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# A posteriori boundary control for FEM approximation of elliptic eigenvalue problems


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# A posteriori boundary control for FEM approximation of elliptic eigenvalue problems

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**Abstract**

We derive new a posteriori error estimates for the finite element solution of an elliptic eigenvalue problem, which take into account also the effects of the polygonal approximation of the domain. This suggests local error indicators that can be used to drive a procedure handling the mesh refinement together with the approximation of the domain.

## 1 Introduction

Adaptive finite element methods (AFEM) for the numerical approximation of partial differential equations (PDE) are nowadays standard tools in science and engineering. The main ingredient for adaptivity is an a posteriori error estimate, which allows the computation of a solution with a prescribed error tolerance. Indeed, the local contributions of the a posteriori error estimate can be used to obtain informations on the error distribution and to eventually refine the computational grid. In this paper, we consider the finite element approximation of the following model eigenvalue problem: let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary, find the eigenpair $(u, \lambda)$ such that

$$-\Delta u = \lambda u \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega. \tag{1}$$

When the computational domain $\Omega_h$ is assumed to coincide with $\Omega$, we refer to [2] for an overview on the optimal error estimates both for eigenvalues and eigenfunctions and to [15, 10, 6, 9], for the corresponding a posteriori error estimates. However, the coincidnce between $\Omega$ and $\Omega_h$ may not be satisfied on coarse computational grids, which are typical, for example, of the early stages of an adaptive procedure. Altough, under some simplifying assumptions, the a priori analysis has been extended to the case in which the original and the computational domains do not coincide [14, 11], the corresponding extension of the a posteriori analysis seems to be open.

In this paper, we will obtain a posteriori error estimates for the eigenvalue and eigenfunction errors, which reduce to the usual a posteriori estimates in the case the discrete and continuous boundaries coincide, but exhibit, in the general case, an extra term, which locally control the Geometric Error, i.e. the mismatch between $\Omega$ and $\Omega_h$. Roughly speaking, our a posteriori error estimates will be derived as the sum of two terms; one measuring the PDE Error, which is related to the accuracy of the finite element approximation of the eigenpair and one measuring the Geometric Error.

The outline of the paper is the following. In Section 2 we present the continuous and discrete problems. In Section 3 we prove the reliability of the error estimator, by deriving some preliminary error bounds and combining them together. In Section 4 we show that, under some geometric saturation assumptions, the introduced error estimator is also (locally) efficient. Finally, in the Appendix we briefly sketch the guidelines of a goal-oriented a posteriori error analysis, which includes the effects of the domain approximation.

## 2 The Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary. Let $\Omega_h$ be a domain with polygonal boundary and $\mathcal{T}_h$ be a conforming and shape-regular triangulation of $\Omega_h$, such that each vertex on $\partial\Omega_h$ belongs to $\partial\Omega$. Moreover, we assume that all points where the condition of smoothness of $\partial\Omega$ is not fulfilled are vertices of $\partial\Omega_h$. Let $h_T$ denote the diameter of an element $T$ of the triangulation $\mathcal{T}_h$. We define by $\mathcal{T}_h^\partial$ the set of elements $T \in \mathcal{T}_h$ that have non-trivial intersection with the discrete boundary $\partial\Omega_h$ and we set $\mathcal{T}_h^0 := \mathcal{T}_h \setminus \mathcal{T}_h^\partial$. We denote by $\Sigma_h$ the set of edges of the triangulation $\mathcal{T}_h$ and by $\Sigma_h^b$ the set of edges that belong to the polygonal boundary $\partial\Omega_h$. Clearly, the set $\Sigma_h^i := \Sigma_h \setminus \Sigma_h^b$ contains the interior edges of the triangulation. Moreover, we introduce the subset $\widehat{\Sigma}_h \subset \Sigma_h^b$

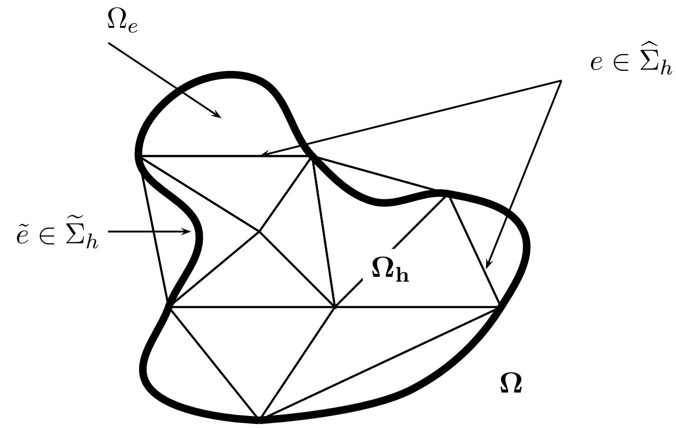


Figure 1: The exact domain $\Omega$ and the computational domain $\Omega_h$.

containing all the boundary edges $e \in \Sigma_h^b$ (or part of boundary edges) such that $e \subset \bar{\Omega}$. Finally, by abuse of notation, we introduce the set $\widetilde{\Sigma}_h$ of "curved" boundary edges $\tilde{e}$, such that $\tilde{e} \subset \partial\Omega$ and $\tilde{e} \subset T$, for some (unique) $T \in \mathcal{T}_h$ (see Figure 1). For the following, we will make the assumption that $\bar{\Omega} \setminus \Omega_h$ is decomposed into disjoint connected closed subset

$$\bar{\Omega} \setminus \Omega_h = \cup_{e \in \widehat{\Sigma}_h} \Omega_e, \qquad \overset{\circ}{\Omega}_e \cap \overset{\circ}{\Omega}_{e'} = \emptyset \ \text{ for } e \neq e'; \tag{2}$$

that is, for every boundary edge $e \in \widehat{\Sigma}_h$ there is a connected non-discretized closed subset $\Omega_e$ of $\bar{\Omega}$ with piecewise smooth boundary (see Figure 1). Furthermore, for every $e \in \Sigma_h^i$ we denote by $\omega_e$ the set of triangles $T' \in \mathcal{T}_h$ sharing the edge $e$, while for every $e \in \widehat{\Sigma}_h$ we set $\omega_e := T \cup \Omega_e$, where $T$ is the unique triangle such that $e \subset \partial T$.

Given any triangle $T \in \mathcal{T}_h$ and an edge $e \subset \partial T$, the symbol $n_e^T$ represents the outward unit normal for the triangle $T$ on the edge $e$. In addition, to each edge $e \in \Sigma_h$ we associate a unit normal $n_e$, its direction fixed arbitrarily once and for all. The only restriction is that the normals $n_e$ associated to boundary edges point outward with respect to the domain $\Omega_h$. Finally, given any $\mathcal{T}_h$-piecewise regular function $\psi$, we define the jump of the normal derivative across an edge $e \in \Sigma_h^i$ as

$$[\![ \frac{\partial \psi}{\partial n_e} ]\!] \,|_e = \frac{\partial \psi|_T}{\partial n_e^T} + \frac{\partial \psi|_{T'}}{\partial n_e^{T'}} \ ,$$

with $T, T' \in \omega_e$ and where we recall $n_e^{T'} = -n_e^T$ by definition.

Given any Lipschitz domain $\omega \subset \Omega \cup \Omega_h$, we denote by

$$(v, \nu)_\omega = \int_\omega v(x)\nu(x)\, dx \qquad \forall v, \nu \in L^2(\omega)$$

the $L^2$-scalar product on $\omega$. Moreover, for all such $\omega$ and functions $v \in H_0^1(\omega)$, we indicate with $\tilde{v}$ the unique function in $H^1(\Omega \cup \Omega_h)$ which is the extension by zero of $v$.

In the following, we will use the symbols $\simeq$, $\lesssim$, $\gtrsim$ to represent equivalences and bounds which hold up to a constant independent of the mesh size.

In this paper, we are interested in solving the following eigenvalue model problem: find eigenpairs $(u, \lambda)$, where $u$ is an eigenvector and $\lambda$ is an eigenvalue, such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3}$$

The weak formulation of (3) reads as follows

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ a(u, v) = \lambda(u, v)_\Omega \qquad \forall v \in H_0^1(\Omega) \ , \end{cases} \tag{4}$$

where

$$a(v, \nu) = \int_\Omega \nabla v \cdot \nabla \nu \qquad \forall v, \nu \in H^1(\Omega) \ . \tag{5}$$

In order to keep the presentation as simple as possible, we assume that the eigenvalue $\lambda$ to be computed is simple.

