widetilde{\mathbb{P}}(F^c)\leq 2\epsilon. \end{aligned}$$

The last inequality, together with (23), concludes the proof. □

Proof of Theorem 2.1.

(i) *Existence.* When the boundary datum $\varphi=\delta$, the existence of a solution $\mathcal{G}^\delta_{(\mu,\nu)}$ is guaranteed by Proposition 3.1, and this is $\mathcal{F}_{(\mu,\nu)}$.

Now let $\varphi\in C([0,1],\mathcal{P}([0,1]))$ and define, for all $(x,y)\in\mathbb{S}$,

$$\mathcal{G}^\varphi_{(\mu,\nu)}(x,y)=\Gamma_\varphi(\mathcal{G}^\delta_{(\mu,\nu)}(x,y));\tag{24}$$

we are going to show that $\mathcal{G}^\varphi_{(\mu,\nu)}$ is indeed a solution to Problem (5) when the bondary datum is φ. In other words, the composition

$$(x,y)\mapsto\mathcal{G}^\delta_{(\mu,\nu)}(x,y)\mapsto\Gamma_\varphi(\mathcal{G}^\delta_{(\mu,\nu)}(x,y))$$

gives a solution to Problem (5), i.e., (6) holds if the map Ψ_φ is defined by setting, for all $\mathcal{G}\in C(\mathbb{S},\mathcal{P}([0,1]))$,

$$\Psi_\varphi(\mathcal{G})(x,y)=\Gamma_\varphi(\mathcal{G}(x,y))=\int_0^1\varphi(t)\ \mathcal{G}(x,y)(dt)\tag{25}$$

with (x,y) ranging over $\mathbb{S}$.

Because of Proposition 3.1, $\mathcal{G}^{\delta}_{(\mu,\nu)}$ satisfies (5b) when the border datum is δ. Since Γ_φ is linear, this implies that, for all $(x,y)\in\mathbb{S}$,

$$\begin{aligned}
&\Psi_\varphi(\mathcal{G}^{\delta}_{(\mu,\nu)})(x,y) = \Gamma_\varphi(\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y))\\
&= \Gamma_\varphi\Big(\frac{x}{x+y}\int \mathcal{G}^{\delta}_{(\mu,\nu)}(x+k,y)\mu(dk) + \frac{y}{x+y}\int \mathcal{G}^{\delta}_{(\mu,\nu)}(x,y+k)\nu(dk)\Big)\\
&= \frac{x}{x+y}\int \Psi_\varphi(\mathcal{G}^{\delta}_{(\mu,\nu)})(x+k,y)\mu(dk) + \frac{y}{x+y}\int \Psi_\varphi(\mathcal{G}^{\delta}_{(\mu,\nu)})(x,y+k)\nu(dk);
\end{aligned}$$

hence $\Psi_\varphi(\mathcal{G}^{\delta}_{(\mu,\nu)})$ satisfies (5b) when the border datum is φ. Now, by Lemma 4.1, $\Psi_\varphi(\mathcal{G}^{\delta}_{(\mu,\nu)})$ is a continuous map from $\mathbb{S}$ to $\mathcal{P}([0,1])$, being the composition of the continuous maps Γ_φ and $\mathcal{G}^{\delta}_{(\mu,\nu)}$; hence (5a), (5c), (5d) and (5e) also hold true.

(ii) *Uniqueness. Sketch of the argument.* Our argument in [1, Section 5] can be easily extended to this more general situation.

Condition (5e) requires $\mathcal{G}$ to be continuous at the projective infinite points. It is therefore convenient to transform the space $\mathbb{S}$ along the projective automorphism τ of $\mathbb{P}^2$ so defined:

$$(x:y:u)\overset{\tau}{\mapsto}(u:x:x+y).$$

The automorphism τ has the following properties:

- the space $\mathbb{S}$ is mapped into the affine space $\mathbb{S}^* = [0,\infty)\times[0,1]$;
- the positive x–axis is mapped into itself by $(x,0)\to(1/x,0)$;
- the positive y–axis is mapped into the semiline $\{y^*=1, x^*>0\}$ by $(0,y)\mapsto(1/y,1)$;
- the projective infinite point relative to the direction $\frac{x}{x+y}=k$ is mapped in the point $(0,k)$;
- the origin is mapped in the projective infinite point $(1:0:0)$.

