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Non-isothermal non-Newtonian fluids: the stationary case

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Abstract

The stationary Navier-Stokes equations for a non-Newtonian incompressible fluid are coupled with the stationary heat equation and subject to Dirichlet type boundary conditions. The viscosity is supposed to depend on the temperature and the stress depends on the strain through a suitable power law depending on $p \in (1, 2)$ (shear thinning case). For this problem we establish the existence of a weak solution as well as we prove some regularity results both for the Navier-Stokes and the Stokes cases. Then, the latter case with the Carreau power law is approximated through a FEM scheme and some error estimates are obtained. Such estimates are then validated through some two-dimensional numerical experiments.

Keywords: Incompressible non-isothermal fluids, power law fluids, shear thinning, existence, regularity, finite elements, a priori error estimates.

MSC 2010: 35Q45, 76A05, 76M10.

1 Introduction

In the last decades, the theoretical and numerical analysis of non-Newtonian fluids has seen a renew of interest, stimulated by the wide spectrum of applied problems, ranging from biological applications to industrial processes. In particular, concerning the latter, we point out that many processes deal with materials exhibiting non-Newtonian behaviors. For instance, food processing, polymer manufacturing, tribology, injection molding of foams, rubber extrusion. Moreover,

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most of them are carried out under non-isothermal conditions. Motivated by these important applications, in the present paper we focus on a class of non-isothermal non-Newtonian fluid models (see, for instance, [3, 4, 25, 32, 37, 35, 13] and references therein). We recall that the description of a non-Newtonian fluid behavior is based on a power-law ansatz. We refer the reader to [12] and [48] for a general continuum mechanical background and to [7, 18, 44, 54, 56, 57] for a detailed discussion of several models for non-Newtonian fluids. In particular, for a large class of non-Newtonian fluids, the dominant departure from a Newtonian behavior is that, in a simple shear flow, the viscosity and the shear rate are not proportional. These are the so-called fluids with shear-dependent viscosity. Here we are interested in the theoretical and numerical analysis of steady flows of such fluids accounting for the presence of a non-negligible temperature field. In particular, we consider the case where the viscosity depends on the temperature. Very similar models, in the Newtonian case, have been analyzed, e.g., in [6]. A slightly more complex model has been derived and investigated in [24] (see also the references therein). In that case, the authors consider a non-stationary model for incompressible homogeneous Newtonian fluids in a fixed bounded three-dimensional domain.

Let us now describe the problem we want to analyze. We consider an incompressible fluid which occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a sufficiently smooth boundary. We denote by $\mathbf{u} : \Omega \to \mathbb{R}^d$ its velocity field and by $\Theta : \Omega \to \mathbb{R}$ its temperature. We suppose that $(\mathbf{u}, \Theta)$ satisfies the following equations

$$-\mathrm{div}\,[\nu(\Theta)\tau(x, \varepsilon(\mathbf{u}))] + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla\pi = \mathbf{f} \qquad \text{in } \Omega \tag{1.1}$$

$$\mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega \tag{1.2}$$

$$-\kappa\Delta\Theta + \mathbf{u} \cdot \nabla\Theta = g \qquad \text{in } \Omega \tag{1.3}$$

endowed with the boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \Theta = \theta \qquad \text{on } \partial\Omega. \tag{1.4}$$

Here $\nu > 0$ is the viscosity coefficient, τ denotes the stress tensor which is a suitable function of the strain rate tensor $\varepsilon(\mathbf{u})$ defined by $\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i)$. Moreover, π is the fluid pressure, $\mathbf{f}$ is a given body force, $\kappa > 0$ is the heat diffusion coefficient, g is a known heat source, and θ is a given boundary temperature.

Concerning the constitutive law which defines τ, we observe that most real non-Newtonian fluids that can be modeled by a constitutive law such that

$$|\tau(x, \mathbf{B})| \leq \tau_2(1 + |\mathbf{B}|)^{p-1}, \qquad x \in \Omega$$

with $\mathbf{B} \in \mathbb{R}^{d\times d}$ symmetric tensor and τ_2 positive constant, are shear thinning fluids, namely, the shear exponent p, is "small", i.e., $p \in (1, 2)$ (cf. [55] for a discussion of such models and further references). This is the case we consider in the present contribution.

The mathematical analysis of the Navier-Stokes problem for non-Newtonian fluids started with the work of Ladyžhenskaya (see,e.g., [48]). After the fundamental works of the "Prague school" led by Nečas et. al. (see [53, 52] and the references therein), the problem has been intensively studied and various existence and regularity properties have been proved in the last years. The literature on this subject is rather vast. Thus we only mention some of the main contributions which are mostly related to the stationary case. In the isothermal case, there are several results on the existence of weak solutions ([34]), interior regularity ([2]) and regularity up-to-the boundary for the Dirichlet problem (see, e.g., [14, 15, 17, 28, 59, 62] and references therein). Moreover, we refer to [45] for some $C^{1,\alpha}$-regularity results. However, there are not so many contribution on the non-isothermal case. In [58], the author obtains the existence of a distributional solution to a steady-state system of equations for incompressible, possibly non-Newtonian of the p-power type, viscous flow coupled with the heat equation in a smooth bounded domain of $\mathbb{R}^d$, $d = 2, 3$. Notice that in this model the fluid viscosity is considered to be independent of the temperature. In [8], the authors analyze a system of equations describing the stationary flow of a quasi-Newtonian fluid, with temperature dependent viscosity and with a viscous heating. Existence of at least one appropriate weak solution is proved. In [27] the existence of weak solutions to the coupled system of stationary equations for a class of general non-Newtonian fluids with energy transfer is established.

Moreover, in the aforementioned work [24], the authors establish the large-data and long-time existence of a suitable class of weak solutions to the corresponding problem regarding unsteady flows of incompressible homogeneous Newtonian fluids. In [6, Sec.2] the authors prove the existence and (conditional) uniqueness of weak solutions, together with some regularity results, to problem (1.1)-(1.4) in the case of Newtonian fluids, which is a simplification of the model treated in [24]. In particular, it describes the stationary flow of a viscous incompressible Newtonian fluid, in the case where the viscosity of the fluid depends on the temperature. The mathematical analysis of a very similar problem in the case of a non-Newtonian fluid, with the exponent p depending on the temperature, can be found in [5]. There, in particular, the existence of a weak solution is obtained, taking p constant, for $p \in (3d/(d + 2), 2]$. Also, a conditional uniqueness for close solutions is shown.

On the numerical side, some pioneering results on steady isothermal non-Newtonian incompressible Stokes fluids, were given in [9, 10, 11, 40, 60]. More recently, in [16] and [43], optimal error estimates for shear thinning non-Newtonian fluids have been addressed, whereas in [61] a finite element method based on a four-field formulation of the nonlinear Stokes equations has been considered. Moreover, without aiming at completeness, we refer to the book [41] and the references therein, and to [29, 36, 46, 47] as recent relevant contributions on the numerical discretization of generalized Stokes problems. We also point out that the approximation of steady isothermal non-Newtonian incompressible Stokes fluid problems on polytopal meshes have been addressed in [19]. Concerning non-isothermal Newtonian fluid problems, we refer to [6], where spectral elements have been employed, and to [30] which considers the finite element approximation of the heat equation coupled with Stokes equations under threshold type boundary condition. It is worth mentioning that a very recent contribution is devoted to the numerical approximation of the steady state problem for an isothermal incompressible heat-conducting fluid with implicit non-Newtonian rheology (see [33]). There, it is proven that the sequence of numerical approximations converge to a weak solution to the original problem. Also, a block preconditioner is introduced to efficiently solve the resulting system of nonlinear equations.

The goal of this paper is twofold: theoretical and numerical. Regarding the former, we establish first the existence of a solution of (1.1)-(1.3) under rather general assumptions on the stress-strain relationship in the shear thinning case. Then we obtain some regularity results. In particular, we extend the regularity results of [45] for the 2D Stokes problem, and the ones of [14] for the 3D Navier-Stokes problem, to the non-isothermal case. Concerning the numerical approximation of (1.1)-(1.3), we consider the Stokes approximation and we choose the Carreau law as constitutive stress-strain law. This choice is motivated by relevant applications in the context of mixing and extrusion processes of polymeric materials. In this kind of problems, the inertia terms in the Navier-Stokes equations can often be neglected, due to the high effective viscosity leading to Reynolds numbers lower than unity (see, e. g., [25, 31, 42]). On account of the existence and regularity results obtained for the continuous problem, we perform an *a priori* error analysis of a FEM approximation scheme of the problem and then we verify the convergence results by means of some numerical experiments.