Given a positive natural number $k$, let

$$V_h = \{v \in H_0^1(\Omega_h) \ : \ v|_T \in \mathcal{P}_k(T) \ \forall T \in T_h\}, \tag{6}$$

where $\mathcal{P}_k(T)$ represents the space of polynomial functions on $T$ of maximum degree $k$. It is worth noticing that the space $V_h$ is defined on $\Omega_h$ and not on $\Omega$. We are now ready to define on $\Omega_h$ the corresponding Galerkin problem, which reads as follows

$$\begin{cases} \text{Find } u_h \in V_h \text{ and } \lambda_h \in \mathbb{R} \text{ such that} \\ a_h(u_h, v_h) = \lambda_h(u_h, v_h)_{\Omega_h} \qquad \forall v_h \in V_h \ , \end{cases} \tag{7}$$

where

$$a_h(v_h, \nu_h) = \int_{\Omega_h} \nabla v_h \cdot \nabla \nu_h \qquad \forall v_h, \nu_h \in H^1(\Omega_h) \ . \tag{8}$$

In the following, we always assume that the continuous and discrete eigensolutions are scaled such that

$$||u||^2_{L^2(\Omega)} = 1 \ , \qquad ||u_h||^2_{L^2(\Omega_h)} = 1 \ , \tag{9}$$

which also implies

$$||\nabla u||^2_{L^2(\Omega)} = \lambda \ , \qquad ||\nabla u_h||^2_{L^2(\Omega_h)} = \lambda_h \ . \tag{10}$$

As the eigenvalue $\lambda$ is assumed to be simple, there follows that for a sufficiently fine triangulation, the approximate eigenvlaue $\lambda_h$ is likewise simple.

In the rest of the paper, we will work under the following saturation type assumption, which is typical in the a posteriori analysis of eigenvalue problems, see for example [9, 6, 10].

**Assumption 2.1** *There exists a constant $K < 1$ independent of $h$ such that*

$$||u - \tilde{u}_h||_{L^2(\Omega)} \leq K \ .$$

*In addition, $|\lambda - \lambda_h|$ is bounded from above by a constant independent of $h$.*

# 3 A posteriori error estimates

In this Section we derive computable upper bounds for the eigenvalue and eigenfunction errors that take into account also the effects of the polygonal approximation of the domain $\Omega$. This is contained in Proposition 3.4, which is proved in four steps.

## 3.1 First Step: three technical lemmas

We start introducing the functional $J_1 : H^1(\Omega) \to \mathbb{R}$ defined by

$$J_1(v) = (\nabla(u - \tilde{u}_h), \nabla v)_\Omega \qquad \forall v \in H^1(\Omega) \ , \tag{11}$$

where $u$ is the solution to (4) and $\tilde{u}_h$ is the Galerkin approximation defined in (7). In the spirit of [9], we introduce the following auxiliary problem

$$\begin{cases} \text{Find } w \in H^1_0(\Omega) \text{ with } (w, u)_\Omega = 0 \text{ such that} \\ a(w, v) - \lambda(w, v)_\Omega = J_1(u)(u, v)_\Omega - J_1(v) \qquad \forall v \in H^1_0(\Omega) \ . \end{cases} \tag{12}$$

The orthogonality condition $(w, u)_\Omega = 0$ is needed to guarantee the uniqueness of $w$, since the operator related to the left hand side has a non trivial kernel given by $span\{u\}$. In this respect, note that the problem is well defined since the right hand side is zero when calculated on $v = u$.

The stability of problem (12) and the identities (9), (10) easily give

$$||w||_{H^1(\Omega)} \lesssim ||\nabla(u - \tilde{u}_h)||_{L^2(\Omega)} \ . \tag{13}$$

We have the following lemma.

**Lemma 3.1** *There holds*

$$||\nabla(u - \tilde{u}_h)||^2_{L^2(\Omega)} \lesssim A_1 + A_2 + A_3 \ , \tag{14}$$

*where the three terms*

$$\begin{aligned} A_1 &= |J_1(u)(u, \tilde{u}_h)_\Omega - J_1(\tilde{u}_h)| \ , \\ A_2 &= \frac{|J_1(u)|}{2} \, ||u - \tilde{u}_h||^2_{L^2(\Omega)} \ , \\ A_3 &= \frac{|J_1(u)|}{2} \, (1 - ||\tilde{u}_h||^2_{L^2(\Omega)}) \ . \end{aligned} \tag{15}$$

*Proof.* From (9) it follows

$$(u - \tilde{u}_h, u - \tilde{u}_h)_\Omega = 2 + (||\tilde{u}_h||^2_{L^2(\Omega)} - 1) - 2(u, \tilde{u}_h)_{L^2(\Omega)} \ , \tag{16}$$

which gives

$$(u, \tilde{u}_h)_\Omega = 1 + \frac{1}{2}(||\tilde{u}_h||^2_{L^2(\Omega)} - 1) - \frac{1}{2}||u - \tilde{u}_h||^2_{L^2(\Omega)} \ . \tag{17}$$

By using (17) we get

$$J_1(u)(u, \tilde{u}_h)_\Omega - J_1(\tilde{u}_h) = J_1(u) - J_1(\tilde{u}_h) - \frac{J_1(u)}{2}||u - \tilde{u}_h||^2_{L^2(\Omega)} - \frac{J_1(u)}{2}(1 - ||\tilde{u}_h||^2_{L^2(\Omega)}) \ . \tag{18}$$

The final result follows from (18) after observing that by definition

$$||\nabla(u - \tilde{u}_h)||^2_{L^2(\Omega)} = J_1(u) - J_1(\tilde{u}_h) \ .$$

□

The second lemma contains an approximation result.

**Lemma 3.2** *Let* $\tilde{u}_h$ *be the solution of* (7). *Then there exists a function* $\tilde{u}_h^0 \in H^1_0(\Omega)$ *such that*

$$||\tilde{u}_h - \tilde{u}_h^0||^2_{H^1(\Omega)} \lesssim \sum_{e \in \tilde{\Sigma}_h} ||\tilde{u}_h||^2_{H^{1/2}(e)} \ .$$

*Proof.* We introduce the problem

$$\begin{cases} \text{Find } \tilde{u}_h^0 \in H^1_0(\Omega) \text{ such that} \\ (\nabla \tilde{u}_h^0, \nabla v)_\Omega = (\nabla \tilde{u}_h, \nabla v)_\Omega \qquad \forall v \in H^1_0(\Omega) \ . \end{cases} \tag{19}$$

Observing that $\tilde{u}_h - \tilde{u}_h^0$ is an harmonic function, a well known result [13] gives

$$||\tilde{u}_h - \tilde{u}_h^0||^2_{H^1(\Omega)} \lesssim ||\tilde{u}_h - \tilde{u}_h^0||^2_{H^{1/2}(\partial\Omega)} = ||\tilde{u}_h||^2_{H^{1/2}(\partial\Omega)} \ .$$

The thesis follows by recalling that $\tilde{u}_h$ is null on all parts of $\partial\Omega$ which are not in $\cup_{e \in \tilde{\Sigma}_h} e$. □

Finally, we present a result of [5], which will be needed in the sequel. To this end, we need to introduce some additional notation. Let $\mathcal{N}_h$ the set of nodes $p$ of the triangulation $\mathcal{T}_h$. For each node $p$ belonging to the polygonal boundary $\partial\Omega_h$, we define by $\Sigma^p$ the set of boundary edges $e \in \Sigma_h^b$ sharing the node $p$. Moreover, for every edge $e \in \Sigma_h$ we denote by $\mathcal{N}(e)$ the set of nodes belonging to $e$. Given any element $T \in \mathcal{T}_h$, we define a mesh size function

$$\tilde{h}_T = h_T + h_T \sum_{e \in (\Sigma_h^b \cap \partial T)} \sum_{p \in \mathcal{N}(e)} \min_{e \in \Sigma^p} C_e h_e^{-1/2} \ , \tag{20}$$

where $C_e = 0$ if $\Omega_e = \emptyset$, and otherwise it is the minimum constant such that

$$||v||_{L^2(\partial\Omega_e)} \le C_e ||\nabla v||_{\Omega_e} \qquad \forall v \in H^1(\Omega_e) \text{ with } v = 0 \text{ on } \partial\Omega \cap \partial\Omega_e \ . \tag{21}$$

Note that for internal triangles the above definition gives $\tilde{h}_T = h_T$. Furthermore, for all $e \in \Sigma_h$ we set

$$\tilde{h}_e = \max_{T \in \omega_e} \tilde{h}_T. \tag{22}$$

**Lemma 3.3** *There exists an interpolation operator on $V_h$ such that for all $v \in H^1_0(\Omega)$ the interpolated function $v_I$ satisfies*

$$\begin{aligned} |(\psi, \tilde{v} - v_I)_{\Omega_h}| &\lesssim \Big( \sum_{T \in T_h} \tilde{h}_T^2 ||\psi||^2_{L^2(T)} \Big)^{1/2} ||\nabla v||_{H^1(\Omega)} \ , \\ |\sum_{e \in \Sigma_h^i} \int_e (\tilde{v} - v_I) [\![ \frac{\partial v_h}{\partial n_e} ]\!] \, ds| &\lesssim \Big( \sum_{T \in T_h} h_T^{-1} \tilde{h}_T^2 \sum_{e \in \partial T \cap \Sigma_h^i} ||[\![ \frac{\partial v_h}{\partial n_e} ]\!]||^2_{L^2(e)} \Big)^{1/2} ||\nabla v||_{H^1(\Omega)} \ , \\ |\sum_{e \in \widehat{\Sigma}_h} \int_e v \, \frac{\partial v_h}{\partial n_e} \, ds \, | &\lesssim \Big( \sum_{e \in \partial \widehat{\Sigma}_h} C_e^2 || \frac{\partial v_h}{\partial n_e} ||^2_{L^2(e)} \Big)^{1/2} ||\nabla v||_{H^1(\Omega)} \end{aligned} \tag{23}$$

*for all $v_h \in V_h$ and $\psi \in L^2(\Omega_h)$.*

Note that the lemma here presented is slightly different from the one in [5], since in our case we need a result in $\Omega_h$ rather than in $\Omega$.