The inverse map of τ is $(x^*:y^*:u^*)\overset{\tau^{-1}}{\mapsto}(y^*:u^*-y^*:x^*)$. Problem (5) can be equivalently formulated on $\mathbb{S}^*$ as follows:
given

$$(\mu,\nu)\in\boldsymbol{\mathcal{P}} \text{ and the boundary datum } \varphi\in C([0,1],\mathcal{P}([0,1])),$$

find

$$\mathcal{G}^*\in C(\mathbb{S}^*,\mathcal{P}([0,1])) \tag{26a}$$

such that, for all $(x, y) \in \mathbb{S}^*$,

$$\begin{aligned}\mathcal{G}^*(x^*, y^*) = y^* \int \mathcal{G}^* \Big(\frac{x^*}{1+kx^*}, \frac{y^*+kx^*}{1+kx^*}\Big)\mu(dk) + \\ + (1-y^*) \int \mathcal{G}^* \Big(\frac{x^*}{1+kx^*}, \frac{y^*}{1+kx^*}\Big)\nu(dk),\end{aligned} \tag{26b}$$

$$\mathcal{G}^*(x^*, 0) = \varphi(0), \tag{26c}$$

$$\mathcal{G}^*(x^*, 1) = \varphi(1), \tag{26d}$$

$$\mathcal{G}^*(0, y^*) = \varphi(y^*). \tag{26e}$$

In fact, (26b) is just (5b) in the new coordinates. Indeed, the transformations

$$\mathcal{G}(x, y) = \mathcal{G}^*(\tau(x, y))$$

$$\mathcal{G}^*(x^*, y^*) = \begin{cases} \mathcal{G}(\tau^{-1}(x^*, y^*)) & \text{if } (x^*, y^*) \in (0, \infty) \times [0, 1]; \\ \lim\limits_{s^* \to 0} \mathcal{G}(\tau^{-1}(s^*, y^*)) & \text{if } x^* = 0, y^* \in [0, 1], \end{cases}$$

show the equivalence of Problem (5) and Problem (26).

Now, let $\mathbb{C}^*_\varphi(\mathbb{S}^*)$ be the space of continuous function $\mathcal{H}^* : \mathbb{S}^* \to \mathcal{P}([0, 1])$ such that, for every $(x^*, y^*) \in \mathbb{S}^*$,

$$\mathcal{H}^*(x^*, 0) = \varphi(0),\ \mathcal{H}^*(x^*, 1) = \varphi(1) \text{ and } \mathcal{H}^*(0, y^*) = \varphi(y^*).$$

Define the following operator A^* mapping $\mathbb{C}^*_\varphi(\mathbb{S}^*)$ into $\mathbb{C}^*_\varphi(\mathbb{S}^*)$:

$$\begin{aligned}A^*(\mathcal{H}^*)(x^*, y^*) = y^* \int \mathcal{H}^* \Big(\frac{x^*}{1+kx^*}, \frac{y^*+kx^*}{1+kx^*}\Big)\mu(dk) + \\ + (1-y^*) \int \mathcal{H}^* \Big(\frac{x^*}{1+kx^*}, \frac{y^*}{1+kx^*}\Big)\nu(dk)\end{aligned}$$

with (x^*, y^*) ranging over $\mathbb{S}^*$.

With the same argument used in [1, Theorem 5.2], one can prove that A^* has at most one fixed point; hence Problem (26) has at most one solution.

(iii) *Continuity with respect to the boundary datum.* We prove this last part by showing that

$$d(\mathcal{G}^{\varphi_1}_{(\mu,\nu)}, \mathcal{G}^{\varphi_2}_{(\mu,\nu)}) \le d_\infty(\varphi_1, \varphi_2) \tag{27}$$

for all $\varphi_1, \varphi_2 \in C([0, 1], \mathcal{P}([0, 1]))$. We recall here that (see, e.g., [3])

$$d_W(\eta_1, \eta_2) = \sup \Big\{ \Big| \int h(t)\eta_1(dt) - \int h(t)\eta_2(dt) \Big| : \|h\|_L \le 1 \Big\} \tag{28}$$

where $\|h\|_L$ is the Lipschitz norm. Then, for h such that $\|h\|_L \le 1$ and $(x, y) \in \mathbb{S}$,

we get

$$\begin{aligned}
&\Big|\int h(s)\,\mathcal{G}^{\varphi_1}_{(\mu,\nu)}(x,y)(ds) - \int h(s)\,\mathcal{G}^{\varphi_2}_{(\mu,\nu)}(x,y)(ds)\Big| \\
&\qquad = \Big|\int h(s)\int \varphi_1(t)(ds)\,\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)(dt) \\
&\qquad\qquad - \int h(s)\int \varphi_2(t)(ds)\,\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)(dt)\Big| \\
&\qquad \le \int \Big|\int h(s)\varphi_1(t)(ds) - \int h(s)\varphi_2(t)(ds)\Big|\,\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)(dt) \\
&\qquad \le \int d_W(\varphi_1(t),\varphi_2(t))\,\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)(dt) \le d_\infty(\varphi_1,\varphi_2);
\end{aligned}$$

hence $d_W(\mathcal{G}^{\varphi_1}_{(\mu,\nu)}(x,y),\mathcal{G}^{\varphi_2}_{(\mu,\nu)}(x,y)) \le d_\infty(\varphi_1,\varphi_2)$, again by (28). Inequality (27) follows easily. □

Remark 4.1 *Given $(\mu,\nu)\in\boldsymbol{\mathcal{P}}$, the inequality* (27) *can be completed as follows: for all $\varphi_1,\varphi_2\in C([0,1],\mathcal{P}([0,1]))$,*

$$d(\mathcal{G}^{\varphi_1}_{(\mu,\nu)},\mathcal{G}^{\varphi_2}_{(\mu,\nu)}) \le d_\infty(\varphi_1,\varphi_2) \le 2d(\mathcal{G}^{\varphi_1}_{(\mu,\nu)},\mathcal{G}^{\varphi_2}_{(\mu,\nu)}). \tag{29}$$

