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A reduced order representation of two-way coupled models through efficient Dirichlet-Neumann data projection across non-conforming interfaces.

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Abstract

In this paper we propose a full model order reduction (MOR) strategy for two-way Dirichlet-Neumann parametric coupled models solved with domain-decomposition sub-structuring methods. At the high-fidelity level, we split the coupled model and its domain in two sub-models with Dirichlet and Neumann interface conditions, respectively, and two sub-domains with a common interface. The high-fidelity solution is, then, found sub-iterating between the sub-models finite element (FE) solutions until interface convergence is reached. At the reduced-order level, we apply reduced basis (RB) algorithms to define a low dimensional representation of the sub-problems solutions on each sub-domains. Moreover, we define a reduced order representation of the Dirichlet and Neumann interface conditions through the discretized empirical interpolation method (DEIM), achieving a fully reduced order representation of the DD techniques implemented. The interface DEIM reduction is also employed to interpolate or project interface data when non-conforming FE interface discretization is considered. The coupled model reduced solution is found sub-iterating between the reduced order sub-models until convergence of the approximated high-fidelity interface solutions. The (MOR) scheme is numerically verified on both steady and unsteady Dirichlet-Neumann coupled models with two test cases.

Keywords: Two-way coupled models, Dirichlet-Neumann, Reduced order models, Discrete empirical interpolation, Interface non-conformity, Domain-decomposition

1. Introduction

The ROM represented represents an efficient numerical strategy to handle parametrized coupled problems that are set on domains split in two (or several) subdomains, considering finite element schemes as high-fidelity FOM.

2. Two-way Dirichlet-Neumann coupled problem

In this Section we present the formulation of the parametrized two-way coupled model objective of this work. The model is solved at the *high-fidelity* level using domain-decomposition techniques based on finite element method. To this end, we defined in a d-dimensional domain Ω, being $d = 2, 3$, a parametrized second order elliptic model in Subsection 2.1 and a corresponding parametrized parabolic model in Subsection 2.2. Moreover, we split Ω in two sub-domains and solve the Dirichlet-Neumann two-way coupled models by means of *iterative sub-structuring methods*, whose algebraic formulations are derive in the corresponding Subsections. We remark that the proposed strategy can be applied on more complex parametrized models.

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The reduced technique will, then, rely on the application of projection-based ROM on the derived coupled model finite element algebraic formulation. In particular, when the interface grids are non-conforming, we choose the following training strategy: (i) we solve the coupled model considering on both sub-domains the same discretization, those of the first interface grids; (ii) we solve again the coupled model discretizing the two sub-domains according to the second interface grids; (iii) we storage the master solution and Dirichlet interface data computed in (i), and the slave solution and Neumann data found in (ii); (iv) we used the stored quantities as snapshots to trained the reduced order coupled model; (v) we solve the ROM with non-conforming interface grids. To summarize, the FOMs are solved twice in the conforming case, while the ROMs are solved once in the non-conforming case.

Thus, in this Subsections 2.1 and 2.2, we present the high fidelity formulation in the conforming case while, in Subsection 2.3, the non-conforming formulation is derived as basic setup for the interface reduction.

Suppose, therefore, that Ω is an open domain with Lipschitz boundary $\partial\Omega$. We denote $\partial\Omega_D$ and $\partial\Omega_N$ suitable disjoint subsets of $\partial\Omega$ such that $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} = \partial\Omega$. Through the splitting of Ω, we define two non-overlapping subdomains Ω_1 and Ω_2 with Lipschitz boundary $\partial\Omega_i$, $i = 1, 2$, such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, and with a common interface $\Gamma := \Omega_1 \cap \Omega_2$. Moreover, for each $i = 1, 2$, we indicate with $\partial\Omega_{i,D} = \partial\Omega_D \cap \partial\Omega_i$ and $\partial\Omega_{i,N} = \partial\Omega_N \cap \partial\Omega_i$. Finally, we denote the problem and the corresponding solution set in Ω_1 as *master model* and *master solution*, and the ones set in Ω_2 as *slave model* and *slave solution*.

2.1. Steady coupled model

Let us start with a steady parameter dependent two-way coupled model. We consider a set of parameters $\boldsymbol{\mu} \in \mathscr{P}^d$, $d \geq 1$ and we search for $\mathbf{u}(\boldsymbol{\mu})$ in Ω such that

$$\begin{cases} \mathcal{L}(\boldsymbol{\mu})\mathbf{u}(\boldsymbol{\mu}) = \mathbf{f}(\boldsymbol{\mu}) & \text{in } \Omega \\ \mathbf{u}(\boldsymbol{\mu}) = \mathbf{g}_D(\boldsymbol{\mu}) & \text{on } \partial\Omega_D \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}(\boldsymbol{\mu}) = \mathbf{g}_N(\boldsymbol{\mu}) & \text{on } \partial\Omega_N, \end{cases} \tag{1}$$

where $\mathcal{L}(\boldsymbol{\mu})$ is a second order elliptic operator, $\mathbf{f}(\boldsymbol{\mu})$, $\mathbf{g}_D(\boldsymbol{\mu})$ and $\mathbf{g}_N(\boldsymbol{\mu})$ are functions defined in Ω, $\partial\Omega_D$ and $\partial\Omega_N$, respectively, and $\partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}(\boldsymbol{\mu})$ is the conormal derivative associated with the operator $\mathcal{L}(\boldsymbol{\mu})$ on $\partial\Omega$.

We set

$$V = H^1_{\partial\Omega_D}(\Omega) = \{\mathbf{v} \in H^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \partial\Omega_D\}, \tag{2}$$

and we define, for each $\boldsymbol{\mu} \in \mathscr{P}^d$, the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu}) : V \times V \to \mathbb{R}$ associated to $\mathcal{L}(\boldsymbol{\mu})$, and the linear functional $\mathcal{F}(\cdot;\boldsymbol{\mu}) : V \to \mathbb{R}$, such that

$$\mathcal{F}(\mathbf{v};\boldsymbol{\mu}) = (\mathbf{f}(\boldsymbol{\mu}), \mathbf{v})_{L^2(\Omega)} + \langle \mathbf{g}_N(\boldsymbol{\mu}), \mathbf{v}\rangle_{\partial\Omega_N}.$$

Here, $(\cdot,\cdot)_{L^2(\Omega)}$ and $\langle\cdot,\cdot\rangle_{\partial\Omega_N}$ are the inner product in $L^2(\Omega)$ and the duality paring between $H^{1/2}(\partial\Omega_N)$ and $H^{-1/2}(\partial\Omega_N))$, respectively.

Then, integrating by part, problem (1) reads: find $\mathbf{u}(\boldsymbol{\mu}) \in H^1(\Omega)$ with $\mathbf{u}(\boldsymbol{\mu}) = \mathbf{g}_D(\boldsymbol{\mu})$ on $\partial\Omega_D$ such that

$$a(\mathbf{u},\mathbf{v};\boldsymbol{\mu}) = \mathcal{F}(\mathbf{v};\boldsymbol{\mu}) \quad \forall \mathbf{v} \in V. \tag{3}$$

Hereon, we suppose that the solution of problem (3) exists and is unique.

A domain decomposition scheme based on *iterative sub-structuring method* [2, 3] is, then, apply to coupled model (1). We introduce the two sequences of functions $\{\mathbf{u}_1^k(\boldsymbol{\mu})\}$ and $\{\mathbf{u}_2^k(\boldsymbol{\mu})\}$ generated starting from the initial guess $\mathbf{u}_1^0(\boldsymbol{\mu})$ and $\mathbf{u}_2^0(\boldsymbol{\mu})$, that will converge to $\mathbf{u}_1(\boldsymbol{\mu})$ and $\mathbf{u}_2(\boldsymbol{\mu})$, respectively. Then, given $\boldsymbol{\lambda}^0$, for each

$k \geq 0$ we search for $\mathbf{u}_1^k(\boldsymbol{\mu})$ in Ω_1 and $\mathbf{u}_2^k(\boldsymbol{\mu})$ in Ω_2 such that

$$\begin{cases} \mathcal{L}(\boldsymbol{\mu})\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \mathbf{f}(\boldsymbol{\mu}) & \text{in } \Omega_1 \quad (4a) \\ \mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \boldsymbol{\lambda}^k & \text{on } \Gamma \quad (4b) \\ \mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \mathbf{g}_D(\boldsymbol{\mu}) & \text{on } \partial\Omega_{1,D} \quad (4c) \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \mathbf{g}_N(\boldsymbol{\mu}) & \text{on } \partial\Omega_{1,N} \quad (4d) \end{cases}$$

and

$$\begin{cases} \mathcal{L}(\boldsymbol{\mu})\mathbf{u}_2^{k+1}(\boldsymbol{\mu}) = \mathbf{f}(\boldsymbol{\mu}) & \text{in } \Omega_2 \quad (5a) \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) + \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_2^{k+1}(\boldsymbol{\mu}) = 0 & \text{on } \Gamma \quad (5b) \\ \mathbf{u}_2^{k+1}(\boldsymbol{\mu}) = \mathbf{g}_D(\boldsymbol{\mu}) & \text{on } \partial\Omega_{2,D} \quad (5c) \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_2^{k+1}(\boldsymbol{\mu}) = \mathbf{g}_N(\boldsymbol{\mu}) & \text{on } \partial\Omega_{2,N} \quad (5d) \end{cases}$$

with

$$\boldsymbol{\lambda}^{k+1} := \theta\mathbf{u}_{2|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}^k,$$

being θ a positive acceleration parameter. Hereon, we consider $\theta = 1$, so that the Dirichlet interface conditions of equation (4b) are

$$\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \mathbf{u}_2^{k+1}(\boldsymbol{\mu}) \quad \text{on } \Gamma.$$

To get the weak formulations of problems (4) and (5), we first define, for each $i = 1, 2$, the local spaces

$$V_i = \{\mathbf{v} \in H^1(\Omega_i) \ \mid \mathbf{v} = 0 \text{ on } \partial\Omega_{i,D}\} \quad \text{and} \quad V_i^0 = \{\mathbf{v} \in V_i \mid \mathbf{v} = 0 \text{ on } \Gamma\}, \tag{6}$$

and the space of traces of the elements of V on the interface Γ, meaning

$$\Lambda = \{\boldsymbol{\lambda} \in H^{1/2}(\Gamma) \ \mid \exists \mathbf{v} \in V : \mathbf{v}_{|\Gamma} = \boldsymbol{\lambda}\}. \tag{7}$$

Moreover, let $a_i(\cdot,\cdot;\boldsymbol{\mu})$ and $\mathcal{F}_i(\cdot;\boldsymbol{\mu})$, $i = 1, 2$, be the restriction of the bilinear form a and of the linear functional $\mathcal{F}$ to Ω_i, than the weak formulation of (4) and (5) become: for each $k \geq 0$, find $\mathbf{u}_i^{k+1}(\boldsymbol{\mu}) \in H^1(\Omega_i)$, with $\mathbf{u}_{i|\partial\Omega_{i,D}}^{k+1}(\boldsymbol{\mu}) = \mathbf{g}_D(\boldsymbol{\mu})$, such that:

$$\begin{cases} a_1(\mathbf{u}_1^{k+1}(\boldsymbol{\mu}), \mathbf{v}_1^0; \boldsymbol{\mu}) = \mathcal{F}_1(\mathbf{v}_1^0; \boldsymbol{\mu}) & \forall \mathbf{v}_1^0 \in V_1^0, \\ \mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \mathbf{u}_2^k(\boldsymbol{\mu}) & \text{on } \Gamma, \end{cases} \tag{8}$$

and

$$\begin{cases} a_2(\mathbf{u}_2^{k+1}(\boldsymbol{\mu}), \mathbf{v}_2^0; \boldsymbol{\mu}) = \mathcal{F}_2(\mathbf{v}_2^0; \boldsymbol{\mu}) & \forall \mathbf{v}_2^0 \in V_2^0, \\ a_2(\mathbf{u}_2^{k+1}(\boldsymbol{\mu}), \mathcal{R}_2\boldsymbol{\eta}; \boldsymbol{\mu}) = \mathcal{F}_2(\mathcal{R}_2\boldsymbol{\eta}; (\boldsymbol{\mu})) + \mathcal{F}_1(\mathcal{R}_1\boldsymbol{\eta}; (\boldsymbol{\mu})) - a_1(\mathbf{u}_1^{k+1}(\boldsymbol{\mu}), \mathcal{R}_1\boldsymbol{\eta}; \boldsymbol{\mu}) & \forall \boldsymbol{\eta} \in \Lambda. \end{cases} \tag{9}$$

Here

$$\mathcal{R}_i : \Lambda \to V_i, \quad \text{s.t.} \quad (\mathcal{R}_i\eta)_{|\Gamma} = \eta \quad \forall \eta \in \Lambda \tag{10}$$

denote any possibile linear and continuous lifting operator from the interface Γ to Ω_i. Indeed, if $\langle\cdot,\cdot\rangle_\Gamma$ is the duality between Λ and its dual Λ', by counter-integration by parts, the interface equations of (9) is equivalent to

$$\langle \partial_{\mathcal{L}_1\boldsymbol{\mu}}\mathbf{u}_1(\boldsymbol{\mu}) + \partial_{\mathcal{L}_2\boldsymbol{\mu}}\mathbf{u}_2(\boldsymbol{\mu}), \boldsymbol{\eta}\rangle_\Gamma = 0 \quad \forall \boldsymbol{\eta} \in \Lambda, \tag{11}$$

which corresponds to the the Neumann conditions in (5b).

