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Canuto, C.; Nochetto, R.H.; Stevenson R,; Verani, M.

MOX, Dipartimento di Matematica
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox-dmat@polimi.it http://mox.polimi.it

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Claudio Canuto[a], Ricardo H. Nochetto[b], Rob Stevenson[c] and Marco Verani[d]

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[a] Dipartimento di Scienze Matematiche, Politecnico di Torino
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy
E-mail: claudio.canuto@polito.it

[b] Department of Mathematics and Institute for Physical Science and Technology
University of Maryland, College Park, MD, USA
E-mail: rhn@math.umd.edu

[c] Korteweg-de Vries Institute for Mathematics, University of Amsterdam
P.O. Box 94248, 1090 GE Amsterdam, The Netherlands
E-mail: r.p.stevenson@uva.nl

[d] MOX, Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo da Vinci 32, I-20133 Milano, Italy
E-mail: marco.verani@polimi.it

Abstract

We design and analyze an adaptive *hp*-finite element method (**hp-AFEM**) in dimensions $n = 1, 2$. The algorithm consists of iterating two routines: **hp-NEARBEST** finds a near-best *hp*-approximation of the current discrete solution and data to a desired accuracy, and **REDUCE** improves the discrete solution to a finer but comparable accuracy. The former hinges on a recent algorithm by Binev for adaptive *hp*-approximation, and acts as a coarsening step. We prove convergence and instance optimality.

1 Introduction

The discovery that elliptic problems with localized singularities, such as corner singularities, can be approximated with exponential accuracy propelled the study and use of *hp*-FEMs, starting with the seminal work of Babuška. The a priori error analysis originated in the late 70's with the earliest attempts to study the adaptive approximation of a univariate function, having a finite number of singularities and otherwise smooth, by means of piecewise polynomials of variable degree [17, 22]. These results influenced Gui and Babuška in their pioneering study of the convergence rate of the *hp*-approximation to a one dimensional model elliptic problem in [26] and in their subsequent work [27], which proves convergence of an adaptive *hp*-algorithm with a predicted rate. However, due to the assumptions on the admissible error estimators, which appear to be overly restrictive, the results in [27] cannot be considered completely satisfactory. Starting from the late 80's the study of a posteriori error estimators and the design of adaptive *hp*-algorithms has been the subject of an intense research. We refer to the book [39], and the survey paper [15], as well as the references therein for more details.

However, despite the interest in *hp*-FEMs, the study of adaptivity is much less developed than for the *h*-version of the FEM, for which a rather complete theory has been

developed in the last decade [23, 36, 6, 16, 44]; we refer to the survey [37]. Regarding the *hp*-FEM, we mention [41, 24, 10, 3] which prove convergence without rates. The purpose of this paper is to bridge this gap: we present a new **hp-AFEM**, which hinges on a recent algorithm by Binev for adaptive *hp*-approximation [4, 5], and prove several properties including instance optimality in dimensions $n = 1, 2$. The theory is complete for $n = 1$ but there are a couple of pending issues for $n = 2$, which we discuss below.

The success of **hp-AFEM**'s hinges on having solutions and data with suitable sparsity structure, as well as practical algorithms that discover such a structure via computation. This is why existing **hp-AFEM** software typically probes the current discrete solution to learn about the local smoothness of the exact solution, but can only search around the current level of resolution. We refer, in particular, to the algorithms presented in [33, 1, 31, 32, 34] for strategies based on analyticity checks or local regularity estimation (see also [41, 24]), to the algorithms in [20, 19, 18, 21] and [38] for strategies based on the use of suitable discrete reference solutions, and to the algorithm in [35] for a strategy based on comparing estimated and predicted errors.

1.1 Challenges of *hp*-Approximation

To shed light on the difficulties to design **hp-AFEM**, we start with the much simpler problem of *hp-approximation* for $n = 1$. Let $\Omega := (0, 1)$ and K be a dyadic interval obtained from $K_0 = \bar{\Omega}$. Let p be the polynomial degree associated with K at a certain stage of the adaptive algorithm, and denote $D = (K, p)$. Given $v \in L^2(\Omega)$ and $p \geq 0$, let

$$e_D(v) := \min_{\varphi \in \mathbb{P}_p(K)} \|v - \varphi\|^2_{L^2(K)} \quad \text{and} \quad Q_D(v) := \operatorname*{argmin}_{\varphi \in \mathbb{P}_p(K)} \|v - \varphi\|_{L^2(K)}, \tag{1.1}$$

the latter function being extended with zero outside K. The following algorithm generates a sequence of *hp*-decompositions $(\mathcal{D}_\ell)_{\ell=0}^\infty$ and corresponding piecewise polynomial approximations $v_\ell = v_{\mathcal{D}_\ell}$. With $v_0 := Q_{K_0,0}(v)$, for $\ell > 0$ and any $D = (K, p) \in \mathcal{D}_\ell$,

- compute $e_{K,p+1}(v - v_\ell)$ as well as $e_{K',p}(v - v_\ell)$ and $e_{K'',p}(v - v_\ell)$ for K' and K'' being the two children of K;
- if $e_{K,p+1}(v - v_\ell) < e_{K',p}(v - v_\ell) + e_{K'',p}(v - v_\ell)$, then replace D by $\widetilde{D} := (K, p + 1)$ in $\mathcal{D}_{\ell+1}$ and set
$$v_{\ell+1} := v_\ell + Q_{\widetilde{D}}(v - v_\ell);$$
- otherwise, replace D by $D' := (K', p)$ and $D'' := (K'', p)$ in $\mathcal{D}_{\ell+1}$ and set
$$v_{\ell+1} := v_\ell + Q_{D'}(v - v_\ell) + Q_{D''}(v - v_\ell).$$

Although this algorithm is deliberately very rudimentary so as to simplify the discussion, it mimics existing schemes that query whether it is more advantageous to refine the element K or increase the polynomial degree p by a fixed amount, say 1. We wonder whether such an algorithm may lead to near-optimal *hp*-partitions. In order to elaborate on this question, we now present two extreme examples that illustrate the role of sparsity for the design of **hp-AFEM**.

Example 1: Lacunary Function. For a given integer $L > 0$, let v be a polynomial of degree $p := 2^L - 1$, such that, on each dyadic interval K of generation $0 \leq \ell < L$, v is L^2-orthogonal to the linear polynomials with vanishing mean. Since we need to impose 2^ℓ orthogonality relations for each level ℓ, we get altogether $1 + 2 + 2^2 + \cdots + 2^{L-1} = 2^L - 1$ constraints. We thus realize that a nontrivial polynomial of degree p does exist because it has 2^L parameters. We also see that the algorithm above bisects all dyadic elements K starting from K_0 until reaching the level L, and that v_ℓ for all $0 \leq \ell < L$ is the

piecewise constant function that takes the mean-value of v on each element in $\mathcal{D}_\ell$. Even if the algorithm stops refining at level L and chooses from then on to raise the polynomial degree by 1 in each of the p elements, then at least p new degrees of freedom have to be added in each interval to represent v exactly. This leads to a total of p^2 degrees of freedom activated to capture a polynomial of degree p, thereby proving that this process is non-optimal. We conclude that to be near-optimal, **hp-AFEM** must be able to backtrack and review decisions made earlier. This process, from now on called *coarsening*, is missing in most algorithms for *hp*-adaptivity except, for example, that of Demkowicz, Oden and Rachowicz [20], for which there are no optimality results. The preceding function is extremely sparse for *hp*-approximation, in fact a single polynomial, but its structure is hard to discover in practice because of the sparsity gap.

Example 2: Non-degenerate Function. We now consider the canonical function $v(x) = x^\alpha$ with $\alpha < 1$ on $\Omega = (0, 1)$, studied by DeVore and Scherer [22] and by Gui and Babuška [26], which does not exhibit a sparsity gap. In fact, the following non-degeneracy property is valid: there exist constants C_1, C_2 such that for all intervals K and polynomial degrees p

$$C_2 \leq \frac{e_{K,p+1}(v)}{e_{K,p}(v)} \leq C_1.$$

The exponential rate of convergence derived a priori in [27], as well as the linear increase of polynomial degrees starting from the origin, are based on this crucial property. Similar results have been derived later for $n = 2$ by Babuška and Guo [28, 29] and for $n = 3$ by Schotzau, Schwab and Wihler [42, 43]; see [39] for a thorough discussion of the cases $n = 1, 2$. It is thus conceivable, as observed in practice, that decisions made by **hp-AFEM**'s with a building block such as that above do not produce unnecessary degrees of freedom for problems such as Example 2. The lack of a coarsening step in most existing *hp*-software could thus be attributed to the very special geometric features of point and edge singularities, this being a special rather than a universal property to design an optimal **hp-AFEM**.

1.2 Our contributions

Since we wish to account for a large class of functions (solutions and data), perhaps exhibiting degeneracies such as in Example 1, our **hp-AFEM** includes a coarsening routine, which we envisage to be unavoidable for obtaining optimality. Our **hp-AFEM** hinges on two routines, **hp-NEARBEST** and **REDUCE**, and the former in turn relies on the adaptive *hp*-approximation routine by Binev [4, 5]. To describe them, let $u = u(f, \lambda) \in H^1_0(\Omega)$ be the solution to a second order elliptic PDE on a domain $\Omega \subset \mathbb{R}^n, n = 1, 2$, with data (f, λ), where f denotes forcing term(s) and λ parameters such as coefficients.

Given a reduction factor $\varrho \in (0, 1)$, a *conforming hp*-partition $\mathcal{D}$, (discontinuous) *hp*-FEM approximations $(f_\mathcal{D}, \lambda_\mathcal{D})$ to (f, λ) over $\mathcal{D}$, the routine **REDUCE** produces a *conforming hp*-refinement $\bar{\mathcal{D}}$ such that the $|\cdot|_{H^1}$-error in the (*continuous*) *hp*-fem Galerkin approximation on $\bar{\mathcal{D}}$ to the exact solution $u(f_\mathcal{D}, \lambda_\mathcal{D})$ is less than ρ times the same Galerkin error relative to the partition $\mathcal{D}$. This routine will be implemented as an AFEM routine that applies under a no-data-oscillation assumption.

The routine **hp-NEARBEST** deals with nonconforming meshes and subordinate discontinuous functions. Given a tolerance $\varepsilon > 0$, a generic function $v \in H^1(\Omega)$, and data (f, λ), **hp-NEARBEST** produces a *nonconforming hp*-partition $\mathcal{D}$ and suitable projections $(f_\mathcal{D}, \lambda_\mathcal{D})$ of the data onto discontinuous *hp*-FEM spaces over $\mathcal{D}$. The output is such that the square root of a specific *error functional* is less than ε. This error functional is defined as the sum of the squared broken $|\cdot|_{H^1}$-error in the best (discontinuous) *hp*-approximation over $\mathcal{D}$ to v and δ^{-1} times the squared *hp*-data oscillation $\mathrm{osc}^2_\mathcal{D}(f, \lambda)$ over

$\mathcal{D}$, for a sufficiently small penalty parameter $\delta > 0$. In turn, $\mathrm{osc}^2_{\mathcal{D}}(f,\lambda)$ measures the errors $f - f_{\mathcal{D}}$ and $\lambda - \lambda_{\mathcal{D}}$ on the partition $\mathcal{D}$ in such squared local norms, that the following bound, expressing the continuous dependence on data of the underlying linear problem, holds:

$$|u - u(f_{\mathcal{D}}, \lambda_{\mathcal{D}})|_{H^1(\Omega)} \lesssim \mathrm{osc}_{\mathcal{D}}(f,\lambda). \tag{1.2}$$

The procedure **hp-NEARBEST** is based on Binev's algorithm and is *instance optimal* for this functional.

Our algorithm **hp-AFEM** consists of a repetition of calls of **hp-NEARBEST** and **REDUCE** with decreasing error tolerances. The calls of **hp-NEARBEST**, with v being the current approximation to the solution u, are made to guarantee *instance optimality* of the coarsened approximations. Coarsening, however, increases the error by a constant factor. This must be compensated by a judicious choice of the reduction factor ϱ of **REDUCE** so that the concatenation of the two routines produces a converging sequence. To realize this idea we must account for the following additional issues.

Making meshes *hp*-conforming: After a call of **hp-NEARBEST**, the generally non-conforming *hp*-partition $\mathcal{D}$ has to be refined to a conforming one $\mathcal{C}(\mathcal{D})$ so that it can serve as input for **REDUCE**. This is obviously an issue for dimension $n = 2$ but not for $n = 1$, in which case one can take $\mathcal{C}(\mathcal{D}) = \mathcal{D}$. One may wonder whether the cardinality of $\mathcal{C}(\mathcal{D})$ can be bounded uniformly by that of $\mathcal{D}$ for $n = 2$. To see that the answer is negative in general consider the following pathological situation: a large triangle of $\mathcal{D}$ with high polynomial degree is surrounded by small triangles with polynomial degree 1. This is the reason why, without further assumptions on the structure of the solution u, we cannot guarantee for $n = 2$ an optimal balance between the accuracy of the *hp*-approximations and the cardinality of the *hp*-partitions at stages intermediate to consecutive calls of **hp-NEARBEST**. Resorting to a discontinuous **hp-AFEM** would cure this gap at the expense of creating other difficulties.

Making functions continuous: In order to quantify the reduction factor ϱ of **REDUCE** we must be able to compare the (broken) $H^1(\Omega)$-errors of the best *continuous* and *discontinuous hp*-FEM approximations over $\mathcal{C}(\mathcal{D})$. We show that the former is bounded by the latter with a multiplicative constant which depends logarithmically on the maximal polynomial degree for $n = 2$. This extends upon a recent result of Veeser for the h-version of the FEM [45]. Such constant does not depend on the polynomial degree for $n = 1$. This construction is needed for the analysis of **hp-AFEM** only but not its implementation.

Dealing with a perturbed problem: When, preceding to a call of **hp-NEARBEST**, the current (continuous) *hp*-approximation to u has a tolerance ε, **hp-NEARBEST** will be called with a tolerance $\eqsim \varepsilon$ in order to guarantee optimality of the coarsened discontinuous *hp*-approximation. In addition, **hp-NEARBEST** produces new approximations $(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$ to the data to be used in the subsequent call of **REDUCE**. The prescribed tolerance ensures, in view of the definition of the error functional, that $\mathrm{osc}_{\mathcal{D}}(f,\lambda) \lesssim \sqrt{\delta}\,\varepsilon$. Hence, concatenating with (1.2), we are guaranteed that $|u - u(f_{\mathcal{D}}, \lambda_{\mathcal{D}})|_{H^1(\Omega)} \lesssim \sqrt{\delta}\,\varepsilon$. The routine **REDUCE** approximates the solution $u(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$, and so cannot be expected to produce an approximation to u that is more accurate than $u(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$. Therefore, in order to obtain convergence of the overall iteration, the condition $|u - u(f_{\mathcal{D}}, \lambda_{\mathcal{D}})|_{H^1(\Omega)} \leq \xi\varepsilon$ is needed for some parameter $\xi \in [0, 1)$, which is achieved by selecting the penalty parameter δ to be sufficiently small.

The routine **REDUCE** will be implemented as an AFEM consisting of the usual loop over **SOLVE**, **ESTIMATE**, **MARK**, and **REFINE**. For $n = 1$, we construct an estimator that is reliable and discretely efficient, uniformly in p. Consequently, the number of iterations to achieve some fixed error reduction ϱ is independent on the maximal polynomial degree.

For $n = 2$, we employ the residual-based a posteriori error estimator analyzed by Melenk and Wohlmuth [35], which turns out to be p-sensitive. We show that in order to achieve a fixed error reduction, it suffices to grow the number of iterations more than quadratically with respect to the maximal polynomial degree. This sub-optimal result is yet another reason for optimality degradation at stages intermediate between two consecutive calls of **hp-NEARBEST**. Nevertheless, our result improves upon a recent one by Bank, Parsania and Sauter [3], which requires the number of iterations to be proportional to the fifth power of the maximal polynomial degree.

Throughout this work, we assume that the arising linear systems are solved exactly. To control the computational cost, optimal iterative solvers, uniformly in the polynomial degree would be required. We refer to [9] for an example.

This work is organized as follows. We present **hp-AFEM** within an abstract setting in Sect. 2 and prove that it converges, and that the sequence of outputs of **hp-NEARBEST** is *instance optimal*. We give a brief description of Binev's algorithm in Sect. 3. In Sect. 4, we apply the abstract setting to the general 1-dimensional elliptic problem. Finally, in Sect. 5 we apply the abstract theory to the Poisson equation in two dimensions.

The following notation will be used thoughout the paper. By $\gamma \lesssim \delta$ we will mean that γ can be bounded by a multiple of δ, independently of parameters which γ and δ may depend on. Likewise, $\gamma \gtrsim \delta$ is defined as $\delta \lesssim \gamma$, and $\gamma \eqsim \delta$ as $\gamma \lesssim \delta$ and $\gamma \gtrsim \delta$.

2 An abstract framework

We now present the **hp-AFEM** in two steps. We first deal with an ideal algorithm and later introduce a practical scheme including **REDUCE**. We also discuss a possible realization of **REDUCE**.

2.1 Definitions and assumptions

On a domain $\Omega \subset \mathbb{R}^n$, we consider a, possibly, parametric PDE

$$A_\lambda u = f. \tag{2.1}$$

Here the forcing f and the parameter λ (representing, e.g., the coefficients of the operator) are taken from some spaces F and $\bar{\Lambda}$ of functions on Ω, such that there exists a unique solution $u = u(f, \lambda)$ living in a space V of functions on Ω. We assume, for simplicity, that V and F are Hilbert spaces over $\mathbb{R}$.

We assume that we are given an essentially disjoint initial partition of $\bar{\Omega}$ into finitely many (closed) subdomains (the 'element domains'). We assume that for each element domain K that we encounter, there exists a unique way in which it can be split into element domains K' and K'', the 'children' of K, such that $K = K' \cup K''$ and $|K' \cap K''| = 0$. The set $\mathfrak{K}$ of all these element domains is therefore naturally organized as an infinite binary 'master tree', having as its roots the element domains of the initial partition of $\bar{\Omega}$. A finite subset of $\mathfrak{K}$ is called a subtree of the master tree when it contains all roots and for each element domain in the subset both its parent and its sibling are in the subset. The leaves of a subtree form an essentially disjoint partition of $\bar{\Omega}$. The set of all such 'h-partitions' will be denoted as $\mathbb{K}$. For $\mathcal{K}, \widetilde{\mathcal{K}} \in \mathbb{K}$, we call $\widetilde{\mathcal{K}}$ a refinement of $\mathcal{K}$, and denoted as $\mathcal{K} \leq \widetilde{\mathcal{K}}$, when any $K \in \widetilde{\mathcal{K}}$ is either in $\mathcal{K}$ or has an ancestor in $\mathcal{K}$.

Our aim is to compute 'hp-finite element' approximations to u, i.e., piecewise polynomial approximations, with variable degrees, w.r.t. partitions from $\mathbb{K}$. In order to do so, it will be needed first to replace the data (f, λ) by approximations from finite dimensional spaces. For that goal as well, we will employ spaces of piecewise polynomials, with variable degrees, w.r.t. partitions from $\mathbb{K}$, as will be described next.

For all $K \in \mathfrak{K}$, let V_K, F_K, Λ_K be (infinite dimensional) spaces of functions on K, such that for any $\mathcal{K} \in \mathbb{K}$, it holds that, possibly up to isomorphisms,

$$V \subseteq \prod_{K\in\mathcal{K}} V_K, \quad F = \prod_{K\in\mathcal{K}} F_K, \quad \Lambda \subseteq \prod_{K\in\mathcal{K}} \Lambda_K \subseteq \bar{\Lambda}.$$

Here Λ is a subset of $\bar{\Lambda}$, which contains all the parameters that will be allowed in our adaptive algorithm **hp-AFEM**, and, for simplicity, has a Hilbert topology. For all $(K,d) \in \mathfrak{K}\times\mathbb{N}$ (hereafter $\mathbb{N}$ stands for the set of all *positive* natural numbers) and $Z \in \{V, F, \Lambda\}$, we assume finite dimensional spaces $Z_{K,d} \subset Z_K$ of functions on K such that $Z_{K,d} \subseteq Z_{K,d+1}$, $Z_{K,d} \subset Z_{K',d} \times Z_{K'',d}$, and $Z \cap \bigcup_{\mathcal{K}\in\mathbb{K},d\in\mathbb{N}} \prod_{K\in\mathcal{K}} Z_{K,d}$ is dense in Z.

In applications, $V_{K,d}$ will be a space of polynomials of dimension $\eqsim d$. For instance, when K is an n-simplex, $V_{K,d}$ may be chosen as $\mathbb{P}_p(K)$, where the associated polynomial degree $p = p(d)$ can be defined as the largest value in $\mathbb{N}$ such that $\dim \mathbb{P}_{p-1}(K) = \binom{n+p-1}{p-1} \leq d$. This definition normalizes the starting value $p(1) = 1$ for all $n \in \mathbb{N}$. Only for $n = 1$, it holds that $p(d) = d$ for all $n \in \mathbb{N}$.

Analogously, the spaces $F_{K,d}$ and $\Lambda_{K,d}$ will be selected as (Cartesian products of) spaces of polynomials as well, of degrees equal to p plus some constant in $\mathbb{Z}$.

In the following, $D \in \mathfrak{K}\times\mathbb{N}$ will denote an *hp-element*: it is a pair (K_D, d_D) consisting of an element domain $K_D \in \mathfrak{K}$, and an integer $d_D \in \mathbb{N}$. We will write $Z_D = Z_{K_D,d_D}$.