Hence, for any given $(\mu,\nu)\in\boldsymbol{\mathcal{P}}$, we have an embedding

$$C([0,1],\mathcal{P}([0,1])) \overset{\Psi_\varphi}{\hookrightarrow} C(\mathbb{S},\mathcal{P}([0,1])).$$

Indeed, for $n=1,2,...,$ (5e) *implies that*

$$d_\infty(\varphi_1,\varphi_2) \le d_n(\mathcal{G}^{\varphi_1}_{(\mu,\nu)},\mathcal{G}^{\varphi_2}_{(\mu,\nu)});$$

since $d_n\le 1$, and thus $d_n\le 2\frac{d_n}{1+d_n}$, this implies the right inequality in (29).

Remark 4.2 *Let m be the common mean of $(\mu,\nu)\in\boldsymbol{\mathcal{P}}$. For $p\in[m_0/m,1]$, set $(\mu',\nu')=(p\mu+(1-p)\delta_0,p\nu+(1-p)\delta_0)$. Then $(\mu',\nu')\in\boldsymbol{\mathcal{P}}$ and*

$$\mathcal{G}^{\varphi}_{(\mu',\nu')} = \mathcal{G}^{\varphi}_{(\mu,\nu)}\,.$$

Remark 4.3 *Let $h:[0,1]\to[0,1]$ be a continuous function and $\varphi\in C([0,1],\mathcal{P}([0,1]))$ a boundary datum. For $\xi\in\mathcal{P}([0,1])$, denote with $h\circ\xi$ the distribution of the random variable $h(W)$, where W is a random variable with probability distribution ξ. Then*

$$\mathcal{G}^{h\circ\varphi}_{(\mu,\nu)}(x,y) = h\circ\mathcal{G}^{\varphi}_{(\mu,\nu)}(x,y),$$

for all $(x,y)\in\mathbb{S}$.

Remark 4.4 *One may notice that the boundary conditions* (5c) *and* (5d) *are redundant. Indeed, if* (5b) *and* (5e) *are true for a* $\mathcal{G} : \mathbb{S} \to \mathcal{P}([0,1])$, *then* $\mathcal{G}$ *satisfies* (5c) *and* (5d). *To see this, let* $x > 0$ *and consider* $\mathcal{G}(x,0)$. *By iteratively applying* (5b), *one obtains*

$$\begin{aligned}\mathcal{G}(x,0) &= \\ &= \int_0^\beta \mathcal{G}(x+k_1,0)\mu(dk_1) \\ &= \int_0^\beta \int_0^\beta \mathcal{G}(x+k_1+k_2,0)\mu(dk_1)\mu(dk_2) \\ &\cdots \\ &= \int_0^\beta \cdots \int_0^\beta \mathcal{G}(x+k_1+\cdots+k_n,0)\mu(dk_1)\cdots\mu(dk_n)\end{aligned}$$

for all $n = 1, 2, \ldots$. *However, because of* (5e), *if* $\sum_{i=1}^n k_i \to \infty$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} d_W(\mathcal{G}(x + \sum_{i=1}^n k_i, 0), \varphi(1)) = 0.$$

Hence, the Law of Large Numbers and the Dominated Convergence Theorem imply that

$$\mathcal{G}(x,0) = \varphi(1).$$

The argument for proving that $\mathcal{G}(0,y) = \varphi(0)$, *if* $y > 0$, *is analogous.*

We are finally in the position to prove Theorem 2.2.

Proof of Theorem 2.2. The theorem is true when the boundary datum is δ, as follows from Proposition 3.2 and the fact that $\boldsymbol{\mathcal{P}}$ is compact. For a general boundary datum $\varphi \in C([0,1], \mathcal{P}([0,1]))$ the result follows once it is proved that the map Ψ_φ is continuous; but this is a consequence of Lemma 4.1. □

5 Diffuse solutions

As an immediate consequence of Proposition 3.1 and [2, Theorem 3.2] we have the following result, which is a particular instance of Theorem 2.3 and represents a tool for proving it.

Proposition 5.1 *For all* (x,y) *in the interior of* $\mathbb{S}$ *and* $z \in [0,1]$,

$$\mathcal{G}^\delta_{(\mu,\nu)}(x,y)(\{z\}) = 0.$$

Proof of Theorem 2.3 Given $z \in [0,1]$, note that $\varphi_z(t) = \varphi(t)(\{z\})$ is a measurable function of t, since it is the monotone limit of the sequence of continuous functions $k_z^{(n)}$ defined by setting, for $n = 1, 2, ...$ and $t \in [0,1]$,

$$k_z^{(n)}(t) = \int h_z^{(n)}(s)\varphi(t)(ds),$$

where

$$h_z^{(n)}(s) = \begin{cases} ns - nz + 1, & \text{if } z - 1/n \le s \le z, \\ nz - ns + 1, & \text{if } z < s \le z + 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