Then, we search for the algebraic formulation of problems (8) and (9). To this end, we define two discretizations $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ on the domains Ω_1 and Ω_2 induced by a global partition $\mathcal{T}_h = \cup_m T_m$ of the domain Ω. $\mathcal{T}_{h_i}$ can be made of simplices (traingles of tetrahedra) or quads (quadrilaterals or hexahedra), depending on a positive parameter $h > 0$. Furthermore, we assume that for any $T_m \in \mathcal{T}_h$, $\partial T_m \cap \partial\Omega$ fully belongs to $\partial\Omega_D$ or $\partial\Omega_N$, while the interface Γ do not cut any $T_m \in \mathcal{T}_h$. This means that the two triangulations $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ are conforming at the interface Γ.

Remark 1. Please note that $h = h_1 = h_2$. However, in Section 4 we will consider different h_i to denote the non-conforming nature of the two discretization $\mathcal{T}_{h_i}$. Thus, for the sake of notation, in this Section we will write in any case h_1 and h_2 and define the relative spaces.

We define the finite element approximation spaces for each partition $\mathcal{T}_{h_i}$ as

$$X_{h_i}^{q_i} = \{\mathbf{v} \in C^0(\overline{\Omega_i}) : \mathbf{v}_{|T_{i,m}} \in \mathcal{Q}_{q_i}, \forall T_{i,m} \in \mathcal{T}_{h_i}\},$$

in which $\mathcal{Q}_{q_i}$ are the quads spaces and q_i are chosen integers.

The finite dimensional spaces to define the discrete formulation of the exploited problems will, then, be

$$V_{h_i} = \{\mathbf{v} \in X_{h_i}^{q_i} : \mathbf{v}_{|\partial\Omega_{i,D}} = 0\}, \quad V_{h_i}^0 = \{\mathbf{v} \in V_{h_i}, \mathbf{v}_{|\Gamma} = 0\}, \quad i = 1, 2, \tag{12}$$

and the spaces of traces on Γ as

$$Y_{h_i} = \{\boldsymbol{\lambda} = \mathbf{v}_{|\Gamma}, \mathbf{v} \in X_{h_i}\} \quad \text{and} \quad \Lambda_{h_i} = \{\boldsymbol{\lambda} = \mathbf{v}_{|\Gamma}, \mathbf{v} \in V_{h_i}.\} \tag{13}$$

Moreover, we define the corresponding space dimensions, *i.e.* $N_i = \dim(V_{h_i})$, $N_i^0 = \dim(V_{h_i}^0)$, $N_{i,Y} = \dim(Y_{h_i})$ and $N_{i,\Lambda} = \dim(\Lambda_{h_i})$.

Then, for $i = 1, 2$, we set the linear and continuous discrete lifting operator

$$\mathcal{R}_{h_i} : \Lambda_{h_i} \to V_{h_i}, \quad \text{s.t.} \quad (\mathcal{R}_{h_i}\eta_{h_i})_{|\Gamma} = \eta_{h_i}, \quad \forall \eta_{h_i} \in \Lambda_{h_i}. \tag{14}$$

In practical implementation, $\mathcal{R}_{h_i}\eta_{h_i}$ is a finite element interpolant that imposes the same values of η_{h_i} on the FE nodes of Γ and zeros on any other FE node of $\mathcal{T}_{h_i}\backslash\Gamma$.

Therefore, with an abuse of notation over $\mathbf{u}_i$ and $\mathbf{g}_D$, we can write the discrete weak forms of (8) and (9): for each $k \geq 0$, $i = 1, 2$, find $\mathbf{u}_i^{k+1}(\boldsymbol{\mu}) \in X_{h_i}$, with $\mathbf{u}_i^{k+1}(\boldsymbol{\mu}) = \mathbf{g}_D(\boldsymbol{\mu})$ on $\partial\Omega_{i,D}$, such that:

$$\begin{cases} a_1(\mathbf{u}_1^{k+1}(\boldsymbol{\mu}), \mathbf{v}_{h_1}^0; \boldsymbol{\mu}) = \mathcal{F}_1(\mathbf{v}_{h_1}^0; \boldsymbol{\mu}) & \forall \mathbf{v}_{h_1}^0 \in V_{h_1}^0, \\ \mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \mathbf{u}_2^k & \text{on } \Gamma, \end{cases} \tag{15}$$

and

$$\begin{cases} a_2(\mathbf{u}_2^{k+1}(\boldsymbol{\mu}), \mathbf{v}_{h_2}^0; \boldsymbol{\mu}) = \mathcal{F}_2(\mathbf{v}_{h_2}^0; \boldsymbol{\mu}) & \forall \mathbf{v}_{h_2}^0 \in V_{h_2}^0, \\ a_2(\mathbf{u}_2^{k+1}(\boldsymbol{\mu}), \mathcal{R}_2\boldsymbol{\eta}_{h_2}; \boldsymbol{\mu}) = \mathcal{F}_2(\mathcal{R}_2\boldsymbol{\eta}_{h_2}; \boldsymbol{\mu}) + \mathcal{F}_1(\mathcal{R}_1\boldsymbol{\eta}_{h_1}; \boldsymbol{\mu}) - a_1(\mathbf{u}_1^{k+1}(\boldsymbol{\mu}), \mathcal{R}_1\boldsymbol{\eta}_{h_1}; \boldsymbol{\mu}) & \forall \boldsymbol{\eta}_{h_i} \in \Lambda_{h_i}. \end{cases} \tag{16}$$

Assuming that $\partial\Gamma_i \cap \partial\Omega_{i,D} = 0$, if we define the discrete residual functional $\mathbf{r}_{h_i}^{k+1}(\boldsymbol{\mu})$ by

$$\mathbf{r}_{h_i}^{k+1}(\boldsymbol{\mu}) = a_i(\mathbf{u}_i^{k+1}(\boldsymbol{\mu}), \mathcal{R}_i\boldsymbol{\eta}_{h_i}; \boldsymbol{\mu}) - \mathcal{F}_i(\mathcal{R}_i\boldsymbol{\eta}_{h_i}\boldsymbol{\mu}) \quad \text{for any } \boldsymbol{\eta}_{h_i} \in \Lambda_{h_i}, i = 1, 2, \tag{17}$$

the second equation of (16) is equivalent to

$$\langle \mathbf{r}_{h_2}^{k+1}(\boldsymbol{\mu}) + \mathbf{r}_{h_1}^{k+1}(\boldsymbol{\mu}), \boldsymbol{\eta}_{h_i} \rangle_\Gamma = 0 \quad \text{for any } \boldsymbol{\eta}_{h_i} \in \Lambda_{h_i}. \tag{18}$$

Note that $\mathbf{r}_{h_i}^{k+1} \in Y_{h_i}$ represent the *residuals* at the interface Γ and they are the discrete approximations of the conormal derivatives $\partial_{\mathcal{L}_i(\boldsymbol{\mu})}\mathbf{u}_i(\boldsymbol{\mu})$, *i.e.* the *discrete fluxes* across the interface. We refer to [4] for a more complete analysis of the residual representation.

To get the algebraic formulation of (15) and (16), we need to define three set of basis functions. First, we call $\{\boldsymbol{\psi}_i^{(j)}\}_{j=1}^{N_{i,Y}}$ the Lagrange basis functions of Y_{h_i}. Moreover, we set $\{\boldsymbol{\phi}_i^{(j)}\}_{j=1}^{N_{i,Y}}$ as the canonical *dual basis* of Y'_{h_i}, the dual space of Y_{h_i}, such that

$$\langle \boldsymbol{\phi}_i^{(j)}, \boldsymbol{\psi}_i^{(k)} \rangle = (\boldsymbol{\phi}_i^{(j)}, \boldsymbol{\psi}_i^{(k)})_{L^2(\Gamma)} = \delta_{jk}, \quad j,k = 1, \dots, N_{i,Y}.$$

By expanding $\mathbf{r}_{h_i} \in Y'_{h_i}$ with respect to the dual basis, we then get

$$\mathbf{r}_{h_i}(\mathbf{x};\boldsymbol{\mu}) = \sum_{j=1}^{N_{i,Y}} \mathbf{r}_i^{(j)}(\boldsymbol{\mu}) \boldsymbol{\phi}_i^{(j)}(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma, i = 1, 2. \tag{19}$$

However, in practical implementation, we compute the coefficients $\mathbf{r}_i^{(j)}(\boldsymbol{\mu})$ considering the discrete and continuous lifting operators

$$\overline{\mathcal{R}}_i = \overline{\mathcal{R}}_{h_i} : Y_{h_i} \to X_{h_i}^{q_i}, \quad \text{s.t. } (\overline{\mathcal{R}}_i \boldsymbol{\lambda}_i)_{|\Gamma} = \boldsymbol{\lambda}_i,$$

i.e. $\overline{\mathcal{R}}_i$ is the operator that coincides with the lifting operator $\mathcal{R}_{h_i}$ of (14) when restricted to Λ_{h_i}. Specifically, if $\boldsymbol{\lambda}_i = \boldsymbol{\eta}^{(j)}$ is the jth Lagrange basis function of Γ, then $\overline{\mathcal{R}}_i \boldsymbol{\eta}^{(j)}$ represents the Lagrange basis function of X_{h_i} whose restriction on Γ is $\boldsymbol{\eta}^{(j)}$. In practical implementation, this can be obtained setting to zero the values of $\overline{\mathcal{R}}_i \boldsymbol{\lambda}_i$ at all nodes of $\mathcal{T}_{h_i}$ not belonging to Γ. Then, we can write

$$\mathbf{r}_i^{(j)}(\boldsymbol{\mu}) = a_i(\mathbf{u}_i(\boldsymbol{\mu}), \overline{\mathcal{R}}_i \boldsymbol{\eta}_i^{(j)}; \boldsymbol{\mu}) - \mathcal{F}_i(\overline{\mathcal{R}}_i \boldsymbol{\eta}_i^{(j)} \boldsymbol{\mu}) \quad j = 1, \dots, N_{i,Y}, \text{ for any } i = 1, 2.$$

Moreover, we denote by $\{\boldsymbol{\varphi}_i^{(j)}\}_{j=1}^{N_i}$ the Lagrange basis functions of V_{h_i}, $i = 1, 2$, so that we can represent each $\mathbf{v}_{h_i} \in V_{h_i}$ as

$$\mathbf{v}_{h_i} = \sum_{j=1}^{N_i} \mathbf{v}_i^{(j)} \boldsymbol{\varphi}_i^{(j)} \quad \text{with } \mathbf{v}_{N_i} = (\mathbf{v}_i^{(1)}, \dots, \mathbf{v}_{N_i}^{(N_i)})^T \in \mathbb{R}^{N_i}.$$

Therefore, the solution is approximated by

$$\mathbf{u}_i(\boldsymbol{\mu}) = \sum_{j=1}^{N_i} \mathbf{u}_i^{(j)}(\boldsymbol{\mu}) \boldsymbol{\varphi}_i^{(j)}.$$

Hereon, we will denote $\mathbf{u}_{N_i}(\boldsymbol{\mu})$ and $\mathbf{r}_{N_i}(\boldsymbol{\mu})$ the vectors of unknown coefficients $\mathbf{u}_i^{(j)}(\boldsymbol{\mu})$ and $\mathbf{r}_i^{(j)}(\boldsymbol{\mu})$, respectively.

However, to define the algebraic formulation, it is useful to consider local vectors and matrices. In particular, we define the following set of indices associated to the nodes $\mathbf{x}_i \in \mathcal{T}_{h_i}$:

$$\begin{aligned} \mathcal{I}_{\overline{\Omega}_i} &= \{1, \dots, \overline{N}_i\}, \\ \mathcal{I}_i &= \{j \in \mathcal{I}_{\overline{\Omega}_i} : \mathbf{x}_j \in \overline{\Omega}_i \backslash (\overline{\partial\Omega_{D,i}} \cup \mathring{\Gamma})\} \\ \mathcal{I}_\Gamma &= \{j \in \mathcal{I}_{\overline{\Omega}_i} : \mathbf{x}_j \in \overline{\Gamma}\} \\ \mathcal{I}_{D_i} &= \{j \in \mathcal{I}_{\overline{\Omega}_i} : \mathbf{x}_j \in \overline{\partial\Omega_{D,i}}\}. \end{aligned} \tag{20}$$

Then, for each $i = 1, 2$, we set the local stiffness matrices

$$\mathbb{A}_i^{kj}(\boldsymbol{\mu}) = a_i(\boldsymbol{\varphi}_i^{(j)}, \boldsymbol{\varphi}_i^{(k)}; \boldsymbol{\mu}), \quad k, j \in \mathcal{I}_{\overline{\Omega}_i},$$

so that with

$$\mathbb{A}_{ii}(\boldsymbol{\mu}) = \mathbb{A}_i(\mathcal{I}_i, \mathcal{I}_i; \boldsymbol{\mu})$$

we indicate the submatrix of $\mathbb{A}_i(\boldsymbol{\mu})$ with rows and columns of $\mathbb{A}_i(\boldsymbol{\mu})$ whose indices belong to $\mathcal{I}_i$, *i.e.* the internal nodes of Ω_i. Similarly, we can define $\mathbb{A}_{\Gamma_i,\Gamma_i}(\boldsymbol{\mu}) = \mathbb{A}_i(\mathcal{I}_\Gamma, \mathcal{I}_\Gamma; \boldsymbol{\mu})$, $\mathbb{A}_{i,\Gamma_i}(\boldsymbol{\mu}) = \mathbb{A}_i(\mathcal{I}_i, \mathcal{I}_\Gamma; \boldsymbol{\mu})$, $\mathbb{A}_{\Gamma,i}(\boldsymbol{\mu}) = \mathbb{A}_i(\mathcal{I}_\Gamma, \mathcal{I}_i; \boldsymbol{\mu})$ and $\mathbb{A}_{i,D}(\boldsymbol{\mu}) = \mathbb{A}_i(\mathcal{I}_i, \mathcal{I}_{D_i}; \boldsymbol{\mu})$.