For all $D \in \mathfrak{K}\times\mathbb{N}$, we assume a projector $Q_D : V \times F \times \Lambda \to V_D \times F_D \times \Lambda_D$, and a local error functional $e_D = e_D(v, f, \lambda) \geq 0$, that, for $(v, f, \lambda) \in V \times F \times \Lambda$, gives a measure for the (squared) distance between $(v|_{K_D}, f|_{K_D}, \lambda|_{K_D})$ and its local approximation $(v_D, f_D, \lambda_D) := Q_D(v, f, \lambda)$. We assume that this error functional is non-increasing under both 'h-refinements' and 'p-enrichments', in the sense that

$$\begin{aligned} e_{D'} + e_{D''} \leq e_D \text{ when } K_{D'}, K_{D''} \text{ are the children of } K_D, \text{ and } d_{D'} = d_{D''} = d_D; \\ e_{D'} \leq e_D \text{ when } K_{D'} = K_D \text{ and } d_{D'} \geq d_D. \end{aligned} \tag{2.2}$$

A collection $\mathcal{D} = \{D = (K_D, d_D)\}$ of *hp*-elements is called an *hp-partition* provided $\mathcal{K}(\mathcal{D}) := \{K_D : D \in \mathcal{D}\} \in \mathbb{K}$. The collection of all *hp*-partitions is denoted as $\mathbb{D}$. For $\mathcal{D} \in \mathbb{D}$, we set the *hp*-approximation spaces

$$Z_{\mathcal{D}} := \prod_{D\in\mathcal{D}} Z_D, \quad (Z \in \{V, F, \Lambda\}),$$

and define

$$\#\mathcal{D} := \sum_{D\in\mathcal{D}} d_D.$$

In our applications, the quantity $\#\mathcal{D}$ is proportional to the dimension of $Z_{\mathcal{D}}$, and $e_D(v, f_D, \lambda_D)$ is the sum of the squared best approximation error of $v|_{K_D}$ from V_D in $|\cdot|_{H^1(K_D)}$ and δ^{-1} times the square of the local data oscillation.

For $\mathcal{D} \in \mathbb{D}$, we set the *global error functional*

$$\mathrm{E}_{\mathcal{D}}(v, f, \lambda) := \sum_{D\in\mathcal{D}} e_D(v, f, \lambda),$$

which is a measure for the (squared) distance between (v, f, λ) and its projection

$$\Big(\prod_{D\in\mathcal{D}} v_D, \prod_{D\in\mathcal{D}} f_D, \prod_{D\in\mathcal{D}} \lambda_D\Big) \in V_{\mathcal{D}} \times F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}. \tag{2.3}$$

For $\mathcal{D}, \widetilde{\mathcal{D}} \in \mathbb{D}$, we call $\widetilde{\mathcal{D}}$ a refinement of $\mathcal{D}$, and write $\mathcal{D} \leq \widetilde{\mathcal{D}}$, when both $\mathcal{K}(\mathcal{D}) \leq \mathcal{K}(\widetilde{\mathcal{D}})$, and $d_{\widetilde{D}} \geq d_D$, for any $D \in \mathcal{D}$, $\widetilde{D} \in \widetilde{\mathcal{D}}$ with K_D being either equal to $K_{\widetilde{D}}$ or an ancestor

of $K_{\widetilde{D}}$. With this notation, (2.2) is equivalent to

$$\mathrm{E}_{\widetilde{\mathcal{D}}}(v,f,\lambda) \le \mathrm{E}_{\mathcal{D}}(v,f,\lambda) \quad \forall \widetilde{\mathcal{D}} \ge \mathcal{D}. \tag{2.4}$$

We will apply a finite element solver that generally operates on a subset $\mathbb{D}^c$ of the set of *hp*-partitions $\mathbb{D}$, typically involving a restriction to those $\mathcal{D} \in \mathbb{D}$ for which the '*h*-partition' $\mathcal{K}(\mathcal{D})$ is 'conforming'. We assume that there exists a mapping $\mathcal{C} : \mathbb{D} \to \mathbb{D}^c$ such that

$$\mathcal{C}(\mathcal{D}) \ge \mathcal{D} \quad \forall \mathcal{D} \in \mathbb{D}. \tag{2.5}$$

We emphasize that even for $\mathcal{D} \in \mathbb{D}^c$, generally the space $V_{\mathcal{D}}$ is not a subspace of V. Conforming subspaces, used e.g. in Galerkin approximations, are defined as

$$V_{\mathcal{D}}^c := V_{\mathcal{D}} \cap V. \tag{2.6}$$

With regard to (2.3), we introduce the notation

$$f_{\mathcal{D}} := \prod_{D\in\mathcal{D}} f_D, \qquad \lambda_{\mathcal{D}} := \prod_{D\in\mathcal{D}} \lambda_D,$$

but reserve the symbol $v_{\mathcal{D}}$ to denote later a suitable near-best approximation to $v \in V$ from $V_{\mathcal{D}}^c$.

2.2 A basic *hp*-adaptive finite element method

Our aim is for given $(f,\lambda) \in F \times \Lambda$ and $\varepsilon > 0$, to find $\mathcal{D}$ with an essentially minimal $\#\mathcal{D}$ such that $\mathrm{E}_{\mathcal{D}}(u(f,\lambda),f,\lambda) \le \varepsilon$. We will achieve this by alternately improving either the efficiency or the accuracy of the approximation. To that end, we begin by considering a *basic* algorithm, which highlights the essential ingredients of a *hp*-adaptive procedure. We make use of the two routines described below. The first routine is available and will be discussed in Sect. 3. Since we are not concerned with complexity now, existence of the second routine is a simple consequence of the density of the union of the *hp*-approximation spaces in V.

- $[\mathcal{D}, f_{\mathcal{D}}, \lambda_{\mathcal{D}}] := \textbf{hp-NEARBEST}(\varepsilon, v, f, \lambda)$

  The routine **hp-NEARBEST** takes as input $\varepsilon > 0$, and $(v,f,\lambda) \in V \times F \times \Lambda$, and outputs $\mathcal{D} \in \mathbb{D}$ as well as $(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$ such that $\mathrm{E}_{\mathcal{D}}(v,f,\lambda)^{\frac{1}{2}} \le \varepsilon$ and, for some constants $0 < b \le 1 \le B$, $\#\mathcal{D} \le B\#\widehat{\mathcal{D}}$ for any $\widehat{\mathcal{D}} \in \mathbb{D}$ with $\mathrm{E}_{\widehat{\mathcal{D}}}(v,f,\lambda)^{\frac{1}{2}} \le b\varepsilon$.

- $[\bar{\mathcal{D}}, \bar{u}] := \textbf{PDE}(\varepsilon, \mathcal{D}, f_{\mathcal{D}}, \lambda_{\mathcal{D}})$

  The routine **PDE** takes as input $\varepsilon > 0$, $\mathcal{D} \in \mathbb{D}^c$, and data $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$. It outputs $\bar{\mathcal{D}} \in \mathbb{D}^c$ with $\mathcal{D} \le \bar{\mathcal{D}}$ and $\bar{u} \in V_{\bar{\mathcal{D}}}^c$ such that $\|u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - \bar{u}\|_V \le \varepsilon$.

The input argument v of **hp-NEARBEST** will be the current approximation to $u(f,\lambda)$. In an '*h*-adaptive' setting, usually the application of such a routine is referred to as 'coarsening'. Since the data $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$ of **PDE** is discrete, it will be said to satisfy a *no-data-oscillation* assumption w.r.t. $\mathcal{D}$.

We make the following abstract assumptions concerning the relation between the error functional, the norm on V, the mapping $(f,\lambda) \mapsto u(f,\lambda)$, and the constant b of **hp-NEARBEST**. We assume the existence of constants $C_1, C_2 > 0$ with

$$C_1 C_2 < b, \tag{2.7}$$

such that

$$\|u(f,\lambda) - u(f_{\mathcal{D}},\lambda_{\mathcal{D}})\|_V \le C_1 \inf_{w\in V} \mathrm{E}_{\mathcal{D}}(w,f,\lambda)^{\frac12} \qquad \forall \mathcal{D}\in\mathbb{D},\ \forall (f,\lambda)\in F\times\Lambda, \tag{2.8}$$

$$\sup_{(f,\lambda)\in F\times\Lambda} |\,\mathrm{E}_{\mathcal{D}}(w,f,\lambda)^{\frac12} - \mathrm{E}_{\mathcal{D}}(v,f,\lambda)^{\frac12}\,| \le C_2\|w-v\|_V \qquad \forall \mathcal{D}\in\mathbb{D},\ \forall v,w\in V. \tag{2.9}$$

The condition (2.9) means that $\mathrm{E}_{\mathcal{D}}(w,f,\lambda)^{\frac12}$ is Lipschitz w.r.t. its first argument. In our applications, we will verify this condition with $C_2 = 1$. The condition (2.8) will be a consequence of the continuous dependence (1.2) of the solution on the data, and the fact that the error functional will contain the square of a data oscillation $\mathrm{osc}_{\mathcal{D}}(f,\lambda)$. Since this term is penalized by a factor δ^{-1}, we will be able to ensure (2.8) with $C_1 \eqsim \sqrt{\delta}$ which yields (2.7) by taking δ sufficiently small.

Our basic *hp*-adaptive finite element routine reads as follows.

hp-AFEM$(\bar u_0, f, \lambda, \varepsilon_0)$
  % Input: $(\bar u_0, f, \lambda) \in V\times F\times\Lambda$, $\varepsilon_0 > 0$ with $\|u(f,\lambda) - \bar u_0\|_V \le \varepsilon_0$.
  % Parameters: $\mu,\omega > 0$ such that $C_1C_2 < b(1-\mu)$, $\omega \in (\frac{C_2}{b}, \frac{1-\mu}{C_1})$, $\mu\in(0,1)$.
  `for` $i = 1,2,\dots$ `do`
    $[\mathcal{D}_i, f_{\mathcal{D}_i}, \lambda_{\mathcal{D}_i}]$:=**hp-NEARBEST**$(\omega\varepsilon_{i-1}, \bar u_{i-1}, f, \lambda)$
    $[\bar{\mathcal{D}}_i, \bar u_i]$:= **PDE**$(\mu\varepsilon_{i-1}, \mathcal{C}(\mathcal{D}_i), f_{\mathcal{D}_i}, \lambda_{\mathcal{D}_i})$
    $\varepsilon_i := (\mu + C_1\omega)\varepsilon_{i-1}$
  `end do`

Note that $b\omega - C_2 > 0$, and that $\varepsilon_i = (\mu + C_1\omega)^i\varepsilon_0$, where $\mu + C_1\omega < 1$.

Theorem 2.1. *Assuming* (2.7)-(2.9)*, for the sequences* $(\bar u_i)$, $(\mathcal{D}_i)$ *produced in* **hp-AFEM***, writing* $u = u(f,\lambda)$*, it holds that*

$$\|u - \bar u_i\|_V \le \varepsilon_i \quad \forall i\ge 0, \qquad \mathrm{E}_{\mathcal{D}_i}(u,f,\lambda)^{\frac12} \le (\omega + C_2)\varepsilon_{i-1} \quad \forall i\ge 1, \tag{2.10}$$

and

$$\#\mathcal{D}_i \le B\#\mathcal{D} \quad \textit{for any } \mathcal{D}\in\mathbb{D} \textit{ with } \mathrm{E}_{\mathcal{D}}(u,f,\lambda)^{\frac12} \le (b\omega - C_2)\varepsilon_{i-1}. \tag{2.11}$$

Proof. The bound $\|u-\bar u_0\|_V \le \varepsilon_0$ is valid by assumption. For $i\ge 1$, the tolerances used for **hp-NEARBEST** and **PDE**, together with (2.8), show that

$$\begin{aligned}\|u-\bar u_i\|_V &\le \|u(f_{\mathcal{D}_i},\lambda_{\mathcal{D}_i}) - \bar u_i\|_V + \|u - u(f_{\mathcal{D}_i},\lambda_{\mathcal{D}_i})\|_V \\ &\le \mu\varepsilon_{i-1} + C_1\,\mathrm{E}_{\mathcal{D}_i}(\bar u_{i-1},f,\lambda)^{\frac12} \le (\mu + C_1\omega)\varepsilon_{i-1} = \varepsilon_i.\end{aligned} \tag{2.12}$$

The first statement follows for all $i\ge 0$. Using this and (2.9) implies the second assertion

$$\mathrm{E}_{\mathcal{D}_i}(u,f,\lambda)^{\frac12} \le \mathrm{E}_{\mathcal{D}_i}(\bar u_{i-1},f,\lambda)^{\frac12} + C_2\|u-\bar u_{i-1}\|_V \le (\omega + C_2)\varepsilon_{i-1} \qquad \forall i\ge 1.$$

Let $\mathcal{D}\in\mathbb{D}$ with $\mathrm{E}_{\mathcal{D}}(u,f,\lambda)^{\frac12} \le (b\omega - C_2)\varepsilon_{i-1}$. Then, again by (2.9), $\mathrm{E}_{\mathcal{D}}(\bar u_{i-1},f,\lambda)^{\frac12} \le b\omega\varepsilon_{i-1}$ and so $\#\mathcal{D}_i \le B\#\mathcal{D}$ because of the optimality property of **hp-NEARBEST**. ∎

The main result of Theorem 2.1 can be summarized by saying that **hp-AFEM** is *instance optimal* for reducing $\mathrm{E}_{\mathcal{D}}(u(f,\lambda),f,\lambda)$ over $\mathcal{D}\in\mathbb{D}$. Recall that in our applications, $\mathrm{E}_{\mathcal{D}}(u(f,\lambda),f,\lambda)$ will be the sum of the squared best approximation error in u from the nonconforming space $V_{\mathcal{D}} = \prod_{D\in\mathcal{D}} V_D$ in the broken H^1-norm and a squared data oscillation term penalized with a factor δ^{-1}.

Additionally, Theorem 2.1 shows linear convergence to u of the sequence $(\bar u_i)$ of conforming approximations, in particular $\bar u_i \in V^c_{\bar{\mathcal{D}}_i}$ where $\mathcal{D}^c \ni \bar{\mathcal{D}}_i \ge \mathcal{C}(\mathcal{D}_i)$. Since

$\varepsilon_i = (\mu + C_1\omega)^i\varepsilon_0$, the infinite loop in **hp-AFEM** can be stopped to meet any desired tolerance.

The preceding algorithm **hp-AFEM** has the minimal structure for convergence and optimality. Since the routine **PDE** neither exploits the current iterate nor work already done, we present a practical **hp-AFEM** in Sect. 2.3 which replaces **PDE** by **REDUCE**.

Finally in this subsection, we comment on the implications of the instance optimality result concerning class optimality. For $N \in \mathbb{N}$, let $\mathcal{D}_N := \operatorname{argmin}\{\mathrm{E}_{\mathcal{D}}(u,f,\lambda)^{\frac{1}{2}} : \mathcal{D} \in \mathbb{D}, \#\mathcal{D} \le N\}$ and let the best approximation error be

$$\sigma_N := \mathrm{E}_{\mathcal{D}_N}(u,f,\lambda)^{\frac{1}{2}}.$$

Remark 2.1 (algebraic decay)**.** If σ_N decays *algebraically* with N, namely $\sup_N N^s\sigma_N < \infty$, then for the sequence $(\mathcal{D}_i)$ produced in **hp-AFEM**, one infers that $\mathrm{E}_{\mathcal{D}_i}(u,f,\lambda)^{\frac{1}{2}}$ decays algebraically with $\#\mathcal{D}_i$ with the optimal rate: $\sup_i(\#\mathcal{D}_i)^s\,\mathrm{E}_{\mathcal{D}_i}(u,f,\lambda)^{\frac{1}{2}} < \infty$. In other words, instance optimality implies algebraic class optimality.

Remark 2.2 (exponential decay)**.** For *hp*-approximation, it is more relevant to consider an *exponential* decay of σ_N, i.e., $\sup_N e^{\eta N^\tau}\sigma_N < \infty$ for some $\eta, \tau > 0$. This is precisely the situation considered in [11, 12, 13] for adaptive Fourier or Legendre methods.

Let us asssume, for convenience, that $\sigma_N = C_\# e^{-\eta N^\tau}$ for some constant $C_\#$ and ignore in subsequent calculations that N has to be an integer. In view of Theorem 2.1, let N and ε_{i-1} be so related that $\sigma_N = (b\omega - C_2)\varepsilon_{i-1}$ Since apparently $\#\mathcal{D}_i \le BN$ and $\mathrm{E}_{\mathcal{D}_i}(u,f,\lambda)^{\frac{1}{2}} \le (\omega + C_2)\varepsilon_{i-1}$, we deduce

$$\sup_i \left(e^{\tilde{\eta}(\#\mathcal{D}_i)^\tau}\,\mathrm{E}_{\mathcal{D}_i}(u,f,\lambda)^{\frac{1}{2}} \right) \le \frac{C_\#(\omega + C_2)}{b\omega - C_2},$$

with $\tilde{\eta} := B^{-\tau}\eta$.

On the other hand, we will see in Corollary 3.1 that the routine **hp-NEARBEST** satisfies its optimality conditions for *any* $B > 1$, at the expense of $b = b(B) \downarrow 0$ when $B \downarrow 1$. Moreover, as we have seen, in our applications we will be able to satisfy (2.7)–(2.9) for any $b > 0$ by taking the penalization parameter δ small enough. Therefore, we conclude that if σ_N decays exponentially, characterised by parameters (η, τ), then so do the errors produced by **hp-AFEM** for parameters $(\tilde{\eta}, \tau)$, where $\tilde{\eta} = B^{-\tau}\eta$ can be chosen arbitrarily close to η (at the expense of increasing the supremum value). This situation is much better than that encountered in [11, 12, 13].

2.3 The practical *hp*-adaptive finite element method

To render **hp-AFEM** more practical we replace the routine **PDE** by **REDUCE**, which exploits the work already carried out within **hp-AFEM** and reads

- $[\bar{\mathcal{D}}, \bar{u}] := \mathbf{REDUCE}(\varrho, \mathcal{D}, f_{\mathcal{D}}, \lambda_{\mathcal{D}})$

  The routine **REDUCE** takes as input a partition $\mathcal{D} \in \mathbb{D}^c$, data $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$, and a desired error reduction factor $\varrho \in (0,1]$, and produces a conforming partition $\bar{\mathcal{D}} = \bar{\mathcal{D}}(\mathcal{D}, \varrho) \in \mathbb{D}^c$ with $\bar{\mathcal{D}} \ge \mathcal{D}$ and a function $\bar{u} \in V^c_{\bar{\mathcal{D}}}$ such that

$$\|u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - \bar{u}\|_V \le \varrho \inf_{v \in V^c_{\mathcal{D}}} \|u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - v\|_V. \tag{2.13}$$

Inside the practical **hp-AFEM**, the routine **REDUCE** will be called with as input partition the result of mapping $\mathcal{C} : \mathcal{D} \to \mathcal{D}^c$ applied to the output partition of the preceding

call of **hp-NEARBEST**. In order to bound the right-hand side of (2.13), we make the following assumption:

$$\inf_{w\in V^c_{\mathcal{C}(\mathcal{D})}} \|v-w\|_V \le C_{3,\mathcal{D}} \inf_{(f,\lambda)\in F\times\Lambda} \mathrm{E}_{\mathcal{D}}(v,f,\lambda)^{\frac12} \qquad \forall \mathcal{D}\in\mathbb{D},\, \forall v\in V. \tag{2.14}$$

In our applications, the infimum on the right-hand side reads as the squared error in the broken H^1-norm of the best approximation to v from $V_{\mathcal{D}} = \prod_{D\in\mathcal{D}} V_D$. The left-hand side reads as the squared error in $H^1_0(\Omega)$ of the best approximation to v from $V^c_{\mathcal{C}(\mathcal{D})} = H^1_0(\Omega)\cap\prod_{D\in\mathcal{D}} V_D$. The constant $C_{3,\mathcal{D}}$ should ideally be independent of $\mathcal{D}$. We will see in Sect. 4 that this is the case for our application in dimension $n=1$. However, for $n=2$ we will show in Sect. 5 that $C_{3,\mathcal{D}}$ depends logarithmically on the largest polynomial degree; this extends a result by A. Veeser [45].