### 3.2 Second Step: preliminary $H^1$-error bound

We now bound the terms $A_1, A_2, A_3$ in Lemma 3.1. Let us start with the term $A_1$. By using definition (11), then adding and subtracting the function $\tilde{u}_h^0$ of Lemma 3.2, finally employing a triangle inequality and recalling (12) we get

$$\begin{aligned} A_1 &= |(\nabla(u - \tilde{u}_h), \nabla u)_\Omega (u, \tilde{u}_h)_\Omega - (\nabla(u - \tilde{u}_h), \nabla \tilde{u}_h)_\Omega| \\ &= |(\nabla(u - \tilde{u}_h), \nabla u)_\Omega (u, \tilde{u}_h - \tilde{u}_h^0)_\Omega - (\nabla(u - \tilde{u}_h), \nabla(\tilde{u}_h - \tilde{u}_h^0)\,)_\Omega \\ &\quad + (\nabla(u - \tilde{u}_h), \nabla u)_\Omega (u, \tilde{u}_h^0)_\Omega - (\nabla(u - \tilde{u}_h), \nabla \tilde{u}_h^0)_\Omega| \\ &\lesssim |(\nabla(u - \tilde{u}_h), \nabla u)_\Omega (u, \tilde{u}_h - \tilde{u}_h^0)_\Omega| + |(\nabla(u - \tilde{u}_h), \nabla(\tilde{u}_h - \tilde{u}_h^0)\,)_\Omega| \\ &\quad + |a(w, \tilde{u}_h^0) - \lambda(w, \tilde{u}_h^0)_\Omega| \ . \end{aligned} \tag{24}$$

The Cauchy-Schwarz inequality, the identities (9) and (10), and the result of Lemma 3.2 easily yield

$$|(\nabla(u - \tilde{u}_h), \nabla u)_\Omega (u, \tilde{u}_h - \tilde{u}_h^0)_\Omega| \lesssim \lambda^{1/2} ||\nabla(u - \tilde{u}_h)||_{L^2(\Omega)} \Big( \sum_{e \in \widetilde{\Sigma}_h} ||\tilde{u}_h||^2_{H^{1/2}(e)} \Big)^{1/2} \tag{25}$$

and

$$|(\nabla(u - \tilde{u}_h), \nabla(\tilde{u}_h - \tilde{u}_h^0)\,)_\Omega| \lesssim ||\nabla(u - \tilde{u}_h)||_{L^2(\Omega)} \Big( \sum_{e \in \widetilde{\Sigma}_h} ||\tilde{u}_h||^2_{H^{1/2}(e)} \Big)^{1/2} \ . \tag{26}$$

Therefore, by introducing

$$T_1 = (\sum_{e\in\widetilde{\Sigma}_h} ||\tilde{u}_h||^2_{H^{1/2}(e)})^{1/2} \ , \tag{27}$$

from (24), (26) and (27) it follows

$$A_1 \ \lesssim \ T_1 \, ||\nabla(u-\tilde{u}_h)||_{L^2(\Omega)} + |a(w,\tilde{u}_h^0) - \lambda(w,\tilde{u}_h^0)_\Omega| \ . \tag{28}$$

In order to treat the second term on the righ-hand side of (28), we add and subtract $\tilde{u}_h$, we use Cauchy-Schwarz inequality, Lemma 3.2 and inequality (13), thus obtaining

$$\begin{aligned} |a(w,\tilde{u}_h^0) - \lambda(w,\tilde{u}_h^0)_\Omega| &\lesssim |a(w,\tilde{u}_h^0-\tilde{u}_h) - \lambda(w,\tilde{u}_h^0-\tilde{u}_h)_\Omega| \\ &\quad + |a(w,\tilde{u}_h) - \lambda(w,\tilde{u}_h)_\Omega| \\ &\lesssim T_1 \, ||\nabla(u-\tilde{u}_h)||_{L^2(\Omega)} + |a(w,\tilde{u}_h) - \lambda(w,\tilde{u}_h)_\Omega| \ . \end{aligned} \tag{29}$$

According to the notation introduced in Section 2, let now $\tilde{w}$ be the extension by zero of $w$. Using that $\tilde{w}$ is null outside $\Omega$ and $\tilde{u}_h$ is null outside $\Omega_h$ yields

$$\begin{aligned} |a(w,\tilde{u}_h) - \lambda(w,\tilde{u}_h)_\Omega| &= |a_h(\tilde{w},u_h) - \lambda(\tilde{w},u_h)_{\Omega_h}| \\ &\lesssim |(\lambda-\lambda_h)(\tilde{w},u_h)_{\Omega_h}| + |a_h(\tilde{w},u_h) - \lambda_h(\tilde{w},u_h)_{\Omega_h}| \ . \end{aligned} \tag{30}$$

We now consider the two terms on the righ-hand side of (30). As for the first term, by using Cauchy-Schwarz inequality, the definition of $\tilde{w}$ and equations (13) and (9), we immediately infer

$$|(\lambda-\lambda_h)(\tilde{w},u_h)_{\Omega_h}| \lesssim |\lambda-\lambda_h| \, ||w||_{L^2(\Omega)}||u_h||_{L^2(\Omega_h)} \lesssim |\lambda-\lambda_h| \, ||\nabla(u-\tilde{u}_h)||_{L^2(\Omega)} \ . \tag{31}$$

As for the second term, let $w_I \in V_h$ be the interpolant of $w$ introduced in Lemma 3.3. Using problem (7) and a standard element-wise integration by parts gives

$$\begin{aligned} |a_h(\tilde{w},u_h) - \lambda_h(\tilde{w},u_h)_{\Omega_h}| &= |a_h(\tilde{w}-w_I,u_h) - \lambda_h(\tilde{w}-w_I,u_h)_{\Omega_h}| \\ &\lesssim |\sum_{T\in T_h} (\Delta u_h + \lambda_h u_h, \tilde{w}-w_I)_T| \\ &\quad + |\sum_{T\in T_h}\sum_{e\in\partial T}\int_e (\tilde{w}-w_I)\frac{\partial u_h}{\partial n_e^T}\, ds| \ . \end{aligned} \tag{32}$$

We note that $w_I$ is null on all edges of $\partial\Omega_h$ while $\tilde{w}$ is null on all edges of $\Omega_h$ except those in $\widehat{\Sigma}_h$. Therefore, a standard collection of the edge terms gives

$$|\sum_{T\in T_h}\sum_{e\in\partial T}\int_e (\tilde{w}-w_I)\frac{\partial u_h}{\partial n_e} ds| \le |\sum_{e\in\Sigma_h^i}\int_e (\tilde{w}-w_I)[\![\frac{\partial u_h}{\partial n_e}]\!]\, ds| + |\sum_{e\in\widehat{\Sigma}_h}\int_e w\frac{\partial u_h}{\partial n_e}\, ds| \ . \tag{33}$$

Applying Lemma 3.3 to (32) and (33), and recalling (13) gives

$$\begin{aligned}|a_h(\tilde{w},u_h)-\lambda_h(\tilde{w},u_h)_{\Omega_h}| &\lesssim (T_2+T_3+T_4)||\nabla w||_{L^2(\Omega)}\\ &\lesssim (T_2+T_3+T_4)||\nabla(u-\tilde{u}_h)||_{L^2(\Omega)}\ ,\end{aligned} \tag{34}$$

where the terms

$$\begin{aligned}T_2 &= \Big(\sum_{T\in T_h}\tilde{h}_T^2||\Delta u_h+\lambda_h u_h||^2_{L^2(T)}\Big)^{1/2}\\ T_3 &= \Big(\sum_{T\in T_h}h_T^{-1}\tilde{h}_T^2\sum_{e\in\partial T\cap\Sigma_h^i}||[\![\frac{\partial u_h}{\partial n_e}]\!]||^2_{L^2(e)}\Big)^{1/2}\\ T_4 &= \Big(\sum_{e\in\widehat{\Sigma}_h}C_e^2||\frac{\partial u_h}{\partial n_e}||^2_{L^2(e)}\Big)^{1/2}\ .\end{aligned} \tag{35}$$

Combining the bounds (28), (29), (30), (31) and (34) we finally get

$$A_1 \lesssim (T_1+T_2+T_3+T_4+|\lambda-\lambda_h|)\,||\nabla(u-\tilde{u}_h)\,||_{L^2(\Omega)}\ . \tag{36}$$

In order to bound the terms $A_2$ and $A_3$, it is sufficient to observe that Cauchy-Schwarz inequality together with (10) yields

$$|J_1(u)|\le\lambda\,||\nabla(u-\tilde{u}_h)\,||_{L^2(\Omega)}\lesssim||\nabla(u-\tilde{u}_h)\,||_{L^2(\Omega)}\ , \tag{37}$$

while identity (9) gives

$$1-||\tilde{u}_h||^2_{L^2(\Omega)}=||u_h||^2_{L^2(\Omega_h)}-||\tilde{u}_h||^2_{L^2(\Omega)}=||u_h||^2_{L^2(\Omega_h/\Omega)}\ . \tag{38}$$

By setting

$$T_5=||u_h||^2_{L^2(\Omega_h/\Omega)} \tag{39}$$

and combining Lemma 3.1 with (36), (37) and (38) we finally obtain

**Proposition 3.1** *It holds*

$$\begin{aligned}||\nabla(u-\tilde{u}_h)||_{L^2(\Omega)} &\lesssim T_1+T_2+T_3+T_4+T_5\\ &\quad+|\lambda-\lambda_h|+||u-\tilde{u}_h||^2_{L^2(\Omega)}\ .\end{aligned} \tag{40}$$

*where the terms $T_i$, $i=1,..,5$ are defined in* (27), (35) *and* (39).