The outline of the paper is the following. In Section 2 we introduce the main assumptions and the functional framework. Section 3 contains the notion of weak solution together with the main analytical results of the paper. Then, in Section 4, we study an approximating problem whose results are exploited in the proof of some analytical results and in the numerical analysis part. Section 5 is devoted to the proofs of the theoretical results. The numerical aspects can be found in Section 6, namely, the analysis of the discrete problem along with *a priori* estimates for the error as well as some numerical experiments.

2 Notation and functional setting

Our basic assumptions are the following:

(H1) $\nu \in W^{1,\infty}(\mathbb{R})$ and there exists $\nu_1 > 0$ such that

$$\nu_1 \leq \nu(s), \qquad \forall s \in \mathbb{R};$$

(H2) $\tau : \Omega \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is a Carathéodory function;

(H3) there exist $\tau_1, \tau_2 > 0$ such that, for all $\mathbf{B} \in \mathbb{R}^{d\times d}_{sym}$ and almost any $x \in \Omega$,

$$\tau(x, \mathbf{B}) \cdot \mathbf{B} \geq \tau_1(|\mathbf{B}|^p - 1),$$
$$|\tau(x, \mathbf{B})| \leq \tau_2(1 + |\mathbf{B}|)^{p-1},$$

where $p \in (1, 2)$ (shear thinning case);

(H4) for all $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{R}^{d\times d}_{sym}$ such that $\mathbf{B}_1 \neq \mathbf{B}_2$ and almost any $x \in \Omega$, we have (strict monotonicity)

$$(\tau(x, \mathbf{B}_1) - \tau(x, \mathbf{B}_2)) \cdot (\mathbf{B}_1 - \mathbf{B}_2) > 0.$$

Here $\cdot$ and $|\cdot|$ denote the scalar product and the Euclidean norm in $\mathbb{R}^{d\times d}_{sym}$ (or in $\mathbb{R}^d$), respectively.

We then set $H := L^2(\Omega)$, $V := H^1(\Omega)$, $V_0 := H^1_0(\Omega)$, and

$$\mathcal{V} := \{\mathbf{v} \in \mathbf{C}^\infty_c(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\},$$
$$\mathbf{L}^r_{div} := \overline{\mathcal{V}}^{\mathbf{L}^r(\Omega)-norm},$$
$$\mathbf{V}^r_{div} := \overline{\mathcal{V}}^{\mathbf{L}^r(\Omega)-gradient\, norm},$$

where $r \in [1, \infty)$.

We denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the norm and the scalar product on both H or $\mathbf{H}$. In particular, we set $\|v\|_{V_0} := \|\nabla v\|$. If X is a (real) Banach space then X' will denote its dual and $\langle \cdot, \cdot \rangle$ will stand for the duality pairing between X' and

X. Moreover, we consider the Stokes operator $\mathbf{A} = -P\Delta$, where P is the Leray orthogonal projector onto $\mathbf{L}^2_{div}$.

Concerning the data, the basic hypotheses are

(H5) $\mathbf{f} \in \mathbf{W}^{-1,p'}(\Omega)$;

(H6) $g \in V'$;

(H7) $\theta \in H^{1/2}(\partial\Omega)$.

Here p' denotes the conjugate index of p.

Denoting by $\Theta_0 \in V$ the Dirichlet lift of θ, namely the (weak) solution to the Dirichlet problem

$$
\begin{aligned}
-\kappa\Delta\Theta_0 = 0 \qquad &\text{in } \Omega\\
\Theta_0 = \theta \qquad &\text{on } \partial\Omega
\end{aligned}
$$

and setting $\vartheta = \Theta - \Theta_0$, we can rewrite (1.1)-(1.4) as follows

$$-\operatorname{div}(\nu(\vartheta + \Theta_0)\tau(\varepsilon(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\pi = \mathbf{f} \qquad \text{in } \Omega \tag{2.1}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega \tag{2.2}$$

$$-\kappa\Delta\vartheta + \mathbf{u}\cdot\nabla\vartheta = g - \mathbf{u}\cdot\nabla\Theta_0 \qquad \text{in } \Omega \tag{2.3}$$

$$\mathbf{u} = \mathbf{0}, \quad \vartheta = 0 \qquad \text{on } \partial\Omega. \tag{2.4}$$

We now introduce the definition of weak solution, namely,

Definition 2.1. *A pair* $(\mathbf{u}, \vartheta) \in \boldsymbol{V}^p_{div} \times V_0$ *is a weak solution to* (1.1)-(1.4) *if*

$$\int_\Omega \nu(\vartheta + \Theta_0)\tau_{ij}(x, \varepsilon(\mathbf{u}))\varepsilon_{ij}(\mathbf{v})dx - \int_\Omega u_j u_i \partial_{x_j} v_i dx = \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in \boldsymbol{V}^p_{div}, \tag{2.5}$$

$$\int_\Omega \kappa\nabla\vartheta\cdot\nabla\phi + \int_\Omega u_j\vartheta\partial_{x_j}\phi = \langle g, \phi\rangle + \int_\Omega u_j\Theta_0\partial_{x_j}\phi \qquad \forall\, \phi \in V_0 \cap W^{1,d}(\Omega). \tag{2.6}$$

Remark 2.1. *Notice that the definition is well posed, since, when* $d = 3$, $\phi \in V_0 \cap W^{1,3}(\Omega)$ *ensures that the convective term in* (2.6) *is finite, by the embeddings* $\boldsymbol{V}^p_{div} \hookrightarrow \boldsymbol{L}^q(\Omega)$, *for some* $q > 2$, *and* $V_0 \hookrightarrow L^6(\Omega)$. *On the contrary, in the case* $d = 2$ *it is enough to consider* $\phi \in V_0$, *by the embeddings* $\boldsymbol{V}^p_{div} \hookrightarrow \boldsymbol{L}^q(\Omega)$, *for some* $q > 2$, *and* $V_0 \hookrightarrow L^r(\Omega)$ *for any* $r \in [2, \infty)$.

3 Main theoretical results

We present here our main theoretical results. The existence of a weak solution is given by

Theorem 3.1. *Let assumptions (H1)-(H7) hold. If* $p \in (2d/(d+2), 2)$ *then there exists a weak solution in the sense of Definition 2.1.*

Remark 3.1. *The pressure* π *(up to a constant) can be recovered from* (2.5) *by means of a suitable version of De Rham's Theorem (see, e.g., [20, Cor.4.1.1]). More precisely, we can find a unique* $\pi \in L^{p'}(\Omega)$ *with zero mean such that*

$$\int_\Omega \nu(\vartheta + \Theta_0)\tau_{ij}(x, \varepsilon(\mathbf{u}))\varepsilon_{ij}(\mathbf{v})dx - \int_\Omega u_j u_i \partial_{x_j} v_i dx = \langle \mathbf{f}, \mathbf{v}\rangle + \int_\Omega \pi \operatorname{div} \mathbf{v} dx$$

for all $\mathbf{v} \in \boldsymbol{C}_c^\infty(\Omega)^d$.

Remark 3.2. *Notice that Theorem 3.1 is still valid if we neglect the convective term* $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$*, i.e., if we consider the Stokes problem,*

We can then establish some regularity properties of a weak solution provided that the stress-strain relationship has the form

$$\tau(x, \mathbf{B}) = \tau(\mathbf{B}) = (1 + |\mathbf{B}|)^{p-2}\mathbf{B} \tag{3.1}$$

and the data are more regular. Actually, we will also consider the so-called Carreau law (see Section 6)

$$\tau(x, \mathbf{B}) = \tau(\mathbf{B}) = (1 + |\mathbf{B}|^2)^{\frac{p-2}{2}}\mathbf{B}. \tag{3.2}$$

For an extensive analysis of power-law fluids, the reader is referred, for instance, to [52, Example 1.73]. We need to distinguish the original Navier-Stokes problem from the Stokes case, namely, when the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is neglected.