Our practical *hp*-adaptive finite element routine reads as follows:

hp-AFEM$(\bar u_0, f, \lambda, \varepsilon_0)$
  % Input: $(\bar u_0, f, \lambda)\in V\times F\times\Lambda$, $\varepsilon_0>0$ with $\|u(f,\lambda)-\bar u_0\|_V\le\varepsilon_0$.
  % Parameters: $\mu,\omega>0$ such that $C_1C_2<b(1-\mu)$, $\omega\in(\frac{C_2}{b},\frac{1-\mu}{C_1})$, $\mu\in(0,1)$.
  `for` $i=1,2,\dots$ `do`
    $[\mathcal{D}_i, f_{\mathcal{D}_i}, \lambda_{\mathcal{D}_i}]$:=**hp-NEARBEST**$(\omega\varepsilon_{i-1}, \bar u_{i-1}, f, \lambda)$
    $[\bar{\mathcal{D}}_i, \bar u_i]$:= **REDUCE**$(\frac{\mu}{1+(C_1+C_{3,\mathcal{D}_i})\omega}, \mathcal{C}(\mathcal{D}_i), f_{\mathcal{D}_i}, \lambda_{\mathcal{D}_i})$
    $\varepsilon_i := (\mu+C_1\omega)\varepsilon_{i-1}$
  `end do`

Corollary 2.1 (convergence and instance optimality)**.** *Assuming* (2.7)-(2.9) *and* (2.14)*, the sequences* $(\bar u_i)$*,* $(\mathcal{D}_i)$ *produced in the practical* **hp-AFEM** *above satisfy properties* (2.10) *and* (2.11) *in Theorem 2.1.*

Proof. In view of the second part of the proof of Theorem 2.1, it is sufficient to prove that $\|u-\bar u_i\|_V\le\varepsilon_i$. We argue by induction. If $\|u-\bar u_i\|_V\le\varepsilon_{i-1}$, which is valid for $i=1$, then, after the ith call of **hp-NEARBEST**, (2.14) and (2.8) imply that

$$\begin{aligned}
&\inf_{v\in V^c_{\mathcal{C}(\mathcal{D}_i)}} \|u(f_{\mathcal{D}_i},\lambda_{\mathcal{D}_i})-v\|_V \\
&\qquad\le \|u-\bar u_{i-1}\|_V + \inf_{v\in V^c_{\mathcal{C}(\mathcal{D}_i)}} \|\bar u_{i-1}-v\|_V + \|u-u(f_{\mathcal{D}_i},\lambda_{\mathcal{D}_i})\|_V \\
&\qquad\le \varepsilon_{i-1} + C_{3,\mathcal{D}_i}\,\mathrm{E}_{\mathcal{D}_i}(\bar u_{i-1},f,\lambda) + C_1\,\mathrm{E}_{\mathcal{D}_i}(\bar u_{i-1},f,\lambda) \\
&\qquad\le (1+(C_{3,\mathcal{D}_i}+C_1)\omega)\varepsilon_{i-1}.
\end{aligned} \tag{2.15}$$

Consequently, after the subsequent call of **REDUCE**, it holds that $\|u(f_{\mathcal{D}_i},\lambda_{\mathcal{D}_i})-\bar u_i\|_V\le\mu\varepsilon_{i-1}$ according to (2.13). This result combined with (2.12) shows that $\|u-\bar u_i\|_V\le\varepsilon_i$. □

Remark 2.3 (complexity of **hp-AFEM**)**.** Let us consider the case that the constants $C_{3,\mathcal{D}}$, defined in (2.14), are insensitive to $\mathcal{D}$, namely,

$$C_3 := \sup_{\mathcal{D}\in\mathbb{D}} C_{3,\mathcal{D}} < \infty. \tag{2.16}$$

This entails that the reduction factor $\varrho_i = \frac{\mu}{1+(C_1+C_{3,\mathcal{D}_i})\omega}$ of **REDUCE** satisfies $\inf_i \varrho_i > 0$. Additionally, suppose that, given a fixed $\varrho\in(0,1]$, **REDUCE** realizes (2.13) with

$$\sup_{\mathcal{D}\in\mathbb{D}^c} \frac{\#\bar{\mathcal{D}}(\mathcal{D},\varrho)}{\#\mathcal{D}} < \infty. \tag{2.17}$$

If, furthermore,

$$C_4 := \sup_{\mathcal{D}\in\mathbb{D}} \frac{\#\mathcal{C}(\mathcal{D})}{\#\mathcal{D}} < \infty, \tag{2.18}$$

then the sequences $(\mathcal{D}_i)_i$ and $(\bar{\mathcal{D}}_i)_i$ produced in **hp-AFEM** are so that $\#\bar{\mathcal{D}}_i \lesssim \#\mathcal{D}_i$. In view of the optimal control over $\#\mathcal{D}_i$, given by Theorem 2.1 and Corollary 2.1, we would have optimal control over the dimension of any *hp*-finite element space created within **hp-AFEM**. This ideal situation only happens in the one-dimensional case.

2.4 A possible realization of REDUCE

Let $A_\lambda \in \mathcal{L}(V, V')$ for all $\lambda \in \bar{\Lambda}$ and define the associated continuous bilinear form $a_\lambda(v, w) := \langle A_\lambda v, w\rangle$ for any $v, w \in V$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and V'. We assume that A_λ is symmetric, which is equivalent to the symmetry of the form a_λ. We furtherly assume that each a_λ is continuous and coercive on V, with continuity and coercivity constants $\alpha^* \geq \alpha_* > 0$ independent of $\lambda \in \bar{\Lambda}$. It is convenient to introduce in V the energy norm $|\!|\!| v |\!|\!|_\lambda = \sqrt{a_\lambda(v, v)}$ associated with the form a_λ, which satisfies $\sqrt{\alpha_*}\|v\|_V \leq |\!|\!| v |\!|\!|_\lambda \leq \sqrt{\alpha^*}\|v\|_V$ for all $v \in V$. Let $F \subset V'$.

Given $\mathcal{D} \in \mathbb{D}$ and data $(f, \lambda) \in F \times \bar{\Lambda}$, the (Galerkin) solution $u_\mathcal{D}(f, \lambda) \in V^c_\mathcal{D}$ of

$$a_\lambda(u_\mathcal{D}(f, \lambda), v) = \langle f, v\rangle \quad \forall v \in V^c_\mathcal{D} \tag{2.19}$$

is the best approximation to $u(f, \lambda)$ from $V^c_\mathcal{D}$ in $|\!|\!|\cdot|\!|\!|_\lambda$. In view of a posteriori error estimation, we will consider Galerkin solutions from $V^c_\mathcal{D}$ only for data in $F_\mathcal{D} \times \Lambda_\mathcal{D}$, i.e., for data without data oscillation w.r.t. $\mathcal{D}$.

For $\mathcal{D} \in \mathbb{D}^c$, $D \in \mathcal{D}$, let us introduce local a posteriori error indicators

$$\eta_{D,\mathcal{D}} : V^c_\mathcal{D} \times F_\mathcal{D} \times \Lambda_\mathcal{D} \to [0, \infty),$$

which give rise to the global estimator

$$\mathcal{E}_\mathcal{D}(v, f_\mathcal{D}, \lambda_\mathcal{D}) := \left(\sum_{D\in\mathcal{D}} \eta^2_{D,\mathcal{D}}(v, f_\mathcal{D}, \lambda_\mathcal{D})\right)^{1/2}. \tag{2.20}$$

Given data $(f_\mathcal{D}, \lambda_\mathcal{D})$ without data oscillation w.r.t. $\mathcal{D}$, $\mathcal{E}_\mathcal{D}(v, f_\mathcal{D}, \lambda_\mathcal{D})$ will be used with $v = u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D})$ as an estimator for the squared error in this Galerkin approximation to $u(f_\mathcal{D}, \lambda_\mathcal{D})$. It should not be confused with $E_\mathcal{D}(v, f, \lambda)$, the latter being the sum of local error functionals $e_D(v, f, \lambda)$, that estimates the squared error in a projection on $V_\mathcal{D} \times F_\mathcal{D} \times \Lambda_\mathcal{D}$ of $(v, f, \lambda) \in V \times F \times \Lambda$.

Given any $\mathcal{M} \subset \mathcal{D}$, it will be useful to associate the estimator restricted to $\mathcal{M}$

$$\mathcal{E}_\mathcal{D}(\mathcal{M}, v, f_\mathcal{D}, \lambda_\mathcal{D}) := \left(\sum_{D\in\mathcal{M}} \eta^2_{D,\mathcal{D}}(v, f_\mathcal{D}, \lambda_\mathcal{D})\right)^{1/2}.$$

We assume that $\mathcal{E}_\mathcal{D}$ satisfies the following assumptions:

- **Reliability:** For $\mathcal{D} \in \mathbb{D}^c$, and $(f_\mathcal{D}, \lambda_\mathcal{D}) \in F_\mathcal{D} \times \Lambda_\mathcal{D}$, there holds

$$\|u(f_\mathcal{D}, \lambda_\mathcal{D}) - u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D})\|_V \lesssim \mathcal{E}_\mathcal{D}(u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D}), f_\mathcal{D}, \lambda_\mathcal{D}). \tag{2.21}$$

- **Discrete efficiency:** For $\mathcal{D} \in \mathbb{D}^c$, $(f_\mathcal{D}, \lambda_\mathcal{D}) \in F_\mathcal{D} \times \Lambda_\mathcal{D}$, and for any $\mathcal{M} \subset \mathcal{D}$, there exists a $\bar{\mathcal{D}}(\mathcal{M}) \in \mathbb{D}^c$ with $\bar{\mathcal{D}}(\mathcal{M}) \geq \mathcal{D}$ and $\#\bar{\mathcal{D}}(\mathcal{M}) \lesssim \#\mathcal{D}$, such that

$$\|u_{\bar{\mathcal{D}}(\mathcal{M})}(f_\mathcal{D}, \lambda_\mathcal{D}) - u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D})\|_V \gtrsim \mathcal{E}_\mathcal{D}(\mathcal{M}, u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D}), f_\mathcal{D}, \lambda_\mathcal{D}). \tag{2.22}$$

Then a valid procedure **REDUCE** is defined as follows.

$[\bar{\mathcal{D}}, u_{\bar{\mathcal{D}}}] = \mathbf{REDUCE}(\varrho, \mathcal{D}, f_{\mathcal{D}}, \lambda_{\mathcal{D}})$
% Input: $\varrho \in (0,1]$, $\mathcal{D} \in \mathbb{D}^c$, $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$.
% Output: $\bar{\mathcal{D}} \in \mathbb{D}^c$ with $\bar{\mathcal{D}} \geq \mathcal{D}$, and the Galerkin solution $u_{\bar{\mathcal{D}}} = u_{\bar{\mathcal{D}}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$.
% Parameters: $\theta \in (0,1]$ fixed.
Compute $M := M(\varrho) \in \mathbb{N}$ sufficiently large, cf. Proposition 2.1.
$\mathcal{D}_0 := \mathcal{D}$; **SOLVE**: compute $u_{\mathcal{D}_0}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$
`for` $i = 1$ `to` M `do`
ESTIMATE: `compute` $\{\eta^2_{D,\mathcal{D}_{i-1}}(u_{\mathcal{D}_{i-1}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \colon D \in \mathcal{D}_{i-1}\}$
MARK: `select` $\mathcal{M}_{i-1} \subseteq \mathcal{D}_{i-1}$ `with`

$$\mathcal{E}^2_{\mathcal{D}_{i-1}}(\mathcal{M}_{i-1}, u_{\mathcal{D}_{i-1}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \geq \theta\, \mathcal{E}^2_{\mathcal{D}_{i-1}}(u_{\mathcal{D}_{i-1}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}})$$

REFINE: $\mathcal{D}_i := \bar{\mathcal{D}}(\mathcal{M}_{i-1})$
SOLVE: compute $u_{\mathcal{D}_i}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$
`end`
$\bar{\mathcal{D}} := \mathcal{D}_M$; $u_{\bar{\mathcal{D}}} = u_{\mathcal{D}_M}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$

Proposition 2.1. *Assuming* (2.21) *and* (2.22)*, the number $M = M(\varrho)$ of iterations that are required so that $[\bar{\mathcal{D}}, u_{\bar{\mathcal{D}}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})] = \mathbf{REDUCE}(\varrho, \mathcal{D}, f_{\mathcal{D}}, \lambda_{\mathcal{D}})$ satisfies*

$$\|u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - u_{\bar{\mathcal{D}}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})\|_V \leq \varrho \inf_{v \in V^c_{\mathcal{D}}} \|u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - v\|_V$$

is at most proportional to $\log \varrho^{-1}$, and $\#\bar{\mathcal{D}} \lesssim \#\mathcal{D}$, both independent of $\mathcal{D} \in \mathbb{D}^c$, and $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$. So both (2.13) *and* (2.17) *are realized.*

Proof. Since $f_{\mathcal{D}}$ and $\lambda_{\mathcal{D}}$ are fixed, for simplicity we drop them from our notations. Applying (2.22) with $\mathcal{D} = \mathcal{D}_{i-1}$ and $\mathcal{D}_i = \bar{\mathcal{D}}(\mathcal{M}_{i-1})$, the definition of **MARK**, and (2.21) we get

$$\begin{aligned}\|u_{\mathcal{D}_i} - u_{\mathcal{D}_{i-1}}\|^2_V &\gtrsim \mathcal{E}^2_{\mathcal{D}_{i-1}}(\mathcal{M}_{i-1}, u_{\mathcal{D}_{i-1}}, f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \\ &\geq \theta \mathcal{E}^2_{\mathcal{D}_{i-1}}(u_{\mathcal{D}_{i-1}}, f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \\ &\gtrsim \theta \|u - u_{\mathcal{D}_{i-1}}\|^2_V.\end{aligned}$$

This and the uniform equivalence of $\|\cdot\|_V$ and $|||\cdot|||_{\lambda_{\mathcal{D}}} =: |||\cdot|||$ give the *saturation property*

$$|||u_{\mathcal{D}_i} - u_{\mathcal{D}_{i-1}}|||^2 \geq C_* \theta |||u - u_{\mathcal{D}_{i-1}}|||^2 \tag{2.23}$$

for some positive constant C_*. Then, using Pythagoras' identity

$$|||u - u_{\mathcal{D}_i}|||^2 = |||u - u_{\mathcal{D}_{i-1}}|||^2 - |||u_{\mathcal{D}_i} - u_{\mathcal{D}_{i-1}}|||^2, \tag{2.24}$$

we obtain the *contraction property*

$$|||u - u_{\mathcal{D}_i}||| \leq \kappa |||u - u_{\mathcal{D}_{i-1}}||| \tag{2.25}$$

for $\kappa = \sqrt{1 - C_* \theta} < 1$. We conclude that

$$\begin{aligned}\|u - u_{\mathcal{D}_M}\|_V &\leq \frac{1}{\sqrt{\alpha_*}} |||u - u_{\mathcal{D}_M}||| \leq \frac{1}{\sqrt{\alpha_*}} \kappa^M |||u - u_{\mathcal{D}}||| \\ &= \frac{1}{\sqrt{\alpha_*}} \kappa^M \inf_{v \in V^c_{\mathcal{D}}} |||u - v||| \leq \sqrt{\frac{\alpha^*}{\alpha_*}} \kappa^M \inf_{v \in V^c_{\mathcal{D}}} \|u - v\|_V.\end{aligned}$$

Enforcing $\sqrt{\frac{\alpha^*}{\alpha_*}}\kappa^M \leq \varrho$ yields $M = \mathcal{O}(\log \varrho^{-1})$. In addition, since $\#\mathcal{D}_i \lesssim \#\mathcal{D}_{i-1}$ for $1 \leq i \leq M$ according to (2.22), the proof is complete. ∎

Remark 2.4. The partition $\bar{\mathcal{D}}(\mathcal{M})$ can be built by an 'h-refinement' or a 'p-enrichment', or both, of the elements $D \in \mathcal{M}$, if necessary followed by a 'completion step' by an application of the mapping $\mathcal{C}$ in order to land in $\mathbb{D}^c$. The estimate $\#\bar{\mathcal{D}}(\mathcal{M}) \lesssim \#\mathcal{D}$ shows no benefit in taking $\theta < 1$, i.e., in taking a local, 'adaptive' refinement. In our algorithm **hp-AFEM**, the adaptive selection of suitable hp partitions takes place in **hp-NEARBEST**. Nevertheless, in a quantitative sense it can be beneficial to incorporate adaptivity in **REDUCE** as well, by selecting, for a $\theta < 1$, a (near) minimal set $\mathcal{M} \subset \mathcal{D}_{i-1}$ in **MARK**.

Remark 2.5. The discrete efficiency of the estimator implies its "continuous" efficiency. Indeed, taking $\mathcal{M} = \mathcal{D}$ in (2.22) and denoting $\bar{\mathcal{D}} = \bar{\mathcal{D}}(\mathcal{D})$, and temporarily dropping $f_{\mathcal{D}}$ and $\lambda_{\mathcal{D}}$ from our notations, we have

$$\begin{aligned}\mathcal{E}_{\mathcal{D}}(u_{\mathcal{D}})^2 &\lesssim \alpha_* \|u_{\bar{\mathcal{D}}} - u_{\mathcal{D}}\|_V^2 \leq |\!|\!| u_{\bar{\mathcal{D}}} - u_{\mathcal{D}} |\!|\!|_\lambda^2 = |\!|\!| u - u_{\mathcal{D}} |\!|\!|_\lambda^2 - |\!|\!| u - u_{\bar{\mathcal{D}}} |\!|\!|_\lambda^2 \leq |\!|\!| u - u_{\mathcal{D}} |\!|\!|_\lambda^2 \\ &= \inf_{v \in V_{\mathcal{D}}^c} |\!|\!| u - v |\!|\!|_\lambda^2 \leq \alpha^* \inf_{v \in V_{\mathcal{D}}^c} \|u - v\|_V^2.\end{aligned}$$

Consequently, recalling (2.21), a stopping criterium for **REDUCE** could be defined as follows

$$\mathcal{E}_{\mathcal{D}_i}(u_{\mathcal{D}_i}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \leq C\varrho \mathcal{E}_{\mathcal{D}}(u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}),$$

where C is a constant in terms of the "hidden constants" in (2.21) and (2.22), and α_* and α^*.

Assumptions (2.21)-(2.22) about reliability and discrete efficiency can be substituted by the following three assumptions concerning the estimator. This will be used for our application in two dimensions in Sect. 5.

- **Reliability and efficiency:** For $\mathcal{D} \in \mathbb{D}^c$, there exists $R_{\mathcal{D}}, r_{\mathcal{D}} > 0$, such that for $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$, and $|\!|\!| \cdot |\!|\!|_{\lambda_{\mathcal{D}}} =: |\!|\!| \cdot |\!|\!|$ one has

$$\begin{aligned} r_{\mathcal{D}}\, \mathcal{E}_{\mathcal{D}}^2(u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) &\leq |\!|\!| u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) |\!|\!|^2 \\ &\leq R_{\mathcal{D}}\, \mathcal{E}_{\mathcal{D}}^2(u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}); \end{aligned} \tag{2.26}$$

- **Stability:** For $\mathcal{D} \in \mathbb{D}^c$, and all $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$, $v, w \in V_{\mathcal{D}}^c$ one has

$$\sqrt{r_{\mathcal{D}}} \Big| \mathcal{E}_{\mathcal{D}}(v, f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - \mathcal{E}_{\mathcal{D}}(w, f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \Big| \leq |\!|\!| v - w |\!|\!|. \tag{2.27}$$

- **Estimator reduction upon refinement:** There exists a constant $\gamma < 1$, such that for any $\mathcal{M} \subset \mathcal{D} \in \mathbb{D}^c$, there exists a $\bar{\mathcal{D}}(\mathcal{M}) \in \mathbb{D}^c$ with $\bar{\mathcal{D}}(\mathcal{M}) \geq \mathcal{D}$, $\#\bar{\mathcal{D}}(\mathcal{M}) \lesssim \#\mathcal{D}$, such that with $\bar{\mathcal{S}} := \{\bar{D} \in \bar{\mathcal{D}}(\mathcal{M}) : \exists D \in \mathcal{M} \text{ with } K_{\bar{D}} \subset K_D\}$,

$$\begin{aligned} \mathcal{E}_{\bar{\mathcal{D}}(\mathcal{M})}^2(\bar{\mathcal{S}}, u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) &\leq \gamma\, \mathcal{E}_{\mathcal{D}}^2(\mathcal{M}, u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \\ \mathcal{E}_{\bar{\mathcal{D}}(\mathcal{M})}^2(\bar{\mathcal{D}}(\mathcal{M}) \setminus \bar{\mathcal{S}}, u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) &\leq \mathcal{E}_{\mathcal{D}}^2(\mathcal{D} \setminus \mathcal{M}, u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}), \end{aligned} \tag{2.28}$$

  for any $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$.