The two functions

$$\varphi_z^-(t) = \sup_{w<z} F_{\varphi(t)}(w), \qquad \varphi_z^+(t) = F_{\varphi(t)}(z)$$

are monotonically nonincreasing in t, since φ is monotone. Moreover, $\varphi_z(t) = \varphi_z^+(t) - \varphi_z^-(t)$. Therefore φ_z is a bounded variation function and it thus has at most a countable number of points of discontinuity. Note that

$$\Phi(\{z\}) = \int_0^1 \phi_z(t)dt = \int_0^1 (\phi_z^+(t) - \phi_z^-(t))dt.$$

Proof of part 1. Let (x_0, y_0) be a point in the interior of $\mathbb{S}$ such that $\mathcal{G}^{\varphi}_{(\mu,\nu)}(x_0, y_0)$ has no point masses in $[0,1]$. By way of contradiction, suppose there is a $z \in [0,1]$ such that $\Phi(\{z\}) > 0$.

Since $\Phi(\{z\}) > 0$, there are $\epsilon > 0, a > 0$ and $z_* \in [0,1]$ such that

$$\varphi_z(t) > \epsilon, \qquad \text{for all } t \in I_* = [z_* - a, z_* + a] \cap [0,1]. \tag{30}$$

Set

$$\begin{aligned} R = \Big\{(x,y) \in \mathbb{S}\colon\, & x \ge \max(2\beta, \frac{64\beta}{a^2}), \\ & y = \big(\frac{1}{\overline{z}} - 1\big)x \quad \big(\overline{z} = \frac{x}{x+y}\big), \\ & \overline{z} \in \big[z_* - \frac{a}{2}, z_* + \frac{a}{2}\big] \cap [0,1]\Big\}. \end{aligned}$$

For all $(x,y) \in R$, Corollary 3.1 with $h_0 = \frac{a}{4}$ implies that

$$\mathbb{P}\Big(Z_\infty(x,y) \notin I_*\Big) \le \mathbb{P}\Big(|Z_\infty(x,y) - \overline{z}| > \frac{a}{2}\Big) \le \frac{4}{a}\sqrt{\frac{\beta}{x}} \le \frac{1}{2}$$

and thus $\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)(I_*) \ge \frac{1}{2}$. Then, because of (30), for all $(x,y) \in R$,

$$\begin{aligned} \mathcal{G}^{\varphi}_{(\mu,\nu)}(x,y)(\{z\}) &= \int_0^1 \varphi_z(t)\, \mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)(dt) \\ &\ge \epsilon \int_{I_*} \mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)(dt) \ge \frac{\epsilon}{2}. \end{aligned}$$

Now, set

$$\tau = \inf\{n \geq 0 \colon (X_n(x_0, y_0), Y_n(x_0, y_0)) \in R\} \qquad (\inf\{\varnothing\} = \infty);$$

it is not difficult to show that $\mathbb{P}(\tau < \infty) = p > 0$. Thus the strong Markov property implies that $\mathcal{G}^{\delta}_{(\mu,\nu)}(x_0, y_0)(I_*) \geq \frac{p}{2}$; therefore $\mathcal{G}^{\varphi}_{(\mu,\nu)}(x_0, y_0)(\{z\}) \geq p\frac{\epsilon}{2}$ contradicting the assumption that $\mathcal{G}^{\varphi}_{(\mu,\nu)}(x_0, y_0)$ has no point masses in $[0, 1]$. This concludes the proof of part *1.*

Proof of part 2. Let now $z \in [0, 1]$; by assumption $\Phi(\{z\}) = 0$. Since φ_z has at most a countable number of points of discontinuity, $\varphi_z \geq 0$ and $\int \varphi_z(t)dt = 0$, there is a sequence $t_1, t_2, ...$ such that $\varphi_z(t) = 0$ for all $t \in F = (\cup_i \{t_i\})^c$. Then, given any (x, y) in the interior of $\mathbb{S}$,

$$\begin{aligned}
\mathcal{G}^{\varphi}_{(\mu,\nu)}(x, y)(\{z\}) &= \int_{[0,1]} \varphi_z(t)\, \mathcal{G}^{\delta}_{(\mu,\nu)}(x, y)(dt) \\
&= \int_{[0,1]\cap F} \varphi_z(t)\, \mathcal{G}^{\delta}_{(\mu,\nu)}(x, y)(dt) + \sum_{t_i} \varphi_z(t_i)\, \mathcal{G}^{\delta}_{(\mu,\nu)}(x, y)(\{t_i\}) \\
&= 0
\end{aligned}$$

the last term being zero because of Proposition 5.1. □

6 Examples

In this section we give explicit descriptions of the solution $\mathcal{G}^{\varphi}_{(\mu,\nu)}$ for some specific choices of the reinforcement distributions $(\mu, \nu) \in \boldsymbol{\mathcal{P}}$ and of the boundary datum $\varphi \in C([0, 1], \mathcal{P}([0, 1]))$. The first example is prototypical since it considers the Polya urn scheme and the family of Beta distributions, whose properties had a central role in originating most of the problems tackled in this paper.