In the case $d = 3$, we suppose

$$\delta = \frac{6p - 9}{3 - p} > 0, \tag{3.3}$$

and assume

$$\mathbf{f} \in \mathbf{L}^{p'}(\Omega), \qquad g \in L^{3+\delta}(\Omega), \qquad \theta \in W^{2-\frac{1}{3+\delta}, 3+\delta}(\partial\Omega) \tag{3.4}$$

or, possibly, the stronger ones

$$g \in W^{1,3+\delta}(\Omega), \qquad \theta \in W^{3-\frac{1}{3+\delta}, 3+\delta}(\partial\Omega). \tag{3.5}$$

Then we can prove the following

Theorem 3.2. *Let $d = 3$ and let assumptions (H1)-(H2),* (3.1), (3.3), *and* (3.4) *hold. If $p \in (p_0, 2)$, with $p_0 = 20/11$, then there exists a weak solution in the sense of Definition 2.1 which enjoys the following additional regularity:*

- $\boldsymbol{u} \in \boldsymbol{W}^{1,\overline{q}}(\Omega) \cap \boldsymbol{W}^{2,l}(\Omega), \qquad \nabla\pi \in \boldsymbol{L}^{l}(\Omega),$
  with $\overline{q} = 4p - 2$ *and* $l = \dfrac{4p-2}{p+1}$,
- $\Theta = \vartheta + \Theta_0 \in W^{2,3+\delta}(\Omega)$.

For the Stokes problem the same regularity holds for any $p \in (3/2, 2)$. Also, in both cases, if extra-assumptions (3.5) *hold then we have*

- $\Theta = \vartheta + \Theta_0 \in W^{3,3+\delta}(\Omega)$.

We also consider the two-dimensional case, since the numerical experiments will be carried on in this setting. We assume

$$\delta = \frac{4(p-1)}{2-p} > 0 \tag{3.6}$$

and

$$\mathbf{f} \in \mathbf{L}^{p'}(\Omega), \qquad g \in L^{2+\delta}(\Omega), \qquad \theta \in W^{2-\frac{1}{2+\delta},2+\delta}(\partial\Omega) \tag{3.7}$$

or, possibly, the stronger ones

$$g \in W^{1,2+\delta}(\Omega), \qquad \theta \in W^{2-\frac{1}{2+\delta},2+\delta}(\partial\Omega). \tag{3.8}$$

Then the following result holds

Theorem 3.3. *Let $d = 2$ and let assumptions (H1)-(H2),* (3.2), (3.6), *and* (3.7) *hold. If $p \in (3/2, 2)$, then there exists a weak solution to the Stokes problem, in the sense of Definition 2.1, which enjoys the following additional regularity:*

- $\boldsymbol{u} \in \boldsymbol{W}^{2,q}(\Omega), \qquad \pi \in W^{1,q}(\Omega),$

for some $q > 2$. 8
We also obtain

- $\Theta = \vartheta + \Theta_0 \in W^{2,2+\delta}(\Omega)$.

Moreover, if assumptions (3.8) *hold then we have*

- $\Theta = \vartheta + \Theta_0 \in W^{3,2+\delta}(\Omega)$.

Remark 3.3. *The lower bound* $3/2$ *for* p *is due to the fact that we have exploited [45, Thm.5.30] which holds for the Navier-Stokes problem. However, we guess that a lower threshold for* p *can be achieved through an ad hoc analysis for the Stokes problem. In particular, as noticed when obtaining [45, (5.5)], if we repeat the proof of regularity without considering the convective term we expect a lower bound for* p *of* $6/5$*, the same as for the interior regularity of [45, Thm 4.26].*

Remark 3.4. *Observe that, by the two-dimensional embedding* $\boldsymbol{W}^{2,q}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\infty}(\Omega)$, $q > 2$*, in the Stokes case with assumption* (3.2)*, we obtain that* $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty}(\Omega)$.

In conclusion, concerning the Stokes problem, i.e., neglecting the term $(\mathbf{u} \cdot \nabla)\mathbf{u}$, we can obtain a conditional uniqueness result for the weak solutions according to Definition 2.1.

Theorem 3.4. *Let assumptions (H1)-(H7) hold. For* $d = 2, 3$*, if we consider* τ *as in* (3.2)*, assuming that, for* $d = 2$, $\boldsymbol{u} \in \boldsymbol{W}^{1,q}(\Omega)$*, with* $q > p$ *and* $p \in (1, 2)$*, whereas, for* $d = 3$, $\boldsymbol{u} \in \boldsymbol{W}^{1,\frac{6p(p-1)}{5p-6}}(\Omega)$ *and* $p \in [3/2, 2)$*, if*

$$\frac{M_3 \nu_1}{2} > \frac{M_4}{\kappa^3} \left(\|g\|_{V'} + \|\Theta_0\|_V \right)^2 \left(1 + \|\boldsymbol{u}\|^{2p-2}_{\boldsymbol{W}^{1,\alpha(p-1)}(\Omega)} \right), \tag{3.9}$$

with $\alpha = \dfrac{q}{p-1}$ *for* $d = 2$*, and* $\alpha = \dfrac{6p}{5p-6}$ *for* $d = 3$*, then the weak solution to the Stokes problem according to Definition 2.1 is unique.* M_3 *and* M_4 *are positive constants, depending on* Ω, τ_1, τ_2*, and* $\|\mathbf{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}$.

In Sections 5.1-5.2 we give the proofs of the above results. In particular, Theorem 3.1 is proven through an approximating problem which is analyzed in next section. This problem will be useful in Section 6 as well. Then, the proof of Theorem 3.4 will be given in Section 5.3.2, together with the proof of some conditional uniqueness results concerning the approximating problem.

4 The approximating problem

Fix $\sigma > 0$ and $r \in [2, \infty)$. Then consider the problem of finding $(\mathbf{u}, \pi, \vartheta)$ such that

$$-\text{div} \left[\nu(\vartheta + \Theta_0) \left(\tau(x, \varepsilon(\mathbf{u})) + \sigma |\varepsilon(\mathbf{u})|^{r-2} \varepsilon(\mathbf{u}) \right) \right] + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{in } \Omega \tag{4.1}$$

$$\text{div}\, \mathbf{u} = 0 \quad \text{in } \Omega \tag{4.2}$$

$$-\kappa \Delta \vartheta + \mathbf{u} \cdot \nabla \vartheta = g - \mathbf{u} \cdot \nabla \Theta_0 \quad \text{in } \Omega \tag{4.3}$$

$$\mathbf{u} = \mathbf{0}, \quad \vartheta = 0 \qquad \text{on } \partial\Omega. \tag{4.4}$$

In this case the definition of weak solution reads

Definition 4.1. *A pair* $(\mathbf{u}, \vartheta) \in \boldsymbol{V}^r_{div} \times V_0$ *is a weak solution if*

$$\int_\Omega \left[\nu(\vartheta + \Theta_0)\left(\tau_{ij}(x, \varepsilon(\mathbf{u})) + \sigma|\varepsilon(\mathbf{u}))|^{r-2}\varepsilon_{ij}(\mathbf{u})\right)\right]\varepsilon_{ij}(\mathbf{v})dx$$

$$- \int_\Omega u_j u_i \partial_{x_j} v_i dx = \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in \boldsymbol{V}^r_{div} \tag{4.5}$$

$$\int_\Omega \kappa \nabla\vartheta \cdot \nabla\phi + \int_\Omega u_j \vartheta \partial_{x_j}\phi = \langle g, \phi\rangle + \int_\Omega u_j \Theta_0 \partial_{x_j}\phi \qquad \forall\, \phi \in V_0. \tag{4.6}$$

We now state some existence and conditional uniqueness results, whose proofs are postponed in Section 5.3. We establish first the existence of a weak solution to the approximating problem.

Theorem 4.1. *Let assumptions (H1)-(H7) hold. Then there exists a weak solution* $(\mathbf{u}_\sigma, \vartheta_\sigma) \in \boldsymbol{V}^r_{div} \times V_0$ *in the sense of Definition 4.1.*

Remark 4.1. *A unique pressure* $\pi \in L^{r'}(\Omega)$ *with zero mean can also be recovered in this case as specified in Remark 3.1.*

Remark 4.2. *In the case* $r = 2$*, we can also consider the case of non-zero divergence, namely*

$$-\text{div}\left[(\nu(\vartheta + \Theta_0)\tau(x, \varepsilon(\mathbf{u}))\right] - \sigma\Delta\mathbf{u} + \nabla\pi = \mathbf{f} \qquad \text{in } \Omega \tag{4.7}$$

$$\text{div } \mathbf{u} = w \qquad \text{in } \Omega \tag{4.8}$$

$$-\kappa\Delta\vartheta + \mathbf{u}\cdot\nabla\vartheta = g - \mathbf{u}\cdot\nabla\Theta_0 \qquad \text{in } \Omega \tag{4.9}$$

$$\mathbf{u} = \mathbf{0}, \quad \vartheta = 0 \qquad \text{on } \partial\Omega. \tag{4.10}$$