With θ from **REDUCE** and γ from (2.28), we set $\bar{\gamma} := (1 - \theta) + \theta\gamma$. For $\mathcal{D} \leq \widehat{\mathcal{D}} \in \mathbb{D}^c$, and $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$, we define the squared *total error* to be

$$\mathfrak{E}_{\widehat{\mathcal{D}}}^2(u_{\widehat{\mathcal{D}}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}) := |\!|\!| u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - u_{\widehat{\mathcal{D}}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) |\!|\!|^2 + (1 - \sqrt{\bar{\gamma}}) r_{\widehat{\mathcal{D}}}\, \mathcal{E}_{\widehat{\mathcal{D}}}^2(u_{\widehat{\mathcal{D}}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), f_{\mathcal{D}}, \lambda_{\mathcal{D}}).$$

Proposition 2.2. *Assume* (2.26), (2.27), *and* (2.28), *and, inside* **REDUCE**, *take* $\bar{\mathcal{D}}(\mathcal{M})$ *as defined in* (2.28). *Let* $\mathcal{D} \in \mathbb{D}^c$, *and* $(f_\mathcal{D}, \lambda_\mathcal{D}) \in F_\mathcal{D} \times \Lambda_\mathcal{D}$. *Then consecutive iterands produced in* **REDUCE**$(\varrho, \mathcal{D}, f_\mathcal{D}, \lambda_\mathcal{D})$ *satisfy*

$$\mathfrak{E}^2_{\mathcal{D}_i}(u_{\mathcal{D}_i}(f_\mathcal{D}, \lambda_\mathcal{D}), f_\mathcal{D}, \lambda_\mathcal{D}) \le \Big[1 - \frac{(1-\sqrt{\bar\gamma})^2}{2} \frac{r_{\mathcal{D}_i}}{R_{\mathcal{D}_{i-1}}}\Big] \mathfrak{E}^2_{\mathcal{D}_{i-1}}(u_{\mathcal{D}_{i-1}}(f_\mathcal{D}, \lambda_\mathcal{D}), f_\mathcal{D}, \lambda_\mathcal{D}).$$

Furthermore, for $\mathcal{D} \in \mathbb{D}^c$ *and* $(f_\mathcal{D}, \lambda_\mathcal{D}) \in F_\mathcal{D} \times \Lambda_\mathcal{D}$,

$$|||u(f_\mathcal{D}, \lambda_\mathcal{D}) - u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D})|||^2 \le \mathfrak{E}^2_\mathcal{D}(u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D}), f_\mathcal{D}, \lambda_\mathcal{D}) \le 2|||u(f_\mathcal{D}, \lambda_\mathcal{D}) - u_\mathcal{D}(f_\mathcal{D}, \lambda_\mathcal{D})|||^2.$$

Therefore, if $\sup_{\mathcal{D}\in\mathbb{D}^c} R_\mathcal{D} < \infty$ *and* $\inf_{\mathcal{D}\in\mathbb{D}^c} r_\mathcal{D} > 0$, *then the statement of Proposition 2.1 is again valid.*

Proof. Since both $f_\mathcal{D}$ and $\lambda_\mathcal{D}$ are fixed, we again drop them from our notations. Applying **MARK** and (2.28) yields

$$\mathcal{E}^2_{\mathcal{D}_i}(u_{\mathcal{D}_{i-1}}) \le \bar\gamma\, \mathcal{E}^2_{\mathcal{D}_{i-1}}(u_{\mathcal{D}_{i-1}}). \tag{2.29}$$

By virtue of (2.27), Young's inequality, and (2.29), we have that for any $\zeta > 0$,

$$\begin{aligned}\mathcal{E}^2_{\mathcal{D}_i}(u_{\mathcal{D}_i}) &\le (1+\zeta)\, \mathcal{E}^2_{\mathcal{D}_i}(u_{\mathcal{D}_{i-1}}) + (1+\zeta^{-1}) r^{-1}_{\mathcal{D}_i} |||u_{\mathcal{D}_i} - u_{\mathcal{D}_{i-1}}|||^2 \\ &\le (1+\zeta)\bar\gamma\, \mathcal{E}^2_{\mathcal{D}_{i-1}}(u_{\mathcal{D}_{i-1}}) + (1+\zeta^{-1}) r^{-1}_{\mathcal{D}_i} |||u_{\mathcal{D}_i} - u_{\mathcal{D}_{i-1}}|||^2.\end{aligned}$$

By multiplying this inequality by $\frac{r_{\mathcal{D}_i}}{(1+\zeta^{-1})}$, substituting $\zeta = \bar\gamma^{-\frac{1}{2}} - 1$, and adding to Pythagoras' identity (2.24), we obtain

$$|||u-u_{\mathcal{D}_i}|||^2 + (1-\sqrt{\bar\gamma}) r_{\mathcal{D}_i}\, \mathcal{E}^2_{\mathcal{D}_i}(\mathcal{D}_i, u_{\mathcal{D}_i}) \le |||u-u_{\mathcal{D}_{i-1}}|||^2 + \sqrt{\bar\gamma}(1-\sqrt{\bar\gamma}) r_{\mathcal{D}_i}\, \mathcal{E}^2_{\mathcal{D}_{i-1}}(\mathcal{D}_{i-1}, u_{\mathcal{D}_{i-1}}).$$

We resort to (2.26) to bound the right-hand side as follows in terms of an arbitrary $\beta \in [0,1]$

$$\beta |||u - u_{\mathcal{D}_{i-1}}|||^2 + \Big((1-\beta)\frac{R_{\mathcal{D}_{i-1}}}{(1-\sqrt{\bar\gamma}) r_{\mathcal{D}_i}} + \sqrt{\bar\gamma}\Big)(1-\sqrt{\bar\gamma}) r_{\mathcal{D}_i}\, \mathcal{E}^2_{\mathcal{D}_{i-1}}(\mathcal{D}_{i-1}, u_{\mathcal{D}_{i-1}}).$$

We now observe that the following function of β attains its minimum at β_*

$$\max_\beta \Big\{\beta, \Big((1-\beta)\frac{R_{\mathcal{D}_{i-1}}}{(1-\sqrt{\bar\gamma}) r_{\mathcal{D}_i}} + \sqrt{\bar\gamma}\Big)\Big\} \ge \beta_* := 1 - \frac{1-\sqrt{\bar\gamma}}{1 + \frac{R_{\mathcal{D}_{i-1}}}{(1-\sqrt{\bar\gamma}) r_{\mathcal{D}_i}}}.$$

The proof of the first statement follows from $\frac{1-\sqrt{\bar\gamma}}{1+\frac{R_{\mathcal{D}_{i-1}}}{(1-\sqrt{\bar\gamma}) r_{\mathcal{D}_i}}} \ge \frac{(1-\sqrt{\bar\gamma})^2}{2} \frac{r_{\mathcal{D}_i}}{R_{\mathcal{D}_{i-1}}}$. The second statement is a direct consequence of (2.26), and the final statement follows directly from the first two.

□

3 The module hp-NEARBEST

In this section we describe briefly the algorithm and theory recently developed by P. Binev for *hp*-adaptive tree approximation [5], which constitutes the building block behind the module **hp-NEARBEST**.

3.1 h-Adaptive Tree Approximation

We first review the algorithm designed and studied by Binev and DeVore [7] for h-adaptive tree approximation. Since, in this subsection, the local approximation spaces do not depend on d, temporarily we identify an element D with the element domain K_D, and $\mathcal{D}$ with the *h-partition* $\mathcal{K}(\mathcal{D})$, the latter being an element of $\mathbb{K}$.

Recall that for any $\mathcal{K} \in \mathbb{K}$, the set of all $K \in \mathcal{K}$ together with their ancestors form a tree $\mathcal{T}$, being a subtree of the master tree $\mathfrak{K}$. Conversely, given such a subtree $\mathcal{T}$, the set $\mathcal{L}(\mathcal{T})$ of its leaves is a partition in $\mathbb{K}$.

For the moment, we will assume that the master tree $\mathfrak{K}$ has only *one root*. In the next subsection, in Remark 3.1, we will deal with the case that it has possibly multiple roots.

For any $K \in \mathfrak{K}$, let $e_K \geq 0$ be some *local h-error functional*. That means that it satisfies the key property (2.2), that in this h-element setting reduces to *subadditivity*:

$$e_{K'} + e_{K''} \leq e_K$$

where K' and K'' denote the children of K. The corresponding *global h-error functional* reads

$$\mathrm{E}_{\mathcal{K}} = \sum_{K \in \mathcal{K}} e_K \quad \forall \mathcal{K} \in \mathbb{K}.$$

The notion of a *best h-partition* w.r.t. this error functional is now apparent: for $N \in \mathbb{N}$, let

$$\sigma_N := \inf_{\#\mathcal{K} \leq N} \mathrm{E}_{\mathcal{K}}\,.$$

This quantity gives the smallest error achievable with h-partitions $\mathcal{K}$ with cardinality $\#\mathcal{K} \leq N$. In spite of the inf being a min, because the minimization is over a finite set, computing a tree that realizes the min has exponential complexity.

A fundamental, but rather surprising, result of Binev and DeVore shows that a *near-best h-adaptive tree* is computable with linear complexity. A key ingredient is a *modified local h-error functional* $\tilde{e}_K$ defined as follows for all $K \in \mathfrak{K}$:

- $\tilde{e}_K := e_K$ if K is the root;
- $\frac{1}{\tilde{e}_K} := \frac{1}{e_K} + \frac{1}{e_{K^*}}$ where K^* is the parent of K and $e_K \neq 0$; otherwise $\tilde{e}_K = 0$.

This harmonic mean has the following essential properties: if $e_K \ll e_{K^*}$, then $\tilde{e}_K \approx e_K$, whereas if $e_K \approx e_{K^*}$, then $\tilde{e}_K \approx \frac{1}{2} e_K$. This means that $\tilde{e}_K$ penalizes the lack of success in reducing the error from K^* to K up to a factor $\frac{1}{2}$, provided $e_K = e_{K^*}$, and always $\frac{1}{2} \leq \frac{\tilde{e}_K}{e_K} < 1$.

The practical method consists of applying a *greedy* algorithm based on $\{\tilde{e}_K\}_{K \in \mathcal{K}}$: given an h-partition $\mathcal{K}_N$, with $\#\mathcal{K}_N = N$, construct $\mathcal{K}_{N+1}$ by bisecting an element domain $K \in \mathcal{K}$ with largest $\tilde{e}_K$. It is worth stressing that if lack of error reduction persists, then the modified error functional $\tilde{e}_K$ diminishes exponentially and forces the greedy algorithm to start refining somewhere else.

For e_K being the squared L^2-error in the best polynomial approximation on K of a function v, this may happen when v has local but strong singularity. The simple, but astute idea to operate on the modified error functionals is responsible alone for the following key result.

Theorem 3.1 (instance optimality of h-trees [7])**.** *Let the master tree $\mathfrak{K}$ have one single root. The sequence of h-partitions $(\mathcal{K}_N)_{N \in \mathbb{N}}$ given by the greedy algorithm based on $(\tilde{e}_K)_{K \in \mathcal{K}}$ provides near-best h-adaptive tree approximations in the sense that*

$$\mathrm{E}_{\mathcal{K}_N} \leq \frac{N}{N - n + 1} \sigma_n \quad \forall n \leq N.$$

The complexity for obtaining $\mathcal{K}_N$ is $\mathcal{O}(N)$.

We can interpret Theorem 3.1 as follows: given N let $n = \lceil \frac{N}{2} \rceil$ be the ceiling of $N/2$, whence $N - n + 1 \geq N/2$ and

$$\mathrm{E}_{\mathcal{K}_N} \leq 2\sigma_{\lceil \frac{N}{2} \rceil}. \tag{3.1}$$

3.2 *hp*-Adaptive Tree Approximation

In this subsection, we return to *hp*-approximations. An element D is a pair $(K, d) = (K_D, d_D)$, with K being the element domain, and d an integer. The local error functional $e_D \geq 0$ is required to satisfy (2.2), i.e., $e_{K',d} + e_{K'',d} \leq e_{K,d}$ when K', K'' are the children of K, and $e_{K,d'} \leq e_{K,d}$ when $d' \geq d$. The corresponding global *hp*-error functional reads as

$$E_{\mathcal{D}} = \sum_{D \in \mathcal{D}} e_D \quad \forall \mathcal{D} \in \mathbb{D}.$$

For $N \in \mathbb{N}$, we set

$$\sigma_N := \inf_{\#\mathcal{D} \leq N} E_{\mathcal{D}}$$

where $\#\mathcal{D} = \sum_{D \in \mathcal{D}} d_D$.

In our applications, d_D is proportional to the dimension of the polynomial approximation space that is applied on K_D so that $\#\mathcal{D}$ is proportional to the dimension of the global *hp*-finite element space. More precisely, given d, we take $p = p(d)$ as the largest integer for which

$$\dim \mathbb{P}_{p-1}(K) = \binom{n+p-1}{p-1} \leq d, \tag{3.2}$$

and corresponding to $D = (K, d)$, we choose $\mathbb{P}_{p(d)}(K)$ as approximation space. Consequently, for $n > 1$, $e_{K,d+1} = e_{K,d}$ whenever $p(d+1) = p(d)$.

We describe an algorithm, designed by Binev [4, 5], that finds a *near-best hp-partition*. It builds two trees: a *ghost h-tree* $\mathcal{T}$, similar to that in Sect. 3.1 but with degree dependent error and modified error functionals, and a *subordinate hp-tree* $\mathcal{P}$. The second tree is obtained by trimming the first one and increasing d as described in the sequel.

Let $\mathcal{K} \in \mathbb{K}$, and let $\mathcal{T}$ denote its corresponding tree. For any $K \in \mathcal{T}$, we denote by $\mathcal{T}(K)$ the *subtree* of $\mathcal{T}$ emanating from K, and let $d(K, \mathcal{T})$ be the number of leaves of $\mathcal{T}(K)$, i.e.

$$d(K, \mathcal{T}) = \#\mathcal{L}(\mathcal{T}(K)). \tag{3.3}$$

The tree-dependent *local hp-error functionals* $e_K(\mathcal{T})$ are defined recursively starting from the leaves and proceeding upwards as follows:

- $e_K(\mathcal{T}) := e_{K,1}$ provided $K \in \mathcal{L}(\mathcal{T})$,
- $e_K(\mathcal{T}) := \min\{e_{K'}(\mathcal{T}) + e_{K''}(\mathcal{T}), e_{K,d(K,\mathcal{T})}\}$ otherwise,

where $K', K'' \in \mathcal{T}$ are the children of K. This local functional carries the information whether it is preferable to enrich the space (increase d) or refine the element (decrease h) to reduce the current error in K. The subordinate *hp*-tree $\mathcal{P}$ is obtained from $\mathcal{T}$ by eliminating the subtree $\mathcal{T}(K)$ of a node $K \in \mathcal{T}$ whenever

$$e_K(\mathcal{T}) = e_{K,d(K,\mathcal{T})}.$$

This procedure is depicted in Figure 3.2.

The *hp*-tree $\mathcal{P}$ gives rise to an *hp*-partition $\mathcal{D}$, namely the collection of *hp*-elements $D = (K, d)$ with K a leaf of $\mathcal{P}$ and $d = d(K, \mathcal{T})$. We have that $\#\mathcal{D} = \#\mathcal{K}$, and $\mathcal{D}$ minimizes $E_{\widetilde{\mathcal{D}}}$ over all $\widetilde{\mathcal{D}} \in \mathbb{D}$ with $\mathcal{K}(\widetilde{\mathcal{D}}) \leq \mathcal{K}$ and $d_D \leq d(K_D, \mathcal{T})$ for all $D \in \widetilde{\mathcal{D}}$, whence $\#\widetilde{\mathcal{D}} \leq \#\mathcal{K}$.

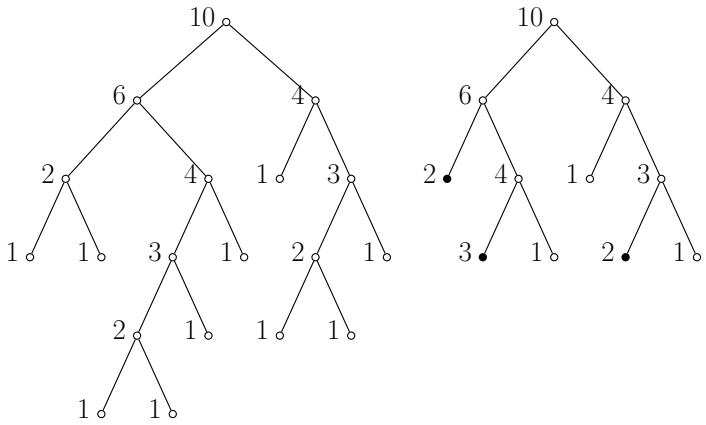

Figure 1: Ghost h-tree $\mathcal{T}$ (left) with 10 leaves ($\#\mathcal{L}(\mathcal{T}) = 10$). The label of each node K is $d(K, \mathcal{T})$. Subordinate hp-tree $\mathcal{P}$ (right) resulting from $\mathcal{T}$ upon trimming 3 subtrees and raising the values of d of the interior nodes of $\mathcal{T}$, now leaves of $\mathcal{P}$, from 1 to $2, 3$, and 2 respectively.

This describes the trimming of the h-tree $\mathcal{T}$, but not how to increase the total cardinality of $\mathcal{T}$. To grow $\mathcal{T}$, P. Binev uses a modified local hp-error functional and a greedy algorithm that selects the leaf of $\mathcal{T}$ that would lead to the largest reduction of the hp-error in $\mathcal{P}$. We refer to [5] for the construction of the full algorithm for hp-adaptive approximation.

Theorem 3.2 (instance optimality of hp-tree [4, 5])**.** *Let the master tree $\mathfrak{K}$ have one single root. For all $N \in \mathbb{N}$, the algorithm sketched above constructs an hp-tree $\mathcal{P}_N$ subordinate to a ghost h-tree $\mathcal{T}_N$ such that the resulting hp-partition $\mathcal{D}_N$ has cardinality $\#\mathcal{D}_N = N$ and global hp-error functional*

$$\mathrm{E}_{\mathcal{D}_N} \leq \frac{2N}{N-n+1}\sigma_n \quad \forall n \leq N.$$

In addition, the cost of the algorithm for obtaining $\mathcal{D}_N$ is bounded by $\mathcal{O}\big(\sum_{K\in\mathcal{T}_N} d(K, \mathcal{T}_N)\big)$, and varies from $\mathcal{O}(N \log N)$ for well balanced trees to $\mathcal{O}(N^2)$ for highly unbalanced trees.

Binev's algorithm gives a routine **hp-NEARBEST** that satisfies the assumptions made in Subsect. 2.2 for any $B > 1$ and $b = \sqrt{\frac{1}{2}(1 - \frac{1}{B})}$:

Corollary 3.1. *Let $B > 1$. Given $\varepsilon > 0$, let $\mathcal{D} \in \mathbb{D}$ be the first partition in the sequence produced by Binev's algorithm for which $E_{\mathcal{D}}^{\frac{1}{2}} \leq \varepsilon$. Then $\#\mathcal{D} \leq B \min\{\#\hat{\mathcal{D}} \colon \hat{\mathcal{D}} \in \mathbb{D}, \mathrm{E}_{\hat{\mathcal{D}}}^{\frac{1}{2}} \leq \sqrt{\frac{1}{2}(1 - \frac{1}{B})}\,\varepsilon\}$.*

Proof. Let $\mathcal{D} = \mathcal{D}_N$, i.e., $\mathcal{D}$ is the Nth partition in the sequence, and $\#\mathcal{D} = N$. For $N = 1$ the statement is true, so let $N > 1$. Suppose there exists a $\hat{\mathcal{D}} \in \mathbb{D}$ with $\mathrm{E}_{\hat{\mathcal{D}}}^{\frac{1}{2}} \leq \sqrt{\frac{1}{2}(1 - \frac{1}{B})}\,\varepsilon$ and $N > B\#\hat{\mathcal{D}}$. Then, with $n := \#\hat{\mathcal{D}}$, we have $\mathrm{E}_{\mathcal{D}_{N-1}} \leq \frac{2(N-1)}{N-1-n+1}\sigma_n \leq \frac{2(N-1)}{N-1-n+1}\,\mathrm{E}_{\hat{\mathcal{D}}} \leq \frac{2(N-1)}{N-1-n+1}\,\frac{1}{2}(1 - \frac{1}{B})\varepsilon^2$. From $\frac{2(N-1)}{N-1-n+1}\,\frac{1}{2}(1 - \frac{1}{B}) \leq 1$, being a consequence of $N \geq Bn$ and $B \geq 1$, we get a contradiction with $\mathcal{D}$ being the first one with $E_{\mathcal{D}}^{\frac{1}{2}} \leq \varepsilon$. ∎

Remark 3.1. *In order to deal with the case that the master tree $\mathfrak{K}$ has $R > 1$ roots, the following approach can be followed.*

We unify the R roots pairwise creating new element domains, each one being the union of two roots. When $R > 2$, this process has to be repeated until only one element domain remains, which will the new, single root. Obviously, this applies only when R is a power of 2. In the other case, we have to introduce at most $\lceil \log_2 R \rceil - 1$ (empty) virtual element domains (and, formally, infinite binary trees of virtual element domains rooted at them). We denote the extended, single rooted master tree by $\widehat{\mathfrak{K}}$.

Next, we extend the definition of $e_{K,d}$ as follows. At first we give a meaning to $e_{K,0}$ for each element domain $K \in \mathfrak{K}$. Typically, for $d \in \mathbb{N}$, $e_{K,d}$ has the meaning of the squared error in the approximation of a quantity from a space of dimension d. Then a natural definition of $e_{K,0}$ is that of the squared error in the zero approximation.

Considering now the elements in $\widehat{\mathfrak{K}} \setminus \mathfrak{K}$, i.e., the newly created element domains, we distinguish between virtual and non-virtual element domains. For each virtual element domain, we set $e_{K,d} := 0$ for any $d \in \mathbb{N} \cup \{0\}$. Finally, for each newly created non-virtual element domain K, being the union of K' and K'' (one of them possibly being a virtual element domain), for $d \in \mathbb{N} \cup \{0\}$ recursively we define

$$e_{K,d} := \min_{\{d',d'' \in \mathbb{N}\cup\{0\}:\ d'+d'' \le d\}} e_{K',d'} + e_{K'',d''}.$$

Note that in the minimum at the right hand side d' or d'' can or has to be zero. In that case, $e_{K',d'} + e_{K'',d''}$ has the interpretation of the squared error in an approximation on K that is zero on K' or K''.

It is easily checked that the error functional $e_{K,d}$ for $(K,d) \in \widehat{\mathfrak{K}} \times \mathbb{N}$ satisfies (2.2), and Theorem 3.2 and Corollary 3.1 apply.

We close the discussion of the module **hp-NEARBEST** with the observation that in dimensions $n > 1$, Binev's algorithm produces *hp*-partitions that are generally non-conforming. Since conformity is required by the module **REDUCE**, a post-processing step which makes the output partition conforming is required. The implementation of such a procedure in dimension 2, and the analysis of its complexity, will be discussed in Sect. 5.1.

4 A self-adjoint elliptic problem in 1D

In this section we apply the abstract framework introduced in Sect. 2 to a one-dimensional self-adjoint elliptic problem.