Moreover, if $(\mathbf{f}_{N_i})_k(\boldsymbol{\mu}) = \mathcal{F}_i(\boldsymbol{\varphi}_i^{(k)};\boldsymbol{\mu})$, $j,k = 1,\ldots,N_i$, we set

$$\mathbf{f}_i(\boldsymbol{\mu}) = \mathbf{f}_{N_i}(\mathcal{I}_i;\boldsymbol{\mu}), \quad \mathbf{f}_{\Gamma_i}(\boldsymbol{\mu}) = \mathbf{f}_{N_i}(\mathcal{I}_\Gamma;\boldsymbol{\mu}),$$

$$\mathbf{u}_i(\boldsymbol{\mu}) = \mathbf{u}_{N_i}(\mathcal{I}_i;\boldsymbol{\mu}), \quad \mathbf{u}_{\Gamma_i}(\boldsymbol{\mu}) = \mathbf{u}_{N_i}(\mathcal{I}_\Gamma;\boldsymbol{\mu}).$$

Then, the algebraic form of (15) is: for each $k \geq 0$, find $\mathbf{u}_1^{k+1}(\boldsymbol{\mu})$ solution of

$$\begin{cases} \mathbb{A}_{11}(\boldsymbol{\mu})\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \mathbf{f}_1(\boldsymbol{\mu}) - \mathbb{A}_{1,D}(\boldsymbol{\mu})\mathbf{g}_{1,D}(\boldsymbol{\mu}) - \mathbb{A}_{1,\Gamma_1}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}) \\ \mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}) = \mathbf{u}_{\Gamma_2}^{k+1}(\boldsymbol{\mu}), \end{cases} \tag{21}$$

where $\mathbf{g}_{1,D}(\boldsymbol{\mu}) = [\mathbf{g}_D(\mathbf{x}_1^j;\boldsymbol{\mu})]_{j\in\mathcal{I}_{D_1}}$.

In practical implementation, we do a lifting operation over the complete vector solution $\mathbf{u}_{N_1}(\boldsymbol{\mu})$, *i.e.* we write

$$\mathbf{u}_{N_1}(\boldsymbol{\mu}) = \tilde{\mathbf{u}}_{N_1}(\boldsymbol{\mu}) + \mathbf{g}_D(\boldsymbol{\mu}) + \mathbf{u}_{\Gamma_1}(\boldsymbol{\mu}), \tag{22}$$

where $\tilde{\mathbf{u}}_{N_1}(\boldsymbol{\mu})$ has null elements in correspondence to the Dirichlet and interface DoFs, while $\mathbf{g}_D(\boldsymbol{\mu})$ and $\mathbf{u}_{1_{\Gamma_1}}(\boldsymbol{\mu})$ are the vectors with all elements equal to zero except that in the Dirichlet and interface DoFs, respectively. Then, problem (21) as

$$\begin{cases} \mathbb{A}_1(\boldsymbol{\mu})\tilde{\mathbf{u}}_{N_1}^{k+1}(\boldsymbol{\mu}) = \mathbf{f}_1(\boldsymbol{\mu}) - \mathbb{A}_1(\boldsymbol{\mu})\mathbf{g}_D(\boldsymbol{\mu}) - \mathbb{A}_1(\boldsymbol{\mu})\mathbf{u}_{1_{\Gamma_1}}^{k+1}(\boldsymbol{\mu}) \\ \mathbf{u}_{1_{\Gamma_1}}^{k+1}(\boldsymbol{\mu}) = \mathbf{u}_{2_{|\Gamma_1}}^{k+1}(\boldsymbol{\mu}). \end{cases} \tag{23}$$

Moreover, we can write the algebraic formulation of problem (16) as: for each $k \geq 0$, find $\mathbf{u}_2^{k+1}(\boldsymbol{\mu})$ such that

$$\begin{cases} \mathbb{A}_{22}(\boldsymbol{\mu})\mathbf{u}_2^{k+1}(\boldsymbol{\mu}) + \mathbb{A}_{2,\Gamma_2}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_2}^{k+1}(\boldsymbol{\mu}) = \mathbf{f}_2(\boldsymbol{\mu}) - \mathbb{A}_{2,D}(\boldsymbol{\mu})\mathbf{g}_{2,D}(\boldsymbol{\mu}) \\ \mathbb{A}_{\Gamma_2,2}(\boldsymbol{\mu})\mathbf{u}_2^{k+1}(\boldsymbol{\mu}) + \mathbb{A}_{\Gamma_2,\Gamma_2}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_2}^{k+1}(\boldsymbol{\mu}) = \mathbf{f}_{\Gamma_2}(\boldsymbol{\mu}) + \mathbf{f}_{\Gamma_1}(\boldsymbol{\mu}) - \mathbb{A}_{\Gamma_1,1}(\boldsymbol{\mu})\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) - \mathbb{A}_{\Gamma_1,\Gamma_1}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}), \end{cases}$$

where $\mathbf{g}_{2,D}(\boldsymbol{\mu}) = [\mathbf{g}_D(\mathbf{x}_2^j;\boldsymbol{\mu})]_{j\in\mathcal{I}_{D_2}}$, as before. Note that

$$\mathbf{r}_{N_1}^{k+1}(\boldsymbol{\mu}) = \mathbb{A}_{\Gamma_1,1}(\boldsymbol{\mu})\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) + \mathbb{A}_{\Gamma_1,\Gamma_1}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}) - \mathbf{f}_{\Gamma_1}(\boldsymbol{\mu}).$$

Then, summing up the two equations, we get

$$\begin{cases} \mathbb{A}_2(\boldsymbol{\mu})\mathbf{u}_{N_2}^{k+1}(\boldsymbol{\mu}) = \mathbf{f}_2(\boldsymbol{\mu}) + \mathbf{r}_{N_2}^{k+1}(\boldsymbol{\mu}) \\ \mathbf{r}_{N_2}^{k+1}(\boldsymbol{\mu}) = -\mathbf{r}_{N_1}^{k+1}(\boldsymbol{\mu}), \end{cases} \tag{24}$$

that corresponds to the final formulation used in practical implementation. Note that $\mathbf{r}_{N_1}(\boldsymbol{\mu})$ can be equally written as

$$\mathbf{r}_{N_1}^{k+1}(\boldsymbol{\mu}) = \left(\mathbb{A}_1(\boldsymbol{\mu})\mathbf{u}_{N_1}^{k+1}(\boldsymbol{\mu}) - \mathbf{f}_1(\boldsymbol{\mu})\right)_{|\Gamma}. \tag{25}$$

Remark 2. The derive formulation is true if $\partial\Gamma \cap \partial\Omega_D \neq 0$, otherwise new index set and corresponding local matrices must be added. See [2, 4] for a detailed description.

2.2. *Unsteady coupled model*

With the same notation of Subsection 2.1, we consider the following unsteady parameter dependent model: for each $\boldsymbol{\mu} \in \mathscr{P}^d$, find $\mathbf{u}(t;\boldsymbol{\mu}) \in \Omega \times \{0,T\}$ such that

$$\begin{cases} \frac{\partial \mathbf{u}(t;\boldsymbol{\mu})}{\partial t} + \mathcal{L}(\boldsymbol{\mu})\mathbf{u}(t;\boldsymbol{\mu}) = \mathbf{f}(t;\boldsymbol{\mu}) & \text{in } \Omega \times \{0,T\} \\ \mathbf{u}(t;\boldsymbol{\mu}) = \mathbf{g}_D(t;\boldsymbol{\mu}) & \text{on } \partial\Omega_D \times \{0,T\} \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}(t;\boldsymbol{\mu}) = \mathbf{g}_N(t;\boldsymbol{\mu}) & \text{on } \partial\Omega_N \times \{0,T\} \\ \mathbf{u}(0;\boldsymbol{\mu}) = \mathbf{u}_0(\boldsymbol{\mu}) & \text{on } \Omega \times \{0\} \end{cases} \tag{26}$$

being $\mathcal{L}(\boldsymbol{\mu})$ a second order elliptic operator time-independent, $\mathbf{f}(t;\boldsymbol{\mu})$, $\mathbf{g}_D(t;\boldsymbol{\mu})$ and $\mathbf{g}_N(t;\boldsymbol{\mu})$ functions defined in $\Omega \times \{0,T\}$, $\partial\Omega_D \times \{0,T\}$ and $\partial\Omega_N \times \{0,T\}$, respectively. As before, $\partial_{\mathcal{L}(\boldsymbol{\mu})}$ is the conormal derivative of the operator $\mathcal{L}(\boldsymbol{\mu})$ on $\partial\Omega$.

Remark 3. Here the elliptic operator $\mathcal{L}(\boldsymbol{\mu})$ is considered as time-independent but the reduced order model presented in this work can be easily extended to the time dependent case.

Similar consideration as for the steady case (see Subsection 2.1) can be stated for this time-dependent model. Thus, in this Subsection we will introduce the weak formulation and the high fidelity discretization of problem (26) using a FE domain-decomposition staggered method.

In particular, with the same *sub-structuring algorithm* [2] used for the steady case, for each $t \in (0,T\}$ we define two sequences of function $\{\mathbf{u}_1^k(t;\boldsymbol{\mu})\}$ and $\{\mathbf{u}_2^k(t;\boldsymbol{\mu})\}$ generated from the initial guess $\mathbf{u}_1^0(t;\boldsymbol{\mu})$ and $\mathbf{u}_2^0(t;\boldsymbol{\mu})$ and converging to $\mathbf{u}_1(t;\boldsymbol{\mu})$ and $\mathbf{u}_1(t;\boldsymbol{\mu})$, respectively. For each $\boldsymbol{\mu} \in \mathscr{P}^d$, $t \in (0,T\}$ and $k \geq 0$, we then search for $\mathbf{u}_1^k(t;\boldsymbol{\mu}) \in \Omega_1$ and $\mathbf{u}_2^k(t;\boldsymbol{\mu}) \in \Omega_2$ solutions of:

$$\begin{cases} \dfrac{\partial \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu})}{\partial t} + \mathcal{L}(\boldsymbol{\mu})\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \mathbf{f}(t;\boldsymbol{\mu}) & \text{in } \Omega_1 \times \{0,T\} & (27a) \\ \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \mathbf{u}_2^{k}(t;\boldsymbol{\mu}) & \text{on } \Gamma \times \{0,T\} & (27b) \\ \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \mathbf{g}_D(t;\boldsymbol{\mu}) & \text{on } \partial\Omega_{1,D} \times \{0,T\} & (27c) \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \mathbf{g}_N(t;\boldsymbol{\mu}) & \text{on } \partial\Omega_{1,N} \times \{0,T\} & (27d) \\ \mathbf{u}_1^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0_{|\Omega_1}}(\boldsymbol{\mu}) & \text{on } \Omega_1 \times \{0\} & (27e) \end{cases}$$

and

$$\begin{cases} \dfrac{\partial \mathbf{u}_2^{k+1}(t;\boldsymbol{\mu})}{\partial t} + \mathcal{L}(\boldsymbol{\mu})\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) = \mathbf{f}(t;\boldsymbol{\mu}) & \text{in } \Omega_2 \times \{0,T\} & (28a) \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) + \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) = 0 & \text{on } \Gamma \times \{0,T\} & (28b) \\ \mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) = \mathbf{g}_D(t;\boldsymbol{\mu}) & \text{on } \partial\Omega_{2,D}\{0,T\} & (28c) \\ \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) = \mathbf{g}_N(t;\boldsymbol{\mu}) & \text{on } \partial\Omega_{2,N}\{0,T\} & (28d) \\ \mathbf{u}_2^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0_{|\Omega_2}}(\boldsymbol{\mu}) & \text{on } \Omega_2 \times \{0\} & (28e) \end{cases}$$

Defining the function spaces V_i, V_i^0, and Λ_{h_i} as in (6) and (7), respectively, the weak formulation of (27) and (28) can be derived as: for each $t \in (0,T\}$, for each $k \geq 0$, find $\mathbf{u}_i^{k+1}(t;\boldsymbol{\mu}) \in H^1(\Omega)$, with $\mathbf{u}^{k+1}_{i_{|\partial\Omega_{i,D}}}(t;\boldsymbol{\mu}) = \mathbf{g}_D(t;\boldsymbol{\mu})$, such that

$$\begin{cases} \int_{\Omega_1} \frac{\partial \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathbf{v}_1^0 d\Omega_1 + a_1(\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}), \mathbf{v}_1^0;\boldsymbol{\mu}) = \mathcal{F}_1(\mathbf{v}_1^0;\boldsymbol{\mu}) & \forall \mathbf{v}_1^0 \in V_1^0 \\ \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \mathbf{u}_2^{k}(t;\boldsymbol{\mu}) & \text{on } \Gamma \\ \mathbf{u}_1^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0_{|\Omega_1}}(\boldsymbol{\mu}) \end{cases} \tag{29}$$

and

$$
\begin{cases}
\int_{\Omega_2} \frac{\partial \mathbf{u}_2^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathbf{v}_2^0 d\Omega_2 + a_2(\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}), \mathbf{v}_2^0;\boldsymbol{\mu}) = \mathcal{F}_2(\mathbf{v}_2^0;\boldsymbol{\mu}) & \forall \mathbf{v}_2^0 \in V_2^0 \\
\int_{\Omega_2} \frac{\partial \mathbf{u}_2^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathcal{R}_2\boldsymbol{\eta} d\Omega_2 + a_2(\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}), \mathcal{R}_2\boldsymbol{\eta};\boldsymbol{\mu}) = \mathcal{F}_2(\mathcal{R}_2\boldsymbol{\eta};(\boldsymbol{\mu}))+ & \\
\quad +\mathcal{F}_1(\mathcal{R}_1\boldsymbol{\eta};(\boldsymbol{\mu})) - \int_{\Omega_1} \frac{\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathcal{R}_1\boldsymbol{\eta} d\Omega_1 - a_1(\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}), \mathcal{R}_1\boldsymbol{\eta};\boldsymbol{\mu}) & \forall \boldsymbol{\eta} \in \Lambda \\
\mathbf{u}_2^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0|\Omega_2}(\boldsymbol{\mu}). &
\end{cases} \tag{30}
$$

Here $a_i(\cdot,\cdot;\boldsymbol{\mu})$ and $\mathcal{F}_i(\cdot;\boldsymbol{\mu})$ are the bilinear form associated to $\mathcal{L}(\boldsymbol{\mu})$ and the usual linear functional. As for the steady case, $\mathcal{R}_i$ denotes the linear and continuous lifting operator from Γ to Ω_i, so that the second equation of (30) corresponds to the Neumann interface conditions, *i.e.*

$$
\langle \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) + \partial_{\mathcal{L}(\boldsymbol{\mu})}\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}), \boldsymbol{\eta}\rangle_\Gamma = 0 \quad \forall \boldsymbol{\eta} \in \Lambda, t \in \times(0,T\},
$$

where $\langle\cdot,\cdot\rangle_\Gamma$ is the duality between Λ and its dual space Λ'.