6.1 The Polya urn scheme and the family of Beta distributions

We indicate with $\mathrm{Beta}(x, y)$ the beta distribution on $[0, 1]$ with parameters $(x, y) \in \mathbb{S}$. If (x, y) is a point in the interior of $\mathbb{S}$, $\mathrm{Beta}(x, y)$ has a density given by

$$f_{\mathrm{Beta}(x,y)}(t) = \frac{\Gamma(x + y)}{\Gamma(x)\Gamma(y)} t^{x-1}(1 - t)^{y-1},$$

for $t \in [0, 1]$; it is convenient to indicate with $\mathrm{Beta}(0, y)$ and $\mathrm{Beta}(x, 0)$ the point mass at 0 or at 1, respectively.

The random limit composition of a Polya urn with initial composition $(x, y) \in \mathbb{S}$ and constant reinforcement equal to 1 has distribution $\mathrm{Beta}(x, y)$. Indeed a Polya urn is a special RRU with reinforcements $\mu = \nu = \delta_1$; hence

$$\mathcal{G}^{\delta}_{(\delta_1,\delta_1)}(x, y) = \mathrm{Beta}(x, y)$$

for all $(x, y) \in \mathbb{S}$.

The Polya urn scheme, where both reinforcement distributions are equal to the same point mass and the boundary datum is δ, may be easily extended by considering a more general boundary datum along the hint given in Remark 4.3. For instance, let $\gamma > 0$, define $\varphi_\gamma : [0, 1] \to \mathcal{P}([0, 1])$ by setting $\varphi_\gamma(t) = \delta(t^{1/\gamma})$, for $t \in [0, 1]$, and consider Problem (5) with $\mu = \nu = \delta_1$ and boundary datum equal to φ_γ. Note that φ_γ is monotone and thus the unique solution to the problem is diffuse. Indeed for (x, y) in the interior $\mathbb{S}$, the distribution $\mathcal{G}^{\varphi_\gamma}_{(\delta_1,\delta_1)}(x, y)$ has density

$$f_{\mathcal{G}^{\varphi_\gamma}_{(\delta_1,\delta_1)}(x,y)}(t) = \gamma \frac{\Gamma(x + y)}{\Gamma(x)\Gamma(y)} t^{\gamma x-1}(1 - t^\gamma)^{y-1},$$

for $t \in [0, 1]$. For $x = 1$ and $y > 0$, the solution $\mathcal{G}^{\varphi_\gamma}_{(\delta_1,\delta_1)}(1, y)$ is called the Kumaraswami distribution with shape parameters γ and y; see [4].

6.2 Bernoulli reinforcements

A more intriguing extension of the Polya urn scheme is obtained by considering reinforcement distributions $(\mu, \nu) \in \boldsymbol{\mathcal{P}}$ different from equal point masses; we here treat the case where μ and ν are scaled Bernoulli distributions with the same mean. Let $m \geq k_\mu \geq k_\nu > 0$, and assume that μ and ν are the distributions of two random variables, say R_X and R_Y, such that R_X/k_μ has distribution $\mathrm{Bernoulli}(m/k_\mu)$ while R_Y/k_ν has distribution $\mathrm{Bernoulli}(m/k_\nu)$.

Equation (1), with (μ, ν) as above, reads

$$\frac{x}{k_\mu}\Big(\mathcal{G}(x, y) - \mathcal{G}(x + k_\mu, y)\Big) + \frac{y}{k_\nu}\Big(\mathcal{G}(x, y) - \mathcal{G}(x, y + k_\nu)\Big) = 0, \qquad (31)$$

which does not depend on m. One easily verifies that the equation is satisfied by the continuous map $\mathcal{G} : \mathbb{S} \to \mathcal{P}([0, 1])$ defined by setting, for all $(x, y) \in \mathbb{S}$,

$$\mathcal{G}(x, y) = \mathrm{Beta}(\frac{x}{k_\mu}, \frac{y}{k_\nu}).$$

Moreover, note that,

$$d_W\big(\mathrm{Beta}(\frac{x}{k_\mu}, \frac{y}{k_\nu}), \delta(\frac{xk_\nu}{xk_\nu + yk_\mu})\big) \underset{(x+y)\to\infty}{\longrightarrow} 0.$$

Hence, if $h : [0,1] \to [0,1]$ is defined by setting

$$h(t) = \frac{tk_\nu}{tk_\nu + (1-t)k_\mu}$$

for all $t \in [0,1]$, then

$$\mathcal{G}^{h\circ\delta}_{(\mu,\nu)}(x,y) = \text{Beta}(\frac{x}{k_\mu}, \frac{y}{k_\nu}),$$

for $(x,y) \in \mathbb{S}$, is the unique solution to Problem (5) when μ and ν are the scaled Bernoulli distributions defined above and the boundary datum is the continuous map $h \circ \delta : [0,1] \to \mathcal{P}([0,1])$ defined by setting

$$h \circ \delta(t) = \delta(\frac{tk_\nu}{tk_\nu + (1-t)k_\mu})$$

for all $t \in [0,1]$.