4.1 The continuous problem and its *hp* discretization

Let $\Omega := (0,1)$. Given $f_1, f_2 \in L^2(\Omega)$ and $\nu, \sigma \in L^\infty(\Omega)$ satisfying

$$0 < \nu_* \le \nu \le \nu^* < \infty \qquad \text{and} \qquad 0 \le \sigma \le \sigma^* < \infty \tag{4.1}$$

for some constants ν_*, ν^* and σ^*, we consider the following model elliptic problem

$$\begin{aligned} -(\nu u')' + \sigma u &= f_1 + f_2' \quad \text{in } \Omega\,, \\ u(0) &= u(1) = 0\,, \end{aligned} \tag{4.2}$$

which can be written as in (2.1) setting $\lambda = (\nu, \sigma)$, $f = f_1 + f_2' \in H^{-1}(\Omega)$ and

$$A_\lambda u := -(\nu u')' + \sigma u \in \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)).$$

Equivalently, $u \in H^1_0(\Omega) =: V$, equipped with the norm $|\cdot|_{H^1(\Omega)}$, satisfies

$$a_\lambda(u,v) = \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega), \tag{4.3}$$

where the bilinear form $a_\lambda : H^1_0(\Omega)\times H^1_0(\Omega) \to \mathbb{R}$ and the linear form $f : H^1_0(\Omega) \to \mathbb{R}$ are defined as

$$a_\lambda(u,v) := \int_\Omega (\nu u'v' + \sigma uv)\, dx\,, \qquad \langle f, v\rangle = \int_\Omega (f_1 v - f_2 v')\, dx\,.$$

In view of the approximation of the operator A_λ we introduce the metric space

$$\bar\Lambda := \{\bar\lambda = (\bar\nu, \bar\sigma) \in L^\infty(\Omega)\times L^\infty(\Omega) : \ \ \bar\nu_* \le \bar\nu \le \bar\nu^*,\ \ -\bar\sigma_* \le \bar\sigma \le \bar\sigma^*\}$$

where $\bar\nu_*, \bar\nu^*, \bar\sigma_*, \bar\sigma^*$ are positive constants defined as follows. Suppose that the pair $(\bar\nu, \bar\sigma)$ approximates (ν, σ) with error

$$\|\nu - \bar\nu\|_{L^\infty(\Omega)} \le \frac{\nu_*}{2}, \qquad \|\sigma - \bar\sigma\|_{L^\infty(\Omega)} \le \frac{\nu_*}{2}; \tag{4.4}$$

then it is easily seen that

$$\bar\nu_* := \frac{\nu_*}{2} \le \bar\nu \le \nu^* + \frac{\nu_*}{2} =: \bar\nu^*, \qquad -\bar\sigma_* := -\frac{\nu_*}{2} \le \bar\sigma \le \sigma^* + \frac{\nu_*}{2} =: \bar\sigma^*.$$

Furthermore, using the Poincaré inequality $\|v\|^2_{L^2(\Omega)} \le \frac12 |v|^2_{H^1(\Omega)}$ we have

$$(\bar\nu_* - \frac12\bar\sigma_*)|v|^2_{H^1(\Omega)} \le a_{\bar\lambda}(v,v) \le (\bar\nu^* + \frac12\bar\sigma^*)|v|^2_{H^1(\Omega)}$$

for all $v \in H^1_0(\Omega)$, $\bar\lambda \in \bar\Lambda$. We conclude that setting $\alpha_* := \bar\nu_* - \frac12\bar\sigma_* = \frac14\nu_*$ and $\alpha^* = \bar\nu^* + \frac12\bar\sigma^* = \nu^* + \frac12\sigma^* + \frac34\nu_*$ it holds

$$\sqrt{\alpha_*}|v|_{H^1(\Omega)} \le |\!|\!| v |\!|\!|_{\bar\lambda} \le \sqrt{\alpha^*}|v|_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega),\ \forall \bar\lambda \in \bar\Lambda \tag{4.5}$$

with $|\!|\!| v |\!|\!|^2_{\bar\lambda} := a_{\bar\lambda}(v,v)$. The space Λ will be a subset of $\bar\Lambda$ containing the coefficients λ of the problem (2.1); it will be defined later on.

Concerning the definition of the space F containing the right-hand side, we write $f = (f_1, f_2) \in L^2(\Omega)\times L^2(\Omega) =: F$ (note that different couples in F may give rise to the same $f \in H^{-1}(\Omega)$).

We now discuss the *hp*-discretization of (4.2). To this end, we specify that the binary master tree $\mathfrak{K}$ is obtained from an initial partition, called the 'root partition', by applying successive dyadic subdivisions to all its elements. Later, cf. Property 4.1, it will be needed to assume that this initial partition is sufficiently fine. Furthermore, with reference to the abstract notation of Section 2, given any $(K,d) \in \mathfrak{K}\times\mathbb{N}$ we have $p(d) = d$. In consideration of this simple relation, throughout this section we will use the notation (K,p) instead of (K,d), i.e., the second parameter of the couple will identify a polynomial degree on the element K. We set

$$V_{K,p} = \mathbb{P}_p(K), \qquad F_{K,p} = \mathbb{P}_{p-1}(K)\times\mathbb{P}_p(K),$$
$$\Lambda_{K,p} = \{\bar\lambda = (\bar\nu,\bar\sigma) \in \mathbb{P}_{p+1}(K)\times\mathbb{P}_{p+1}(K) : \ \ \bar\nu_* \le \bar\nu \le \bar\nu^*,\ \ -\bar\sigma_* \le \bar\sigma \le \bar\sigma^*\}.$$

Thus

$$V^c_{\mathcal{D}} = \{v \in H^1_0(\Omega) : v_{|K_D} \in \mathbb{P}_{p_D}(K_D)\ \forall D \in \mathcal{D}\}$$

will be the discretization space associated with the *hp*-partition $\mathcal{D}$. Furthermore, we have $F_{\mathcal{D}} \subset F$ and $\Lambda_{\mathcal{D}} \subset \bar\Lambda$, with F and $\bar\Lambda$ defined above. The difference in polynomial degrees between the various components of the approximation spaces for data is motivated by the need of balancing the different terms entering in the local error estimators, see (4.18) below.

At this point, we have all the ingredients that determine a Galerkin approximation as in (2.19).

4.2 Computable a posteriori error estimator

Given data $(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in F_{\mathcal{D}} \times \Lambda_{\mathcal{D}}$, let $u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in V_{\mathcal{D}}^c$ be the solution of the Galerkin problem (2.19) with such data. To it, we associate the residual $r = r(u_{\mathcal{D}}, f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in H^{-1}(\Omega)$, defined by

$$\langle r, v\rangle = \langle f_{\mathcal{D}}, v\rangle - a_{\lambda_{\mathcal{D}}}(u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}}), v) \qquad \forall v \in H_0^1(\Omega)\ , \tag{4.6}$$

and satisfying $\langle r, v_{\mathcal{D}}\rangle = 0$ for all $v_{\mathcal{D}} \in V_{\mathcal{D}}^c$. The dual norm of the residual is a natural a posteriori error estimator, since one has

$$\frac{1}{\sqrt{\alpha^*}}\,\|r\|_{H^{-1}(\Omega)} \le |\!|\!| u(f_{\mathcal{D}}, \lambda_{\mathcal{D}}) - u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})|\!|\!|_{\lambda_{\mathcal{D}}} \le \frac{1}{\sqrt{\alpha_*}}\,\|r\|_{H^{-1}(\Omega)}\ ; \tag{4.7}$$

in one dimension, such norm can be expressed in terms of independent contributions coming from the elements K_D of the partition $\mathcal{D}$, which are easily and exactly computable if, e.g., the residual is locally polynomial. To see this, let us introduce the subspace of $H_0^1(\Omega)$ of the piecewise linear functions on $\mathcal{D}$, i.e.,

$$V_{\mathcal{D}}^L = \{v \in H_0^1(\Omega) \mid v_{|K_D} \in \mathbb{P}_1(K_D) \quad \forall D \in \mathcal{D}\} \subseteq V_{\mathcal{D}}^c$$

and let us first notice that $H_0^1(\Omega)$ admits the orthogonal decomposition (with respect to the inner product associated with the norm $|\cdot|_{H^1(\Omega)}$)

$$H_0^1(\Omega) = V_{\mathcal{D}}^L \oplus \bigoplus_{D\in\mathcal{D}} H_0^1(K_D)\ ,$$

where functions in $H_0^1(K_D)$ are assumed to be extended by 0 outside the interval K_D; indeed, for any $v \in V$, we have the orthogonal splitting

$$v = v_L + \sum_{D\in\mathcal{D}} v_{K_D}\ ,$$

where $v_L \in V_{\mathcal{D}}^L$ is the piecewise linear interpolant of v on $\mathcal{D}$ and $v_{K_D} = (v - v_L)_{|K_D} \in H_0^1(K_D)$. Recalling that $\langle r, v_L\rangle = 0$ for all $v_L \in V_{\mathcal{D}}^L$, it is easily seen that the following expression holds:

$$\|r\|_{H^{-1}(\Omega)}^2 = \sum_{D\in\mathcal{D}} \|r_{K_D}\|_{H^{-1}(K_D)}^2\ , \tag{4.8}$$

where r_{K_D} denotes the restriction of r to $H_0^1(K_D)$.

The computability of the terms on the right-hand side is assured by the following representation: for any $D \in \mathcal{D}$, one has

$$\|r_{K_D}\|_{H^{-1}(K_D)}^2 = |z_{K_D}|_{H^1(K_D)}^2,$$

where $z_{K_D} \in H_0^1(K_D)$ satisfies

$$(z'_{K_D}, v')_{L_2(K_D)} = \langle r_{K_D}, v\rangle \quad \forall v \in H_0^1(K_D). \tag{4.9}$$

Writing $u_{\mathcal{D}} = u_{\mathcal{D}}(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$ and $K_D = (a, b)$, and noting that, since $f_{2,\mathcal{D}}$ is a polynomial in K_D,

$$\langle r_{K_D}, v\rangle = \int_{K_D} \big(f_{1,\mathcal{D}} + f'_{2,\mathcal{D}} + (\nu_{\mathcal{D}} u'_{\mathcal{D}})' - \sigma_{\mathcal{D}} u_{\mathcal{D}}\big) v\, dx = (r_{K_D}, v)_{L_2(K_D)}, \tag{4.10}$$

it is easily seen that the solution z_{K_D} has the following analytic expression

$$z_{K_D}(x) = \int_{K_D} G(x, y) r_{K_D}(y) dy\ , \tag{4.11}$$

where $G(x,y) := \begin{cases} \frac{(a-x)(b-y)}{b-a} & x < y \\ \frac{(a-y)(b-x)}{b-a} & x > y \end{cases}$ is the Green's function of our local problem (4.9). Thus, the squared norm $\|r_{K_D}\|^2_{H^{-1}(K_D)}$ of the local residual can be explicitly computed, since r_{K_D} is a polynomial.

Summarizing, defining for any $D \in \mathcal{D}$ the local error estimator

$$\eta^2_{D,\mathcal{D}}(u_{\mathcal{D}}(f_{\mathcal{D}},\lambda_{\mathcal{D}}),f_{\mathcal{D}},\lambda_{\mathcal{D}}) := |z_{K_D}|^2_{H^1(K_D)} \tag{4.12}$$

and defining the global error estimator as in (2.20), we have by (4.7)

$$\begin{aligned} \frac{1}{\sqrt{\alpha^*}}\,\mathcal{E}_{\mathcal{D}}(u_{\mathcal{D}}(f_{\mathcal{D}},\lambda_{\mathcal{D}}),f_{\mathcal{D}},\lambda_{\mathcal{D}}) &\le |\!|\!| u(f_{\mathcal{D}},\lambda_{\mathcal{D}}) - u_{\mathcal{D}}(f_{\mathcal{D}},\lambda_{\mathcal{D}}) |\!|\!|_{\lambda_{\mathcal{D}}} \\ &\le \frac{1}{\sqrt{\alpha_*}}\,\mathcal{E}_{\mathcal{D}}(u_{\mathcal{D}}(f_{\mathcal{D}},\lambda_{\mathcal{D}}),f_{\mathcal{D}},\lambda_{\mathcal{D}}), \end{aligned} \tag{4.13}$$

which in particular implies the reliability assumption (2.21).

4.3 The module REFINE

Hereafter, we present a realization of the module **REFINE**, that guarantees the discrete efficiency property (2.22), hence the contraction property of **REDUCE**. For every $D \in \mathcal{M} \subseteq \mathcal{D}$ the module raises the local polynomial degree to some higher value, whereas for $D \in \mathcal{D}\setminus\mathcal{M}$ the local polynomial degree remains unchanged. No h-refinement is performed.

To be precise, consider an element $D = (K_D, p_D) \in \mathcal{M}$. Suppose that the local polynomial degree of the data is related to some $\hat{p}_D$, in the sense that

$$f_{1,\mathcal{D}}|_{K_D} \in \mathbb{P}_{\hat{p}_D-1}(K_D), \quad f_{2,\mathcal{D}}|_{K_D} \in \mathbb{P}_{\hat{p}_D}(K_D), \quad \nu_{\mathcal{D}}|_{K_D},\ \sigma_{\mathcal{D}}|_{K_D} \in \mathbb{P}_{\hat{p}_D+1}(K_D).$$

Recall that $u_{\mathcal{D}} = u_{\mathcal{D}}(f_{\mathcal{D}},\lambda_{\mathcal{D}})$ satisfies $u_{\mathcal{D}}|_{K_D} \in \mathbb{P}_{p_D}(K_D)$. Then it is easily seen that the residual $r = r(u_{\mathcal{D}},f_{\mathcal{D}},\lambda_{\mathcal{D}})$ is such that its restriction r_{K_D} to K_D is a polynomial of degree $\hat{p}_D + p_D + 1$, while the function z_{K_D} defined in (4.9) is a polynomial of degree

$$\bar{p}_D := \hat{p}_D + p_D + 3\,. \tag{4.14}$$

Therefore, the module **REFINE** builds $\bar{\mathcal{D}} = \bar{\mathcal{D}}(\mathcal{M}) \in \mathbb{D}^c = \mathbb{D}$ with $\bar{\mathcal{D}}(\mathcal{M}) \ge \mathcal{D}$ as follows:

$$\bar{\mathcal{D}} = \{\bar{D}\} \quad \text{with } \bar{D} = \begin{cases} (K_D,\bar{p}_D) & \text{for } D \in \mathcal{M} \\ D & \text{for } D \in \mathcal{D}\setminus\mathcal{M}. \end{cases}$$

In order to prove (2.22), consider a marked element $D \in \mathcal{M}$. Setting $\mathbb{P}^0_{\bar{p}_D}(K_D) := \mathbb{P}_{\bar{p}_D}(K_D) \cap H^1_0(K_D)$ and recalling that $z_{K_D} \in \mathbb{P}^0_{\bar{p}_D}(K_D)$ we have

$$\eta_{D,\mathcal{D}}(u_{\mathcal{D}},f_{\mathcal{D}},\lambda_{\mathcal{D}}) = |z_{K_D}|_{H^1(K_D)} = \sup_{w\in\mathbb{P}^0_{\bar{p}_D}(K_D)} \frac{(z'_{K_D},w')_{L^2(K_D)}}{|w|_{H^1(K_D)}} = \sup_{w\in\mathbb{P}^0_{\bar{p}_D}(K_D)} \frac{\langle r_{K_D},w\rangle}{|w|_{H^1(K_D)}}\,. \tag{4.15}$$

On the other hand, the Galerkin solution $u_{\bar{\mathcal{D}}} = u_{\bar{\mathcal{D}}}(f_{\mathcal{D}},\lambda_{\mathcal{D}})$ is such that its residual $\bar{r} = r(u_{\bar{\mathcal{D}}},f_{\mathcal{D}},\lambda_{\mathcal{D}})$ satisfies $\langle \bar{r}_{K_D},w\rangle = 0$ for all $w \in \mathbb{P}^0_{\bar{p}_D}(K_D)$. Thus, denoting by $a_{\lambda_{\mathcal{D}},K_D}(\cdot,\cdot)$ the restriction of the form $a_{\lambda_{\mathcal{D}}}(\cdot,\cdot)$ to $H^1(K_D)\times H^1(K_D)$, and setting $|\!|\!| v |\!|\!|^2_{\lambda_{\mathcal{D}},K_D} = a_{\lambda_{\mathcal{D}},K_D}(v,v)$, we get

$$\begin{aligned} \eta_{D,\mathcal{D}}(u_{\mathcal{D}},f_{\mathcal{D}},\lambda_{\mathcal{D}}) &= \sup_{w\in\mathbb{P}^0_{\bar{p}_D}(K_D)} \frac{\langle r_{K_D}-\bar{r}_{K_D},w\rangle}{|w|_{H^1(K_D)}} \\ &= \sup_{w\in\mathbb{P}^0_{\bar{p}_D}(K_D)} \frac{a_{\lambda_{\mathcal{D}},K_D}(u_{\mathcal{D}}-u_{\bar{\mathcal{D}}},w)}{|w|_{H^1(K_D)}} \le \sqrt{\alpha^*}\,|\!|\!| u_{\mathcal{D}}-u_{\bar{\mathcal{D}}} |\!|\!|_{\lambda_{\mathcal{D}},K_D}\,. \end{aligned}$$

Squaring and summing-up over all $D \in \mathcal{M}$, we obtain

$$\mathcal{E}^2_{\mathcal{D}}(\mathcal{M}, u_{\mathcal{D}}, f_{\mathcal{D}}, \lambda_{\mathcal{D}}) \leq \alpha^* |\!|\!| u_{\mathcal{D}} - u_{\bar{\mathcal{D}}} |\!|\!|^2_{\lambda_{\mathcal{D}}} , \tag{4.16}$$

which immediately implies (2.22).

Remark 4.1. *The choice of the error estimator and the refinement strategy indicated above guarantees that the reliability assumption* (2.21) *and the efficiency assumption* (2.22) *are fulfilled, hence the conclusions of Proposition 2.1 hold true. Actually, one can be more precise, since using* (4.13) *and* (4.16) *and following the steps of the proof of Proposition 2.1, we get that the sequence of Galerkin approximations built by a call of* **REDUCE** *satisfies the contraction property* (2.25) *with contraction factor* $\kappa = \sqrt{1 - \frac{\alpha_*}{\alpha^*}\theta}$.

4.4 Convergence and optimality properties of hp-AFEM

In this section we discuss the convergence and optimality properties of our adaptive algorithm **hp-AFEM** in the present one-dimensional setting. To this end, we first specify the abstract functional framework introduced in Sect. 2. We already set $V := H^1_0(\Omega)$ and $F := L^2(\Omega) \times L^2(\Omega)$. Concerning the space Λ containing the coefficients of the operator, we assume stronger regularity than just $L^\infty(\Omega)$ in order to guarantee that the piecewise polynomial approximations of the coefficients still define a coercive variational problem.

To be precise, from now on we assume that $\lambda = (\nu, \sigma)$ belongs to the space

$$\Lambda := \{\lambda = (\nu, \sigma) \in H^1(\Omega) \times H^1(\Omega) : \quad \nu_* \leq \nu \leq \nu^*, \ 0 \leq \sigma \leq \sigma^*\}. \tag{4.17}$$

Here, in view of (2.2), we choose to work with a smoothness space of Sobolev type with summability index 2, so that squared best approximation errors are non-increasing under h-refinements. We notice that it would be sufficient to require the coefficients to be piecewise H^1 on the initial partition. We decide to work under stronger assumptions just for the sake of simplicity.

We now define the projectors $Q_{K,p}$ introduced in Sect. 2.1. To this end, let $\Pi^0_{K,p} \in \mathcal{L}(L^2(K), \mathbb{P}_p(K))$ be the L^2-orthogonal projection and $\Pi^1_{K,p} \in \mathcal{L}(H^1(K), \mathbb{P}_p(K))$ be the H^1-type orthogonal projection defined as follows: if $v \in H^1(K)$ with $K = [a, b]$ then

$$\left(\Pi^1_{K,p} v\right)(x) := c + \int_a^x \left(\Pi^0_{K,p-1} v'\right)(t)\, dt$$

where the constant c is such that $\int_K \Pi^1_{K,p} v \, dx = \int_K v \, dx$.

Then we define $Q_{K,p} \in \mathcal{L}(V \times F \times \Lambda, \mathbb{P}_p(K) \times (\mathbb{P}_{p-1}(K) \times \mathbb{P}_p(K)) \times (\mathbb{P}_{p+1}(K) \times \mathbb{P}_{p+1}(K))$ by setting

$$Q_{K,p}(v, f, \lambda) := (\Pi^1_{K,p} v_{|K}, \ \Pi^0_{K,p-1} f_{1|K}, \ \Pi^0_{K,p} f_{2|K}, \ \Pi^1_{K,p+1} \nu_{|K}, \ \Pi^1_{K,p+1} \sigma_{|K}).$$

At last, we define the local error functionals $e_{K,p}$. We set

$$e_{K,p}(v, f, \lambda) := |(\mathrm{I} - \Pi^1_{K,p}) v_{|K}|^2_{H^1(K)} + \delta^{-1} \mathrm{osc}^2_{K,p}(f, \lambda) \tag{4.18}$$

where $\delta > 0$ is a positive penalization parameter to be chosen later and

$$\begin{aligned} \mathrm{osc}^2_{K,p}(f, \lambda) := \| \frac{h}{p} (\mathrm{I} - \Pi^0_{K,p-1}) f_{1|K} \|^2_{L^2(K)} + \| (\mathrm{I} - \Pi^0_{K,p}) f_{2|K} \|^2_{L^2(K)} \\ + \ |(\mathrm{I} - \Pi^1_{K,p+1}) \nu_{|K}|^2_{H^1(K)} + |(\mathrm{I} - \Pi^1_{K,p+1}) \sigma_{|K}|^2_{H^1(K)} \end{aligned} \tag{4.19}$$

where $h = |K|$. Note that the choice of polynomial degrees is such that for smooth data the four addends above scale in the same way with respect to the parameters h and p.

Furthermore, the data oscillation that appears in (4.18) is of higher order with respect to the projection error for the function v.