Considering the same FE spatial discretization as for the steady case (see Subsection 2.1), for each time instant t, the high fidelity formulations of problems (29) and (30) are represented by two dynamical systems. Indeed, if we set the spaces V_{h_i}, $V_{h_i}^0$, Y_{h_i} and Λ_{h_i} as in (12) and (13), with an abuse of notation over $\mathbf{u}_i^{k+1}(t;\boldsymbol{\mu})$, we can write the Galerking approximation of problems (29) and (30) as: for each $t \in (0,T\}$, for each $k \geq 0$, search for $\mathbf{u}_i^{k+1}(t;\boldsymbol{\mu}) \in X_{h_i}$, $i = 1,2$, with $\mathbf{u}_i^{k+1}(\boldsymbol{\mu}) = \mathbf{g}_D(\boldsymbol{\mu})$ on $\partial\Omega_{i,D}$, such that

$$
\begin{cases}
\int_{\Omega_1} \frac{\partial \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathbf{v}_{h_1}^0 d\Omega_1 + a_1(\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}), \mathbf{v}_{h_1}^0;\boldsymbol{\mu}) = \mathcal{F}_1(\mathbf{v}_{h_1}^0;\boldsymbol{\mu}) & \forall \mathbf{v}_{h_1}^0 \in V_{h_1}^0 \\
\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \mathbf{u}_2^k(t;\boldsymbol{\mu}) & \text{on } \Gamma \\
\mathbf{u}_1^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0|\Omega_1}(\boldsymbol{\mu}) &
\end{cases} \tag{31}
$$

and

$$
\begin{cases}
\int_{\Omega_2} \frac{\partial \mathbf{u}_2^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathbf{v}_{h_2}^0 d\Omega_2 + a_2(\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}), \mathbf{v}_{h_2}^0;\boldsymbol{\mu}) = \mathcal{F}_2(\mathbf{v}_{h_2}^0;\boldsymbol{\mu}) & \forall \mathbf{v}_{h_2}^0 \in V_{h_2}^0 \\
\int_{\Omega_2} \frac{\partial \mathbf{u}_2^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathcal{R}_2\boldsymbol{\eta}_{h_2} d\Omega_2 + a_2(\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}), \mathcal{R}_2\boldsymbol{\eta}_{h_2};\boldsymbol{\mu}) = \mathcal{F}_2(\mathcal{R}_2\boldsymbol{\eta}_{h_2};\boldsymbol{\mu})+ & \\
\quad +\mathcal{F}_1(\mathcal{R}_1\boldsymbol{\eta}_{h_1};\boldsymbol{\mu}) - \int_{\Omega_1} \frac{\partial \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathcal{R}_1\boldsymbol{\eta}_{h_1} d\Omega_1 - a_1(\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}), \mathcal{R}_1\boldsymbol{\eta}_{h_1};\boldsymbol{\mu}) & \forall \boldsymbol{\eta}_{h_1} \in \Lambda_{h_1} \\
\mathbf{u}_2^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0|\Omega_2}(\boldsymbol{\mu}). &
\end{cases} \tag{32}
$$

To get the algebraic formulation, we consider the set of Lagrange basis functions $\{\boldsymbol{\varphi}_i^{(j)}\}_{j=1}^{N_i}$ of V_{h_i}, $i = 1,2$. Then, for each $t \in (0,T\}$, we write the Galerkin representation of a vector $\mathbf{v}_{h_i} \in V_{h_i}$ as

$$
\mathbf{v}_i(\mathbf{x},t) = \sum_{j=1}^{N_i} \mathbf{v}_i^{(j)}(t)\boldsymbol{\varphi}_i^{(j)}(\mathbf{x}) \quad \text{with } \mathbf{v}_{N_i}(t) = (\mathbf{v}_i^{(1)}(t),\ldots,\mathbf{v}_i^{(N_i)}(t))^T \in \mathbb{R}^{N_i}.
$$

Moreover, we introduce the mass matrices $(\mathbb{M}_i)_{k,j} = \int_{\Omega_i} \boldsymbol{\varphi}_i^{(j)} \cdot \boldsymbol{\varphi}_i^{(k)} d\Omega_i$. Using the notation of the previous Section for the local matrices, we can then write $\mathbb{M}_{ii} = \mathbb{M}_i(\mathcal{I}_i,\mathcal{I}_i)$, $\mathbb{M}_{i,\Gamma_i} = \mathbb{M}_i(\mathcal{I}_i,\mathcal{I}_\Gamma)$ and $\mathbb{M}_{\Gamma_i,i} = \mathbb{M}_i(\mathcal{I}_\Gamma,\mathcal{I}_i)$.

Then, problem (31) becomes: for each $k \geq 0$ and $t \in (0,T\}$, find $\mathbf{u}_{N_1}^{k+1}(t;\boldsymbol{\mu})$ such that:

$$
\begin{cases}
\mathbb{M}_{11}\dfrac{d}{dt}\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) + \mathbb{A}_{11}(\boldsymbol{\mu})\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \mathbf{f}_1(t;\boldsymbol{\mu}) - \mathbb{M}_{1,D}\dfrac{d}{dt}\mathbf{g}_{1,D}(t;\boldsymbol{\mu}) - \mathbb{A}_{1,D}(\boldsymbol{\mu})\mathbf{g}_{1,D}(t;\boldsymbol{\mu}) \\
\qquad\qquad - \mathbb{M}_{1,\Gamma_1}\dfrac{d}{dt}\mathbf{u}_{\Gamma_1}^{k+1}(t;\boldsymbol{\mu}) - \mathbb{A}_{1,\Gamma_1}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_1}^{k+1}(t;\boldsymbol{\mu}), \\
\mathbf{u}_{\Gamma_1}^{k+1}(t;\boldsymbol{\mu}) = \mathbf{u}_{\Gamma_2}^k(t;\boldsymbol{\mu}) \\
\mathbf{u}_1^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0|\Omega_1}(\boldsymbol{\mu}).
\end{cases} \tag{33}
$$

However, as for the steady case, considering the lifting operation (22), for each $t \in (0,T\}$, in practical implementation we solve the following system:

$$\begin{cases} \left(\mathbb{M}_1 \frac{d}{dt} + \mathbb{A}_1(\boldsymbol{\mu})\right) \tilde{\mathbf{u}}_{N_1}^{k+1}(t;\boldsymbol{\mu}) = \mathbf{f}_1(t;\boldsymbol{\mu}) - \left(\mathbb{M}_1 \frac{d}{dt} + \mathbb{A}_1(\boldsymbol{\mu})\right) \mathbf{g}_D(t;\boldsymbol{\mu}) - \left(\mathbb{M}_1 \frac{d}{dt} + \mathbb{A}_1(\boldsymbol{\mu})\right) \mathbf{u}_{\Gamma_1}^{k+1}(t;\boldsymbol{\mu}) \\ \mathbf{u}_{\Gamma_1}^{k+1}(t;\boldsymbol{\mu}) = \mathbf{u}_{\Gamma_2}^{k}(t;\boldsymbol{\mu}) \\ \tilde{\mathbf{u}}_{N_1}^{k+1}(0;\boldsymbol{\mu}) = \tilde{\mathbf{u}}_{0,N_1}(\boldsymbol{\mu}), \end{cases} \tag{34}$$

where $\tilde{\mathbf{u}}_{0,N_1}(\boldsymbol{\mu}) = \mathbf{u}_{0_{|\Omega_1}}(\boldsymbol{\mu}) - \mathbf{u}_{0_{|\Gamma_1}}(\boldsymbol{\mu})$.

Similarly, problem (32) is: for each $k \geq 0$ and $t \in (0, T\}$, find $\mathbf{u}_{N_2}^{k+1}(t;\boldsymbol{\mu})$, such that

$$\begin{cases} \mathbb{M}_{22}\dfrac{d}{dt}\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) + \mathbb{A}_{22}(\boldsymbol{\mu})\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) + \mathbb{M}_{2,\Gamma_2}\dfrac{d}{dt}\mathbf{u}_{\Gamma_2}^{k+1}(t;\boldsymbol{\mu}) + \mathbb{A}_{2,\Gamma_2}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_2}^{k+1}(t;\boldsymbol{\mu}) \\ \qquad\qquad = \mathbf{f}_2(t;\boldsymbol{\mu}) - \mathbb{M}_{2,D}\dfrac{d}{dt}\mathbf{g}_{2,D}(t;\boldsymbol{\mu}) - \mathbb{A}_{2,D}(\boldsymbol{\mu})\mathbf{g}_{2,D}(t;\boldsymbol{\mu}) \\ \mathbb{M}_{\Gamma_2,2}\dfrac{d}{dt}\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) + \mathbb{M}_{\Gamma_2,\Gamma_2}\dfrac{d}{dt}\mathbf{u}_{\Gamma_2}^{k+1}(t;\boldsymbol{\mu}) + \mathbb{A}_{\Gamma_2,2}(\boldsymbol{\mu})\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}) + \mathbb{A}_{\Gamma_2,\Gamma_2}(t;\boldsymbol{\mu})\mathbf{u}_{\Gamma_2}^{k+1}(t;\boldsymbol{\mu}) = \\ \qquad \mathbf{f}_{\Gamma_2}(t;\boldsymbol{\mu}) + \mathbf{f}_{\Gamma_1}(t;\boldsymbol{\mu}) - \mathbb{M}_{\Gamma_1,1}\dfrac{d}{dt}\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) - \mathbb{M}_{\Gamma_1,\Gamma_1}\dfrac{d}{dt}\mathbf{u}_{\Gamma_1}^{k+1}(t;\boldsymbol{\mu}) \\ \qquad - \mathbb{A}_{\Gamma_1,1}(\boldsymbol{\mu})\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) - \mathbb{A}_{\Gamma_1,\Gamma_1}(\boldsymbol{\mu})\mathbf{u}_{\Gamma_1}^{k+1}(t;\boldsymbol{\mu}) \\ \mathbf{u}_2^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0_{|\Omega_2}}(\boldsymbol{\mu}) \end{cases} \tag{35}$$

Furthermore, we define the set of Lagrange basis functions of Y'_{h_i}, *i.e.* $\{\boldsymbol{\phi}_i^{(j)}\}_{j=1}^{N_{i,Y}}$, and we write the residual $\mathbf{r}_{h_i} \in Y'_{h_i}$ with respect to them, as for the steady case:

$$\mathbf{r}_{h_i}(t,\mathbf{x};\boldsymbol{\mu}) = \sum_{j=1}^{N_{i,Y}} \mathbf{r}_i^{(j)}(t;\boldsymbol{\mu})\boldsymbol{\phi}_i^{(j)}(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma_i, i = 1, 2,$$

where the coefficients $\mathbf{r}_i^{(j)}(t;\boldsymbol{\mu})$ can be obtained through

$$\mathbf{r}_i^{(j)}(t;\boldsymbol{\mu}) = \int_{\Omega_i} \frac{\partial \mathbf{u}_i^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathcal{R}_i\boldsymbol{\eta}_{h_i} d\Omega_i + a_i(\mathbf{u}_i, \overline{\mathcal{R}}_i\boldsymbol{\eta}_i^{(j)};\boldsymbol{\mu}) - \mathcal{F}_i(\overline{\mathcal{R}}_i\boldsymbol{\eta}_i^{(j)}\boldsymbol{\mu}) \quad j = 1, \dots, N_{i,Y}, i = 1, 2. \tag{36}$$

The algebraic formulation of problem (32) is finally found summing up the equations of (35), as for the steady case, *i.e.*

$$\begin{cases} \left(\mathbb{M}_2 \frac{d}{dt} + \mathbb{A}_2\right) \mathbf{u}_{N_2}^{k+1}(t;\boldsymbol{\mu}) = \mathbf{f}_2(t;\boldsymbol{\mu}) + \mathbf{r}_{N_2}^{k+1}(t;\boldsymbol{\mu}) \\ \mathbf{r}_{N_2}^{k+1}(t;\boldsymbol{\mu}) = -\mathbf{r}_{N_1}^{k+1}(t;\boldsymbol{\mu}) \\ \mathbf{u}_{N_2}^{k+1}(0;\boldsymbol{\mu}) = \mathbf{u}_{0,N_2}(\boldsymbol{\mu}), \end{cases} \tag{37}$$

where $\mathbf{u}_{0,N_2}(\boldsymbol{\mu}) = \mathbf{u}_{0_{\Omega_2}}(\boldsymbol{\mu})$ and $\mathbf{r}_{N_1}^{k+1}(t;\boldsymbol{\mu})$ can be computed as

$$\mathbf{r}_{N_1}^{k+1}(t;\boldsymbol{\mu}) = \left(\mathbb{M}_1 \frac{d}{dt}\mathbf{u}_{N_1}^{k+1}(t;\boldsymbol{\mu}) + \mathbb{A}_1(\boldsymbol{\mu})\mathbf{u}_{N_1}^{k+1}(t;\boldsymbol{\mu}) - \mathbf{f}_1(\boldsymbol{\mu})\right)_{|\Gamma}. \tag{38}$$