We now want to find the distribution of the limit composition of a RRU whose reinforcements are distributed according to the scaled Bernoulli distributions μ and ν. Note that h is continuous, monotonically increasing and its inverse is

$$h^{-1}(u) = \frac{uk_\mu}{uk_\mu + (1-u)k_\nu}$$

for $u \in [0,1]$. Then it follows from Remark 4.3 that,

$$G^{\delta}_{(\mu,\nu)}(x,y) = G^{h^{-1}\circ h\circ\delta}_{(\mu,\nu)}(x,y) = h^{-1} \circ \text{Beta}(\frac{x}{k_\mu}, \frac{y}{k_\nu}),$$

for all $(x,y) \in \mathbb{S}$. For (x,y) in the interior of $\mathbb{S}$, $\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)$ has a density and this is

$$f_{\mathcal{G}^{\delta}_{(\mu,\nu)}(x,y)}(t) = k_\mu^{\frac{y}{k_\nu}} k_\nu^{\frac{x}{k_\mu}} \frac{\Gamma(x/k_\mu + y/k_\nu)}{\Gamma(x/k_\mu)\Gamma(y/k_\nu)} \frac{t^{\frac{x}{k_\mu}-1}(1-t)^{\frac{y}{k_\nu}-1}}{[tk_\nu + (1-t)k_\mu]^{\frac{x}{k_\mu}+\frac{y}{k_\nu}}} \tag{32}$$

for $t \in [0,1]$.

A Doob decomposition of the RRU process

This Appendix provides a series of auxiliary results necessary to prove Propositions 3.1 and 3.2. We will refer to the notations introduced in Section 3. For $n = 1, 2, ...$ let $\mathcal{A}_n = \sigma(\delta_1, R_X(1), R_Y(1), \ldots, \delta_n, R_X(n), R_Y(n))$

and consider the filtration $\{\mathcal{A}_n\}$; then, given the initial urn composition $(x, y) \in \mathbb{S}$, the Doob's semi-martingale decomposition of $Z_n(x, y)$ is

$$Z_n(x, y) = Z_0(x, y) + M_n(x, y) + A_n(x, y)$$

where $\{M_n\}$ is a zero mean martingale and the previsible process $\{A_n\}$ is eventually increasing (decreasing), again by [7, Theorem 2]. We also denote by $\{\langle M\rangle_n\}$ the bracket process associated to $\{M_n\}$, i.e. the previsible process obtained by the Doob's decomposition of M_n^2.

We first provide some auxiliary inequalities. As a consequence of [2, Lemma 4.1], we can bound the increments ΔA_n of the Z_n-compensator process and the increments $\Delta\langle M\rangle_n$ of the bracket process associated to $\{M_n\}$. In fact, an easy computation gives

$$\Delta A_{n+1} = \mathrm{E}(\Delta Z_{n+1}|\mathcal{A}_n) = Z_n(1 - Z_n)A^*_{n+1}$$

and

$$\mathrm{E}((\Delta Z_{n+1})^2|\mathcal{A}_n) = Z_n(1 - Z_n)Z^*_{n+1}.$$

where

$$A^*_{n+1} = \mathrm{E}\Big(\frac{\frac{R_X(n+1)}{D_n}}{1 + \frac{R_X(n+1)}{D_n}} - \frac{\frac{R_Y(n+1)}{D_n}}{1 + \frac{R_Y(n+1)}{D_n}}\Big|\mathcal{A}_n\Big),$$

and

$$Z^*_{n+1} = \mathrm{E}\Big((1 - Z_n)\Big(\frac{\frac{R_X(n+1)}{D_n}}{1 + \frac{R_X(n+1)}{D_n}}\Big)^2 + Z_n\Big(\frac{\frac{R_Y(n+1)}{D_n}}{1 + \frac{R_Y(n+1)}{D_n}}\Big)^2\Big|\mathcal{A}_n\Big).$$

Now, [2, Lemma 4.2] with $m = \int_0^\beta k\mu(dk) = \int_0^\beta k\nu(dk)$ gives

$$|A^*_{n+1}| \le \frac{m}{m + D_n} - \frac{m}{\beta + D_n}. \tag{A.33}$$

By applying [2, Lemma 4.1] with $h(x, t) = (\frac{x}{x+t})^2$, $B_D = [2\beta, \infty)$, $D = D_n$, $R = R_X(n + 1)$ or $R = R_Y(n + 1)$ and $\mathcal{A} = \mathcal{A}_n$, one obtains:

$$Z_n(1 - Z_n)\Big(\frac{m}{m + D_n}\Big)^2 \le \mathrm{E}((\Delta Z_{n+1})^2|\mathcal{A}_n) \le Z_n(1 - Z_n)\frac{m\beta}{(\beta + D_n)^2}, \tag{A.34}$$