It is straightforward to check the validity of (2.2). We recall that given a partition $\mathcal{D} \in \mathbb{D}$, we denote by $f_{\mathcal{D}} = (f_{1,\mathcal{D}}, f_{2,\mathcal{D}})$ and $\lambda_{\mathcal{D}} = (\nu_{\mathcal{D}}, \sigma_{\mathcal{D}})$ the piecewise polynomial function obtained by projecting f and λ, respectively, element by element as indicated above. Note that while $f_{\mathcal{D}} \in F_{\mathcal{D}} \subset F$, $\lambda_{\mathcal{D}}$ need not belong to $\bar{\Lambda}$. Given a partition $\mathcal{D} \in \mathbb{D}$, we will set

$$\mathrm{osc}^2_{\mathcal{D}}(f, \lambda) := \sum_{D \in \mathcal{D}} \mathrm{osc}^2_D(f, \lambda),$$

where $\mathrm{osc}^2_D(f, \lambda) = \mathrm{osc}^2_{K_D, p_D}(f, \lambda)$.

The following result provides a uniform bound on the approximation error of the coefficients of the operator, assuring that $\lambda_{\mathcal{D}} \in \bar{\Lambda}$.

Property 4.1. *Let $\hat{\mathcal{D}}$ be the root partition with polynomial degree equal to one on each element domain. Assume that $\mathcal{K}(\hat{\mathcal{D}})$ is sufficiently fine for the given data $\lambda \in \Lambda$, in the sense that for each $K \in \mathcal{K}(\hat{\mathcal{D}})$ it holds*

$$|(\mathrm{I} - \Pi^1_{K,1})\nu_{|K}|_{H^1(K)} \leq \frac{\nu_*}{2}, \qquad |(\mathrm{I} - \Pi^1_{K,1})\sigma_{|K}|_{H^1(K)} \leq \frac{\nu_*}{2}.$$

Then for any $\mathcal{D} \in \mathbb{D}$ we have (4.4), i.e.,

$$\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} \leq \frac{\nu_*}{2}, \qquad \|\sigma - \sigma_{\mathcal{D}}\|_{L^\infty(\Omega)} \leq \frac{\nu_*}{2}.$$

Consequently, $\lambda_{\mathcal{D}} \in \Lambda_{\mathcal{D}} \subset \bar{\Lambda}$.

Proof. For any $D = (K_D, p_D)$, let $\hat{K} \in \mathcal{K}(\hat{\mathcal{D}})$ the element of the root partition containing K_D. Then, we have

$$|(\mathrm{I} - \Pi^1_{K_D, p_D+1})\nu_{|K_D}|_{H^1(K_D)} \leq |(\mathrm{I} - \Pi^1_{\hat{K},1})\nu_{|\hat{K}}|_{H^1(\hat{K})} \leq \frac{\nu_*}{2}.$$

On the other hand, set $\psi = (\mathrm{I} - \Pi^1_{K_D, p_D+1})\nu|_{K_D}$; recalling that ψ has zero mean-value in K_D, it vanishes at some point $x_0 \in K_D$ since it is a continuous function. Writing $\psi(x) = \psi(x_0) + \int_{x_0}^x \psi'(t)dt$ for any $x \in K_D$ yields

$$|\psi(x)| \leq |x - x_0|^{1/2} \|\psi'\|_{L^2(K_D)} \leq |K_D|^{1/2} |\psi|_{H^1(K_D)} \,,$$

whence the result immediately follows after observing $|K_D| \leq 1$. ∎

We now focus on the abstract assumptions (2.7)-(2.9).

Proposition 4.1. *In the present setting, assumptions (2.8)-(2.9) hold true. Furthermore, if δ is chosen sufficiently small, then (2.7) is fulfilled.*

Proof. We start by verifying condition (2.9). For any $v, w \in H^1_0(\Omega)$ and for any $\mathcal{D} \in \mathbb{D}$ and any $D \in \mathcal{D}$, it holds that

$$\begin{aligned}
|(\mathrm{I} - \Pi^1_{K_D, p_D})w_{|K_D}|_{H^1(K_D)} &= \inf_{\varphi \in \mathbb{P}_{p_D}(K_D)} |w_{|K_D} - \varphi|_{H^1(K_D)} \\
&\leq \inf_{\varphi \in \mathbb{P}_{p_D}(K_D)} |v_{|K_D} - \varphi|_{H^1(K_D)} + |(v - w)_{|K_D}|_{H^1(K_D)} \\
&= |(\mathrm{I} - \Pi^1_{K_D, p_D})v_{|K_D}|_{H^1(K_D)} + |(v - w)_{|K_D}|_{H^1(K_D)}.
\end{aligned}$$

Two applications of a triangle inequality show that

$$\begin{aligned}
&\left| \mathrm{E}_{\mathcal{D}}(v,f,\lambda)^{\frac{1}{2}} - \mathrm{E}_{\mathcal{D}}(w,f,\lambda)^{\frac{1}{2}} \right| \\
&\leq \left(\sum_{D\in\mathcal{D}} \left(\left(\left| (\mathrm{I}-\Pi_D^1) v_{|K_D} \right|^2_{H^1(K_D)} + \delta^{-1}\mathrm{osc}_D^2(f,\lambda) \right)^{\frac{1}{2}} \right.\right. \\
&\qquad\qquad \left.\left. - \left(\left| (\mathrm{I}-\Pi_D^1) v_{|K_D} \right|^2_{H^1(K_D)} + \delta^{-1}\mathrm{osc}_D^2(f,\lambda) \right)^{\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{D\in\mathcal{D}} \left(\left| (\mathrm{I}-\Pi_D^1) v_{|K_D} \right|_{H^1(K_D)} - \left| (\mathrm{I}-\Pi_D^1) v_{|K_D} \right|_{H^1(K_D)} \right)^2 \right)^{\frac{1}{2}} \leq \|v-w\|_V,
\end{aligned}$$

i.e., (2.9) holds true with constant $C_2 = 1$.

Let us now verify assumption (2.8). Note that $u(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$ is well defined since $\lambda_{\mathcal{D}} \in \overline{\Lambda}$. Setting for simplicity $u = u(f,\lambda)$ and $\bar{u} = u(f_{\mathcal{D}}, \lambda_{\mathcal{D}})$, it is straightforward to check that $u - \bar{u}$ satisfies for any $v \in V$

$$a_\lambda(u-\bar{u}, v) = \langle f - f_{\mathcal{D}}, v\rangle - \int_\Omega (\nu - \nu_{\mathcal{D}}) \bar{u}' v' \, dx - \int_\Omega (\sigma - \sigma_{\mathcal{D}}) \bar{u} v \, dx \tag{4.20}$$

whence, using the Poincaré inequality $\|v\|_{L^2(\Omega)} \leq 2^{-\frac{1}{2}} |v|_{H_0^1(\Omega)}$, and selecting $v = u - \bar{u}$, we obtain

$$\begin{aligned}
\alpha_* |u-\bar{u}|_{H^1(\Omega)} \quad &\leq \quad \|f_1 - f_{1,\mathcal{D}}\|_{H^{-1}(\Omega)} + \|f_2 - f_{2,\mathcal{D}}\|_{L^2(\Omega)} \\
&\quad + \left(\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} + \frac{1}{2} \, \|\sigma - \sigma_{\mathcal{D}}\|_{L^\infty(\Omega)} \right) |\bar{u}|_{H^1(\Omega)}.
\end{aligned} \tag{4.21}$$

We now bound the quantity on the right hand side of (4.21) in terms of $\mathrm{osc}^2_{\mathcal{D}}(f,\lambda)$. To this end, starting with the first term, we have for any $v \in H_0^1(\Omega)$

$$\begin{aligned}
(f_1 - f_{1,\mathcal{D}}, v)_{L^2(\Omega)} \quad &= \quad \sum_{D\in\mathcal{D}} ((\mathrm{I} - \Pi^0_{K_D,p_D-1}) f_{1|K_D}, v)_{L^2(K_D)} \\
&= \quad \sum_{D\in\mathcal{D}} ((\mathrm{I} - \Pi^0_{K_D,p_D-1}) f_{1|K_D}, (\mathrm{I} - \Pi^0_{K_D,p_D-1}) v_{|K_D})_{L^2(K_D)} \\
&\leq \quad \sum_{D\in\mathcal{D}} \|(\mathrm{I} - \Pi^0_{K_D,p_D-1}) f_{1|K_D}\|_{L^2(K_D)} \|(\mathrm{I} - \Pi^0_{K_D,p_D-1}) v_{|K_D})\|_{L^2(K_D)}
\end{aligned} \tag{4.22}$$

By the classical *hp*-error estimate for the orthogonal L^2-projection upon $\mathbb{P}_{p_D}(K_D)$ (see, e.g., [39, Corollary 3.12]) we have $\|(\mathrm{I} - \Pi^0_{K_D,p_D-1}) v_{|K_D})\|_{L^2(K_D)} \leq \hat{C} \frac{h_{\mathcal{D}}}{p_{\mathcal{D}}} |v|_{H^1(K_D)}$ for some constant $\hat{C} > 0$. Thus, we get

$$\|f_1 - f_{1,\mathcal{D}}\|_{H^{-1}(\Omega)} \leq \hat{C} \left(\sum_{D\in\mathcal{D}} \| \frac{h_D}{p_D} (\mathrm{I} - \Pi^0_{K_D,p_D-1}) f_{1|K_D} \|^2_{L^2(K_D)} \right)^{\frac{1}{2}}. \tag{4.23}$$

Concerning the second term on the right hand side of (4.21), we simply write it as

$$\|f_2 - f_{2,\mathcal{D}}\|_{L^2(\Omega)} = \left(\sum_{D\in\mathcal{D}} \|(\mathrm{I} - \Pi^0_{K_D,p_D}) f_{2|K_D}\|^2_{L^2(K_D)} \right)^{\frac{1}{2}}. \tag{4.24}$$

Coming to the third and fourth terms, we first observe that

$$\begin{aligned} |\bar{u}|_{H^1(\Omega)} &\leq \frac{1}{\alpha_*}\left(2^{-\frac{1}{2}}\|f_{1,\mathcal{D}}\|_{L^2(\Omega)} + \|f_{2,\mathcal{D}}\|_{L^2(\Omega)}\right) \\ &\leq \frac{1}{\alpha_*}\left(2^{-\frac{1}{2}}\|f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)}\right) =: C(f), \end{aligned} \tag{4.25}$$

since $f_{i,\mathcal{D}}$, $i = 1, 2$ is locally an L^2-projection of f_i. On the other hand, using the same argument as in the proof of Property 4.1 we get

$$\begin{aligned} \|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} &= \max_{D\in\mathcal{D}} \|(\mathrm{I} - \Pi^1_{K_D,p_D+1})\nu_{|K_D}\|_{L^\infty(K_D)} \\ &\leq \max_{D\in\mathcal{D}} |K_D|^{\frac{1}{2}} |(\mathrm{I} - \Pi^1_{K_D,p_D+1})\nu_{|K_D}|_{H^1(K_D)} \\ &\leq \left(\sum_{D\in\mathcal{D}} |(\mathrm{I} - \Pi^1_{K_D,p_D+1})\nu_{|K_D}|^2_{H^1(K_D)}\right)^{\frac{1}{2}}. \end{aligned} \tag{4.26}$$

A similar result holds for $\|\sigma - \sigma_{\mathcal{D}}\|_{L^\infty(\Omega)}$. Substituting (4.23)-(4.26) into (4.21) and recalling (4.19) we get

$$\alpha_*|u - \bar{u}|_{H^1(\Omega)} \leq \left(\frac{3}{2}C(f) + \hat{C} + 1\right)\left(\sum_{D\in\mathcal{D}} \mathrm{osc}^2_D(f,\lambda)\right)^{\frac{1}{2}}. \tag{4.27}$$

Thus, setting $\bar{C} := \frac{1}{\alpha_*}\left(\frac{3}{2}C(f) + \hat{C} + 1\right)$ and recalling (4.18), we conclude that

$$|u(f,\lambda) - u(f_{\mathcal{D}},\lambda_{\mathcal{D}})|_{H^1(\Omega)} \leq \bar{C}\delta^{\frac{1}{2}}\left(\sum_{D\in\mathcal{D}} e_D(w,f,\lambda)\right)^{\frac{1}{2}} = \bar{C}\delta^{\frac{1}{2}} E_{\mathcal{D}}(w,f,\lambda)^{\frac{1}{2}} \tag{4.28}$$

for any $w \in H^1_0(\Omega)$. This proves that (2.8) is fulfilled with $C_1 = \bar{C}\delta^{\frac{1}{2}}$. Finally, choosing any δ such that $C_1 < b$ we fulfill (2.7). ∎

We conclude that choosing δ sufficiently small we may apply Theorem 2.1. This leads to the conclusion that for solving (4.2), where $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$, and $\lambda = (\nu, \sigma) \in \Lambda$ defined in (4.17), and with a root partition $\hat{\mathcal{D}}$ that is sufficiently fine such that it satisfies Property 4.1, **hp-AFEM** is an instance optimal reducer, in the sense of Theorem 2.1, of the error functional

$$E_{\mathcal{D}}(u(f,\lambda),f,\lambda) = \sum_{D\in\mathcal{D}} \inf_{\varphi\in\mathbb{P}_{p_D}(K_D)} |u(f,\lambda)_{|K_D} - \varphi|^2_{H^1(K_D)} + \delta^{-1}\mathrm{osc}^2_D(f,\lambda),$$

over all $\mathcal{D} \in \mathbb{D}$, where $\mathrm{osc}^2_D(f,\lambda)$ is defined in (4.19).

Finally, we consider assumption (2.14). At first, we note that in one dimension all partitions are trivially conforming, i.e., $\mathbb{D}^c = \mathbb{D}$. Next, we observe that the following result holds.

Lemma 4.1. *For any $\mathcal{D} \in \mathbb{D}$ and any $v \in H^1_0(\Omega)$ there holds*

$$\inf_{w_{\mathcal{D}}\in V^c_{\mathcal{D}}} |v - w_{\mathcal{D}}|^2_{H^1(\Omega)} = \sum_{D\in\mathcal{D}} \inf_{\varphi\in\mathbb{P}_{p_D}(K_D)} |v - \varphi|^2_{H^1(K_D)}. \tag{4.29}$$

Proof. For $D \in \mathcal{D}$, let $q_D \in \mathbb{P}_{p_D}(K_D)$ be such that $|v - q_D|_{H^1(K_D)} = \inf_{\varphi \in \mathbb{P}_{p_D}(K_D)} |v - \varphi|_{H^1(K_D)}$. Define $g \in L_2(\Omega)$ by $g|_{K_D} = q_D'$ for all $D \in \mathcal{D}$, and $w_{\mathcal{D}} \in H^1(\Omega)$ by $w_{\mathcal{D}}(x) = \int_0^x g(s)ds$. From $\int_{K_D} q_D' = \int_{K_D} v'$, we infer that $w_{\mathcal{D}}(0) = w_{\mathcal{D}}(1) = 0$, and so $w_{\mathcal{D}} \in V_{\mathcal{D}}^c$. Moreover, $|v - w_{\mathcal{D}}|^2_{H^1(\Omega)} = \sum_{D \in \mathcal{D}} |v - q_D|^2_{H^1(K_D)}$. ∎

Observing that

$$\inf_{\varphi \in \mathbb{P}_{p_D}(K_D)} |v - \varphi|^2_{H^1(K_D)} = |(\mathrm{I} - \Pi^1_{K_D, p_D}) v_{|K_D}|^2_{H^1(K_D)} \leq e_D(v, f, \lambda)$$

for any $f \in F$, $\lambda \in \Lambda$, we obtain the following result.

Proposition 4.2. *For all $\mathcal{D} \in \mathbb{D}$ and all $v \in H^1_0(\Omega)$, one has*

$$\inf_{w_{\mathcal{D}} \in V_{\mathcal{D}}^c} |v - w_{\mathcal{D}}|_{H^1(\Omega)} \leq \inf_{(f,\lambda) \in F \times \Lambda} \mathrm{E}_{\mathcal{D}}(v, f, \lambda)^{\frac{1}{2}},$$

i.e., for $\mathcal{C} := I$ assumption (2.14) *is fulfilled with $C_{3,\mathcal{D}} = 1$.*

As a consequence, (2.16) and (2.18) are fulfilled with $C_3 = C_4 = 1$. Since **hp-AFEM** calls the routine **REDUCE** with the fixed value $\varrho = \frac{\mu}{1+(C_1+1)\omega}$, and by Proposition 2.1 the number of iterations in **REDUCE** is bounded by $\mathcal{O}(\log \varrho^{-1})$, we are guaranteed that the number of iterations performed by **REDUCE** at any call from **hp-AFEM** is uniformly bounded. On the other hand, recalling (4.14), for each iteration in **REDUCE** the polynomial degree in each marked element is increased by a constant value depending only on the local polynomial degree in the input partition. Thus, even in the worst-case scenario that at each iteration all elements are marked for enrichment, we conclude that the output partition of **REDUCE** has a cardinality which is bounded by a fixed multiple of the one of the input partition, which is optimal as it is produced by **hp-NEARBEST**.

Another obvious, but relevant application of Lemma 4.1 is that **hp-AFEM** is an instance optimal reducer over $\mathcal{D} \in \mathbb{D}$ of the error functional written in the more common form

$$\inf_{w_{\mathcal{D}} \in V_{\mathcal{D}}^c} |u(f,\lambda) - w_{\mathcal{D}}|^2_{H^1(\Omega)} + \delta^{-1} \mathrm{osc}^2_{\mathcal{D}}(f, \lambda).$$

5 The Poisson problem in two dimensions

On a polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the Poisson problem

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in standard variational form. We consider right-hand sides $f \in L^2(\Omega)$, and so take $V = H^1_0(\Omega)$, $F = L^2(\Omega)$, and $\Lambda = \emptyset$. We equip $H^1_0(\Omega)$ with $|\cdot|_{H^1(\Omega)}$, and $H^{-1}(\Omega)$ with the corresponding dual norm.

Let $\mathcal{K}_0$ be an initial conforming triangulation of $\bar{\Omega}$, and let in each triangle in $\mathcal{K}_0$ one of its vertices be selected as its newest vertex, in such a way that if an internal edge of the triangulation is opposite to the newest vertex of the triangle on one side of the edge, then it is also opposite to the newest vertex of the triangle on the other side. As shown in [6, Lemma 2.1], such an assignment of the newest vertices can always be made.

Now let $\mathbb{K}$ be the collection of all triangulations that can be constructed from $\mathcal{K}_0$ by *newest vertex bisection*, i.e., a repetition of bisections of triangles by connecting their newest vertex by the midpoint of the opposite edge. With each bisection, two new triangles are generated, being 'children' of the triangle that was just bisected, with their newest vertices being defined as the midpoint of the edge that has been cut. The set of all triangles

that can be produced in this way is naturally organized as a binary master tree $\mathfrak{K}$, having as roots the triangles from $\mathcal{K}_0$. The triangles from $\mathfrak{K}$ are uniformly shape regular. The collection $\mathbb{K}$ of triangulations of Ω is equal to the sets of leaves of all possible subtrees of $\mathfrak{K}$.

For $K \in \mathfrak{K}$, we set $V_K = H^1(K)$ and $F_K = L^2(K)$, and for $d \in \mathbb{N}$, we set

$$V_{K,d} := \mathbb{P}_{p(d)}(K), \quad F_{K,d} := \mathbb{P}_{p(d)-1}(K), \tag{5.1}$$

with, as in Sect. 2.1, $p = p(d)$ being the largest value in $\mathbb{N}$ such that $\dim \mathbb{P}_{p-1}(K) = \binom{2+p-1}{p-1} \leq d$. For example, for $d = 1, \dots, 10$, we have $p = 1, 1, 2, 2, 2, 3, 3, 3, 3, 4$.

Remark 5.1. *Alternatively, one can select sequences of strictly nested spaces $(V_{K,d})_d$, $(F_{K,d})_d$ with the condition that for the values of d of the form $\binom{2+p-1}{p-1}$ for some $p =: p(d) \in \mathbb{N}$, definitions in* (5.1) *hold.*

For $D = (K_D, d_D) \in \mathcal{K} \times \mathbb{N}$, we write $V_D = V_{K_D,d_D}$, $F_D = F_{K_D,d_D}$ and $p_D = p(d_D)$. Note that with the current definition of V_D, this space is uniquely determined by specifying K_D and p_D. For some constant $\delta > 0$ that will be determined later, we set the local error functional

$$e_D(w, f) := e_D(w) + \delta^{-1} \frac{|K|}{p_D^2} \inf_{f_D \in \mathbb{P}_{p_D-1}(K_D)} \|f - f_D\|^2_{L^2(K_D)},$$

where

$$e_D(w) := \inf_{\{w_D \in \mathbb{P}_{p_D}(K_D):\ \int_{K_D} w_D = \int_{K_D} w\}} |w - w_D|^2_{H^1(K_D)}. \tag{5.2}$$

We define

$$Q_D(w, f) := (w_D, f_D) \tag{5.3}$$

as the pair of functions for which the infima are attained.

Having specified the master tree $\mathfrak{K}$, the local approximation spaces V_D and F_D, the error functional $e_D(w, f)$, and the projection $Q_D(w, f) = (w_D, f_D)$, we have determined, according to Sect. 2.1, the collection of *hp*-partitions $\mathbb{D}$, the approximation spaces $V_{\mathcal{D}}$ and $F_{\mathcal{D}}$ for $\mathcal{D} \in \mathbb{D}$, the global error functional

$$\mathrm{E}_{\mathcal{D}}(w, f) = \sum_{D \in \mathcal{D}} e_D(w) + \delta^{-1} \mathrm{osc}^2_{\mathcal{D}}(f), \tag{5.4}$$

where

$$\mathrm{osc}^2_{\mathcal{D}}(f) := \sum_{D \in \mathcal{D}} \frac{|K|}{p_D^2} \inf_{f_D \in \mathbb{P}_{p_D-1}(K_D)} \|f - f_D\|^2_{L^2(K_D)},$$

as well as the projection $f_{\mathcal{D}} := \prod_{D \in \mathcal{D}} f_D$.