The time discretization can instead be handled using different numerical schemes [2, 5, 6]. In this work we consider a backward differentiation formula (BDF). Indeed, calling n the index accounting for a fixed time instant $t^n = n\Delta t$, we set the total number of selected time instants as N_t and with $\Delta t = \frac{T}{N_t}$ the time step, so that we can approximate

$$\mathbf{u}_i(t^n;\boldsymbol{\mu}_i) \simeq \mathbf{u}_i^n(\boldsymbol{\mu}_i) \quad \forall n = 0, \dots, N_t.$$

The time derivative is, therefore,

$$\frac{\partial \mathbf{u}_i^{n+1}(\boldsymbol{\mu}_i)}{\partial t} \simeq \frac{\mathbf{u}_i^{n+1}(\boldsymbol{\mu}_i) - \mathbf{u}_i^{n}(\boldsymbol{\mu}_i)}{\Delta t} \qquad \forall n = 0, \dots, N_t - 1.$$

Finally, we can define a new formulation of (34) and (37), the one used in the reduction step, *i.e.*: for each $k \geq 0$, $n = 0, \dots, N_t - 1$, find $\mathbf{u}_{N_i}^{n+1,k+1}(\boldsymbol{\mu}) \in \mathbb{R}^{N_i}$, such that

$$\begin{cases} \left(\dfrac{\mathbb{M}_1}{\Delta t} + \mathbb{A}_1(\boldsymbol{\mu})\right) \tilde{\mathbf{u}}_{N_1}^{n+1,k+1}(\boldsymbol{\mu}) = \mathbf{f}_1^{n+1}(\boldsymbol{\mu}) - \left(\dfrac{\mathbb{M}_1}{\Delta t} + \mathbb{A}_1(\boldsymbol{\mu})\right) \mathbf{g}_D^{n+1}(\boldsymbol{\mu}) \\ \qquad\qquad - \left(\dfrac{\mathbb{M}_1}{\Delta t} + \mathbb{A}_1(\boldsymbol{\mu})\right) \mathbf{u}_{1_{\Gamma_1}}^{n+1,k+1}(\boldsymbol{\mu}) + \left(\dfrac{\mathbb{M}_1}{\Delta t} + \mathbb{A}_1(\boldsymbol{\mu})\right) \tilde{\mathbf{u}}_{N_1}^{n}(\boldsymbol{\mu}) \\ \mathbf{u}_{\Gamma_1}^{n+1,k+1}(\boldsymbol{\mu}) = \mathbf{u}_{\Gamma_2}^{n+1,k}(\boldsymbol{\mu}) \\ \tilde{\mathbf{u}}_{N_1}^{0,k+1}(0;\boldsymbol{\mu}) = \tilde{\mathbf{u}}_{0,N_1}(\boldsymbol{\mu}), \end{cases} \tag{39}$$

and

$$\begin{cases} \left(\frac{\mathbb{M}_2}{\Delta t} + \mathbb{A}_2\right) \mathbf{u}_{N_2}^{n+1,k+1}(\boldsymbol{\mu}) = \mathbf{f}_2^{n+1}(\boldsymbol{\mu}) + \left(\frac{\mathbb{M}_2}{\Delta t} + \mathbb{A}_2\right) \mathbf{u}_{N_2}^{n}(\boldsymbol{\mu}) + \mathbf{r}_{N_2}^{n+1,k+1}(\boldsymbol{\mu}) \\ \mathbf{r}_{N_2}^{n+1,k+1}(\boldsymbol{\mu}) = -\mathbf{r}_{N_1}^{n+1,k+1}(\boldsymbol{\mu}) \\ \mathbf{u}_{N_2}^{0,k+1}(\boldsymbol{\mu}) = \mathbf{u}_{0,N_2}(\boldsymbol{\mu}), \end{cases} \tag{40}$$

where $\mathbf{r}_{N_i}^{n+1,k+1}$ are the residual at time instant t^n. Hereon, with an abuse of notation, we will call $\mathbf{u}_i$ and $\mathbf{r}_i$ the algebraic counterpart of $\mathbf{u}_{N_i}$ and $\mathbf{r}_{N_i}$.

2.3. Non-conforming discretization

In this Subsection we present the algebraic formulations of both steady and unsteady coupled models when interface non-conforming is considered. The principal difference with respect to previous descriptions is in the interpolation or projection of the interface conditions.

We define, therefore, two a-priori independent and different discretizations on Ω_i through two families of triangulations $\mathcal{T}_{h_1} = \cup_m T_{1,m}$ in Ω_1 and $\mathcal{T}_{h_2} = \cup_m T_{2,m}$ in Ω_2, meaning that different simplices or quads, or both of them, can be used to create the two meshes. Moreover, different mesh size h_1 and h_2 or different polynomial degrees p_1 or p_2 can be considered. Then, we call Γ_1 and Γ_2 the internal interfaces induced by $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ of Ω_1 and Ω_2, respectively: we talk of signal interpolation if $\Gamma_1 = \Gamma_2$ and of signal projection if $\Gamma_1 \neq \Gamma_2$.

The FE spaces $X_{h_i}^{q_i}$, V_{h_i}, $V_{h_i}^0$, Y_{h_i} and Λ_{h_i}, and their relative dimensions can be defined as in Subsection 2.1. The same can be done also for the linear and continuous discrete lifting operator $\mathcal{R}_{h_i}$.

We need, instead, to introduce two independent interpolation or projection operators able to exchange information between the independent grids on the interface Γ, namely

$$\Pi_{12} : Y_{h_2} \to Y_{h_1} \quad \text{and} \quad \Pi_{21} : Y_{h_1} \to Y_{h_2}.$$

In the non-conforming case, if Γ_1 and Γ_2 coincide, such operators could be the classical Lagrange interpolation operators, while when the mesh are conforming Π_{jk} are the identity operators. Instead, if the mesh are non-conforming and $\Gamma_1 \neq \Gamma_2$, Π_{12} and Π_{21} could be *e.g.* Rescaled Localized Radial Basis Function, as for the INTERNODES [4, 17, 18].

Therefore, in the steady case, recalling the discrete weak formulations 15 and 16 of the two sub-problems, we can define the Dirichlet and Neumann interface conditions as: for each $k \geq 0$,

$$\mathbf{u}_1^{k+1}(\boldsymbol{\mu}) = \Pi_{12} \mathbf{u}_2^{k}(\boldsymbol{\mu}) \tag{41}$$

and

$$a_2(\mathbf{u}_2^{k+1}(\boldsymbol{\mu}), \mathcal{R}_2 \boldsymbol{\eta}_{h_2}; \boldsymbol{\mu}) = \mathcal{F}_2(\mathcal{R}_2 \boldsymbol{\eta}_{h_2}; \boldsymbol{\mu}) + \Pi_{21} \left(\mathcal{F}_1(\mathcal{R}_1 \boldsymbol{\eta}_{h_1}; \boldsymbol{\mu}) - a_1(\mathbf{u}_1^{k+1}(\boldsymbol{\mu}), \mathcal{R}_1 \boldsymbol{\eta}_{h_1}; \boldsymbol{\mu}) \right), \tag{42}$$

respectively. Moreover, according to (17), we can substitute equation (42) with

$$\mathbf{r}_{N_2}^{k+1} = -\Pi_{21}\mathbf{r}_{N_1}^{k+1} \quad \text{on } \Gamma_2. \tag{43}$$

In a similar way, since the discrete weak formulation of the time-dependent sub-problems are 31 and 32, we get the Dirichlet and Neumann interface data for the unsteady case: for each $t \in (0, T\}$ and $k \geq 0$

$$\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}) = \Pi_{12}\mathbf{u}_2^{k}(t;\boldsymbol{\mu}) \tag{44}$$

and

$$\int_{\Omega_2} \frac{\partial \mathbf{u}_2^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathcal{R}_2\boldsymbol{\eta}_{h_2} d\Omega_2 + a_2(\mathbf{u}_2^{k+1}(t;\boldsymbol{\mu}), \mathcal{R}_2\boldsymbol{\eta}_{h_2};\boldsymbol{\mu}) = \mathcal{F}_2(\mathcal{R}_2\boldsymbol{\eta}_{h_2};\boldsymbol{\mu})$$
$$+ \Pi_{21}\left(\mathcal{F}_1(\mathcal{R}_1\boldsymbol{\eta}_{h_1};\boldsymbol{\mu}) - \int_{\Omega_1} \frac{\partial \mathbf{u}_1^{k+1}(t;\boldsymbol{\mu})}{\partial t}\mathcal{R}_1\boldsymbol{\eta}_{h_1} d\Omega_2 - a_1(\mathbf{u}_1^{k+1}(t;\boldsymbol{\mu}), \mathcal{R}_1\boldsymbol{\eta}_{h_1};\boldsymbol{\mu})\right). \tag{45}$$

As before, according to the residual definition (36), we can write (45) as

$$\mathbf{r}_{N_2}^{k+1}(t;\boldsymbol{\mu}) = -\Pi_{21}\mathbf{r}_{N_1}^{k+1}(t;\boldsymbol{\mu}) \quad \text{on } \Gamma_2. \tag{46}$$

Remark 4. When $\partial\Gamma_i \cap \partial\Omega_{i,D} \neq 0$, the residual $\mathbf{r}_{N_i}$ should be corrected to take into account the interpolation process on all the degrees of freedom of Γ_i, including those on $\partial\Gamma_i$ (see *e.g.* [4]). Even if the reduced technique presented in this paper will work in both cases, hereon we will consider only the $\partial\Gamma_i \cap \partial\Omega_{i,D} = 0$ one.

To get an algebraic representation of the residuals, we need to use the *interface mass matrices* $\mathbb{M}_{\Gamma_i}$, that we define through the Lagrange basis functions of Y_{h_i}, *i.e* $\{\boldsymbol{\psi}_i^{(j)}\}_{j=1}^{N_{i,Y}}$. Indeed, we can set

$$(\mathbb{M}_{\Gamma_i})_{jk} = (\boldsymbol{\psi}_i^{(k)}, \boldsymbol{\psi}_i^{(j)})_{L^2(\Gamma_i)}, \quad j,k = 1,\dots,N_{i,Y}, i = 1,2.$$

Considering the canonical *dual basis* of Y'_{h_i}, it can be seen [19] that Y'_{h_i} and Y_{h_i} are the same finite dimensional linear space and

$$\boldsymbol{\phi}_i^{(j)} = \sum_{k=1}^{N_{i,Y}} (\mathbb{M}_{\Gamma_i}^{-1})_{kj}\boldsymbol{\psi}_k^{(j)}, \quad j = 1,\dots,N_{i,Y}.$$

Therefore, by expanding $\mathbf{r}_{N_i} \in Y'_{h_i}$ with respect to the dual basis (19), we can finally express the residual through a corresponding vector in the dual space Y'_{h_i}, meaning

$$\mathbf{r}_{N_i}(\mathbf{x};\boldsymbol{\mu}) = \sum_{j=1}^{N_{i,Y}} \left(\sum_{k=1}^{N_{i,Y}} (\mathbb{M}_{\Gamma_i}^{-1})_{jk}\mathbf{r}_i^{(k)}(\boldsymbol{\mu})\right)\boldsymbol{\psi}_i^{(j)}(\mathbf{x}) = \sum_{j=1}^{N_{i,Y}} \mathbf{z}_i^{(j)}(\boldsymbol{\mu})\boldsymbol{\psi}_i^{(j)}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_i,\ i = 1,2.$$

Calling $\mathbf{z}_i(\boldsymbol{\mu}) = (\mathbf{z}_i^{(1)}(\boldsymbol{\mu}),\dots,\mathbf{z}_i^{(N_{i,Y})}(\boldsymbol{\mu}))^T \in \mathbb{R}^{N_{i,Y}}$ and $\mathbf{r}_i(\boldsymbol{\mu}) = (\mathbf{r}_i^{(1)}(\boldsymbol{\mu}),\dots,\mathbf{r}_i^{(N_{i,Y})})^T \in \mathbb{R}^{N_{i,Y}(\boldsymbol{\mu})}$, the interface mass matrix $\mathbb{M}_{\Gamma_i}$ and its inverse become the transfer matrices form the Lagrange basis to the dual one and viceversa, *i.e.*

$$\mathbf{z}_i(\boldsymbol{\mu}) = \mathbb{M}_{\Gamma_i}^{-1}\mathbf{r}_i(\boldsymbol{\mu}). \tag{47}$$

The same results can be extended also to the unsteady case, including the time dependency of the residual terms.