Indeed, we can use the method of "subtracting the divergence". More precisely, observe that we can find a solution $\mathbf{v}$ *to the problem (see, e.g., [63, Lemma 2.1.1])*

$$\text{div } \mathbf{v} = w \qquad \text{in } \Omega$$

$$\mathbf{u} = \mathbf{0}, \qquad \text{on } \partial\Omega.$$

provided that $w \in H$ *has zero spatial average. Then, setting* $\mathbf{U} := \mathbf{u} - \mathbf{v}$*, we can rewrite problem* (4.7)-(4.10) *in the following form*

$$-\text{div}\left[(\nu(\vartheta + \Theta_0)\tau(x, \varepsilon(\mathbf{U} + \mathbf{v}))\right] - \sigma\Delta\mathbf{U} + \nabla\pi = \mathbf{f} + \sigma\Delta\mathbf{v} \qquad \text{in } \Omega \tag{4.11}$$

$$\text{div } \mathbf{U} = 0 \qquad \text{in } \Omega \tag{4.12}$$

$$-\kappa\Delta\vartheta + \mathbf{U}\cdot\nabla\vartheta = g - \mathbf{u}\cdot\nabla\Theta_0 - \mathbf{v}\cdot\nabla\vartheta \qquad \text{in } \Omega \tag{4.13}$$

$$\mathbf{u} = \mathbf{0}, \quad \vartheta = 0 \qquad \text{on } \partial\Omega. \tag{4.14}$$

We expect to be able to prove Theorem 4.1 also in this case.

We can also prove a first conditional uniqueness result. This is given by

Theorem 4.2. *Let assumptions (H1)-(H7) hold. Suppose $d = 3$ and $p \in (1, 5/3]$. Suppose, in addition, that $\mathbf{u} \in \boldsymbol{W}^{1,q}(\Omega)$ where $q > d$. Set $N = \|\nabla\mathbf{u}\|_{\boldsymbol{L}^q(\Omega)}$. There exist two positive constants M_1 and M_2, depending on Ω, τ_1, τ_2, and $\|\mathbf{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}$, such that if*

$$\sigma\nu_1 > M_1 \frac{\|\nu'\|_{L^\infty(\mathbb{R})}}{\kappa^{3/2}} \left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)\left(1 + \frac{1}{(\sqrt{\nu_1\sigma})^{p-1}} + \sigma N\right) + \frac{M_2}{\sqrt{\nu_1\sigma}} \tag{4.15}$$

then the weak solution is unique.

If we assume some additional hypotheses, we can prove a more refined conditional uniqueness result. In particular, let us suppose

$$\theta \in H^{3/2}(\partial\Omega), \qquad g \in H. \tag{4.16}$$

We analyze the case with $r = 2$ and $d = 3$, which is the critical one. We have

Theorem 4.3. *Let assumptions (H1)-(H7), together with* (4.16)*, hold. Suppose $p \in (1, 2)$. Then there exist four positive constants M_1, M_2, M_3 and M_4, depending on Ω, τ_1, τ_2, and $\|\mathbf{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}$, such that if*

$$\begin{aligned}
\frac{\sigma\nu_1}{2} &< \frac{M_1}{\kappa^2}\|\nu'\|^2_{L^\infty(\mathbb{R})}\left(1 + \frac{1}{(\sqrt{\nu_1\sigma})^{p-1}} + \frac{\sigma}{\sqrt{\nu_1\sigma}}\right)^2 \\
&\times\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'} + \frac{M_2\|\theta\|_{H^{1/2}(\partial\Omega)}}{\sqrt{\nu_1\sigma}}\right) + \frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right)^2 \\
&+ M_3\left(1 + \frac{1}{\sqrt{\nu_1\sigma}} + \sigma\left(\|\nu\|_{L^\infty(\mathbb{R})}\right.\right. \\
&+\|\nu'\|_{L^\infty(\mathbb{R})}\left(\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'} + \frac{M_2\|\theta\|_{H^{1/2}(\partial\Omega)}}{\sqrt{\nu_1\sigma}}\right) + \frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)}\right.\right. \\
&\left.\left.\left.\left. + \frac{1}{\kappa}\|g\|\right) + \|\theta\|_{H^{3/2}(\partial\Omega)}\right)\right)\right),
\end{aligned} \tag{4.17}$$

then the solution is unique.

Remark 4.3. *Notice that condition* (4.17) *is better than the one of Theorem 4.2, since it does not depend on the solution $\boldsymbol{u}$, but only on the data and on Ω.*

5 Proofs of the main theoretical results

5.1 Proof of Theorem 3.1

Assume for the moment that the Dirichlet lift Θ_0 belongs to $H^2(\Omega)$ (i.e. $\theta \in H^{3/2}(\partial\Omega)$). Then choose $\sigma = \frac{1}{n}$, $n \in \mathbb{N}_0$, in (4.5) and denote by $(\mathbf{u}_n, \vartheta_n)$ a solution to (4.5)-(4.6). Take $\mathbf{v} = \mathbf{u}_n$ in (4.5) and $\phi = \vartheta$ in (4.6). This yields, on account of div $\mathbf{u}_n = 0$, the identities

$$\int_\Omega \nu(\vartheta_n + \Theta_0) \left(\tau_{ij}(x, \varepsilon(\mathbf{u}_n))\varepsilon_{ij}(\mathbf{u}_n) + \frac{1}{n}|\varepsilon(\mathbf{u}_n))|^r \right) dx = \langle \mathbf{f}, \mathbf{u}_n \rangle \tag{5.1}$$

$$\int_\Omega \kappa |\nabla \vartheta_n|^2 = \langle g, \vartheta_n \rangle + \int_\Omega (u_n)_j \Theta_0 \partial_{x_j} \vartheta_n. \tag{5.2}$$

From identity (5.1), recalling (H1)-(H3) and using Young's and Korn's inequalities, we deduce

$$\|\mathbf{u}_n\|^p_{\mathbf{V}^p_{div}} + \frac{C_{19}}{n} \|\mathbf{u}_n\|^r_{\mathbf{V}^r_{div}} \le C_{20} \left(\|\mathbf{f}\|^{p'}_{(\mathbf{V}^p_{div})'} + 1 \right). \tag{5.3}$$

Here C_{19} and C_{20} are positive constants depending at most on Ω, d, ν_1, and τ_1. From identity (5.2) we infer

$$\|\vartheta_n\|^2_{V_0} \le C_{21} \left(\|g\|^2_{V'} + \|\mathbf{u}_n\|^2_{\mathbf{H}_{div}} \|\Theta_0\|^2_{H^2(\Omega)} \right). \tag{5.4}$$

Here we have used Poincaré's and Young's inequalities as well as the embedding $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$. The constant $C_{21} > 0$ depends at most on κ, Ω, and d.

Combining (5.3) and (5.4), we can find a subsequence $(\mathbf{u}_{n_h}, \vartheta_{n_h})$ and a pair $(\mathbf{u}, \vartheta) \in \mathbf{V}^p_{div} \times V_0$ such that

$$\mathbf{u}_{n_h} \rightharpoonup \mathbf{u} \text{ in } \mathbf{V}^p_{div}, \quad \mathbf{u}_{n_h} \to \mathbf{u} \text{ in } \mathbf{L}^2_{div}, \tag{5.5}$$

$$\vartheta_{n_h} \rightharpoonup \vartheta \text{ in } V_0, \quad \vartheta_{n_h} \to \vartheta \text{ in } L^{5+\delta}(\Omega) \tag{5.6}$$

for any given $\delta \in (0, 1)$. Note that this is possible due to the compact embedding $\mathbf{V}^p_{div} \hookrightarrow \mathbf{L}^2(\Omega)$ given by the assumption $p \in (2d/(d+2), 2)$. We also have (cf. (5.3))

$$\frac{1}{n} \|\mathbf{u}_n\|^{r-1}_{\mathbf{V}^r_{div}} \to 0. \tag{5.7}$$

First of all, observe that, on account of (5.5)-(5.6), we can pass to the limit in

$$\int_\Omega \kappa \nabla \vartheta_{n_h} \cdot \nabla \phi + \int_\Omega (u_{n_h})_j \vartheta_{n_h} \partial_{x_j} \phi = \langle g, \phi \rangle + \int_\Omega (u_{n_h})_j \Theta_0 \partial_{x_j} \phi \qquad \forall\, \phi \in C^\infty_c(\Omega). \tag{5.8}$$

This gives

$$\int_\Omega \kappa \nabla \vartheta \cdot \nabla \phi + \int_\Omega u_j \vartheta \partial_{x_j} \phi = \langle g, \phi \rangle + \int_\Omega u_j \Theta_0 \partial_{x_j} \phi \qquad \forall\, \phi \in C^\infty_c(\Omega) \tag{5.9}$$

and it is easy to realize that this variational identity also holds if $\Theta_0 \in V$ and the test functions are taken in $V_0 \cap W^{1,d}(\Omega)$ (see Remark 2.1).