The first five terms in (40) will appear directly in the error estimator (see Proposition 3.4). The last two terms will instead be bounded in the following two Sections.

### 3.3 Third Step: preliminary $L^2$-error bound

In this Section we derive an upper (suboptimal) bound for the eigenfunction error $\|u-\tilde{u}_h\|_{L^2(\Omega)}$. Similarly to the case of the $H^1$-error, we introduce functional $J_0 : H^1(\Omega) \to \mathbb{R}$ defined by

$$J_0(v) = (u - \tilde{u}_h, v)_\Omega \qquad \forall v \in H^1(\Omega) \ , \tag{41}$$

and consider the following auxiliary problem

$$\begin{cases} \text{Find } z \in H^1_0(\Omega) \text{ with } (z,u)_\Omega = 0 \text{ such that} \\ a(z,v) - \lambda(z,v)_\Omega = J_0(u)(u,v)_\Omega - J_0(v) \qquad \forall v \in H^1_0(\Omega) \ . \end{cases} \tag{42}$$

It is easy to show the following stability result

$$\|z\|_{H^1(\Omega)} \lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)} \ , \tag{43}$$

and the counterpart of Lemma 3.1 which is contained in the following lemma.

**Lemma 3.4** *There holds*

$$\|u - \tilde{u}_h\|^2_{L^2(\Omega)} \lesssim B_1 + B_2 + B_3 \ , \tag{44}$$

*where*

$$\begin{aligned} B_1 &= |J_0(u)(u, \tilde{u}_h)_\Omega - J_0(\tilde{u}_h)| \ , \\ B_2 &= \frac{|J_0(u)|}{2} \, ||u - \tilde{u}_h||^2_{L^2(\Omega)} \ , \\ B_3 &= \frac{|J_0(u)|}{2} \, (1 - ||\tilde{u}_h||^2_{L^2(\Omega)}) \ . \end{aligned} \tag{45}$$

By mimicking the proof of Proposition 3.1, it is straightforward to obtain the following result

**Proposition 3.2** *There holds*

$$\begin{aligned} \|u - \tilde{u}_h\|_{L^2(\Omega)} \lesssim\ & T_1 + T_2 + T_3 + T_4 + T_5 \\ & + |\lambda - \lambda_h| + ||u - \tilde{u}_h||^2_{L^2(\Omega)} \ , \end{aligned} \tag{46}$$

*where the terms $T_i$, $i = 1, .., 5$ are defined in* (27), (35) *and* (39).

Furthermore, using Assumption 2.1, from the above Proposition we immediately infer

**Corollary 3.1** *There holds*

$$\|u - \tilde{u}_h\|_{L^2(\Omega)} \lesssim T_1 + T_2 + T_3 + T_4 + T_5 + |\lambda - \lambda_h| \ . \tag{47}$$

### 3.4 Fourth Step: eigenvalue error bounds

In this Section we derive (see Proposition 3.3) an upper bound for $|\lambda - \lambda_h|$ that will enter in (40). Using (9) we get

$$-1 \leq (||\tilde{u}_h||^2_{L^2(\Omega)} - 1) \leq 0 \ . \tag{48}$$

By employing(17) together with (48) and Assumption 2.1, we obtain

$$\begin{aligned} |(\lambda - \lambda_h)\,(u, \tilde{u}_h)_\Omega| &= |(\lambda - \lambda_h)\,\big(1 + \frac{1}{2}(||\tilde{u}_h||^2_{L^2(\Omega)} - 1) - \frac{1}{2}||u - \tilde{u}_h||^2_{L^2(\Omega)}\big)| \\ &\geq \alpha|\lambda - \lambda_h| \ , \end{aligned} \tag{49}$$

where the constant $\alpha = (1 - K)/2 > 0$. Let us denote by $\Delta_h\tilde{u}$ the $L^2$ function given by the extension by zero of $\Delta u \in L^2(\Omega)$ to $\Omega \cup \Omega_h$. Recalling (4) and using the fact that the support of $u$ is in $\Omega$ while the support of $\tilde{u}_h$ is in $\Omega_h$, we get

$$(\lambda - \lambda_h)\,(u, \tilde{u}_h)_\Omega = -(\Delta u, \tilde{u}_h)_\Omega - \lambda_h(u, \tilde{u}_h)_\Omega = -(\Delta_h\tilde{u}, u_h)_{\Omega_h} - \lambda_h(\tilde{u}, u_h)_{\Omega_h} \ . \tag{50}$$

Combining (49) and (50) with a piece-wise integration by parts easily yields

$$|\lambda - \lambda_h| \lesssim |(\nabla\tilde{u}, \nabla u_h)_{\Omega_h} - \lambda_h(\tilde{u}, u_h)_{\Omega_h}| + |\sum_{e\in\widetilde{\Sigma}_h}\int_e \frac{\partial u}{\partial n} u_h \, ds| \ . \tag{51}$$

Since $\nabla u \in (L^2(\Omega))^2$ and $\mathrm{div}\nabla u = \Delta u$ is also in $L^2(\Omega)$, a classical trace result [7] implies that $\frac{\partial u}{\partial n}$ is in $H^{-1/2}(\partial\Omega)$ and

$$||\frac{\partial u}{\partial n}||_{H^{-1/2}(\partial\Omega)} \lesssim ||\nabla u||_{L^2(\Omega)} + ||\Delta u||_{L^2(\Omega)} \leq \lambda^{1/2} + \lambda \ , \tag{52}$$

where in the last bound we used (10) and (4). Therefore, using an $H^{1/2}$ duality on the boundary, for the second term in (51) we infer

$$|\sum_{e\in\widetilde{\Sigma}_h}\int_e \frac{\partial u}{\partial n} u_h \, ds| = |\int_{\partial\Omega} \frac{\partial u}{\partial n}\tilde{u}_h \, ds| \leq ||\frac{\partial u}{\partial n}||_{H^{-1/2}(\partial\Omega)}||\tilde{u}_h||_{H^{1/2}(\partial\Omega)} \ . \tag{53}$$

Applying (52) and using that $\tilde{u}_h$ is null on part of $\partial\Omega$ yields

$$|\sum_{e\in\widetilde{\Sigma}_h}\int_e \frac{\partial u}{\partial n} u_h \, ds| \lesssim \big(\sum_{e\in\widetilde{\Sigma}_h} ||u_h||^2_{H^{1/2}(e)}\big)^{1/2} = \ T_1 \ . \tag{54}$$

Let $u_I \in V_h$ be the approximation of $u$ introduced in Lemma 3.3. Then, using (7) we obtain

$$|(\nabla\tilde{u}, \nabla u_h)_{\Omega_h} - \lambda_h(\tilde{u}, u_h)_{\Omega_h}| = |a_h(\tilde{u} - u_I, u_h) - \lambda_h(\tilde{u} - u_I, u_h)_{\Omega_h}| \ . \tag{55}$$

The term on the right-hand side in (55) is treated exactly as the term appearing in (32) up to a substitution of $\tilde{w}$ with $\tilde{u}$. Therefore, we get the same result as in the first line of (34), i.e.

$$|a_h(\tilde{u}-u_I,u_h)-\lambda_h(\tilde{u}-u_I,u_h)_{\Omega_h}| \lesssim (T_2+T_3+T_4)\,||\nabla u||_{L^2(\Omega)} \ , \tag{56}$$

which, by using (55) and (10), gives

$$|(\nabla\tilde{u},\nabla u_h)_{\Omega_h}-\lambda_h(\tilde{u},u_h)_{\Omega_h}| \lesssim T_2+T_3+T_4 \ , \tag{57}$$

where the terms $T_1, T_2$ and $T_3$ have been defined in (35). Combining (51), (54) and (57) we finally get the following result.