Then, calling $\mathbb{R}_{jk}$ the rectangular matrices associated with Π_{jk}, we can finally obtained the algebraic formulation of the Dirichlet and Neumann terms for the steady case, *i.e.* (41) and (42): for each $k \geq 0$

$$\mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}) = \mathbb{R}_{12}\mathbf{u}_{\Gamma_2}^{k+1}(\boldsymbol{\mu}) \tag{48}$$

and

$$\mathbf{r}_{N_2}^{k+1}(\boldsymbol{\mu}) = -\mathbb{M}_{\Gamma_2}\mathbb{R}_{21}\mathbb{M}_{\Gamma_1}^{-1}\mathbf{r}_{N_1}^{k+1}(\boldsymbol{\mu}), \tag{49}$$

The same results can be stated also for the unsteady terms, including the time dependency of the considered quantity, *i.e.* we impose for each $n = 0, \dots, N_t - 1$ and $k \geq 0$ that

$$\mathbf{u}_{\Gamma_1}^{n+1,k+1}(\boldsymbol{\mu}) = \mathbb{R}_{12}\mathbf{u}_{\Gamma_2}^{n+1,k+1}(\boldsymbol{\mu}) \tag{50}$$

and

$$\mathbf{r}_{N_2}^{n+1,k+1}(\boldsymbol{\mu}) = -\mathbb{M}_{\Gamma_2}\mathbb{R}_{21}\mathbb{M}_{\Gamma_1}^{-1}\mathbf{r}_{N_1}^{n+1,k+1}, \tag{51}$$

In Section 4 we will use DEIM matrices instead of $\mathbb{R}_{jk}$ as interpolation and/or projection method, while we refer to [4] for the corresponding INTERNODES interpretation. Note that the conforming interface case can be recovered by taking $\mathbb{R}_{12}$ and $\mathbb{M}_{\Gamma_2}\mathbb{R}_{21}\mathbb{M}_{\Gamma_1}^{-1}$ equal to the identity matrix.

3. Reduce order formulation

The proposed strategy aims at reducing separately the two sub-problems obtained from the application of the *sub-structuring domain decomposition method*, including the interface conditions. In particular, this model order reduction technique can be seen as an extension of the one proposed in [1] and is, therefore, modular, *i.e.* it combines different RB method on each sub-problems and, in particular, the DEIM to treat both interface conditions, defining independent reduced order representation of the involved quantities.

Indeed, for each $k \geq 0$, we first approximate the FOM solution of the master and slave models by means of a small number of basis functions selected through a POD-DEIM procedure on the FOM solution snapshots. Moreover, using the DEIM, we identify a suitable set of basis functions for the Dirichlet and Neumann data snaphosts, and use them to interpolate or project such data across conforming and non-conforming interface grids. Lastly, we sub-iterate between the two sub-problems reduced forms until convergence of the FOM approximated solutions. In particular, the convergence criteria is defined through the l_2-norm of FOMs solution difference at the domains interface, *i.e.* for a chosen tolerance ϵ, the sub-iterations convergence is reached when

$$\|\mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}) - \mathbf{u}_{\Gamma_2}^{k+1}(\boldsymbol{\mu})\|_2 < \epsilon. \tag{52}$$

Note that the same criteria can be applied also for the unsteady case for each time-instant t^n, including the n-index in the equality.

The reduced form of the master and slave problems in the steady and unstedy case can be found in Subsections 3.1 and 3.2, respectively, while we derive the parameters dependent Dirichlet and Neumann data reductions in Section 4.

For the sake of notation, we remark that hereon $\mathbf{u}_i$, $i = 1, 2$ refer to the algebraic representation of the FOM solutions, meaning $\mathbf{u}_{N_i}$.

3.1. Steady master and slave reduced order problems

We define the reduced version of problems (23) and (24) through POD-Galerkin approach [7]. Therefore, in the offline stage we collect the set of snapshots solving the sub-FOMs for a suitable set of parameter values. In particular, we choose as snapshots the FOM master and slave solutions *at convergence* of the sub-iterations, *i.e.* $\tilde{\mathbf{S}}_1 = \{\tilde{\mathbf{u}}_1(\boldsymbol{\mu}_\ell), \boldsymbol{\mu}_\ell \in \mathscr{P}^d\}$ and $\mathbf{S}_2 = \{\mathbf{u}_2(\boldsymbol{\mu}_\ell), \boldsymbol{\mu}_\ell \in \mathscr{P}^d\}$, respectively. The paramaters samplings are usually done considering a latine hypercube method [8, 9].

Remark 5. The master and slave solution snapshots can be directly collected from the FOM computations when *(i)* conforming discretizations are considered in the two sub-domains or *(ii)* when interpolation/projection methods are implemented to handle the discretization non-conformity, *e.g.* MORTAT methods or INTERNODES. Instead, as already stated in Section 2, in case of interface non-conformity, in this paper in the offline stage we solved the FOMs twice, one for each chosen discretizations. Then, the snapshots are the master solutions obtained from the first computation, *i.e.* considering the discretization chosen for the master domain, and the slave solutions obtained from the second computation, *i.e.* with the discretization chosen for the slave domain.

Remark 6. The k index is omitted when quantity at convergence of the sub-iterations are considered.

The POD techniques is, then, applied to each set of snapshots and a corresponding set of reduced basis functions is computed and stored. Defining as $\mathbb{V}_i \in \mathbb{R}^{N_i \times n_i}$, $n_i \ll N_i$, the matrices whose column yield the obtained basis functions, in the online phase we can approximate the FOM solutions as

$$\tilde{\mathbf{u}}_1(\boldsymbol{\mu}) \approx \mathbb{V}_1 \tilde{\mathbf{u}}_{n_1}(\boldsymbol{\mu})$$

and

$$\mathbf{u}_2(\boldsymbol{\mu}) \approx \mathbb{V}_2 \mathbf{u}_{n_2}(\boldsymbol{\mu}).$$

Projecting problems (23) and (24) onto the reduced spaces defined by $\mathbb{V}_i$, starting from an initial guess $\tilde{\mathbf{u}}^0_{n_i}(\boldsymbol{\mu})$ and $\mathbf{u}^0_{n_2}(\boldsymbol{\mu})$, in the online phase, for each $k \geq 0$, we search for the reduced solutions $\tilde{\mathbf{u}}^{k+1}_{n_1}(\boldsymbol{\mu}) \in \mathbb{R}^{n_1}$ and $\mathbf{u}^{k+1}_{n_2}(\boldsymbol{\mu}) \in \mathbb{R}^{n_2}$ such that

$$\begin{cases} \mathbb{A}_{n_1}(\boldsymbol{\mu})\tilde{\mathbf{u}}^{k+1}_{n_1}(\boldsymbol{\mu}) = \mathbf{f}_{n_1}(\boldsymbol{\mu}) - \mathbb{V}_1^T \mathbb{A}_1(\boldsymbol{\mu})\mathbf{g}_D(\boldsymbol{\mu}) - \mathbb{V}_1^T \mathbb{A}_1(\boldsymbol{\mu})\mathbf{u}^{k+1}_{\Gamma_1}(\boldsymbol{\mu}) \\ \mathbf{u}^{k+1}_{\Gamma_1}(\boldsymbol{\mu}) = \mathbb{R}_{12}\mathbf{u}^k_{\Gamma_2}(\boldsymbol{\mu}). \end{cases} \tag{53}$$

and

$$\begin{cases} \mathbb{A}_{n_2}(\boldsymbol{\mu})\mathbf{u}^{k+1}_{n_2}(\boldsymbol{\mu}) = \mathbf{f}_{n_2}(\boldsymbol{\mu}) + \mathbb{V}_2^T \mathbf{r}^{k+1}_{N_2}(\boldsymbol{\mu}) \\ \mathbf{r}^{k+1}_{N_2}(\boldsymbol{\mu}) = -\mathbb{M}_{\Gamma_2}\mathbb{R}_{21}\mathbb{M}^{-1}_{\Gamma_1}\mathbf{r}^{k+1}_{N_1}(\boldsymbol{\mu}), \end{cases} \tag{54}$$

where $\mathbb{A}_{n_i}(\boldsymbol{\mu}) = \mathbb{V}_i^T \mathbb{A}_i(\boldsymbol{\mu})\mathbb{V}_i$ and $\mathbf{f}_{n_i}(\boldsymbol{\mu}) = \mathbb{V}_i^T \mathbf{f}_i(\boldsymbol{\mu})$, $i = 1, 2$.

Remark 7. The presence of nonlinear terms in the master and slave formulations can be handle through suitable hyper-reduction techniques, *e.g* DEIM [10, 11, 12]. For simplicity, in this paper we consider only the linear case.

3.2. Unsteady master and slave reduced order problems

As for the steady case, we search for the reduced formulation of problems (39) and (40) applying POD-Galerkin approach to the single sub-problem. Then, in the online phase, for each time instant t^n, we sub-iterate between the solution of the obtained ROMs until the convergence criteria (52) is satisfied. Moreover, the time variable can be considered as an additional parameter of the two sub-problems.

Thus, we select as snapshots the FOMs solution at convergence of the sub-iterations, for each time step of the simulation, *i.e.* $\tilde{\mathbf{S}}_1 = \{\tilde{\mathbf{u}}_1^{t^1}(\boldsymbol{\mu}_\ell), \dots \tilde{\mathbf{u}}_1^{t^{N_t}}(\boldsymbol{\mu}_\ell); \boldsymbol{\mu}_\ell \in \mathscr{P}^d\}$ for the master problem and $\mathbf{S}_2 = \{\tilde{\mathbf{u}}_2^{t^1}(\boldsymbol{\mu}_\ell), \dots \tilde{\mathbf{u}}_2^{t^{N_t}}(\boldsymbol{\mu}_\ell); \boldsymbol{\mu}_\ell \in \mathscr{P}^d\}$ for the slave problem. The total number of snapshots for each set will be, therefore, the product between the number of time steps N_t and the number of selected parameters values.

As for the time-indepedent case, POD is applied to each set of snapshots and the $\mathbb{V}_i$ matrices are computed. Then, operating a Galerkin projection on the sub-spaces defined from $\mathbb{V}_i$, we can obtained the reduced forms of problems (39) and (40), *i.e*, for each $k \geq 0$ and $n = 0, \dots, N_t - 1$, find $\tilde{\mathbf{u}}^{n+1,k+1}_{n_1}(\boldsymbol{\mu}) \in \mathbb{R}^{n_1}$

and $\mathbf{u}_{n_2}^{n+1,k+1}(\boldsymbol{\mu}) \in \mathbb{R}^{n_2}$ such that

$$
\begin{cases}
\left(\dfrac{\mathbb{M}_{n_1}}{\Delta t} + \mathbb{A}_{n_1}(\boldsymbol{\mu})\right) \tilde{\mathbf{u}}_{n_1}^{n+1,k+1}(\boldsymbol{\mu}) = \mathbf{f}_{n_1}^{n+1}(\boldsymbol{\mu}) - \mathbb{V}_1^T\left(\dfrac{\mathbb{M}_1}{\Delta t} + \mathbb{A}_1(\boldsymbol{\mu})\right)\mathbf{g}_D^{n+1}(\boldsymbol{\mu}) \\
\qquad\qquad - \mathbb{V}_1^T\left(\dfrac{\mathbb{M}_1}{\Delta t} + \mathbb{A}_1(\boldsymbol{\mu})\right)\mathbf{u}_{\Gamma_1}^{n+1,k+1}(\boldsymbol{\mu}) + \left(\dfrac{\mathbb{M}_{n_1}}{\Delta t} + \mathbb{A}_{n_1}(\boldsymbol{\mu})\right)\tilde{\mathbf{u}}_{n_1}^{n}(\boldsymbol{\mu}) \\
\mathbf{u}_{\Gamma_1}^{n+1,k+1}(\boldsymbol{\mu}) = \mathbb{R}_{12}\mathbf{u}_{\Gamma_2}^{n+1,k}(\boldsymbol{\mu}) \\
\tilde{\mathbf{u}}_{n,1}^{0,k+1}(0;\boldsymbol{\mu}) = \tilde{\mathbf{u}}_{0,n_1}(\boldsymbol{\mu}),
\end{cases}
\tag{55}
$$

and

$$
\begin{cases}
\left(\frac{\mathbb{M}_{n_2}}{\Delta t} + \mathbb{A}_{n_2}\right)\mathbf{u}_{n_2}^{n+1,k+1}(\boldsymbol{\mu}) = \mathbf{f}_{n_2}^{n+1}(\boldsymbol{\mu}) + \left(\frac{\mathbb{M}_{n_2}}{\Delta t} + \mathbb{A}_{n_2}\right)\mathbf{u}_{n_2}^{n}(\boldsymbol{\mu}) + \mathbb{V}_2^T\mathbf{r}_{N_2}^{n+1,k+1}(\boldsymbol{\mu}) \\
\mathbf{r}_{N_2}^{n+1,k+1}(\boldsymbol{\mu}) = -\mathbb{M}_{\Gamma_2}\mathbb{R}_{21}\mathbb{M}_{\Gamma_1}^{-1}\mathbf{r}_{N_1}^{n+1,k+1}(\boldsymbol{\mu}) \\
\mathbf{u}_{n_2}^{0,k+1}(\boldsymbol{\mu}) = \mathbf{u}_{0,n_2}(\boldsymbol{\mu}),
\end{cases}
\tag{56}
$$

where $\mathbb{M}_{n_i} = \mathbb{V}_i^T\mathbb{M}_i\mathbb{V}_i$, $\mathbb{A}_{n_i}(\boldsymbol{\mu}) = \mathbb{V}_i^T\mathbb{A}_i(\boldsymbol{\mu})\mathbb{V}_i$, $\mathbf{f}_{n_i}(\boldsymbol{\mu}) = \mathbb{V}_i^T\mathbf{f}_i(\boldsymbol{\mu})$ and $\tilde{\mathbf{u}}_{0,n_1}(\boldsymbol{\mu})$ $\mathbf{u}_{0,n_2}(\boldsymbol{\mu})$ are the projection of the initial solution $\mathbf{u}_{0,N_1}(\boldsymbol{\mu})$ and $\mathbf{u}_{0,N_2}(\boldsymbol{\mu})$ on the master and slave reduced basis, respectively.

Differently from the time-independent case, here $\mathbf{f}_i^{n+1}(\boldsymbol{\mu})$ is time dependent and must, therefore, reassemble for each time step. Instead, the mass and stiffness matrices $\mathbb{M}_i$ and $\mathbb{A}_i(\boldsymbol{\mu})$ are time-independent and can be assemble only ones. In particular, $\mathbb{M}_i$ is reduced and stored during the offline stage, while $\mathbb{A}_i(\boldsymbol{\mu})$ is reduced and stored at the first time step.