Consider now

$$\int_\Omega \nu(\vartheta_{n_h} + \Theta_0)\tau_{ij}(x, \varepsilon(\mathbf{u}_{n_h}))\varepsilon_{ij}(\mathbf{v})dx + \frac{1}{n_h}\int_\Omega |\varepsilon(\mathbf{u}_{n_h})|^{r-2}\varepsilon_{ij}(\mathbf{u}_{n_h})\varepsilon_{ij}(\mathbf{v})dx$$
$$- \int_\Omega (u_{n_h})_j(u_{n_h})_i\partial_{x_j}v_i dx = \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in \mathcal{V}. \tag{5.10}$$

Recalling (H3) we have

$$\tau(\cdot, \varepsilon(\mathbf{u}_{n_h})) \rightharpoonup \chi \text{ in } \mathbf{L}^{p'}_{div}. \tag{5.11}$$

Moreover, due to (5.7), we get

$$\frac{1}{n_h}\int_\Omega \nu(\vartheta_{n_h} + \Theta_0)|\varepsilon(\mathbf{u}_{n_h}))|^{r-2}\varepsilon_{ij}(\mathbf{u}_{n_h})\varepsilon_{ij}(\mathbf{v})dx \to 0. \tag{5.12}$$

The strong convergence in (5.5) entails

$$\int_\Omega (u_{n_h})_j(u_{n_h})_i\partial_{x_j}v_i dx \to \int_\Omega u_j u_i \partial_{x_j}v_i dx.$$

Observe now that, also recalling the strong convergence in (5.6), we obtain

$$\int_\Omega \left(\nu(\vartheta_{n_h} + \Theta_0)\tau_{ij}(x, \varepsilon(\mathbf{u}_{n_h})) - \nu(\vartheta + \Theta_0)\chi_{ij}\right)\varepsilon_{ij}(\mathbf{v})dx$$
$$= \int_\Omega \left(\nu(\vartheta_{n_h} + \Theta_0) - \nu(\vartheta + \Theta_0)\right)\tau_{ij}(x, \varepsilon(\mathbf{u}_{n_h}))\varepsilon_{ij}(\mathbf{v})dx$$
$$+ \int_\Omega \nu(\vartheta + \Theta_0)\left(\tau_{ij}(x, \varepsilon(\mathbf{u}_{n_h})) - \chi_{ij}\right)\varepsilon_{ij}(\mathbf{v})dx \to 0 \qquad \forall\, \mathbf{v} \in \mathcal{V}. \tag{5.13}$$

Therefore, we get

$$\int_\Omega \nu(\vartheta + \Theta_0)\chi_{ij}\varepsilon_{ij}(\mathbf{v})dx - \int_\Omega u_j u_i \partial_{x_j} v_i dx = \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in \mathcal{V} \tag{5.14}$$

which also holds, by density, for all $\mathbf{v} \in \mathbf{V}^p_{div}$.

It remains to show that $\chi(x) = \tau(x, \varepsilon(\mathbf{u}(x)))$ for almost any $x \in \Omega$. Let us set, for all $\mathbf{v} \in \mathcal{V}$,

$$\Xi_{n_h} := \int_\Omega \nu(\vartheta_{n_h} + \Theta_0)\left(\tau_{ij}(x, \varepsilon(\mathbf{u}_{n_h})) - \tau_{ij}(x, \varepsilon(\mathbf{v}))\right)\left(\varepsilon_{ij}(\mathbf{u}_{n_h}) - \varepsilon_{ij}(\mathbf{v})\right)dx. \tag{5.15}$$

Note that $\Xi_{n_h} \geq 0$ for all $h \in \mathbb{N}_0$, because of (H1) and (H4). In addition, observe that (cf. (5.1))

$$\Xi_{n_h} = -\frac{1}{n_h}\int_\Omega \nu(\vartheta_{n_h} + \Theta_0)|\varepsilon(\mathbf{u}_{n_h})|^r dx + \langle \mathbf{f}, \mathbf{u}_{n_h}\rangle$$

$$- \int_\Omega \nu(\vartheta_{n_h} + \Theta_0)\tau_{ij}(x, \varepsilon(\mathbf{u}_{n_h}))\varepsilon_{ij}(\mathbf{v})$$

$$- \int_\Omega \nu(\vartheta_{n_h} + \Theta_0)\tau_{ij}(x, \varepsilon(\mathbf{v}))(\varepsilon_{ij}(\mathbf{u}_{n_h}) - \varepsilon_{ij}(\mathbf{v}))dx. \tag{5.16}$$

Taking the above convergences into account (cf., in particular, (5.11)-(5.13)) and using (5.14), we deduce

$$\lim_{h\to\infty} \Xi_{n_h} = \int_\Omega \nu(\vartheta + \Theta_0)(\chi_{ij} - \tau_{ij}(x, \varepsilon(\mathbf{v}))(\varepsilon_{ij}(\mathbf{u}) - \varepsilon_{ij}(\mathbf{v}))dx. \tag{5.17}$$

Hence, using density, we have

$$\int_\Omega \nu(\vartheta + \Theta_0)(\tau_{ij}(x, \varepsilon(\mathbf{v})) - \chi_{ij})(\varepsilon_{ij}(\mathbf{v}) - \varepsilon_{ij}(\mathbf{u}))dx \geq 0 \qquad \forall\, \mathbf{v} \in \mathbf{V}^p_{div}. \tag{5.18}$$

Then Minty's trick (see, e.g. [49, Chap.2.2.1]) entails that

$$\chi = \tau(\cdot, \varepsilon(\mathbf{u})) \qquad \text{a.e. in } \Omega.$$

The proof is finished.

Remark 5.1. *In order to apply the Minty's trick, we recall that, for any* $\mathbf{u} \in \boldsymbol{V}^p_{div}$*, recalling (H2), we can define* $A(\mathbf{u}) \in (\boldsymbol{V}^p_{div})'$ *by setting*

$$\langle A(\mathbf{u}), \mathbf{v}\rangle := \int_\Omega \nu(\vartheta + \Theta_0)\tau_{ij}(x, \varepsilon(\mathbf{u}))\varepsilon_{ij}(\mathbf{v})dx \qquad \forall\, \mathbf{v} \in \boldsymbol{V}^p_{div}.$$

We thus have a nonlinear strictly monotone operator $A : \boldsymbol{V}^p_{div} \to (\boldsymbol{V}^p_{div})'$ *which is hemicontinuous (cf. (H2)-(H4)). Hence* $A(\mathbf{u}) = F$ *for a given* $F \in (\boldsymbol{V}^p_{div})'$ *if and only if* $\langle A(\mathbf{u}) - F, \mathbf{v} - \mathbf{u}\rangle \geq 0$*, that is,* (5.18)*. Indeed, take* $\mathbf{v} = \mathbf{u} - \lambda\mathbf{w}$ *for some* $\lambda > 0$*, Then* (5.18) *gives*

$$\lambda \int_\Omega \nu(\vartheta + \Theta_0)(\tau_{ij}(x, \varepsilon(\mathbf{u} - \lambda\mathbf{w})) - \chi_{ij})\varepsilon_{ij}(\mathbf{w})dx \geq 0,$$

that is,

$$\int_\Omega \nu(\vartheta + \Theta_0)(\tau_{ij}(x, \varepsilon(\mathbf{u} - \lambda\mathbf{w})) - \chi_{ij})\varepsilon_{ij}(\mathbf{w})dx \geq 0.$$

We can now let λ *go to* 0*. Using dominated convergence theorem, we find*

$$\int_\Omega \nu(\vartheta + \Theta_0)(\tau_{ij}(x, \varepsilon(\mathbf{u})) - \chi_{ij})\varepsilon_{ij}(\mathbf{w})dx \geq 0 \qquad \forall\, \mathbf{w} \in \boldsymbol{V}^p_{div}.$$

Thus we deduce

$$\int_\Omega \nu(\vartheta + \Theta_0)(\tau_{ij}(x, \varepsilon(\mathbf{u})) - \chi_{ij})\varepsilon_{ij}(\mathbf{w})dx = 0 \qquad \forall\, \mathbf{w} \in \boldsymbol{V}^p_{div}$$

which means $\langle A(\mathbf{u}) - F, \mathbf{v}\rangle = 0$*, where*

$$\langle F, \mathbf{v}\rangle = \int_\Omega \nu(\vartheta + \Theta_0)\chi_{ij}\varepsilon_{ij}(\mathbf{v})dx \qquad \forall\, \mathbf{v} \in \boldsymbol{V}^p_{div}.$$

5.2 Proof of Theorems 3.2 and 3.3

5.2.1 Preliminary results

We first introduce some technical tools necessary to carry out the proof.