**Proposition 3.3** *There holds*

$$|\lambda-\lambda_h| \lesssim T_1+T_2+T_3+T_4 \ , \tag{58}$$

*with the above terms defined in* (27) *and* (35)*.*

As for the construction of an optimal computable error estimate for the eigenvalue error $|\lambda-\lambda_h|$, we will need a parallel result to Proposition 3.3. Indeed, the upper estimate directly built from (58) would result to be suboptimal [14].

We show the simple result rather briefly.

**Lemma 3.5** *There holds*

$$|\lambda-\lambda_h| \lesssim ||u-\tilde{u}_h||^2_{H^1(\Omega)}+T_1+T_6 \ , \tag{59}$$

*where*

$$T_6 = ||u_h||^2_{H^1(\Omega_h/\Omega)}. \tag{60}$$

*Proof.* From bounds (51), (54) and recalling (7) it follows

$$|\lambda-\lambda_h| \lesssim |a_h(\tilde{u},u_h)-\lambda_h(\tilde{u},u_h)_{\Omega_h}|+T_1 = |a_h(\tilde{u}-u_h,u_h)-\lambda_h(\tilde{u}-u_h,u_h)_{\Omega_h}|+T_1 \ . \tag{61}$$

Adding and subtracting terms in (61) and using (9), we easily obtain

$$\begin{aligned} |\lambda-\lambda_h| &\lesssim |a_h(\tilde{u}-u_h,\tilde{u}-u_h)|+|a_h(\tilde{u}-u_h,\tilde{u})-\lambda(\tilde{u}-u_h,\tilde{u})_{\Omega_h}| \\ &+|\lambda(\tilde{u}-u_h,\tilde{u}-u_h)_{\Omega_h}|+|(\lambda-\lambda_h)\,(\tilde{u}-u_h,u_h)_{\Omega_h}|+T_1 \\ &\lesssim ||\nabla(\tilde{u}-u_h)||^2_{L^2(\Omega_h)}+|a_h(\tilde{u}-u_h,\tilde{u})-\lambda(\tilde{u}-u_h,\tilde{u})_{\Omega_h}| \\ &+||\tilde{u}-u_h||^2_{L^2(\Omega_h)}+|\lambda-\lambda_h|\,||\tilde{u}-u_h||_{L^2(\Omega_h)}+T_1 \ , \end{aligned} \tag{62}$$

which, thanks to some simple algebra and to Assumption 2.1, yields

$$|\lambda-\lambda_h| \lesssim ||\tilde{u}-u_h||^2_{H^1(\Omega_h)}+|a_h(\tilde{u}-u_h,\tilde{u})-\lambda(\tilde{u}-u_h,\tilde{u})_{\Omega_h}|+T_1 \tag{63}$$

Bearing in mind the relation between the supports of $\tilde{u}$ and $\tilde{u}_h$, it can be checked that

$$|a_h(\tilde{u}-u_h,\tilde{u})-\lambda(\tilde{u}-u_h,\tilde{u})_{\Omega_h}| \lesssim |a(u-\tilde{u}_h,u)-\lambda(u-\tilde{u}_h,u)_\Omega|+||u-\tilde{u}_h||^2_{H^1(\Omega/\Omega_h)} . \quad (64)$$

Applying bound (64) in (63) and rearranging the integrals in the norms, we now get

$$|\lambda-\lambda_h| \lesssim ||\tilde{u}-u_h||^2_{H^1(\Omega)} + |a(u-\tilde{u}_h,u)-\lambda(u-\tilde{u}_h,u)_\Omega| + T_1 + T_6 \ , \quad (65)$$

where the term $T_6$ is defined in (60). From (4), since both $u$ and $\tilde{u}_h^0$ of Lemma 3.2 are in $H^1_0(\Omega)$, using also (9), (10) we obtain

$$\begin{aligned} |a(u-\tilde{u}_h,u)-\lambda(u-\tilde{u}_h,u)_\Omega| &= |a(\tilde{u}_h^0-\tilde{u}_h,u)-\lambda(\tilde{u}_h^0-\tilde{u}_h,u)_\Omega| \\ &\lesssim \left(||\nabla u||_{L^2(\Omega)}+\lambda||u||_{L^2(\Omega)}||\right)||\tilde{u}_h^0-\tilde{u}_h||_{H^1(\Omega)} \\ &\lesssim ||\tilde{u}_h^0-\tilde{u}_h||_{H^1(\Omega)}. \end{aligned} \quad (66)$$

The thesis follows by first applying Lemma 3.2 to (66) and by then combining the result with (65).
□

### 3.5 Main result

Combining Proposition 3.1, Corollary 3.1 and Proposition 3.3 with Lemma 3.5 and Assumption 2.1 yields the following a posteriori error estimates

**Proposition 3.4** *There holds*

$$\begin{aligned} \|\nabla(u-\tilde{u}_h)\|^2_{L^2(\Omega)} \ &\lesssim \ \sum_{T\in\mathcal{T}_h}\left\{\tilde{h}_T^2||\Delta u_h+\lambda_h u_h||^2_{L^2(T)} + h_T^{-1}\tilde{h}_T^2\sum_{e\in\partial T\cap\Sigma_h^i}||[\![\frac{\partial u_h}{\partial n_e}]\!]||^2_{L^2(e)}\right\} \\ &+\sum_{e\in\widetilde{\Sigma}_h}||\tilde{u}_h||^2_{H^{1/2}(e)}+\sum_{e\in\widehat{\Sigma}_h}C_e^2||\frac{\partial u_h}{\partial n_e}||^2_{L^2(e)}+||u_h||^4_{L^2(\Omega_h/\Omega)} \quad (67) \\ |\lambda-\lambda_h| \ &\lesssim \ \sum_{T\in\mathcal{T}_h}\left\{\tilde{h}_T^2||\Delta u_h+\lambda_h u_h||^2_{L^2(T)} + h_T^{-1}\tilde{h}_T^2\sum_{e\in\partial T\cap\Sigma_h^i}||[\![\frac{\partial u_h}{\partial n_e}]\!]||^2_{L^2(e)}\right\} \\ &+\sum_{e\in\widehat{\Sigma}_h}C_e^2||\frac{\partial u_h}{\partial n_e}||^2_{L^2(e)}+(\sum_{e\in\widetilde{\Sigma}_h}||\tilde{u}_h||^2_{H^{1/2}(e)})^{1/2} \\ &+||u_h||^2_{H^1(\Omega_h/\Omega)}+||u_h||^4_{L^2(\Omega_h/\Omega)} \quad (68) \end{aligned}$$

**Remark 3.1** *Noting that $u_h\in V_h$ yields the computability of the term $T_1 := ||u_h||_{H^{1/2}(e)}$, by using the definition of the norm $||\cdot||_{H^{1/2}(e)}$ and a suitable quadrature formula.*

**Remark 3.2** *From Proposition 3.1 and Corollary 3.1, it is evident that the arguments in the previous section can also be employed to build an a posteriori error estimate for the $L^2$ error $\|u-\tilde{u}_h\|^2_{L^2(\Omega)}$. However, this estimate will be suboptimal, unless, for example, the eigenfunction $u$ is assumed to be $H^2$-regular (see e.g. [14, 9]).*

We now state a result of consistency-type for the error estimates appearing in Proposition 3.4; that is when $\Omega = \Omega_h$, the a posteriori error bounds (67)-(68) reduce to the standard ones [6, 10, 9].

**Corollary 3.2** *Let $\Omega = \Omega_h$. Then there holds*

$$\begin{aligned}\|\nabla(u-u_h)\|^2_{L^2(\Omega)} &\lesssim \sum_{T\in T_h}\Big\{h_T^2\|\Delta u_h+\lambda_h u_h\|^2_{L^2(T)} \\ &\quad + h_T\sum_{e\in\partial T\cap\Sigma_h^i}\|[\![\frac{\partial u_h}{\partial n_e}]\!]\|^2_{L^2(e)}\Big\} \qquad (69)\\ |\lambda-\lambda_h| &\lesssim \sum_{T\in T_h}\Big\{h_T^2\|\Delta u_h+\lambda_h u_h\|^2_{L^2(T)} \\ &\quad + h_T\sum_{e\in\partial T\cap\Sigma_h^i}\|[\![\frac{\partial u_h}{\partial n_e}]\!]\|^2_{L^2(e)}\Big\}. \qquad (70)\end{aligned}$$

From Proposition 3.4 and Corollary 3.2, it is evident the role of the different terms appearing on the right-hand sides of (67) and (68), which deal with two different sources of error: the approximation of the PDE (PDE-Error) and the approximation of the domain (Geometric-Error). In particular, the term

$$\sum_{T\in T_h}\Big\{\tilde{h}_T^2\|\Delta u_h+\lambda_h u_h\|^2_{L^2(T)} + h_T^{-1}\tilde{h}_T^2\sum_{e\in\partial T\cap\Sigma_h^i}\|[\![\frac{\partial u_h}{\partial n_e}]\!]\|^2_{L^2(e)}\Big\} \qquad (71)$$

deals with the PDE-error and it is related with the accuracy of the Galerkin approximation $(u_h,\lambda_h)$ in the polygonal domain $\Omega_h$. The term

$$\sum_{e\in\widehat{\Sigma}_h} C_e^2\|\frac{\partial u_h}{\partial n_e}\|^2_{L^2(e)} + (\sum_{e\in\widetilde{\Sigma}_h}\|\tilde{u}_h\|^2_{H^{1/2}(e)})^{1/2} + \|u_h\|^2_{H^1(\Omega_h/\Omega)} + \|u_h\|^4_{L^2(\Omega_h/\Omega)} \qquad (72)$$

deals with the Geometric Error and it measures the influence on the accuracy of the discrete eigenpair of the mismatch between $\Omega_h$ and $\Omega$.