4. Parametric interface data reduction

Dealing with interface conditions is usually an expensive task, especially when non-conforming grids and/or large domains and very fine discretizations are involved. Since the sub-problems are parameters dependent, the interface data naturally inherit the parameter dependency and DEIM [10, 11, 12, 13, 14, 15, 16] can be applied to reduce the dimension of such data but also as an interpolation methods to transfer the information across the interface grids.

Moreover, in the conforming case the DEIM can be used directly on the quantity of interest, *i.e.* on the interface solution - for Dirichlet data - and on the interface residual - for Neumann data - while, for non-conforming interface grids, the residual dual vectors must be involved to treat the Neumann terms. Recalling the interpolation operators $\mathbb{R}_{jk}$ defined in Subsection 2.3, for the sake of generalizations, in Subsection 4.1 and 4.2 we define the reduction of the Dirichlet and Neumann data, respectively, when non-conforming interface grids are considered.

4.1. Parametric Dirichlet data

The parametric Dirichlet data interpolation method used in this work is the same presented in [1]. Such techniques relies on the DEIM and can be applied on the same quantity in case of both conforming and non-conforming interface grids.

First, in the offline phase we collect the snapshots, *i.e.* we extract the interface Dirichlet data obtained for different instances of the parameters vectors from the first of the FOM computations, the one corresponding to the master domain discretizations in the non-conforming case. Moreover, we select only the Dirichlet data at convergence of the sub-iterations, as for the solution reductions - therefore, we omit the k dependency of the interface Dirichlet data in what follows -, namely

$$\mathbf{S}_D = \{\mathbf{u}_{\Gamma_1}(\boldsymbol{\mu}_\ell),\ (\boldsymbol{\mu}_\ell) \in \mathscr{P}^d\}.$$

A low-dimensional representation of the Dirichlet data, then, can be computed applying the POD basis functions $\boldsymbol{\Phi}_D$. Calling $M_1 \ll N_{1,\Lambda}$ the small dimension of such set, we can approximate the Dirichlet data as

$$\mathbf{u}_{\Gamma_1}(\boldsymbol{\mu}) \approx \boldsymbol{\Phi}_D\mathbf{u}_{1,M_1}(\boldsymbol{\mu}),$$

where $\mathbf{u}_{1,M_1}(\boldsymbol{\mu})$ is a vector of M_1 coefficients. Furthermore, with a greedy algorithm [14], we select iteratively M_1 indices

$$\mathcal{I}_{1,D} \subset \{1, \dots, N_{1,\Lambda}\}, \quad |\mathcal{I}_{1,D}| = M_1 \tag{57}$$

from the basis $\mathbf{\Phi}_D$ which minimize the interpolation error over the snapshots set according to the maximum norm. These indices, which are usually referred to as *magic points*, represents the degrees of freedom from which to extract the FOM Dirichlet data $\mathbf{u}_{\Gamma_{1,|\mathcal{I}_{1,D}}}$ to compute $\mathbf{u}_{1,M}$. In particular, in the online phase, $\mathbf{u}_{1,M}^{k+1}$ can be found solving the following linear system

$$\mathbf{\Phi}_{D_{|\mathcal{I}_{1,D}}} \mathbf{u}_{1,M}^{k+1}(\boldsymbol{\mu}) = \mathbf{u}_{\Gamma_{1_{|\mathcal{I}_{1,D}}}}^{k+1},$$

where $\mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}} \in \mathbb{R}^{M_1 \times M_1}$ represents the matrix with the $\mathcal{I}_{1,D}$ rows of $\mathbf{\Phi}_D$. Therefore, we can approximate the Dirichlet interface data as

$$\mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}) \approx \mathbf{\Phi}_D \mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}}^{-1} \mathbf{u}_{\Gamma_{1_{|\mathcal{I}_{1,D}}}}^{k+1}.$$

The interpolation or projection operation is get substituting $\mathbf{u}_{\Gamma_{1_{|\mathcal{I}_{1,D}}}}^{k+1}$ with the slave solution extracted in the DoFs corresponding to the magic points. Thus, given the position $\mathbf{p}_1$ of the DoFs corresponding to each index $i_{1,D} \in \mathcal{I}_{1,D}$ in Cartesian coordinates, we search for the corresponding DoFs in the slave interface, *i.e.* for the points $\mathbf{p}_2$ such that

$$\mathbf{p}_2 = \min_{\mathbf{p}_2^j \in \mathrm{DoFs}_{\Gamma_2}} (\mathrm{dist}(\mathbf{p}_1 - \mathbf{p}_2^j)).$$

Then, we can define the set of indices on the slave grids $\mathcal{I}_{2,D}$ corresponding to the indices in $\mathcal{I}_{1,D}$ that identify the DoFs of the slave interface in position $\mathbf{p}_2$, *i.e.*

$$\mathcal{I}_{2,D} = \{i_{2,D}^{i_{1,D}}\}_{i_{1,D} \in \mathcal{I}_{1,D}}.$$

Finally, in the online phase, the M_1 needed interpolants point to approximate the interface Dirichlet data directly on the master interface are the slave solutions values extracted on the DoFs with indices in $\mathcal{I}_{2,D}$, meaning

$$\mathbf{u}_{1_{\Gamma_1}}^{k+1}(\boldsymbol{\mu}) \approx \mathbf{\Phi}_D \mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}}^{-1} \mathbf{u}_{2_{|\mathcal{I}_{2,D}}}^{k}.$$

Note that $\mathbf{u}_{2_{|\mathcal{I}_{2,D}}}^{k}$ refers to the FOM solution of the slave models that must, therefore, be computed from the ROM solution $\mathbf{u}_{n_2}^k$ during the online phase. However, only part of the FOM slave solution is needed, meaning the one in the magic points. Therefore, one can assemble the FOM solution only in such points multiplying the ROM slave solutions for those rows of $\mathbb{V}_2$ corresponding to the magic points, *i.e.* for the matrix $\mathbb{V}_{2_{|\mathcal{I}_{2,D}}}$.

In this way, recalling the master ROM (53), the lifting term can be approximated as

$$\mathbb{V}_1^T \mathbb{A}_1(\boldsymbol{\mu}) \mathbf{u}_{\Gamma_1}^{k+1}(\boldsymbol{\mu}) \approx \mathbb{V}_1^T \mathbb{A}_1 \mathbf{\Phi}_D \mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}}^{-1} \mathbb{V}_{2_{|\mathcal{I}_{2,D}}} \mathbf{u}_{n_2}^k, \tag{58}$$

where the matrix product $\mathbb{V}_1^T \mathbb{A}_1 \mathbf{\Phi}_D \mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}}^{-1} \mathbb{V}_{2_{|\mathcal{I}_{2,D}}}$ does not depend on the solution and can be pre-computed and stored in the offline phase. Therefore, the $\mathbb{R}_{12}$ operator, in *both* conforming and non-conforming case, corresponds to the matrix product

$$\mathbb{R}_{12} = \mathbf{\Phi}_D \mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}}^{-1}.$$

Remark 8. Even if the snapshots are selected as the Dirichlet data at convergence of the sub-iterations, the reduced coupled model is solved sub-iterating between the reduced master and slave models. Therefore, we have added the k index to quantities computed in the online stage.

Remark 9. As for the FOM computation, an initial guess of the Dirichlet boundary conditions must be considered, but only on the magic points. Therefore, for $k = 0$, the approximated FOM solution on the magic points can be substituted with the FOM initial guess

$$\mathbb{V}_{2|\mathcal{I}_{2,D}} \mathbf{u}_{n_2}^0 = \mathbf{u}_{2|\mathcal{I}_{2,D}}^0,$$

for the time-independent case, and

$$\mathbb{V}_{2|\mathcal{I}_{2,D}} \mathbf{u}_{n_2}^{n+1,0} = \mathbf{u}_{2|\mathcal{I}_{2,D}}^{n+1,0}, \quad n = 0, \ldots, N_t - 1$$

for the time-dependent one.

Remark 10. Differently from the master and slave reductions, the DEIM interpolation is time-independent. Therefore, if the coupled model is unsteady and to take into account the time variations of the solution, the interface data at convergence of the sub-iterations for each time instant $n = 1, \ldots, N_t$ must be collected in the set of snapshots. Then, the interpolation of the Dirichlet data is the same as of the steady case.

4.2. Parametric Neumann data reduction

The DEIM used to interpolate the parametric Dirichlet interface conditions can be applied also the parametric Neumann interface conditions. Starting from the steady case, if the interface grids are conforming, for each $k \geq 0$,

$$\mathbf{r}_{N_2}^{k+1}(\boldsymbol{\mu}) = \mathbf{r}_{N_1}^{k+1}(\boldsymbol{\mu})$$

so that DEIM can be used on the interface residual. Instead, in the non-conforming case, the interface mass matrices are involved, as in (51). However, recalling definition (47) of $\mathbf{z}_i(\boldsymbol{\mu})$, *i.e.* the dual vectors of the interface residual, equation (51) can be substituted with

$$\mathbf{z}_2^{k+1}(\boldsymbol{\mu}) = -\mathbb{R}_{21}\mathbf{z}_1^{k+1}(\boldsymbol{\mu}), \tag{59}$$

which corresponds to equation (48). Therefore, vector $\mathbf{z}_2^{k+1}$ is the quantity to be reduced and reconstructed using the DEIM.

For the sake of generality, in this Subsection we derive the Neumann data approximation for the non-conforming case, but the same procedure holds also for the interface residual of the conforming case.

As in Subsection 4.1, for each $\boldsymbol{\mu}$ we compute the snapshots, *i.e.* the dual interface residual $\mathbf{z}_2(\boldsymbol{\mu})$ at convergence of the sub-iterations

$$\mathbf{S}_N = \{\mathbf{z}_2(\boldsymbol{\mu}_\ell), \boldsymbol{\mu}_\ell \in \mathscr{P}^d\}.$$

Applying the POD, a set of M_2 basis functions $\mathbf{\Phi}_N$ is found, being $M_2 \ll N_{2,\Lambda}$. The dual vector of the residual is, therefore, approximated as

$$\mathbf{z}_2(\boldsymbol{\mu}) \approx \mathbf{\Phi}_N \mathbf{z}_{2,M_2}(\boldsymbol{\mu}).$$

Furthermore, with a greedy algorithm, M_2 magic points are selected and their indices in the slave grid numbering are collected in the set

$$\mathcal{I}_{2,N} \subset \{1, \ldots, N_{2,\Lambda}\}, \quad |\mathcal{I}_{2,N}| = M_2.$$

Therefore, in the online phase we need to find $\mathbf{z}_{2,M_2}(\boldsymbol{\mu})$ solution of the linear system

$$\mathbf{\Phi}_{N|\mathcal{I}_{2,N}} \mathbf{z}_{2,M_2}^{k+1}(\boldsymbol{\mu}) = \mathbf{z}_{2|\mathcal{I}_{2,N}}^{k+1}(\boldsymbol{\mu}),$$

where $\mathbf{\Phi}_{N|\mathcal{I}_{2,N}}$ is the restriction of $\mathbf{\Phi}_N$ on the magic points, and the dual interface residual vector can be approximated as

$$\mathbf{z}_2^{k+1}(\boldsymbol{\mu}) \approx \mathbf{\Phi}_N \mathbf{\Phi}_{N|\mathcal{I}_{2,N}}^{-1} \mathbf{z}_{2|\mathcal{I}_{2,N}}^{k+1}(\boldsymbol{\mu}).$$

Then, $\mathbf{z}_{2|\mathcal{I}_{2,N}}^{k+1}(\boldsymbol{\mu})$ is substituted with the values of $\mathbf{z}_1^{k+1}(\boldsymbol{\mu})$ extracted on the magic points. To do this, we need first to select the set of indices $\mathcal{I}_{1,N}$ on the master interface Γ_1 corresponding to $\mathcal{I}_{2,N}$. Therefore, calling $\mathbf{p}_2$ the DoF in cartesian coordinated corresponding to the i-th index in $\mathcal{I}_{2,N}$, we search for

$$\mathbf{p}_1 = \min_{\mathbf{p}_i^j \in \mathrm{DoFs}_{\Gamma_1}} (\mathrm{dist}(\mathbf{p}_2 - \mathbf{p}_1^j)),$$

and we collect the corresponding index in the master grid numbering in

$$\mathcal{I}_{1,N} = \{i_{1,N}^{i_{2,N}}\}_{i_{2,N} \in \mathcal{I}_{2,N}}.$$

Thus, in the online phase, we get

$$\mathbf{z}_2^{k+1}(\boldsymbol{\mu}) \approx -\boldsymbol{\Phi}_N \boldsymbol{\Phi}_{N|\mathcal{I}_{2,N}}^{-1} \mathbf{z}_{1|\mathcal{I}_{1,N}}^{k+1}(\boldsymbol{\mu}).$$

Finally, considering the definition of $\mathbf{z}_2(\boldsymbol{\mu})$ in (47), we can recover the Neumann interface residual term of (54) as

$$\mathbb{V}_2^T \mathbf{r}_{N_2}^{k+1}(\boldsymbol{\mu}) \approx -\mathbb{V}_2^T \mathbb{M}_{\Gamma_2} \boldsymbol{\Phi}_N \boldsymbol{\Phi}_{N|\mathcal{I}_{2,N}}^{-1} \mathbb{M}_{\Gamma_{1|\mathcal{I}_{2,N}}} \mathbf{r}_{N_{1|\mathcal{I}_{1,N}}}^{k+1}(\boldsymbol{\mu}). \tag{60}$$

Note that matrix product $\mathbb{V}_2^T \mathbb{M}_{\Gamma_2} \boldsymbol{\Phi}_N \boldsymbol{\Phi}_{N|\mathcal{I}_{2,N}}^{-1} \mathbb{M}_{\Gamma_{1|\mathcal{I}_{2,N}}}$ does not dependent on the parameters and can be computed and stored in the offline phase.