Regularity for the Stokes system. We consider the $d-$dimensional system (in which we consider the tensor τ expressed in (3.1) or (3.2)):

$$\begin{cases} -\mathrm{div}(\nu(\phi)\tau(D\mathbf{u})) + \nabla\pi = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \end{cases} \tag{5.19}$$

with no-slip boundary conditions, for $1 < p < 2$, where ϕ is a generic function and $\mathbf{f} \in \mathbf{L}^{p'}(\Omega)$. For the case with constant viscosity ν and τ as in (3.1), when $p \in (3/2, 2)$ and $d = 3$, there exists a unique (considering zero mean for the pressure) couple $(\mathbf{u}, \pi)$ with $\mathbf{u} \in \mathbf{W}^{1,\overline{q}}(\Omega) \cap \mathbf{W}^{2,l}(\Omega)$ and $\nabla\pi \in \mathbf{L}^{l}(\Omega)$ and it holds (see, e.g. [15])

$$\|\mathbf{u}\|_{\mathbf{W}^{1,\overline{q}}(\Omega)} \le C(1 + \|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)}^{\frac{3}{2p-1}}), \tag{5.20}$$

$$\|\mathbf{u}\|_{\mathbf{W}^{2,l}(\Omega)} \le C(\|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)}^{\frac{5-p}{2p-1}}), \tag{5.21}$$

where $\overline{q} = 4p - 2$ and $l = \dfrac{4p-2}{p+1}$. Note that the constants C in these two estimates depend on the parameters of the problem and only on $\|\nabla\mathbf{u}\|_{\mathbf{L}^p(\Omega)}$ and $\|\pi\|_{L^{p'}(\Omega)}$, which in this case are *a priori* proven to be bounded and depending only on $\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}$. In the case $d = 2$ and $p \in (3/2, 2)$ we have instead the following result, which is valid in the case of Navier-Stokes problem, thus *a fortiori* in this case (see [45, Thm.5.30]). Indeed, one could repeat the same proof of [45, Thm.5.30] without considering the convective term. As already noticed in Remark 3.3, we actually expect this regularity result to hold at least also in $p \in \left(\frac{6}{5}, 2\right)$. We have that, considering τ as in (3.2), if $\mathbf{f} \in \mathbf{L}^{p'}(\Omega)$, given $(\mathbf{u}, \pi)$ a weak solution to the problem, then

$$\mathbf{u} \in \mathbf{W}^{2,q}(\Omega), \quad \pi \in W^{1,q}(\Omega), \tag{5.22}$$

for some $q > 2$. Notice that this is possible since in the Stokes problem uniqueness comes directly from [49, Ch.2, Thm. 2.2], exploiting a lower bound similar to the one in Lemma 6.1 (see, e.g., [52, Lemma 1.19]) to comply with the hypotheses, whereas in the convective case this is not ensured and we only have the existence of a strong solution, not implying that any weak solution is actually strong.

Following the ideas of [1, Lemma 4], we consider the case with variable viscosity, showing that we can reduce the problem to the weak formulation of (5.19), with $\mathbf{f}$ substituted by a suitable $\mathbf{F} \in \mathbf{L}^{p'}(\Omega)$. We state the following

Theorem 5.1. *Assume that* $\nu \in W^{1,\infty}(\mathbb{R})$ *and let* $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ *be a bounded domain of class* C^2, *such that* $0 < \nu_* \leq \nu(\cdot) \leq \nu^*$ *in* $\mathbb{R}$, $\phi \in W^{1,\infty}(\Omega)$ *and* $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$. *Consider the (unique) weak solution to* (5.19). *If* $p \in (3/2, 2)$ *we have, for* $d = 3$ *and assuming* (3.1),

$$\boldsymbol{u} \in \boldsymbol{W}^{1,\overline{q}}(\Omega) \cap \boldsymbol{W}^{2,l}(\Omega), \qquad \nabla\pi \in \boldsymbol{L}^l(\Omega),$$

with $\overline{q} = 4p - 2$ *and* $l = \dfrac{4p-2}{p+1}$, *whereas, for* $d = 2$ *and assuming* (3.2),

$$\boldsymbol{u} \in \boldsymbol{W}^{2,q}(\Omega), \qquad \pi \in W^{1,q}(\Omega),$$

for some $q > 2$.

Proof. We consider $d = 2, 3$ and (3.1). In order to begin the proof, we first observe that, appealing to the theory of monotone operators we know that there exists a unique weak solution to (5.19) (see, e.g., [49, Ch.2, Thm.2.1-2.2]). Moreover, we preliminarily note that, by basic techniques, we get

$$\int_\Omega \nu(\phi)(1 + |D\mathbf{u}|)^{p-2}|D\mathbf{u}|^2 dx \geq \nu_* 2^{p-2} \left(\int_\Omega |D\mathbf{u}|^p dx - |\Omega| \right), \tag{5.23}$$

thus, testing the weak formulation against $\mathbf{u}$,

$$\|D\mathbf{u}\|^p_{\mathbf{L}^p(\Omega)} \leq \frac{2^{2-p}}{\nu_*}(\mathbf{f}, \mathbf{u}) + |\Omega|,$$

hence,

$$\|\nabla\mathbf{u}\|^{p-1}_{\mathbf{L}^p(\Omega)} \leq C(\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)} + 1). \tag{5.24}$$

Note that C is independent of ϕ and depends on ν_*. Then, applying a well known Lemma (see, e.g., [15, Lemma 3.1]), since we have, in distribution,

$$\nabla\pi = \mathrm{div}(\nu(\phi)(1 + |D\mathbf{u}|)^{p-2} D\mathbf{u}) + \mathbf{f},$$

we deduce, by (5.24),

$$\|\pi\|_{L^{p'}(\Omega)} \leq C(1 + \|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)} + \|\mathbf{f}\|^{\frac{1}{p-1}}_{\mathbf{W}^{-1,p'}(\Omega)}), \tag{5.25}$$

were C depends also on ν^* and ν_*. For convenience we can fix π by assuming that its mean value in Ω vanishes.

Therefore, we have obtained the necessary bounds on the two norms involved in the higher-order estimates. We need now to find a weak formulation appealing to (5.19) with constant viscosity. We denote by B the Bogovskii operator. We know that $B : L^q_{(0)}(\Omega) \to W^{1,q}_0(\Omega)$ (the (0) stands for zero integral mean of the functions, whereas $W^{1,q}_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in $W^{1,q}(\Omega)$), $1 < q < \infty$, such that $\mathrm{div} Bf = f$. Moreover, we have, for all $q \in (1, \infty)$,

$$\|Bf\|_{W^{1,q}(\Omega)} \leq C\|f\|_{L^q(\Omega)} \tag{5.26}$$

and, if $f = \mathrm{div}\mathbf{g}$, where $\mathbf{g} \in \mathbf{L}^q(\Omega)$ is such that $\mathrm{div}\mathbf{g} \in L^q(\Omega)$ and $\mathbf{g} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we have

$$\|Bf\|_{L^q(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^q(\Omega)}. \tag{5.27}$$

In order to exploit these properties, we consider $\mathbf{v} \in \mathcal{V}$ and define $\mathbf{w} = \dfrac{\mathbf{v}}{\nu(\phi)} - B\left[\mathrm{div}\left(\dfrac{\mathbf{v}}{\nu(\phi)}\right)\right]$, which clearly gives $\mathbf{w} \in \mathbf{V}^p_{div}$. Taking then $\mathbf{w}$ in the weak formulation we obtain

$$\begin{aligned}\int_\Omega (1 + |\nabla\mathbf{u}|)^{p-2} D\mathbf{u} \cdot D\mathbf{v} dx = &\left(\mathbf{f}, \frac{\mathbf{v}}{\nu(\phi)} - B\left[\mathrm{div}\left(\frac{\mathbf{v}}{\nu(\phi)}\right)\right]\right) \\ &- \left(\nu(\phi)(1 + |D\mathbf{u}|)^{p-2} D\mathbf{u}, \mathbf{v} \otimes \nabla\left(\frac{1}{\nu(\phi)}\right)\right) \\ &+ \left(\nu(\phi)(1 + |D\mathbf{u}|)^{p-2} D\mathbf{u}, \nabla B\left[\mathrm{div}\left(\frac{\mathbf{v}}{\nu(\phi)}\right)\right]\right).\end{aligned} \tag{5.28}$$