**Remark 3.3** *Roughly speaking, having in mind the upper estimate* (54)*, the term*

$$\sum_{e\in\widehat{\Sigma}_h} C_e^2\|\frac{\partial u_h}{\partial n_e}\|^2_{L^2(e)} + (\sum_{e\in\widetilde{\Sigma}_h}\|\tilde{u}_h\|^2_{H^{1/2}(e)})^{1/2}$$

*can be seen as an approximation to the quantity* $\int_{\partial\Omega} |\frac{\partial u}{\partial n}|^2 V \cdot n$*, which is the expression of the shape derivative of the eigenvalue* $\lambda$*; i.e. the variation of* $\lambda$ *with respect to the domain deformation of* $\Omega$ *into* $\Omega_h$*, induced by a suitable vector field* $V$ *(see [8]). For further comments on possible connections between shape calculus tools and a posteriori error estimators including the effects of domain approximation, see Remark 5.1.*

# 4 Efficiency of a posteriori error estimators

In this Section we will discuss the efficiency of the error estimators appearing in Proposition 3.4 in the "geometric saturated" state, i.e. by working under the following assumption. The first three points where firstly introduced in [5].

**Assumption 4.1** *Every triangle* $T \in T_h^\partial$ *has at most one edge* $e \in \Sigma_h^b$*. Moreover the approximation* $\Omega_h$ *to* $\Omega$ *has reached the saturated state; i.e. there holds*

1. *for every* $T \in T_h^\partial$ *with* $T \not\subset \Omega$ *there is a triangle* $T' \subset T$*, constructed from* $T$ *by a parallel displacement of the edge* $e \in \Sigma_h^b$ *such that* $T' \subset \Omega$ *and* $h_{T'} \geq \frac{1}{2} h_T$*,*
2. *there is a triangle* $T''$*, with* $T \subset T''$*, constructed from* $T$ *by a parallel displacement of the edge* $e \in \Sigma_h^b$*, such that* $\Omega^e \subset T''$ *and* $h_{T''} \leq 2h_T$*,*
3. $\tilde{h}_T \leq 2h_T$ *for all* $T \in T_h$*,*
4. *for all* $\Omega_e$*,* $e \in \widehat{\Sigma}_h$*, the curvature* $\mathcal{C}(x)$ *of* $\partial\Omega \cap \partial\Omega_e$ *is essentially constant. Namely, it exist two global positive constants* $c_1, c_2$ *such that* $c_1\, \mathcal{C}(y) \leq \mathcal{C}(x) \leq c_2\, \mathcal{C}(y)$ *for all* $x, y \in \partial\Omega \cap \partial\Omega_e$*,* $\forall e \in \widehat{\Sigma}_h$*.*

Note that the last condition is quite demanding, but it is used only to bound term $T_4$ in a sufficiently simple way. According to [16, 1], we introduce two types of bubble functions; namely the interior-bubble function and the edge-bubble function. In particular for every triangle $T \in T_h$ we denote by $\mathfrak{b}_T$ the interior-bubble function supported on $T$ and for every edge $e \in \Sigma_h^i \cup \widehat{\Sigma}_h$ we denote by $\mathfrak{b}_e$ the edge-bubble function supported on $\omega_e$.
We recall [16, 1] two useful results on the bubble and edge functions.

**Lemma 4.1** *Let* $T \in T_h$ *be an element of the triangulation and* $\mathcal{P}(T) \subset H^1(T)$ *be a finite dimensional space defined on* $T$*. Let* $\mathfrak{b}_T$ *be the interior bubble function over* $T$*. Then there exists a constant* $C$ *independent of* $v$ *and* $T$ *such that for every* $v \in \mathcal{P}(T)$ *the following inequalities hold:*

$$C^{-1}\|v\|^2_{L^2(T)} \leq \int_T \mathfrak{b}_T v^2 \leq C\|v\|^2_{L^2(T)}, \tag{73}$$

$$C^{-1}\|v\|_{L^2(T)} \leq \|\mathfrak{b}_T v\|_{L^2(T)} + h_T|\mathfrak{b}_T v|_{H^1(T)} \leq C\|v\|_{L^2(T)} \tag{74}$$

**Lemma 4.2** *Let $e \in \Sigma_h^i$ and $\mathfrak{b}_e$ be the corresponding edge bubble function defined over $\omega_e$. Let $\mathcal{P}(e) \subset H^1(e)$ be a finite dimensional space defined on $e$. Then there exists a constant $c$ independent of $v$ and $e$ such that for every $v \in \mathcal{P}(e)$ the following inequalities hold:*

$$c^{-1}\|v\|^2_{L^2(e)} \leq \int_e \mathfrak{b}_e v^2 \leq c\|v\|^2_{L^2(e)}, \tag{75}$$

$$h_T^{-1/2}\|\mathfrak{b}_e v\|_{L^2(T)} + h_T^{1/2}|\mathfrak{b}_e v|_{H^1(T)} \leq c\|v\|_{L^2(e)}, \tag{76}$$

*with $T \in \omega_e$.*
*Let $e \in \widehat{\Sigma}_h$ and $\mathfrak{b}_e$ be the corresponding edge bubble function defined over $\omega_e := T \cup \Omega_e$. Let $R_e$ be a rectangle containing $\Omega_e$ with edges parallel and orthogonal to $e$. We assume without loss of generality that $R_e := [0, h_e] \times [0, H_e]$, with $H_e > 0$. Then there exists another constant $c$ independent of $v$ and $e$ such that for every $v \in \mathcal{P}(e)$ the following inequalities hold:*

$$c^{-1}\|v\|^2_{L^2(e)} \leq \int_e \mathfrak{b}_e v^2 \leq c\|v\|^2_{L^2(e)}, \tag{77}$$

$$h_e^{-1/2}\|\mathfrak{b}_e v\|_{L^2(T)} + h_e^{1/2}|\mathfrak{b}_e v|_{H^1(T)} \leq c\|v\|_{L^2(e)}, \tag{78}$$

$$H_e^{-1/2}\|\mathfrak{b}_e v\|_{L^2(\Omega_e)} + \Big(\frac{\min(h_e^2, H_e^2)}{H_e}\Big)^{1/2}|\mathfrak{b}_e v|_{H^1(\Omega_e)} \leq c\|v\|_{L^2(e)}. \tag{79}$$

*Moreover, under the Assumption 4.1, there holds $H_e \leq h_e$. Hence, combining* (78) *and* (79) *it follows*

$$h_e^{-1/2}\|\mathfrak{b}_e v\|_{L^2(\omega_e)} + H_e^{1/2}|\mathfrak{b}_e v|_{H^1(\omega_e)} \leq c\|v\|_{L^2(e)}. \tag{80}$$

Note that bound (79) above (and therefore (80)) is the only non standard one. Nevertheless, it follows easily using the fourth point in Assumption 4.1, a simple minded bubble and a scaling argument. For every triangle $T \in \mathcal{T}_h$, we set $R_T := \Delta u_h + \lambda_h u_h$ and $z := \mathfrak{b}_T R_T$. We distinguish two cases: (i) $T \subset \Omega$ and (ii) $T \not\subset \Omega$. Let us first consider the case $T \subset \Omega$. Using Lemma 4.1 yields

$$\begin{aligned}
\|R_T\|^2_{L^2(T)} &\lesssim \|\mathfrak{b}_T^{1/2} R_T\|^2_{L^2(T)} = (R_T, z)_T \\
&= (\Delta u_h + \lambda u, z)_T + ((\lambda_h - \lambda)u_h, z)_T + (\lambda(u_h - u), z)_T \\
&= (\nabla(u - u_h), \nabla z)_T + (\lambda_h - \lambda)(u_h, z)_T + \lambda(u_h - u, z) \\
&\lesssim h_T^{-1}\|\nabla(u - u_h)\|_{L^2(T)}\|z\|_{L^2(T)} + |\lambda - \lambda_h|\|u_h\|_{L^2(T)}\|z\|_{L^2(T)} \\
&\quad +|\lambda|\|u - u_h\|_{L^2(T)}\|z\|_{L^2(T)} \\
&\lesssim \Big(h_T^{-1}\|\nabla(u - u_h)\|_{L^2(T)} + |\lambda - \lambda_h|\|u_h\|_{L^2(T)} \\
&\quad +|\lambda|\|u - u_h\|_{L^2(T)}\Big)\|R_T\|_{L^2(T)}.
\end{aligned}$$