We proceed with verifying assumptions (2.7), (2.8) and (2.9).

Proposition 5.1. *There holds*

$$\sup_{f \in F} |\, \mathrm{E}_{\mathcal{D}}(w, f)^{\frac{1}{2}} - \mathrm{E}_{\mathcal{D}}(v, f)^{\frac{1}{2}} | \leq \|w - v\|_V \qquad \forall \mathcal{D} \in \mathbb{D}, \quad \forall v, w \in V,$$

i.e., (2.9) *is valid with $C_2 = 1$.*

Proof. For $v, w \in V$, it holds that $e_D(w)^{\frac{1}{2}} \leq e_D(v)^{\frac{1}{2}} + |v - w|_{H^1(K_D)}$, which yields the proof using the same arguments as in the proof of Proposition 4.1. ∎

Proposition 5.2. *There holds*

$$|u(f)-u(f_{\mathcal{D}})|_{H^1(\Omega)} \lesssim \sqrt{\delta} \inf_{w\in H^1_0(\Omega)} \mathrm{E}_{\mathcal{D}}(w,f)^{\frac{1}{2}} \quad \forall \mathcal{D}\in\mathbb{D}, \quad \forall f\in L^2(\Omega),$$

i.e., (2.8) *is valid with* $C_1 \eqsim \sqrt{\delta}$*, and, when* δ *is chosen to be sufficiently small, so is* (2.7)*.*

Proof. Since $f\mapsto u(f)\in\mathcal{L}(H^{-1}(\Omega),H^1_0(\Omega))$ is an isomorphism, it is enough to estimate $\|f-f_{\mathcal{D}}\|_{H^{-1}(\Omega)}$. To this end, we note that for K being a triangle and $p\in\mathbb{N}$, it holds that [14]

$$\sup_{0\neq w\in H^1(K)} \inf_{v\in\mathbb{P}_p(K)} \frac{\|w-v\|_{L^2(K)}}{|w|_{H^1(K)}} \lesssim \frac{\mathrm{diam}(K)}{p+1},$$

only dependent on a lower bound for the smallest angle in K. Consequently, we have that

$$\begin{aligned} \|f-f_{\mathcal{D}}\|_{H^{-1}(\Omega)} &= \sup_{w\in H^1_0(\Omega)} \frac{\inf_{v\in F_{\mathcal{D}}}\langle f-f_{\mathcal{D}}, w-v\rangle_{L^2(\Omega)}}{|w|_{H^1(\Omega)}} \\ &\lesssim \sup_{w\in H^1_0(\Omega)} \frac{\sum_{D\in\mathcal{D}} \frac{|K_D|^{\frac{1}{2}}}{p_D}\|f-f_D\|_{L^2(K_D)}|w|_{H^1(K_D)}}{|w|_{H^1(\Omega)}} \leq \sqrt{\mathrm{osc}^2_{\mathcal{D}}(f)}. \end{aligned} \tag{5.5}$$

∎

5.1 Conforming h-partitions, and conforming hp finite element spaces

For the design of a routine **REDUCE**, in particular, for a posteriori error estimation, it is preferable to work with h-partitions that are conforming. Let

$$\mathbb{K}^c := \{\mathcal{K}\in\mathbb{K}\colon \mathcal{K} \text{ is conforming}\}.$$

As shown in [6, Lemma 2.5], for $\mathcal{K}\in\mathbb{K}$, its smallest refinement $\mathcal{K}^c\in\mathbb{K}^c$ satisfies $\#\mathcal{K}^c \lesssim \#\mathcal{K}$.

With the subclass

$$\mathbb{D}^c := \{\mathcal{D}\in\mathbb{D}\colon \mathcal{K}(\mathcal{D})\in\mathbb{K}^c\},$$

we define $\mathcal{C}:\mathbb{D}\to\mathbb{D}^c$ by setting $\mathcal{C}(\mathcal{D})=\underline{\mathcal{D}}$, where $\underline{\mathcal{D}}$ is defined as the partition in $\mathbb{D}^c$ with *minimal* $\#\underline{\mathcal{D}}$ for which $\underline{\mathcal{D}}\geq\mathcal{D}$. That is, $\mathcal{K}(\underline{\mathcal{D}})=\mathcal{K}(\mathcal{D})^c$, and $p_{\underline{D}}=p_D$ for $\underline{D}\in\underline{\mathcal{D}}$, $D\in\mathcal{D}$ with $K_{\underline{D}}\subseteq K_D$.

Unfortunately, $\sup_{\mathcal{D}\in\mathbb{D}}\frac{\#\mathcal{C}(\mathcal{D})}{\#\mathcal{D}}=\infty$, i.e., (2.18) is not valid. Indeed, as an example, consider $\mathcal{K}_0$ to consist of two triangles K_1 and K_2. Let $\mathcal{D}\in\mathbb{D}$ be such that $K_1\in\mathcal{K}(\mathcal{D})$, with corresponding polynomial degree $p(d)$, and that in $\mathcal{K}(\mathcal{D})$, K_2 has been replaced by 2^N triangles of generation N, each with polynomial degree 1. Then $\#\mathcal{D}\eqsim d+2^N$. Since $\mathcal{K}(\mathcal{C}(\mathcal{D}))=\mathcal{K}(\mathcal{D})^c$ contains in any case $\eqsim 2^{N/2}$ triangles inside K_1, so with polynomial degrees $p(d)$, we conclude that $\#\mathcal{C}(\mathcal{D})\gtrsim 2^N+2^{N/2}d$. By taking say $d\eqsim 2^N$, we conclude the above claim.

The fact that (2.18) does not hold implies that, unlike for an h-method, we will not have a proper control on the dimension of the finite element spaces that are created inside **REDUCE**.

From (2.6), recall the definition $V^c_{\mathcal{D}}=V_{\mathcal{D}}\cap H^1_0(\Omega)$ for $\mathcal{D}\in\mathbb{D}^c$, and from (5.2)-(5.3), recall the definition of $e_D(w)$ and w_D for $D\in\mathcal{D}$ and $w\in H^1(K_D)$. The main task in this section will be the proof of the following result.

Theorem 5.1. *Setting, for* $\mathcal{D} \in \mathbb{D}$, $\|p_{\mathcal{D}}\|_\infty := \max_{D\in\mathcal{D}} p_D$ *, for* $\mathcal{D} \in \mathbb{D}^c$ *it holds that*

$$\inf_{v\in V_{\mathcal{D}}^c} |w-v|^2_{H^1(\Omega)} \lesssim (1+\log\|p_{\mathcal{D}}\|_\infty)^3 \sum_{D\in\mathcal{D}} e_D(w) \quad \forall w \in H^1_0(\Omega).$$

Since, for $\mathcal{D} \in \mathbb{D}$, obviously $\sum_{D\in\mathcal{C}(\mathcal{D})} e_D(w) \le \sum_{D\in\mathcal{D}} e_D(w)$, Theorem 5.1 implies (2.14) with

$$C_{3,\mathcal{D}} \eqsim (1+\log\|p_{\mathcal{D}}\|_\infty)^{\frac{3}{2}}.$$

For an underlying h-partition that is conforming, Theorem 5.1 says that the error in H^1-norm of the best conforming hp-approximation of a $w \in H^1_0(\Omega)$, is at most slightly larger than the error in the broken H^1-norm of the best nonconforming hp-approximation.

The proof of this remarkable result will be based on Veeser's proof in [45] of the corresponding result in the 'h'-setting. In [45], the result is shown by taking v to be the Scott-Zhang ([40]) quasi-interpolant of w. This Scott-Zhang quasi-interpolation is constructed in terms of the nodal basis, and the proof relies on an inverse inequality applied to these basis functions, which inequality involves a multiplicative factor that is known to degrade seriously, i.e. not logarithmically, with increasing polynomial degree.

In our proof the role of the nodal basis on a triangle will be played by the union of the three linear nodal basis functions associated to the vertices, the polynomials on each edge that vanish at the endpoints, that will be boundedly extended to polynomials on the interior of the triangle, and, finally, the polynomials on the triangle that vanish at its boundary. We will construct a $\Pi_{\mathcal{D}} \in \mathcal{L}(H^1_0(\Omega), V^c_{\mathcal{D}})$ such that, with

$$p_{D,\mathcal{D}} := \max_{\{D'\in\mathcal{D}: K_{D'}\cap K_D\neq\emptyset\}} p_{D'} \quad \forall D \in \mathcal{D}, \tag{5.6}$$

it holds that

$$|(w-\Pi_{\mathcal{D}}w)|_{K_D}|^2_{H^1(K_D)} \le (1+\log p_{D,\mathcal{D}})^3 \sum_{\{D'\in\mathcal{D}: K_{D'}\cap K_D\neq\emptyset\}} e_{D'}(w) \quad \forall D \in \mathcal{D}, \tag{5.7}$$

which obviously implies the statement of the theorem. Since the right-hand side of (5.7) vanishes for $w \in V^c_{\mathcal{D}}$, because it even vanishes for $w \in V_{\mathcal{D}}$, the mapping $\Pi_{\mathcal{D}}$ is a *projector*.

Proof. (*Theorem 5.1*) Let $\mathcal{D} \in \mathbb{D}^c$. In order to show (5.7), it is sufficient to show

$$|(\Pi_{\mathcal{D}}w)|_{K_D} - w_D|^2_{H^1(K_D)} \le (1+\log p_{D,\mathcal{D}})^3 \sum_{\{D'\in\mathcal{D}: K_{D'}\cap K_D\neq\emptyset\}} e_{D'}(w) \quad \forall D \in \mathcal{D},\ w \in H^1_0(\Omega). \tag{5.8}$$

Let $\mathcal{N}(\mathcal{D})$ and $\mathcal{E}(\mathcal{D})$ denote the collection of vertices (or nodes), and (closed) edges of $\mathcal{K}(\mathcal{D})$. To construct $\Pi_{\mathcal{D}}$, for $e \in \mathcal{E}(\mathcal{D})$ we set

$$p_{e,\mathcal{D}} := \min\{p_D : D \in \mathcal{D},\ e \subset \partial K_D\}, \quad \bar{p}_{e,\mathcal{D}} := \max\{p_D : D \in \mathcal{D},\ e \subset \partial K_D\}. \tag{5.9}$$

With the mesh skeleton $\partial\mathcal{K}(\mathcal{D}) := \cup_{D\in\mathcal{D}}\partial K_D$, we set

$$V_{\partial\mathcal{D}} := \{v \in C(\partial\mathcal{K}(\mathcal{D})) : v|_e \in \mathbb{P}_{p_{e,\mathcal{D}}}(e)\ \ \forall e \in \mathcal{E}(\mathcal{D})\},$$
$$\bar{V}_{\partial\mathcal{D}} := \{v \in C(\partial\mathcal{K}(\mathcal{D})) : v|_e \in \mathbb{P}_{\bar{p}_{e,\mathcal{D}}}(e)\ \ \forall e \in \mathcal{E}(\mathcal{D})\}.$$

We construct $\Pi_{\partial\mathcal{D}} \in \mathcal{L}(\prod_{e\in\mathcal{E}(\mathcal{D})} H^{\frac{1}{2}}(e), V_{\partial\mathcal{D}})$, and an auxiliary $\bar{\Pi}_{\partial\mathcal{D}} \in \mathcal{L}(\prod_{e\in\mathcal{E}(\mathcal{D})} H^{\frac{1}{2}}(e), \bar{V}_{\partial\mathcal{D}})$, such that

$$(\Pi_{\partial\mathcal{D}}v)|_{\partial\Omega} = 0 \quad \text{for all } v = (v_e)_{e\in\mathcal{E}(\mathcal{D})} \in \prod_{e\in\mathcal{E}(\mathcal{D})} H^{\frac{1}{2}}(e) \text{ with } v_e = 0 \text{ for } e \subset \partial\Omega. \tag{5.10}$$

For any triangle K with edges e_1, e_2, e_3, there exists an extension $E_K \in \mathcal{L}(H^{\frac{1}{2}}(\partial K), H^1(K))$ that, for any $p \in \mathbb{N}$, maps $C(\partial K) \cap \prod_{i=1}^3 \mathbb{P}_p(e_i)$ into $\mathbb{P}_p(K)$ (see e.g. [2, Sect.7]). Defining $\Pi_{\mathcal{D}}$ by

$$(\Pi_{\mathcal{D}} w)|_{K_D} := E_{K_D}((\Pi_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_{\partial K_D}) + w_D - E_{K_D}(w_D|_{\partial K_D}), \tag{5.11}$$

in view of the definition of $V_{\partial\mathcal{D}}$ and (5.10), we have $\Pi_{\mathcal{D}} \in \mathcal{L}(H_0^1(\Omega), V_{\mathcal{D}}^c)$.

To construct $\Pi_{\partial\mathcal{D}}$, $\bar{\Pi}_{\partial\mathcal{D}}$, for each $\nu \in \mathcal{N}(\mathcal{D})$ we select some

$$e_\nu \in \mathcal{E}(\mathcal{D}) \text{ with } \nu \in e_\nu \text{ and } e_\nu \subset \partial\Omega \text{ when } \nu \in \partial\Omega. \tag{5.12}$$

For $\nu \in \mathcal{N}(\mathcal{D})$, by ϕ_ν we denote the nodal hat function, i.e., ϕ_ν is continuous piecewise linear w.r.t. $\mathcal{K}(\mathcal{D})$ and $\phi_\nu(\hat{v}) = \delta_{v,\hat{v}}\ \forall v, \hat{v} \in \mathcal{N}(\mathcal{D})$. For $e \in \mathcal{E}(\mathcal{D})$, let

$$\begin{aligned}
\bar{Q}_e &: H^{\frac{1}{2}}(e) \to H^{\frac{1}{2}}(e) \text{ be the } H^{\frac{1}{2}}(e)\text{-orthogonal projector onto } \mathbb{P}_{\bar{p}_{e,\mathcal{D}}}(e),\\
Q_{0,e} &: H^{\frac{1}{2}}(e) \to H^{\frac{1}{2}}(e) \text{ be the } H^{\frac{1}{2}}(e)\text{-orthogonal projector onto } \mathbb{P}_{p_{e,\mathcal{D}}}(e) \cap H_0^1(e),\\
\bar{Q}_{0,e} &: H^{\frac{1}{2}}(e) \to H^{\frac{1}{2}}(e) \text{ be the } H^{\frac{1}{2}}(e)\text{-orthogonal projector onto } \mathbb{P}_{\bar{p}_{e,\mathcal{D}}}(e) \cap H_0^1(e).
\end{aligned}$$

Denoting the endpoints of an $e \in \mathcal{E}(\mathcal{D})$ by $\nu_{1,e}, \nu_{2,e}$, we now define $\Pi_{\partial\mathcal{D}}$ and $\bar{\Pi}_{\partial\mathcal{D}}$ by setting, for $v = (v_e)_{e\in\mathcal{E}(\mathcal{D})} \in \prod_{e\in\mathcal{E}(\mathcal{D})} H^{\frac{1}{2}}(e)$,

$$\begin{aligned}
(\Pi_{\partial\mathcal{D}} v)|_e &:= \sum_{i=1}^2 (\bar{Q}_{e_{\nu_{i,e}}} v_{e_{\nu_{i,e}}})(\nu_{i,e})\phi_{\nu_{i,e}}|_e + Q_{0,e}\Big(v_e - \sum_{i=1}^2 (\bar{Q}_{e_{\nu_{i,e}}} v_{e_{\nu_{i,e}}})(\nu_{i,e})\phi_{\nu_{i,e}}|_e\Big),\\
(\bar{\Pi}_{\partial\mathcal{D}} v)|_e &:= \sum_{i=1}^2 (\bar{Q}_{e_{\nu_{i,e}}} v_{e_{\nu_{i,e}}})(\nu_{i,e})\phi_{\nu_{i,e}}|_e + \bar{Q}_{0,e}\Big(v_e - \sum_{i=1}^2 (\bar{Q}_{e_{\nu_{i,e}}} v_{e_{\nu_{i,e}}})(\nu_{i,e})\phi_{\nu_{i,e}}|_e\Big),
\end{aligned}$$

for any $e \in \mathcal{E}(\mathcal{D})$. It is clear that $\Pi_{\partial\mathcal{D}}$ maps into $V_{\partial\mathcal{D}}$, and, thanks to (5.12), that it satisfies (5.10). Similarly, $\bar{\Pi}_{\partial\mathcal{D}}$ maps into $\bar{V}_{\partial\mathcal{D}}$

These definitions show that, for $D \in \mathcal{D}$, $(\Pi_{\mathcal{D}} w)|_{K_D}$ depends only on $w|_{\cup\{K_{D'}: D'\in\mathcal{D},\, K_{D'}\cap K_D\neq\emptyset\}}$. Therefore, in order to prove (5.8), a homogeneity argument shows that we may assume that K_D is a uniformly shape regular triangle with

$$|K_D| = 1.$$

Since the extension $E_{K_D} \in \mathcal{L}(H^{\frac{1}{2}}(\partial K_D), H^1(K_D))$ can be chosen to be uniformly bounded over all such K_D, in view of (5.11) in order to arrive at (5.8), and so at the statement of the theorem, what remains to be proven is that

$$\begin{aligned}
&\|(\Pi_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_{\partial K_D} - w_D|_{\partial K_D}\|^2_{H^{\frac{1}{2}}(\partial K_D)}\\
&\qquad \lesssim (1+\log p_{D,\mathcal{D}})^3 \sum_{\{D'\in\mathcal{D}: K_{D'}\cap K_D\neq\emptyset\}} e_{D'}(w) \quad \forall D \in \mathcal{D},\, w \in H^1(\Omega).
\end{aligned} \tag{5.13}$$

In [2, Thms. 6.2 and 6.5], it was shown that on an interval I of length $\eqsim 1$, it holds that

$$\|z\|_{L_\infty(I)} \lesssim (1+\log p)^{\frac{1}{2}} \|z\|_{H^{\frac{1}{2}}(I)} \quad \forall z \in \mathbb{P}_p(I), \tag{5.14}$$

$$\|z\|_{H^{\frac{1}{2}}_{00}(I)} \lesssim (1+\log p) \|z\|_{H^{\frac{1}{2}}(I)} \quad \forall z \in \mathbb{P}_p(I) \cap H_0^1(I). \tag{5.15}$$

These estimates will be used hereafter.