Now by means of the assumptions on ν and (5.27) we immediately get

$$\begin{aligned}&\left|\left(\mathbf{f}, \frac{\mathbf{v}}{\nu(\phi)} - B\left[\mathrm{div}\left(\frac{\mathbf{v}}{\nu(\phi)}\right)\right]\right)\right| \leq \|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)} \left(\frac{1}{\nu_*}\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} + C\left\|\frac{\mathbf{v}}{\nu(\phi)}\right\|_{\mathbf{L}^p(\Omega)}\right) \\ &\leq C\|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)}.\end{aligned}$$

Then we have, by Holder's inequality,

$$\begin{aligned}&\left|\left(\nu(\phi)(1 + |D\mathbf{u}|)^{p-2} D\mathbf{u}, \mathbf{v} \otimes \nabla\left(\frac{1}{\nu(\phi)}\right)\right)\right| \\ &= \left|\left(\nu(\phi)(1 + |D\mathbf{u}|)^{p-2} D\mathbf{u}, \mathbf{v} \otimes \left(\frac{\nu'(\phi)}{\nu^2(\phi)}\nabla\phi\right)\right)\right| \\ &\leq C\int_\Omega (1 + |D\mathbf{u}|)^{p-1}|\mathbf{v}||\nabla\phi| dx \\ &\leq C\|\nabla\phi\|_{\mathbf{L}^{p'}(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} + C\|\nabla\phi\|_{\mathbf{L}^\infty(\Omega)}\|D\mathbf{u}\|^{p-1}_{\mathbf{L}^p(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} \\ &\leq C(1 + \|D\mathbf{u}\|^{p-1}_{\mathbf{L}^p(\Omega)})\|\nabla\phi\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)},\end{aligned}$$

and analogously, appealing to (5.26),

$$
\begin{aligned}
&\left|\left(\nu(\phi)(1+|D\mathbf{u}|)^{p-2}D\mathbf{u}, \nabla B\left[\operatorname{div}\left(\frac{\mathbf{v}}{\nu(\phi)}\right)\right]\right)\right| \\
&\leq C\|1+|D\mathbf{u}|\|_{\mathbf{L}^p(\Omega)}^{p-1}\left\|\nabla B\left[\operatorname{div}\left(\frac{\mathbf{v}}{\nu(\phi)}\right)\right]\right\|_{\mathbf{L}^p(\Omega)} \\
&\leq C\left(1+\|D\mathbf{u}\|_{\mathbf{L}^p(\Omega)}^{p-1}\right)\left\|\operatorname{div}\left(\frac{\mathbf{v}}{\nu(\phi)}\right)\right\|_{\mathbf{L}^p(\Omega)} \\
&= C\left(1+\|D\mathbf{u}\|_{\mathbf{L}^p(\Omega)}^{p-1}\right)\left\|\left(\nabla\frac{1}{\nu(\phi)}\cdot\mathbf{v}\right)\right\|_{\mathbf{L}^p(\Omega)} \\
&= C\left(1+\|D\mathbf{u}\|_{\mathbf{L}^p(\Omega)}^{p-1}\right)\left\|\left(\frac{\nu'(\phi)}{\nu^2(\phi)}\nabla\phi\cdot\mathbf{v}\right)\right\|_{\mathbf{L}^p(\Omega)} \\
&\leq C\left(1+\|D\mathbf{u}\|_{\mathbf{L}^p(\Omega)}^{p-1}\right)\|\nabla\phi\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)}.
\end{aligned}
$$

Therefore, we have found a linear continuous operator $\mathbf{F}$ over $\mathcal{V}$ such that

$$\int_\Omega (1+|D\mathbf{u}|)^{p-2}D\mathbf{u}\cdot\nabla\mathbf{v}dx =< \mathbf{F},\mathbf{v} > \qquad \forall\mathbf{v}\in\mathcal{V},$$

and $\mathbf{F}$ can be uniquely extended by density to a linear continuous operator (not relabeled) over $\mathbf{L}^p(\Omega)$. Thus, again by a density argument, we obtain

$$\int_\Omega (1+|D\mathbf{u}|)^{p-2}D\mathbf{u}\cdot\nabla\mathbf{v}dx =< \mathbf{F},\mathbf{v} > \qquad \forall\mathbf{v}\in\mathbf{V}^p_{div}, \tag{5.29}$$

where, identifying $\mathbf{L}^{p'}(\Omega)$ with the dual of $\mathbf{L}^p(\Omega)$,

$$\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} \leq C\left(\|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)} + \left(1+\|D\mathbf{u}\|_{\mathbf{L}^p(\Omega)}^{p-1}\right)\|\nabla\phi\|_{\mathbf{L}^\infty(\Omega)}\right), \tag{5.30}$$

which means, recalling (5.24),

$$\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} \leq C\left(\|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)} + \|\nabla\phi\|_{\mathbf{L}^\infty(\Omega)}\right). \tag{5.31}$$

If we now notice that (5.29) coincides (substituting $\mathbf{f}$ with $\mathbf{F}$) with the weak formulation of (5.19) with constant viscosity, being $\phi\in W^{1,\infty}(\Omega)$, we can apply the aforementioned regularity results to obtain, for $d=3$, from (5.20)-(5.21),

$$\mathbf{u}\in\mathbf{W}^{1,\overline{q}}(\Omega)\cap\mathbf{W}^{2,l}(\Omega), \qquad \nabla\pi\in\mathbf{L}^l(\Omega),$$

with $\overline{q}=4p-2$ and $l=\dfrac{4p-2}{p+1}$. Note that, due to (5.20), the exponent of the $\mathbf{L}^{p'}$-norm of the forcing term $\mathbf{F}$ appearing in the estimate is $\dfrac{3}{2p-1}$, i.e.,

$$\|\mathbf{u}\|_{\mathbf{W}^{1,\overline{q}}(\Omega)} \leq C(1+\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)}^{\frac{3}{2p-1}}). \tag{5.32}$$

We stress again that this has been possible since we have obtained the *a priori* estimates (5.24)-(5.25) independently of the presence of the variable viscosity: indeed, the constant C in (5.32) depends on $\|\nabla \mathbf{u}\|_{\mathbf{L}^p(\Omega)}$ and $\|\pi\|_{L^{p'}(\Omega)}$. For the case $d = 2$, notice that, up to minor modifications (e.g., the constant in (5.23) becoming $\nu_* 2^{\frac{p-2}{2}}$), the same results (5.29)-(5.31) still hold also for the case of τ as in (3.2). Indeed, observe that $(1 + |D\mathbf{u}|^2)^{\frac{p-2}{2}} \leq C(1 + |D\mathbf{u}|)^{p-2}$. Therefore, the regularity result comes analogously from (5.24) (indeed the weak solution needs to belong to $\mathbf{W}^{1,p}(\Omega)$ independently of ϕ), (5.31) and the regularity result (5.22). The proof is ended. $\square$

Regularity for the Navier-Stokes system. We consider the $d-$dimensional system

$$\begin{cases} -\mathrm{div}(\nu(\phi)(1 + |D\mathbf{u}|)^{p-2} D\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \end{cases} \tag{5.33}$$

in Ω, with no-slip boundary conditions, for $1 < p < 2$, where ϕ is a generic function and $\mathbf{f} \in \mathbf{L}^{p'}(\Omega)$.

Following [14, Sec.6], we can prove the following Theorem for the case $d = 3$:

Theorem 5.2. *Assume that $\nu \in W^{1,\infty}(\mathbb{R})$, such that $0 < \nu_* \leq \nu(\cdot) \leq \nu^*$ in $\mathbb{R}$, $\phi \in W^{1,\infty}(\Omega)$ and $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$. Consider a weak solution to* (5.33). *If $p > p_0 = 20/11$ we have*

$$\boldsymbol{u} \in \boldsymbol{W}^{1,\overline{q}}(\Omega) \cap \boldsymbol{W}^{2,l}(\Omega), \qquad \nabla \pi \in \boldsymbol{L}^l(\Omega),$$

with $\overline{q} = 4p - 2$ and $l = \dfrac{4p-2}{p+1}$.