Hence it follows

$$\begin{aligned}
h_T\|\Delta u_h + \lambda_h u_h\|_{L^2(T)} \leq\ & \|u - u_h\|_{H^1(T)} + h_T|\lambda - \lambda_h|\|u_h\|_{L^2(T)} \\
& +|\lambda|h_T\|u - u_h\|_{L^2(T)}.
\end{aligned} \tag{81}$$

In the case $T \not\subset \Omega$ we can use the same arguments as before on $T'$ instead of $T$ to get

$$\begin{aligned} h_{T'}\|\Delta u_h + \lambda_h u_h\|_{L^2(T')} &\lesssim \|u - u_h\|_{H^1(T')} + h_{T'}|\lambda - \lambda_h|\|u_h\|_{L^2(T')} \\ &\quad + |\lambda| h_{T'}\|u - u_h\|_{L^2(T')}. \end{aligned} \tag{82}$$

By Assumption 4.1 and since $\Delta u_h + \lambda_h u_h$ is a polynomial function, we can estimate the above term on $T'$ from below by the corresponding term on $T$, thus obtaining

$$\begin{aligned} h_T\|\Delta u_h + \lambda_h u_h\|_{L^2(T)} &\lesssim \|u - u_h\|_{H^1(T\cap\Omega)} + h_T|\lambda - \lambda_h|\|u_h\|_{L^2(T\cap\Omega)} \\ &\quad + |\lambda| h_T\|u - u_h\|_{L^2(T\cap\Omega)}. \end{aligned} \tag{83}$$

For every edge $e \in \Sigma_h$ we set

$$R_e = \begin{cases} [\![\nabla u_h \cdot n_e]\!]|_e & e \in \Sigma_h^i \\ (\nabla u_h \cdot n_e)|_e & e \in \widehat{\Sigma}_h, \end{cases} \tag{84}$$

and $w = \mathfrak{b}_e R_e$. We first consider the case $e \in \Sigma_h^i$ such that $\omega_e \subset \Omega$. Using Lemma 4.2 yields

$$\begin{aligned} \|R_e\|^2_{L^2(e)} &\lesssim \|\mathfrak{b}_e^{1/2} R_e\|^2_{L^2(e)} = (R_e, w)_e = (\Delta u_h, w)_{\omega_e} + (\nabla u_h, \nabla w)_{\omega_e} \\ &= (\Delta u_h + \lambda_h u_h, w)_{\omega_e} + (\nabla(u_h - u), \nabla w)_{\omega_e} + \lambda(u, w)_{\omega_e} - \lambda_h(u_h, w)_{\omega_e} \\ &\lesssim \sum_{T\in\omega_e} \|\Delta u_h + \lambda_h u_h\|_{L^2(T)}\|w\|_{L^2(T)} + h_e^{-1}\|\nabla(u - u_h)\|_{L^2(\omega_e)}\|w\|_{L^2(\omega_e)} \\ &\quad + |\lambda - \lambda_h|\|u_h\|_{L^2(\omega_e)}\|w\|_{L^2(\omega_e)} + |\lambda|\|u - u_h\|_{L^2(\omega_e)}\|w\|_{L^2(\omega_e)} \\ &\lesssim \sum_{T\in\omega_e} h_e^{1/2}\|\Delta u_h + \lambda_h u_h\|_{L^2(T)}\|R_e\|_{L^2(e)} + h_e^{-1/2}\|\nabla(u - u_h)\|_{L^2(\omega_e)}\|R_e\|_{L^2(e)} \\ &\quad + h_e^{1/2}|\lambda - \lambda_h|\|u_h\|_{L^2(\omega_e)}\|R_e\|_{L^2(e)} + h_e^{1/2}|\lambda|\|u - u_h\|_{L^2(\omega_e)}\|R_e\|_{L^2(e)}. \end{aligned}$$

Hence it follows

$$\begin{aligned} h_e^{1/2}\|[\![\nabla u_h \cdot n_e]\!]\|_{L^2(e)} &\lesssim \sum_{T\in\omega_e} h_e\|\Delta u_h + \lambda_h u_h\|_{L^2(T)} + \|\nabla(u - u_h)\|_{L^2(\omega_e)} \\ &\quad + h_e|\lambda|\|u - u_h\|_{L^2(\omega_e)} \\ &\quad + h_e|\lambda - \lambda_h|\|u_h\|_{L^2(\omega_e)} \end{aligned} \tag{85}$$

In the case $e \in \Sigma_h^i$ with $\omega_e \not\subset \Omega$, we consider $e' \subset e$ with $\omega_{e'} \subset \omega_e$ such that $\omega_{e'} \subset \Omega$ and we use the same arguments as before on $e'$ instead of $e$ to get

$$\begin{aligned} h_{e'}^{1/2}\|R_{e'}\|_{L^2(e')} &\lesssim \sum_{T'\in\omega_{e'}} h_{e'}\|\Delta u_h + \lambda_h u_h\|_{L^2(T')} + \|\nabla(u - u_h)\|_{L^2(\omega_{e'})} \\ &\quad + h_{e'}|\lambda|\|u - u_h\|_{L^2(\omega_{e'})} \\ &\quad + h_{e'}|\lambda - \lambda_h|\|u_h\|_{L^2(\omega_{e'})} \end{aligned} \tag{86}$$

By Assumption 4.1 and since $\nabla u_h \cdot n$ is a polynomial function on each edge, we can estimate the above term on $e'$ from below by the corresponding term on $e$, thus obtaining

$$\begin{aligned} h_e^{1/2}\|[\![\nabla u_h \cdot n_e]\!]\|_{L^2(e)} \lesssim & \sum_{T\in\omega_e} h_e\|\Delta u_h + \lambda_h u_h\|_{L^2(T\cap\Omega)} + \|\nabla(u-u_h)\|_{L^2(\omega_e\cap\Omega)} \\ & + h_e|\lambda|\|u-u_h\|_{L^2(\omega_e\cap\Omega)} \\ & + h_e|\lambda-\lambda_h|\|u_h\|_{L^2(\omega_e\cap\Omega)} \end{aligned} \quad (87)$$

Finally, for every edge $e \in \widehat{\Sigma}_h$, with $e \subset \partial T$, for a certain $T \in \mathcal{T}_h$, we set $\omega_e := \Omega_e \cup T$. Using bound (80) in Lemma 4.2 under the Assumption 4.1 (i.e. $H_e \leq h_e$) and repeating the same argument as before yield

$$\begin{aligned} H_e^{1/2}\|\nabla u_h \cdot n_e\|_{L^2(e)} \lesssim & \; h_e\|\Delta u_h + \lambda_h u_h\|_{L^2(\omega_e)} + \|\nabla(u-u_h)\|_{L^2(\omega_e)} \\ & + h_e|\lambda| \, \|u-u_h\|_{L^2(\omega_e)} \\ & + h_e|\lambda-\lambda_h| \, \|u_h\|_{L^2(\omega_e)}. \end{aligned} \quad (88)$$

Finally, as it is easy to check that the constant $C_e$ in (21) satisfies $C_e \leq H_e^{1/2}$, we have

$$\begin{aligned} C_e\|\nabla u_h \cdot n_e\|_{L^2(e)} \lesssim & \; h_e\|\Delta u_h + \lambda_h u_h\|_{L^2(T\cap\Omega)} + h_e\|\Delta u_h + \lambda_h u_h\|_{L^2(\Omega_e\cap\Omega)} \\ & + \|\nabla(u-u_h)\|_{L^2(\omega_e\cap\Omega)} + h_e|\lambda|\|u-u_h\|_{L^2(\omega_e\cap\Omega)} \\ & + h_e|\lambda-\lambda_h|\|u_h\|_{L^2(\omega_e\cap\Omega)}. \end{aligned} \quad (89)$$

Combining the above bounds yields the following result

**Proposition 4.1** *Let the Assumption 4.1 be valid. For every $T \in \mathcal{T}_h$, there holds*

$$\begin{aligned} h_T^2||\Delta u_h + \lambda_h u_h||^2_{L^2(T)} + \sum_{e\subset\partial T\cap\Sigma_h^i} h_e||[\![\frac{\partial u_h}{\partial n_e}]\!]||^2_{L^2(e)} \lesssim & \; ||u-u_h||^2_{H^1(T^*\cap\Omega)} \\ & + h_T^2||u-u_h||^2_{L^2(T^*\cap\Omega)} + h_T^2|\lambda-\lambda_h|^2, \end{aligned} \quad (90)$$

*where $T^*$ is the union of the triangle $T$ and all the neighboring triangles $T'$ sharing a face with $T$.*

*Moreover, for every edge $e \in \widehat{\Sigma}_h$, there holds*

$$C_e^2||\frac{\partial u_h}{\partial n_e}||^2_{L^2(e)} \lesssim ||u-u_h||^2_{H^1(\omega_e\cap\Omega)} + h_T^2||u-u_h||^2_{L^2(\omega_e\cap\Omega)} + h_T^2|\lambda-\lambda_h|^2, \quad (91)$$