Moreover, recalling the definition (25) of the interface residual, we can recover the dependency of such vectors with the reduced master solution, *i.e*

$$\mathbf{r}_{N_{1|\mathcal{I}_{1,N}}}^{k+1}(\boldsymbol{\mu}) \approx \left((\mathbb{A}_1(\boldsymbol{\mu})\mathbb{V}_1 \mathbf{u}_{n_1}^{k+1}(\boldsymbol{\mu}) - \mathbf{f}_1(\boldsymbol{\mu}))_{|\Gamma_1}\right)_{|\mathcal{I}_{1,N}}. \tag{61}$$

However, only the interface residual on the magic points is needed and must be computed, reducing the dimension $N_1 \times n_1$ of the matrix-vector operations $\mathbb{A}_1(\boldsymbol{\mu})\mathbb{V}_1\mathbf{u}_{n_1}^{k+1}(\boldsymbol{\mu})$ to $M_2 \times n_1$. Thus, similar to the Dirichlet interpolation or projection, the operator $\mathbb{R}_{21}$ here is represented by

$$\mathbb{R}_{21} = \boldsymbol{\Phi}_N \boldsymbol{\Phi}_{N|\mathcal{I}_{2,N}}^{-1},$$

and can be used with *both* conforming or non-conforming interface grids.

We summarize the interface DEIM reduction, considering both Dirichlet and Neumann processing, in algorithm 1, while the complete reduction of the two-way coupled model can be found in algorithm 2.

Remark 11. Note that similar results can be obtained for time-dependent coupled problems. In particular, equation (59) can be equally derived including the time index n, while the parametric Neumann data for each t^n must be included in the set of snapshots for the DEIM reduction. Then, the same procedure leads to equation (60), where the master residual definition (61) is to be substituted with the equivalent definition (38) for the unsteady case. In this regards, the DEIM interpolation can be considered as time-independent also for the Neumann data.

Remark 12. We remark that when the interface grids are conforming, meaning that $\Gamma_1 = \Gamma_2 = \Gamma$, a perfect match between the corresponding DoFs on the master and slave interface is found. However, this does not happened in the non-conforming case, *i.e.* when $\Gamma_1 \neq \Gamma_2$. In this regards, we are introducing an error both in Dirichlet and Neumann data approximations, especially when the interface discretizations are very different. Considering the numerical tests of Section 5, to minimized such error we suggest to consider a finer discretization on the slave domain than in the master one, since the Dirichlet approximation seems to suffer more from the interface difference than the Neumann one.

Moreover, given the smaller number of DoFs in the master interface than in the slave one - considering a coarser discretizations in the master domains, as just stated - as convergence criteria during the ROM solution we choose to consider the difference between the approximated FOM solution restricted at the master interface DoFs, *i.e.*

$$\|\mathbf{u}_{1|\mathcal{I}_{1,D}} - \mathbf{u}_{2|\mathcal{I}_{2,D}}\|_2 < \epsilon.$$

Algorithm 1 Interface DEIM procedure

procedure [ROM ARRAYS] = OFFLINE(FOM arrays, $\mathscr{P}_{train}$, ϵ_{tol_D}, ϵ_{tol_N}, tol)
    Dirichlet and Neumann data snapshots
    for $\boldsymbol{\mu} \in \mathscr{P}_{train}$ **do**
        while $\|\mathbf{u}_{\Gamma_1} - \mathbf{u}_{\Gamma_2}\|_2 >$tol **do**
            $\mathbf{u}_1 \leftarrow$ solve the master model;
            $\mathbf{u}_2 \leftarrow$ solve the slave model;
        end while
        $\mathbf{u}_{|\Gamma_1} \leftarrow$ extract the master interface solution;
        $\mathbf{z}_2 \leftarrow$ extract the slave dual vector of the residual.
        $\mathbf{S}_D = [\mathbf{S}_D, \mathbf{u}_{\Gamma_1}]$;
        $\mathbf{S}_N = [\mathbf{S}_N, \mathbf{z}_2]$;
    end for
    DEIM reduced-order arrays:
    $\mathbf{\Phi}_D \leftarrow \mathrm{POD}(\mathbf{S}_D, \epsilon_{tol_D})$; $\mathcal{I}_{1,D} \leftarrow$ DEIM-indices($\mathbf{\Phi}_D$);
    $\mathbf{\Phi}_N \leftarrow \mathrm{POD}(\mathbf{S}_N, \epsilon_{tol_N})$; $\mathcal{I}_{2,N} \leftarrow$ DEIM-indices($\mathbf{\Phi}_N$);
    Dirichlet magic points:
    for $i_{1,D} \in \mathcal{I}_{1,D}$ **do**
        $p_1 \leftarrow$ get Cartesian coordinates of $i_{1,D}$ DoF;
        $p_2 = \min_{p_2^j \in DoF_{\Gamma_2}}(\mathrm{dist}(p_1 - p_2^j)) \leftarrow$ search the nearest DoF of p_1 in Γ_1;
        $i_{2,D} \leftarrow$ get the Dirichlet index for p_2;
        $\mathcal{I}_{2,D} = [\mathcal{I}_{2,D}, i_{2,D}]$;
    end for
    Neumann magic points:
    for $i_{2,N} \in \mathcal{I}_{2,N}$ **do**
        $p_2 \leftarrow$ get Cartesian coordinates of $i_{2,N}$ DoF;
        $p_1 = \min_{p_1^j \in DoF_{\Gamma_1}}(\mathrm{dist}(p_2 - p_1^j)) \leftarrow$ search the nearest DoF of p_2 in Γ_2;
        $i_{1,N} \leftarrow$ get the Neumann index for p_1;
        $\mathcal{I}_{1,N} = [\mathcal{I}_{1,N}, i_{1,N}]$;
    end for
end procedure

procedure $[\mathbf{u}_1, \mathbf{u}_2]$ = ONLINE QUERY(ROM arrays, FOM arrays, $\boldsymbol{\mu}$, tol)
    while $\|\mathbf{u}_{1|\mathcal{I}_{1,D}} - \mathbf{u}_{2|\mathcal{I}_{2,D}}\|_2 >$tol **do**
        $\mathbf{u}_{2|\mathcal{I}_{2,D}} \leftarrow$ extract Dirichlet magic points;
        $\mathbf{u}_{\Gamma_1} \approx \mathbf{\Phi}_D \mathbf{\Phi}^{-1}_{D|\mathcal{I}_{1,D}} \mathbf{u}_{2|\mathcal{I}_{2,D}} \leftarrow$ Dirichlet DEIM approximation;
        apply $\mathbf{u}_{\Gamma_1}$ and solve the master problem with $\boldsymbol{\mu}$;
        $\mathbf{r}_{N_{1|\mathcal{I}_{1,N}}} \leftarrow$ extract the master interface residual on the magic points;
        $\mathbf{r}_{N_{2|\Gamma_2}} \approx \mathbb{M}_{\Gamma_2} \mathbf{\Phi}_N \mathbf{\Phi}^{-1}_{N|\mathcal{I}_{2,N}} \mathbb{M}^{-1}_{\Gamma_1} \mathbf{r}_{N_{1|\mathcal{I}_{1,N}}} \leftarrow$ Neumann DEIM approximation ;
        $\mathbf{r}_{N_2} \leftarrow$ recover the interface residual, *i.e.* the Neumann term of the second problem;
        apply $\mathbf{r}_{N_2}$ and solve the slave model with $\boldsymbol{\mu}$;
    end while
end procedure

5. Numerical results

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Algorithm 2 ROM procedure

procedure [ROM ARRAYS] = OFFLINE(FOM arrays, $\mathscr{P}_{train}$,ϵ_{tol_1},ϵ_{tol_2}, ϵ_{tol_D}, ϵ_{tol_N}, tol)
  Solution, Dirichlet and Neumann data snapshots
  for $\boldsymbol{\mu} \in \mathscr{P}_{train}$ **do**
    while $\|\mathbf{u}_{\Gamma_1} - \mathbf{u}_{\Gamma_2}\|_2$ >tol **do**
      $\mathbf{u}_1 \leftarrow$ solve the master model;
      $\mathbf{u}_2 \leftarrow$ solve the slave model;
    end while
    $\mathbf{u}_{1_{|\Gamma_1}} \leftarrow$ extract the master interface solution;
    $\mathbf{z}_2 \leftarrow$ extract the slave dual vector of the residual.
    $\mathbf{S}_1 = [\mathbf{S}_1, \tilde{\mathbf{u}}_1]$;
    $\mathbf{S}_2 = [\mathbf{S}_2, \mathbf{u}_2]$;
    $\mathbf{S}_D = [\mathbf{S}_D, \mathbf{u}_{\Gamma_1}]$;
    $\mathbf{S}_N = [\mathbf{S}_N, \mathbf{z}_2]$;
  end for
  POD reduced-order arrays:
  $\mathbb{V}_1 \leftarrow \mathrm{POD}(\mathbf{S}_1, \epsilon_{tol_1})$;
  $\mathbb{V}_2 \leftarrow \mathrm{POD}(\mathbf{S}_2, \epsilon_{tol_2})$;
  $\{\mathbb{A}_{n_1}, \mathbf{f}_{n_1}\} \leftarrow$ Galerkin projection of the FOM master arrays onto $\mathbb{V}_1$;
  $\{\mathbb{A}_{n_2}, \mathbf{f}_{n_2}\} \leftarrow$ Galerkin projection of the FOM slave arrays onto $\mathbb{V}_2$;
  DEIM reduced-order arrays:
  $\mathbf{\Phi}_D \leftarrow \mathrm{POD}(\mathbf{S}_D, \epsilon_{tol_D})$; $\mathcal{I}_{1,D} \leftarrow$ DEIM-indices($\mathbf{\Phi}_D$);
  $\mathbf{\Phi}_N \leftarrow \mathrm{POD}(\mathbf{S}_N, \epsilon_{tol_N})$; $\mathcal{I}_{2,N} \leftarrow$ DEIM-indices($\mathbf{\Phi}_N$);
  Dirichlet magic points:
  for $i_{1,D} \in \mathcal{I}_{1,D}$ **do**
    $p_1 \leftarrow$ get Cartesian coordinates of $i_{1,D}$ DoF;
    $p_2 = \min_{p_2^j \in DoF_{\Gamma_2}}(\mathrm{dist}(p_1 - p_2^j)) \leftarrow$ search the nearest DoF of p_1 in Γ_1;
    $i_{2,D} \leftarrow$ get the Dirichlet index for p_2;
    $\mathcal{I}_{2,D} = [\mathcal{I}_{2,D}, i_{2,D}]$;
  end for
  Neumann magic points:
  for $i_{2,N} \in \mathcal{I}_{2,N}$ **do**
    $p_2 \leftarrow$ get Cartesian coordinates of $i_{2,N}$ DoF;
    $p_1 = \min_{p_1^j \in DoF_{\Gamma_1}}(\mathrm{dist}(p_2 - p_1^j)) \leftarrow$ search the nearest DoF of p_2 in Γ_2;
    $i_{1,N} \leftarrow$ get the Neumann index for p_1;
    $\mathcal{I}_{1,N} = [\mathcal{I}_{1,N}, i_{1,N}]$;
  end for
  $\mathbb{V}_1^T \mathbb{A}_1 \mathbf{\Phi}_D \mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}}^{-1} \mathbb{V}_{2_{|\mathcal{I}_{2,D}}} \leftarrow$ save matrix product for master lifting term;
  $\mathbb{V}_2^T \mathbb{M}_{\Gamma_2} \mathbf{\Phi}_N \mathbf{\Phi}_{N_{|\mathcal{I}_{2,N}}}^{-1} \mathbb{M}_{\Gamma_{1_{|\mathcal{I}_{2,N}}}} \leftarrow$ save matrix product for slave residual term;
end procedure

procedure [$\mathbf{u}_1$,$\mathbf{u}_2$] = ONLINE QUERY(ROM arrays, FOM arrays, $\boldsymbol{\mu}$, tol)
  while $\|\mathbf{u}_{1_{|\mathcal{I}_{1,D}}} - \mathbf{u}_{2_{|\mathcal{I}_{2,D}}}\|_2$ >tol **do**
    $\mathbf{u}_{2|\mathcal{I}_{2,D}} \leftarrow$ extract Dirichlet magic points;
    $\mathbb{V}_1^T \mathbb{A}_1 \mathbf{\Phi}_D \mathbf{\Phi}_{D_{\mathcal{I}_{1,D}}}^{-1} \mathbb{V}_{2_{|\mathcal{I}_{2,D}}} \mathbf{u}_{n_2}^{k+1} \leftarrow$ assemble the lifting term;
    $\tilde{\mathbf{u}}_{n_1} \leftarrow$ solve the master reduced order problem with $\boldsymbol{\mu}$;
    $\mathbf{r}_{N_{1_{|\mathcal{I}_{1,N}}}} \leftarrow$ extract the master interface residual on the magic points;
    $\mathbb{V}_2^T \mathbb{M}_{\Gamma_2} \mathbf{\Phi}_N \mathbf{\Phi}_{N_{|\mathcal{I}_{2,N}}}^{-1} \mathbb{M}_{\Gamma_{1_{|\mathcal{I}_{2,N}}}} \mathbf{r}_{N_{1_{|\mathcal{I}_{1,N}}}}^{k+1}(\boldsymbol{\mu}) \leftarrow$ assemble the interface residual term;
    $\mathbf{u}_{n_2} \leftarrow$ solve the slave reduced order problem with $\boldsymbol{\mu}$;
  end while
end procedure

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