Lemma 5.1. *For $\nu \in \mathcal{N}(\mathcal{D}) \cap \partial K_D$, $e, e' \in \mathcal{E}(\mathcal{D})$ with $e \cap e' = \nu$, we have*

$$|(\bar{Q}_e w|_e - \bar{Q}_{e'} w|_{e'})(\nu)|^2 \lesssim \sum_{\{D'\in\mathcal{D}:\ K_{D'}\ni\nu\}} (1+\log p_{D'}) e_{D'}(w) \qquad \forall w \in H^1(\Omega).$$

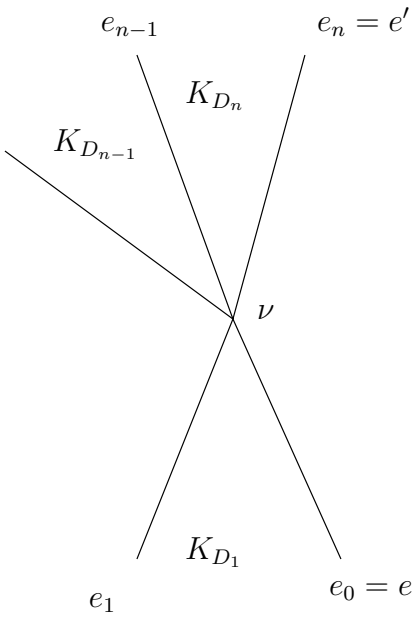

Figure 2: Notations relative to the proof of Lemma 5.1

Proof. Consider the notations as in Figure 2. Using that for $1 \leq i \leq n$, $(\bar{Q}_{e_i} w_{D_i}|_{e_i})(\nu) = (\bar{Q}_{e_{i-1}} w_{D_i}|_{e_{i-1}})(\nu)$, we have

$$(\bar{Q}_{e_n} w|_{e_n} - \bar{Q}_{e_0} w|_{e_0})(\nu) =$$
$$\Big(\bar{Q}_{e_n}(w - w_{K_{D_n}})|_{e_n} + \sum_{i=1}^{n-1} \bar{Q}_{e_i}(w_{K_{D_{i+1}}} - w + w - w_{K_{D_i}})|_{e_i} + \bar{Q}_{e_0}(w_{K_{D_1}} - w)|_{e_0}\Big)(\nu),$$

and so, using (5.14) and the trace inequality,

$$|(\bar{Q}_{e_n} w|_{e_n} - \bar{Q}_{e_0} w|_{e_0})(\nu)| \lesssim \sum_{i=1}^{n} \big[(1 + \log \bar{p}_{e_{i-1}})^{\frac{1}{2}} + (1 + \log \bar{p}_{e_i})^{\frac{1}{2}}\big] e_{D_i}(w)^{\frac{1}{2}}.$$

□

We continue with the proof of Theorem 5.1. As a first application of this lemma, we show that it suffices to prove (5.13) with $\Pi_{\partial\mathcal{D}}$ reading as $\bar{\Pi}_{\partial\mathcal{D}}$. To this end, for $e \in \mathcal{E}(\mathcal{D}) \cap \partial K_D$, let $D' \in \mathcal{D}$ such that $e \subset \partial K_{D'}$ and $p_{e,\mathcal{D}} = p_{D'}$. Then

$$(\Pi_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_e - (\bar{\Pi}_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_e = (Q_{0,e} - \bar{Q}_{0,e})\Big(w|_e - \sum_{i=1}^{2} (\bar{Q}_{e_{\nu_{i,e}}} w|_{e_{\nu_{i,e}}})(\nu_{i,e})\phi_{\nu_{i,e}}|_e\Big)$$
$$= (Q_{0,e} - \bar{Q}_{0,e})\Big(w|_e - w_{D'}|_e - \sum_{i=1}^{2} (\bar{Q}_{e_{\nu_{i,e}}} w|_{e_{\nu_{i,e}}} - w_{D'}|_e)(\nu_{i,e})\phi_{\nu_{i,e}}|_e\Big).$$

From (5.15), the trace theorem and the property $\|\phi_{\nu_{i,e}}\|_{H^{\frac12}(e)} \lesssim 1$, we infer that

$$\begin{aligned}\|(\Pi_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_e - (\bar\Pi_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_e\|_{H^{\frac12}_{00}(e)} \\ \lesssim (1+\log \bar p_{e,\mathcal{D}})\Big(e_{D'}(w)^{\frac12} + \sum_{i=1}^2 |(\bar Q_{e_{\nu_{i,e}}} w|_{e_{\nu_{i,e}}} - w_{D'}|_e)(\nu_{i,e})|\Big).\end{aligned} \tag{5.16}$$

Writing

$$\bar Q_{e_{\nu_{i,e}}} w|_{e_{\nu_{i,e}}} - w_{D'}|_e = \bar Q_{e_{\nu_{i,e}}} w|_{e_{\nu_{i,e}}} - \bar Q_e w|_e + \bar Q_e(w|_e - w_{D'}|_e),$$

and applying (5.14) as well as the trace theorem, shows that

$$\begin{aligned}|(\bar Q_{e_{\nu_{i,e}}} w|_{e_{\nu_{i,e}}} - w_{D'}|_e)(\nu_{i,e})| \\ \lesssim |(\bar Q_{e_{\nu_{i,e}}} w|_{e_{\nu_{i,e}}} - \bar Q_e w|_e)(\nu_{i,e})| + (1+\log(\bar p_{e,\mathcal{D}}))^{\frac12} e_{D'}(w)^{\frac12}.\end{aligned} \tag{5.17}$$

By combining (5.16) and (5.17), and applying Lemma 5.1 to the first term on the right-hand side of (5.17), we conclude that

$$\begin{aligned}\|\big((\Pi_{\partial\mathcal{D}} - \bar\Pi_{\partial\mathcal{D}}) w|_{\partial\mathcal{K}(\mathcal{D})}\big)|_{\partial K_D}\|^2_{H^{\frac12}(\partial K_D)} \\ \lesssim (1+\log p_{D,\mathcal{D}})^3 \sum_{\{D'\in\mathcal{D}: K_{D'}\cap K_D\neq\emptyset\}} e_{D'}(w) \quad \forall D\in\mathcal{D},\, w\in H^1(\Omega).\end{aligned} \tag{5.18}$$

As a consequence, what remains to show is (5.13) with $\Pi_{\partial\mathcal{D}}$ reading as $\bar\Pi_{\partial\mathcal{D}}$, that is, to show that

$$\begin{aligned}\|(\bar\Pi_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_{\partial K_D} - w_D|_{\partial K_D}\|^2_{H^{\frac12}(\partial K_D)} \\ \lesssim (1+\log p_{D,\mathcal{D}})^3 \sum_{\{D'\in\mathcal{D}: K_{D'}\cap K_D\neq\emptyset\}} e_{D'}(w) \quad \forall D\in\mathcal{D},\, w\in H^1(\Omega).\end{aligned} \tag{5.19}$$

Let us first consider the situation that $e_\nu \subset \partial K_D$ for all $\nu \in \mathcal{N}(\mathcal{D})\cap\partial K_D$. Then $((I-\bar\Pi_{\partial\mathcal{D}}) w_D|_{\partial\mathcal{K}(\mathcal{D})})|_{\partial K_D} = 0$ (this is generally not true for $\Pi_{\partial\mathcal{D}}$), and so

$$\|(\bar\Pi_{\partial\mathcal{D}} w|_{\partial\mathcal{K}(\mathcal{D})})|_{\partial K_D} - w_D|_{\partial K_D}\|_{H^{\frac12}(\partial K_D)} = \|(\bar\Pi_{\partial\mathcal{D}}(w-w_D)|_{\partial K_D})|_{\partial K_D}\|_{H^{\frac12}(\partial K_D)}. \tag{5.20}$$

To bound the right-hand side, let us write $v = (w-w_D)|_{\partial K_D}$. For edges e_1, e_2 of ∂K_D, and $\nu := e_1\cap e_2$, an application of (5.14) shows that

$$\|(\bar Q_{e_\nu} v_{e_\nu})(\nu)\phi_\nu\|_{H^{\frac12}(\partial K_D))} \lesssim (1+\log \bar p_{e_\nu,\mathcal{D}})^{\frac12}\|v_{e_\nu}\|_{H^{\frac12}(e_\nu)}. \tag{5.21}$$

For an edge $e\subset\partial K_D$, applications of (5.15) and (5.14) show that

$$\begin{aligned}&\|\bar Q_{0,e}\Big(v|_e - \sum_{i=1}^2 (\bar Q_{e_{\nu_{e,i}}} v|_{e_{\nu_{e,i}}})(\nu_{e,i})\phi_{\nu_{e,i}}|_e\Big)\|_{H^{\frac12}_{00}(e)} \\ &\lesssim (1+\log\bar p_{e,\mathcal{D}})\|\bar Q_{0,e}\Big(v|_e - \sum_{i=1}^2 (\bar Q_{e_{\nu_{e,i}}} v|_{e_{\nu_{e,i}}})(\nu_{e,i})\phi_{\nu_{e,i}}|_e\Big)\|_{H^{\frac12}(e)} \\ &\lesssim (1+\log\bar p_{e,\mathcal{D}})\Big(\|v|_e\|_{H^{\frac12}(e)} + \max_{i=1,2}(1+\log\bar p_{e_{\nu_{e,i}},\mathcal{D}})^{\frac12}\|v|_{e_{\nu_{e,i}}}\|_{H^{\frac12}(e_{\nu_{e,i}})}\Big).\end{aligned} \tag{5.22}$$

Combination of (5.20), (5.21), and (5.22), together with an application of the trace theorem, show that, in the situation of $e_\nu\subset\partial K_D$ for all $\nu\in\mathcal{N}(\mathcal{D})\cap\partial K_D$,

$$\|(\bar\Pi_{\partial\mathcal{D}} w|_{\partial K_D})|_{\partial K_D} - w_D|_{\partial K_D}\|^2_{H^{\frac12}(\partial K_D)} \lesssim (1+\log p_{D,\mathcal{D}})^3 e_D(w) \quad \forall w\in H^1(K_D),$$

which implies (5.19).

Consider now the situation that for one (or similarly more) $\nu \in \mathcal{N}(\mathcal{D}) \cap \partial K_D$, $e_\nu \not\subset \partial K_D$. We estimate the difference, in $H^{\frac{1}{2}}(\partial K_D)$-norm, with the situation that e_ν is equal to some edge $\bar{e} \subset \partial K_D$. Applications of (5.15) and Lemma 5.1 show that

$$\begin{aligned}\| \sum_{\{e\in\mathcal{E}(\mathcal{D})\cap\partial K_D : e\ni\nu\}} & (I-\bar{Q}_{0,e})\big((\bar{Q}_{e_\nu} w|_{e_\nu} - \bar{Q}_{\bar{e}} w|_{\bar{e}})(\nu)\phi_\nu|_e\big)\|_{H^{\frac{1}{2}}(\partial K_D)} \\ &\lesssim \big(1+\sum_{\{e\in\mathcal{E}(\mathcal{D})\cap\partial K_D : e\ni\nu\}} (1+\log \bar{p}_{e,\mathcal{D}})\big)|(\bar{Q}_{e_\nu} w|_{e_\nu} - \bar{Q}_{\bar{e}} w|_{\bar{e}})(\nu)| \\ &\lesssim (1+\log p_{D,\mathcal{D}})^{\frac{3}{2}} \sum_{\{D'\in\mathcal{D} : K_{D'}\cap K_D\neq\emptyset\}} e_{D'}(w)^{\frac{1}{2}},\end{aligned}$$

which completes the proof of (5.19), and thus of the theorem. ∎

5.2 The routine REDUCE

For $\mathcal{D} \in \mathbb{D}^c$, with $w_\mathcal{D}$ we will denote the best approximation to w from $V^c_\mathcal{D} = V_\mathcal{D} \cap H^1_0(\Omega)$ w.r.t. $|\cdot|_{H^1(\Omega)}$. For $w = u(f)$, being the solution of the Poisson problem with right-hand side f, $u_\mathcal{D}(f)$ turns out to be the Galerkin approximation to $u(f)$ from $V^c_\mathcal{D}$.

In this section, we will apply results from [35] on residual based a posteriori error estimators in the *hp* context. These results were derived under the condition that the polynomial degrees p_D and p'_D for $D, D' \in \mathcal{D} \in \mathbb{D}^c$ with $K_D \cap K_{D'} \neq \emptyset$ differ not more than an arbitrary, but constant factor. Fixing such a factor, let $\breve{\mathbb{D}}^c$ denote the subset of those $\mathcal{D} \in \mathbb{D}^c$ that satisfy this condition. Obviously, for each $\mathcal{D} \in \mathbb{D}^c$, there exists a $\breve{\mathcal{D}} \in \breve{\mathbb{D}}^c$ with $\mathcal{K}(\breve{\mathcal{D}}) = \mathcal{K}(\mathcal{D})$ and $\breve{\mathcal{D}} \geq \mathcal{D}$. Unfortunately, even for the smallest possible of such $\breve{\mathcal{D}}$, let us write it as $\breve{\mathcal{D}}(\mathcal{D})$, the ratio $\#\breve{\mathcal{D}}(\mathcal{D})/\#\mathcal{D}$ cannot be bounded uniformly in $\mathcal{D} \in \mathbb{D}^c$.

In view of the replacement of $\mathbb{D}^c$ by $\breve{\mathbb{D}}^c$, the mapping $\mathcal{C} : \mathbb{D} \to \mathbb{D}^c$ constructed in the previous subsection has to be replaced by $\breve{\mathcal{C}} := \mathbb{D} \to \breve{\mathbb{D}}^c : \mathcal{D} \mapsto \breve{\mathcal{D}}(\mathcal{C}(\mathcal{D}))$. From now on, we will denote $\breve{\mathbb{D}}^c$ as $\mathbb{D}^c$, and $\breve{\mathcal{C}}$ as $\mathcal{C}$. Since obviously $\breve{\mathcal{D}}$ can be constructed such that $\|p_{\breve{\mathcal{D}}}\|_\infty = \|p_\mathcal{D}\|_\infty$, with these new definitions (2.14) is still valid with $C_{3,\mathcal{D}} \eqsim (1+\log(\|p_\mathcal{D}\|_\infty))^{\frac{3}{2}}$, and, as before, unfortunately $\sup_{\mathcal{D}\in\mathbb{D}} \#\mathcal{C}(\mathcal{D})/\#\mathcal{D} = \infty$.

We note that in the present application, for $\mathcal{D} \in \mathbb{D}^c$, $f_\mathcal{D} \in F_\mathcal{D}$, and a desired reduction factor $\varrho \in (0,1]$, **REDUCE**$(\varrho, \mathcal{D}, f_\mathcal{D})$ has to produce a $\mathcal{D} \leq \bar{\mathcal{D}} \in \mathbb{D}^c$ such that $|u(f_\mathcal{D}) - u_{\bar{\mathcal{D}}}(f_\mathcal{D})|_{H^1(\Omega)} \leq \varrho |u(f_\mathcal{D}) - u_\mathcal{D}(f_\mathcal{D})|_{H^1(\Omega)}$. As explained in Section 2.3, the i-th iteration of **hp-AFEM** performs a call of **REDUCE**$(\frac{\mu}{1+(C_1+C_{3,\mathcal{D}_i})\omega}, \mathcal{C}(\mathcal{D}_i), f_{\mathcal{D}_i})$. The scalars μ and ω are parameters as set in **hp-AFEM**. They depend on the constant b from **hp-NEARBEST**, cf. Sect.2.2, the constant C_2, here being equal to 1, cf. Proposition 5.1, and the constant C_1, here being $\eqsim \sqrt{\delta}$, see Proposition 5.2. The scalar δ is a parameter in the definition of the error functional E, see (5.4), that is chosen such that $C_1 C_2 < b$, cf. (2.7). The only possible dependence of the required reduction factor $\frac{\mu}{1+(C_1+C_{3,\mathcal{D}_i})\omega}$ on $\mathcal{D}_i$ is via the value of $C_{3,\mathcal{D}_i}$. As we have seen, $C_{3,\mathcal{D}_i} \eqsim (1+\log\|\bar{p}_{\mathcal{D}_i}\|_\infty)^{\frac{3}{2}}$, meaning that when the maximum polynomial degree in $\mathcal{D}_i$ tends to infinity, this reduction factor tends to zero, but only very slowly.

The construction of **REDUCE** will follow the general template given in Sect. 2.4. We will verify the assumptions (2.26), (2.27), and (2.28). For $(w,f) \in H^1_0(\Omega) \times L^2(\Omega)$, $\mathcal{D} \in \mathbb{D}^c$, and $D \in \mathcal{D}$, we set the residual based (squared) a posteriori error indicator

$$\eta^2_{D,\mathcal{D}}(w,f) := \frac{|K_D|}{p^2_D}\|f+\triangle w\|^2_{L^2(K_D)} + \sum_{\{e\in\mathcal{E}(\mathcal{D}) : e\subset\partial K_D\cap\Omega\}} \frac{|e|}{2p_{e,\mathcal{D}}}\|[\![\nabla w\cdot\mathbf{n}_e]\!]\|^2_{L^2(e)},$$

where $p_{e,\mathcal{D}}$ is from (5.9), and define

$$\mathcal{E}_{\mathcal{D}}(w, f_{\mathcal{D}}) := \left(\sum_{D \in \mathcal{D}} \eta^2_{D,\mathcal{D}}(w, f_{\mathcal{D}}) \right)^{1/2}.$$

The following theorem stems from [35, Theorem 3.6]. Inspection of the proof given therein shows that the local lower bound provided by the (squared) a posteriori error indicator applies to any $w \in V^c_{\mathcal{D}}$ and so not only to the Galerkin solution.

Theorem 5.2 ('reliability and efficiency'). *There exists a constant $R > 0$ such that for $\mathcal{D} \in \mathbb{D}^c$ and $f_{\mathcal{D}} \in F_{\mathcal{D}}$,*

$$|u(f_{\mathcal{D}}) - u_{\mathcal{D}}(f_{\mathcal{D}})|^2_{H^1(\Omega)} \leq R\, \mathcal{E}^2_{\mathcal{D}}(u_{\mathcal{D}}(f_{\mathcal{D}}), f_{\mathcal{D}}). \tag{5.23}$$

For any $\varepsilon > 0$, and all $\mathcal{D} \in \mathbb{D}^c$, there exists an $r_{\mathcal{D},\varepsilon} \eqsim \|p_{\mathcal{D}}\|_\infty^{-2-2\varepsilon}$, such that for all $f_{\mathcal{D}} \in F_{\mathcal{D}}$, and $w \in V^c_{\mathcal{D}}$,

$$r_{\mathcal{D},\varepsilon}\, \mathcal{E}^2_{\mathcal{D}}(w, f_{\mathcal{D}}) \leq |u(f_{\mathcal{D}}) - w|^2_{H^1(\Omega)}. \tag{5.24}$$

Corollary 5.1 ('stability'). *With $r_{\mathcal{D},\varepsilon}$ as in Theorem 5.2, for all $\mathcal{D} \in \mathbb{D}^c$, $f_{\mathcal{D}} \in F_{\mathcal{D}}$, and $v, w \in V^c_{\mathcal{D}}$, it holds that*

$$\sqrt{r_{\mathcal{D},\varepsilon}} \Big| \mathcal{E}_{\mathcal{D}}(v, f_{\mathcal{D}}) - \mathcal{E}_{\mathcal{D}}(w, f_{\mathcal{D}}) \Big| \leq |v - w|_{H^1(\Omega)}. \tag{5.25}$$

Proof. A repeated application of the triangle inequality, first in ℓ_2 sequence spaces and then in L^2 function spaces, shows that

$$\begin{aligned}
&|\mathcal{E}_{\mathcal{D}}(v, f_{\mathcal{D}}) - \mathcal{E}_{\mathcal{D}}(w, f_{\mathcal{D}})| \\
&\leq \left(\sum_{D \in \mathcal{D}} \frac{|K_D|}{p_D^2} \|\triangle(v - w)\|^2_{L^2(K_D)} + \sum_{\{e \in \mathcal{E}(\mathcal{D}) : e \subset \partial K_D \cap \Omega\}} \frac{|e|}{2p_{e,\mathcal{D}}} \|[\![\nabla(v - w) \cdot \mathbf{n}_e]\!]\|^2_{L^2(e)} \right)^{\frac{1}{2}} \\
&\leq r_{\mathcal{D},\varepsilon}^{-\frac{1}{2}} |v - w|_{H^1(\Omega)},
\end{aligned}$$

where the last inequality follows from an application of (5.24) with "$f_{\mathcal{D}}$" reading as 0, and "w" reading as $v - w$. ∎

What is left is to establish the 'estimator reduction by refinement', i.e. (2.28). Given $\mathcal{M} \subset \mathcal{D} \in \mathbb{D}^c$, we define $\bar{\mathcal{D}}(\mathcal{M}) \in \mathbb{D}^c$ as follows: The h-partition $\mathcal{K}(\bar{\mathcal{D}}(\mathcal{M}))$ is the smallest in $\mathbb{K}^c$ in which each K_D for $D \in \mathcal{M}$ has been replaced by its four grandchildren in $\mathfrak{K}$; and for $D \in \bar{\mathcal{D}}(\mathcal{M})$, it holds that $p_D = p_{D'}$ where $D' \in \mathcal{D}$ is such that $K_{D'}$ be either equal to K_D, or its ancestor in $\mathcal{K}(\mathcal{D})$.

Proposition 5.3 ('estimator reduction by refinement'). *For $\mathcal{M} \subset \mathcal{D} \in \mathbb{D}^c$, and $\bar{\mathcal{D}}(\mathcal{M}) \in \mathbb{D}^c$ defined above, it holds that $\#\bar{\mathcal{D}}(\mathcal{M}) \lesssim \#\mathcal{D}$. For any $f_{\mathcal{D}} \in F_{\mathcal{D}}$, the estimator reduction property* (2.28) *is valid for $\gamma = \frac{1}{2}$.*

Proof. This follows easily from the fact that each K_D ($D \in \mathcal{M}$) is subdivided into four subtriangles that have equal area, that each $e \in \mathcal{E}(\mathcal{M})$ is cut into two equal parts, and that the jump of the normal derivative of $w \in V^c_{\mathcal{D}}$ over a newly created edge, i.e., an edge interior to a K_D for $D \in \mathcal{D}$, is zero. ∎

Given $\mathcal{D} \in \mathbb{D}^c$ and $f \in F_{\mathcal{D}}$, let $\mathcal{D} = \mathcal{D}_0 \leq \mathcal{D}_1 \leq \cdots \subset \mathbb{D}^c$ be the sequence of hp-partitions produced by **REDUCE**$(\varrho, \mathcal{D}, f_{\mathcal{D}})$. We have established (5.23), (5.24), and

(5.25), for any fixed $\varepsilon > 0$, as well as Proposition 5.3. Observing that $\|p_{\bar{\mathcal{D}}(\mathcal{M})}\|_\infty = \|p_{\mathcal{D}}\|_\infty$, an application of Proposition 2.2 now shows that in each iteration the quantity

$$|u(f_{\mathcal{D}}) - u_{\mathcal{D}_i}(f_{\mathcal{D}})|^2_{H^1(\Omega)} + (1 - \sqrt{\bar{\gamma}}) r_{\mathcal{D},\varepsilon}\, \mathcal{E}_{\mathcal{D}_i}(u_{\mathcal{D}_i}, f_{\mathcal{D}}),$$

where $\bar{\gamma} = (1-\theta) + \theta/2$, is reduced by at least a factor $1 - \frac{(1-\sqrt{\bar{\gamma}})^2}{2} \frac{r_{\mathcal{D},\varepsilon}}{R}$, and that this quantity is equivalent to $|u(f_{\mathcal{D}}) - u_{\mathcal{D}_i}(f_{\mathcal{D}})|^2_{H^1(\Omega)}$. In view of $r_{\mathcal{D},\varepsilon} \eqsim \|p_{\mathcal{D}}\|_\infty^{-2-2\varepsilon}$, we conclude that in order to reduce the initial error $|u(f_{\mathcal{D}}) - u_{\mathcal{D}}(f_{\mathcal{D}})|_{H^1(\Omega)}$ by a factor ϱ by an application of **REDUCE**, the number of iterations that are required is

$$M \eqsim \log(1/\varrho) \|p_{\mathcal{D}}\|_\infty^{2+2\varepsilon}.$$

Remark 5.2. *This result is not satisfactory because the number of iterations grows more than quadratically with the maximal polynomial degree. Yet, it improves upon the result stated in [3], where the number of iterations scales with the fifth power of the maximal polynomial degree.*

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