on the set $\{D_n \ge 2\beta\}$. Since

$$\mathrm{E}((\Delta Z_{n+1})^2|\mathcal{A}_n) = \mathrm{E}((\Delta A_{n+1} + \Delta M_{n+1})^2|\mathcal{A}_n) = (\Delta A_{n+1})^2 + \Delta\langle M\rangle_{n+1},$$

if $D_0 \geq 2\beta$, and thus $\beta + D_n \geq 3\beta$, (A.33) together with (A.34) yields

$$\begin{aligned}\Delta\langle M\rangle_{n+1} &\geq Z_n(1-Z_n)\Big(\frac{m}{m+D_n}\Big)^2\Big(1-\Big(\frac{\beta-m}{\beta+D_n}\Big)^2\Big)\\ &\geq \frac{8}{9}Z_n(1-Z_n)\Big(\frac{m}{m+D_n}\Big)^2, \qquad\qquad (A.35)\\ \Delta\langle M\rangle_{n+1} &\leq Z_n(1-Z_n)\frac{m\beta}{(\beta+D_n)^2}.\end{aligned}$$

Lemma A.1 *For all $k = 1, 2, ...,$*

$$\mathrm{E}(\frac{1}{D_k}) \leq \frac{1+(\beta-m)/D_0}{D_0+m(k-1)+\beta}. \qquad (A.36)$$

If, in addition, $D_0 \geq 2\beta$ then, for all $k, n = 1, 2, ...,$

$$\Big|\,\mathrm{E}\Big(\frac{1}{c+D_{k+n}} - \frac{1}{d+D_{k+n}}\Big|\mathcal{A}_n\Big)\Big| \leq \frac{\beta-m+d}{m}\Big(\frac{1}{b_k}-\frac{1}{b_{k+1}}\Big), \qquad (A.37)$$

when $d \geq c \geq 0$ and $b_k = c + D_n - \beta + mk$.

Proof Let η^* be a random variabile independent of $\mathcal{A}_\infty$ and let η_1 be a random variable independent of $\sigma(\mathcal{A}_\infty, \eta^*)$ and such that η_1/β has distribution Binomial$(1, m/\beta)$. Define $\mathcal{A}^*_{k+n^-} = \sigma(\eta^*, \mathcal{A}_{k+n-1}, \mathbb{I}(k+n))$; by [2, Lemma 4.1], if $D > 0$ is $\mathcal{A}^*_{k+n^-}$-measurable and $0 \leq R \leq \beta$ with $\mathrm{E}(R) = m$ is independent of $\mathcal{A}^*_{k+n^-}$, one obtains

$$\mathrm{E}\Big(\frac{1}{D+R}\Big|\mathcal{A}^*_{k+n^-}\Big) \leq \frac{m}{\beta}\frac{1}{D+\beta} + \frac{\beta-m}{\beta}\frac{1}{D} = \mathrm{E}\Big(\frac{1}{D+\eta_1}\Big|\mathcal{A}^*_{k+n^-}\Big),$$

and thus

$$\begin{aligned}&\mathrm{E}\Big(\frac{1}{D_{k+n}+\eta^*}\Big|\mathcal{A}^*_{k+n^-}\Big)\\ &\quad= \mathbb{I}(k+n)\,\mathrm{E}\Big(\frac{1}{D_{k+n-1}+\eta^*+R_X(k+n)}\Big|\mathcal{A}^*_{k+n^-}\Big)\\ &\qquad+(1-\mathbb{I}(k+n))\,\mathrm{E}\Big(\frac{1}{D_{k+n-1}+\eta^*+R_Y(k+n)}\Big|\mathcal{A}^*_{k+n^-}\Big)\\ &\quad\leq \mathbb{I}(k+n)\,\mathrm{E}\Big(\frac{1}{D_{k+n-1}+\eta^*+\eta_1}\Big|\mathcal{A}^*_{k+n^-}\Big) \qquad (A.38)\\ &\qquad+(1-\mathbb{I}(k+n))\,\mathrm{E}\Big(\frac{1}{D_{k+n-1}+\eta^*+\eta_1}\Big|\mathcal{A}^*_{k+n^-}\Big)\\ &\quad= \mathrm{E}\Big(\frac{1}{D_{k+n-1}+\eta^*+\eta_1}\Big|\mathcal{A}^*_{k+n^-}\Big).\end{aligned}$$

Therefore, for $c \geq 0$, by applying (A.38) k-times, we get

$$\mathrm{E}\Big(\frac{1}{D_{k+n}+c}\Big|\mathcal{A}_n\Big) \leq \mathrm{E}\Big(\frac{1}{D_n+c+\eta_k}\Big|\mathcal{A}_n\Big) \tag{A.39}$$

where η_k is independent of $\sigma(\mathcal{A}_\infty)$ and η_k/β has distribution Binomial$(k, m/\beta)$. Equation (A.36) is now a consequence of [9, Eq. (21)]: if $\tilde{\eta_k} \sim$ Binomial(k, r) and $l > 0$,

$$\mathrm{E}\Big(\frac{1}{l+\tilde{\eta_k}}\Big) \leq \Big(1+\frac{1-r}{l}\Big)\frac{1}{l+kr+(1-r)}\,.$$

Apply this to (A.39) with $n = 0$, $\tilde{\eta_k} = \eta_k/\beta$, $l = D_0/\beta$ and $r = m/\beta$ to obtain (A.36).