*where $\omega_e = \Omega_e \cap T$.*

**Remark 4.1** *To complete the discussion on the efficiency of the error estimators appearing in Proposition 3.4, we need to comment on the terms $T_1$ and $T_5$. It is straightforward to note that for every $e \in \widetilde{\Sigma}_h$ there holds*

$$T_1 := ||u_h||_{H^{1/2}(e)} = ||u - u_h||_{H^{1/2}(e)} \leq C||u - u_h||_{H^1(T)} \quad (92)$$

*with $e \in \partial T$. Therefore also the term $T_1$ is bounded by the error.*

*Moreover, again for $e \in \widetilde{\Sigma}_h$, the following equality is true on $\Omega_h \setminus \Omega$*

$$T_5 := ||u_h||_{L^2(\Omega_h \setminus \Omega)} = ||\tilde{u} - u_h||_{L^2(\Omega_h \setminus \Omega)}. \quad (93)$$

*and the same holds for the term $||u_h||_{H^1(\Omega_h \setminus \Omega)}$ appearing in* (68)*. Therefore such pieces are also bounded by the error, although outside $\Omega$.*

The local error indicators in (67) and (68) can be used to drive an adaptive finite element method (AFEM) of the form

$$\ldots \rightarrow (\Omega_h^{(k)}, V_h^{(k)}) \rightarrow \texttt{SOLVE} \rightarrow \texttt{ESTIMATE} \rightarrow \texttt{MARK} \rightarrow \texttt{REFINE} \rightarrow (\Omega_h^{(k+1)}, V_h^{(k+1)}) \rightarrow \ldots$$

where $\Omega_h^{(k)}$ and $V_h^{(k)}$ are the $k$-th computational domain and the corresponding finite element space built by the adaptive procedure, respectively. We remark that the module `REFINE` builds the new computational grid (possibly on a new computational domain) depending on the elements that have been marked for refinement in the module `MARK`. We refer to [5] for a detailed description of the practical implementation of such modules.

# 5 Appendix: goal-oriented analysis including the effects of boundary approximation

In this Appendix, by applying the basic philosophy employed in the context of the eigenvalue problem, we derive a posteriori error estimators, incorporating also the effects of the boundary appproximation, for the approximation of quantities of interest related to the solution of partial differential equations. In order to make the presentation as simple as possible, suppose we are given the prototype elliptic boundary value problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma = \partial\Omega, \quad (94)$$

and a linear ouput functional $G$ such that $G(u)$ is a quantity of physical, engineering or scientific interest. In order to compute $G(u)$, one may want to compute $G(u_h)$, where $u_h$ is the finite element approximation to $u$ over a conforming triangulation $\mathcal{T}_h$ of the polygonal approximation $\Omega_h$ of $\Omega$. We are interested in building an a posteriori estimator for the approximation error $|G(u - \tilde{u}_h)|$, being $\tilde{u}_h$ the extension by zero of $u_h$ in $\Omega \cup \Omega_h$, which takes into account also the effects of the domain approximation. The following result contains a non-standard Dual Weighted Residual (DWR) a posteriori error estimator (see [4] for an introduction on classical DWR method and [3] for a similar result).

**Proposition 5.1** *Let $\Omega \subset \mathbb{R}^2$ a bounded domain with piecewise smooth boundary. Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ be the solution to (94). Let $V_h$ be the finite element space defined in (6) over a conforming triangulation $\mathcal{T}_h$ of $\Omega_h$ and $u_h \in V_h$ be the Galerkin approximation to $u$ satisying*

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h = \int_{\Omega_h} f_h v_h \qquad \forall v_h \in V_h, \tag{95}$$

*with $f_h \in L^2(\Omega_h)$. Let $G(u) = \int_\Omega gu$, with $g \in L^2(\Omega)$. Let $z \in H_0^1(\Omega)$ be the solution to the adjoint problem*

$$-\Delta z = g \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{96}$$

*Let $z_h \in V_h$ be the finite element approximation to $z$ satisfying*

$$\int_{\Omega_h} \nabla z_h \cdot \nabla v_h = \int_{\Omega_h} g_h v_h \qquad \forall v_h \in V_h, \tag{97}$$

*with $g_h \in L^2(\Omega_h)$. Then the following DWR-type a posteriori error estimator holds*

$$\begin{aligned} \big|G(u-\tilde{u}_h)\big| \;\; \leq \;\; & \big|\sum_{T\in\mathcal{T}_h} (R, z-z_h)_T - (J, z-z_h)_{\partial T}\big| + \big|\sum_{\Omega_e \subset \Omega\setminus\Omega_h} (f,z)_{\Omega_e}\big| \\ & + \big|\int_{\Omega\cap\Omega_h} (f-f_h)z\big| + \big|\sum_{\tilde{e}\in\widetilde{\Sigma}_h} (\frac{\partial z}{\partial n}, u_h)_e\big|, \end{aligned} \tag{98}$$

*where $\Omega_e$ and $\widetilde{\Sigma}_h$ have been defined in Section 2 and*

$$R := f + \Delta u_h \qquad \forall\, T \in \mathcal{T}_h,$$

$$J := \begin{cases} \frac{1}{2}[\![\nabla u_h \cdot n]\!] & \forall\, e \subset \partial T \setminus \partial\Omega_h \\ \nabla u_h \cdot n & \forall\, e \subset \partial T \cap \partial\Omega_h, \end{cases}$$

*Proof.* By using the definition of the adjoint solution $z$ we obtain

$$\begin{aligned} G(u-\tilde{u}_h) &= \int_\Omega g(u-\tilde{u}_h) = \int_\Omega -\Delta z(u-\tilde{u}_h) \\ &= \int_\Omega \nabla z \cdot \nabla(u-\tilde{u}_h) - \int_{\partial\Omega} \frac{\partial z}{\partial n}(u-\tilde{u}_h) \\ &= \int_\Omega \nabla u \cdot \nabla z - \int_\Omega \nabla \tilde{u}_h \cdot \nabla z - \int_{\partial\Omega} \frac{\partial z}{\partial n}(u-\tilde{u}_h) \\ &= \int_\Omega fz - \int_{\Omega_h} \nabla \tilde{u}_h \cdot \nabla z - \int_{\partial\Omega} \frac{\partial z}{\partial n}(u-\tilde{u}_h) \\ &= \int_\Omega fz - \int_{\Omega_h} \nabla u_h \cdot \nabla(z-z_h) - \int_{\Omega_h} f_h z_h - \int_{\partial\Omega} \frac{\partial z}{\partial n}(u-\tilde{u}_h) \\ &= \int_{\Omega_h} f_h(z-z_h) - \int_{\Omega_h} \nabla u_h \cdot \nabla(z-z_h) + \int_{\Omega\setminus\Omega_h} fz \\ &\quad + \int_{\Omega\cap\Omega_h} (f-f_h)z - \int_{\partial\Omega} \frac{\partial z}{\partial n}(u-\tilde{u}_h). \end{aligned} \tag{99}$$

Finally, integrating by parts over the elements of the triangulation $\mathcal{T}_h$ and recalling that $u = 0$ on $\partial\Omega$ yields the thesis. □

**Remark 5.1** *Strictly speaking,* (98)*, as it stands is not really a classical a posteriori error bound, because it involves the unknown adjoint solution $z$ and the value of $u_h$ on (a subset of) the boundary $\partial\Omega$. Ad hoc numerical strategies for the approximation of $z$ (see e.g. [4]) and of $u_h$ on $\partial\Omega$ are needed in order to obtain computable bounds. However, it is worth noticing that the four terms on the right-hand side of* (98) *have a precise interpretation: the first term is the standard DWR-type estimator that we would get in the case $\Omega = \Omega_h$, the second term is the corresponding DWR estimator in $\Omega\backslash\Omega_h$, the third term quantifies the mismatch between $f$ and $f_h$, while the fourth term measures the geometric distance between the domain $\Omega$ and $\Omega_h$ in terms of the functional $G$. Roughly speaking, since $u_h|_{\partial\Omega_h} = 0$ the fourth term in* (98) *can be seen as an approximation of the quantity $\int_{\partial\Omega} \frac{\partial z}{\partial n}\frac{\partial u}{\partial n} V \cdot n$, which is, if we assume for example $f, g \in L^2(\mathbb{R}^2)$, the expression of the shape derivative of $G(u)$, i.e. the variation of the functional $G(u)$ with respect to the domain deformation, induced by a suitable vector field $V$, of $\Omega$ into $\Omega_h$ (see [12] for more details). This may shed new light on the use of shape calculus tools to construct effective estimators for controlling the approximation of the domain, both in the context of the numerical solution of PDEs and Shape Optimization problems.*

**Remark 5.2** *If $G$ is a functional in $H^{-1}$, then one can recover a result similar to* (98) *using Lemma 3.2 and following the approach shown in Section 3.2.*

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