Equation (A.37) is a consequence of [9, Eq. (25)]: if $\tilde{\eta_k} \sim$ Bino(k, r) and $l > 1$,

$$\mathrm{E}\Big(\frac{1}{l+\tilde{\eta_k}}\Big) \leq \frac{1}{l+kr-(1-r)}\,.$$

Apply this to (A.39) with $\tilde{\eta_k} = \eta_k/\beta$, $l = D_n + c/\beta$ (which is greater than 2) and $r = m/\beta$ to obtain

$$\mathrm{E}\Big(\frac{1}{c+D_{n+k}}\Big|\mathcal{A}_n\Big) \leq \frac{1}{c+D_n+m(k+1)-\beta}.$$

Jensen's inequality yields $\mathrm{E}((d+D_{n+k})^{-1}|\mathcal{A}_n) \geq (d+D_n+mk)^{-1}$, and thus

$$\Big|\,\mathrm{E}\Big(\frac{1}{c+D_{n+k}}-\frac{1}{d+D_{n+k}}\Big|\mathcal{A}_n\Big)\Big| \leq \frac{\beta-m+d-c}{(c+D_n+m(k+1)-\beta)(d+D_n+mk)}.$$

Since

$$\frac{1}{b_k}-\frac{1}{b_{k+1}} = \frac{m}{(c+D_n-\beta+mk)(c+D_n-\beta+m(k+1))}$$

we get (A.37):

$$\begin{aligned}\frac{\Big|\,\mathrm{E}\Big(\frac{1}{c+D_{n+k}}-\frac{1}{d+D_{n+k}}\Big|\mathcal{A}_n\Big)\Big|}{\frac{1}{b_k}-\frac{1}{b_{k+1}}} &\leq \frac{\beta-m+d-c}{m}\frac{c+D_n-\beta+mk}{d+D_n+mk}\\ &\leq \frac{\beta-m+d-c}{m}.\end{aligned}$$

□

The following Lemma A.2 and Lemma A.3 provide inequalities which control the previsible and the martingale part of the process Z_n respectively; they require that the initial composition of the urn is sufficiently large.

Lemma A.2 *If $D_0 \geq 2\beta$, then*

$$\mathrm{E}(\sup_r |A_r|) \leq \frac{\beta}{D_0}.$$

Proof Apply (A.37) with $n = 0$, $c = m$, $d = \beta$. Equation (A.33) then reads

$$\mathrm{E}(|A^*_{k+1}|) \leq (2\beta - m)\Big(\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big),$$

if $b_k = km + D_0 - (\beta - m)$. Since $A_0 = 0$,

$$\begin{aligned}
\mathrm{E}(\sup_r |A_r|) &\leq \mathrm{E}\Big(\sum_k |\Delta A_{k+1}|\Big) \leq \sum_k \frac{1}{4}\,\mathrm{E}(|A^*_{k+1}|) \\
&\leq \frac{2\beta - m}{4} \sum_k \Big(\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big) = \frac{2\beta - m}{4} \frac{1}{D_0 - (\beta - m)} \\
&\leq \frac{\beta}{D_0},
\end{aligned}$$

where the last inequality is true beacause $\beta - m \leq \beta \leq D_0/2$. □

Lemma A.3 *Let $D_0 \geq 2\beta$. For all $n \geq 0$,*

$$\mathrm{E}(\langle M\rangle_\infty - \langle M\rangle_n | \mathcal{A}_n) \leq \frac{\beta}{D_0}.$$

Proof Since $Z_{n+k}(1 - Z_{n+k}) \leq 1/4$, by (A.35), one gets

$$\Delta\langle M\rangle_{n+k+1} \leq \frac{m\beta}{4(\beta + D_{n+k})^2} \leq \frac{m}{4}\Big(\frac{1}{D_{n+k}} - \frac{1}{\beta + D_{n+k}}\Big).$$

Apply (A.37) with $c = 0$ and $d = \beta$, obtaining

$$\mathrm{E}(\Delta\langle M\rangle_{n+k+1} | \mathcal{A}_n) \leq \frac{m}{4}\frac{2\beta - m}{m}\Big(\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big),$$

if $b_k = km + D_n - \beta$. Thus

$$\begin{aligned}
\mathrm{E}(\langle M\rangle_\infty - \langle M\rangle_n | \mathcal{A}_n) &= \mathrm{E}\Big(\sum_{k\geq 0} \Delta\langle M\rangle_{k+n+1} \Big| \mathcal{A}_n\Big) \\
&\leq \frac{2\beta - m}{4} \sum_{k\geq 0} \Big(\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big) \\
&\leq \frac{2\beta}{2}\frac{1}{2(D_n - \beta)} \leq \frac{\beta}{D_0},
\end{aligned}$$

since $2(D_n - \beta) \geq 2(D_0 - \beta) \geq D_0$. □

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