Proof. We observe that if we consider $\tilde{\mathbf{f}} = \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u}$ we can recast the problem exactly as in (5.19), with $\tilde{\mathbf{f}}$ in place of $\mathbf{f}$. Before doing this, we note that, as in the Stokes case, the crucial point is to show that the *a priori* estimates of $\|\nabla \mathbf{u}\|_p$ and $\|\pi\|_{p'}$ are independent of the extra term $(\mathbf{u} \cdot \nabla)\mathbf{u}$. Indeed, the constant bounding the more regular norms in Theorem 5.1 depends on them. This is quite immediate, as noticed in [15, Appendix A], since $((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u}) = 0$ and thus this term does not appear in the energy estimate leading to (5.24), which can be carried out identically. Concerning the pressure π, we notice that if $p \geq 9/5$, which is already guaranteed, being $p_0 > 9/5$, we have by Sobolev embeddings,

$$\|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{\mathbf{W}^{-1,p'}(\Omega)} \leq C\|\mathbf{u}\|^2_{\mathbf{L}^{2p'}(\Omega)} \leq C\|\nabla \mathbf{u}\|^2_{\mathbf{L}^p(\Omega)},$$

thus, by (5.24) and this result we get

$$\begin{aligned}\|\pi\|_{L^{p'}(\Omega)} &\le C(\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbf{W}^{-1,p'}(\Omega)} + \|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)} + 1)\\ &\le C(\|\mathbf{f}\|^{\frac{2}{p-1}}_{\mathbf{W}^{-1,p'}(\Omega)} + \|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)} + 1),\end{aligned} \tag{5.34}$$

which is enough for our purposes. Thus, we only need to estimate $\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbf{L}^{p'}(\Omega)}$ in order to follow the proof Theorem 5.1. We can repeat word by word the estimates devised in [14, Sec.6] to obtain in the end, in force of the validity of (5.24), that, if $p > p_0$,

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbf{L}^{p'}(\Omega)} \le C\|\nabla\mathbf{u}\|^{\gamma}_{\mathbf{L}^{\overline{q}}(\Omega)},$$

where $\overline{q} = 4p-2$ and $0 \le \gamma < \dfrac{2p-1}{3}$. We thus follow the proof of Theorem 5.1: in particular, from (5.32) we have in $\mathbf{F}$ the extra term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ (indeed, we have $\tilde{\mathbf{f}}$ in place of $\mathbf{f}$), therefore, being $\phi \in W^{1,\infty}(\Omega)$, we deduce

$$\|\mathbf{u}\|_{\mathbf{W}^{1,\overline{q}}(\Omega)} \le C\left(1 + \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|^{\frac{3}{2p-1}}_{\mathbf{L}^{p'}(\Omega)}\right) \le C\left(1 + \|\nabla\mathbf{u}\|^{\frac{3\gamma}{2p-1}}_{\mathbf{L}^{\overline{q}}(\Omega)}\right) \tag{5.35}$$

and, since $\dfrac{3\gamma}{2p-1} < 1$, by Young's inequality we infer that $\|\mathbf{u}\|_{\mathbf{W}^{1,\overline{q}}(\Omega)}$ is bounded. From this result, which bounds $\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbf{L}^{p'}(\Omega)}$, we also get the $\mathbf{W}^{2,l}$-regularity of $\mathbf{u}$ given in the statement of this theorem, exploiting (5.21). The proof is finished. □

Useful estimates in 2D and 3D. We now recall some important estimates used in the sequel. We start with a Sobolev embedding in 3D (see, e.g., [23]): let $p \in (3/2, 2)$, then

$$W^{1,p}(\Omega) \hookrightarrow L^{3+\delta}(\Omega), \tag{5.36}$$

for $\delta = \dfrac{6p-9}{3-p} > 0$. Moreover, we also have

$$W^{2,3+\delta}(\Omega) \hookrightarrow W^{1,\infty}(\Omega),$$

and the following Gagliardo-Nirenberg's inequality follows:

$$\|f\|_{W^{1,\infty}(\Omega)} \le C\|f\|^{\chi_3}_{W^{1,2}(\Omega)}\|f\|^{1-\chi_3}_{W^{2,3+\delta}(\Omega)}, \tag{5.37}$$

with $\chi_3 = \dfrac{2\delta}{9+5\delta} < 1$.

We continue with a similar Sobolev embedding in 2D (see again [23]): let $p \in (1,2)$, then

$$W^{1,p}(\Omega) \hookrightarrow L^{2+\delta}(\Omega), \tag{5.38}$$

for $\delta = \dfrac{4(p-1)}{2-p} > 0$. Moreover, the following Gagliardo-Nirenberg's inequality follows:

$$\|f\|_{W^{1,\infty}(\Omega)} \leq C\|f\|^{\chi_2}_{W^{1,2}(\Omega)}\|f\|^{1-\chi_2}_{W^{2,2+\delta}(\Omega)}, \tag{5.39}$$

with $\chi_2 = \dfrac{\delta}{2(1+\delta)} < 1$. We can thus prove Theorems 3.2 and 3.3.

5.2.2 Proof of Theorems 3.2 and 3.3

The existence of a weak solution is the result of Theorem 3.1. In order to gain the additional regularity we want to apply Theorem 5.2 to this specific weak solution, which can be considered as a weak solution to (5.33), neglecting the equation for Θ (with $\phi = \Theta$). Therefore, the only assumption to be verified is that $\vartheta \in W^{1,\infty}(\Omega)$ and then the proof is finished, since, due to (3.4) and (3.7) for the 3D and 2D case, respectively, $\Theta_0 \in W^{2,d+\delta}(\Omega)$, for δ depending on the dimension d, defined in (3.3) and (3.6). By well-known elliptic regularity results, we have

$$\|\vartheta\|_{W^{2,d+\delta}(\Omega)} \leq \frac{1}{\kappa}\left(\|g\|_{L^{d+\delta}(\Omega)} + \|\mathbf{u}\cdot\nabla\vartheta\|_{L^{d+\delta}(\Omega)} + \|\mathbf{u}\cdot\nabla\Theta_0\|_{L^{d+\delta}(\Omega)}\right). \tag{5.40}$$

We need to observe that, due to (5.36) and (5.37) for $d=3$ and (5.38) and (5.39) for $d=2$, by Young's inequality,

$$\begin{aligned}
&\frac{1}{\kappa}\|\mathbf{u}\cdot\nabla\vartheta\|_{L^{d+\delta}(\Omega)} \leq \frac{1}{\kappa}\|\mathbf{u}\|_{\mathbf{L}^{d+\delta}(\Omega)}\|\nabla\vartheta\|_{\mathbf{L}^{\infty}(\Omega)}\\
&\leq C\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}\|\vartheta\|^{\chi_d}_{W^{1,2}(\Omega)}\|\vartheta\|^{1-\chi_d}_{W^{2,d+\delta}(\Omega)}\\
&\leq C\|\mathbf{u}\|^{1/\chi_d}_{\mathbf{W}^{1,p}(\Omega)}\|\vartheta\|_{W^{1,2}(\Omega)} + \frac{1}{2}\|\vartheta\|_{W^{2,d+\delta}(\Omega)}.
\end{aligned}$$

Note that this is possible since $1-\chi_d < 1$. Moreover, we have, again by (5.36) and the properties of the lift operator,

$$\begin{aligned}
&\frac{1}{\kappa}\|\mathbf{u}\cdot\nabla\Theta_0\|_{\mathbf{L}^{d+\delta}(\Omega)} \leq \frac{1}{\kappa}\|\mathbf{u}\|_{\mathbf{L}^{d+\delta}(\Omega)}\|\nabla\Theta_0\|_{\mathbf{L}^{\infty}(\Omega)}\\
&\leq C\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}\|\Theta_0\|_{W^{2,d+\delta}(\Omega)}\\
&\leq C\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}\|\theta\|_{W^{2-\frac{1}{d+\delta},d+\delta}(\partial\Omega)}.
\end{aligned}$$

Collecting these results and using them in (5.40), we immediately deduce that $\|\vartheta\|_{W^{2,d+\delta}(\Omega)}$ is bounded, meaning that, by the already mentioned Sobolev embedding, $\Theta = \vartheta + \Theta_0 \in W^{2,d+\delta}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. Note that in the case $d = 3$ with (3.1) this is enough to apply Theorem 5.2 and conclude the proof, whereas for $d = 2$, if we consider the Stokes problem and (3.2), neglecting the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$, by Theorem 5.1 the proof is concluded. Note that, in the case of the Stokes problem also for $d = 3$, we can repeat the same proof and conclude in force of Theorem 5.1, which holds also for any $p \in (3/2, 2)$. The first part of the proof is thus concluded.

If we consider the additional stronger hypotheses (3.5) and (3.8), for $d = 3$ and $d = 2$, respectively, we first note that $\Theta_0 \in W^{3,d+\delta}(\Omega)$. In the case $d = 3$ we consider both the Navier-Stokes problem with the restriction $p \in (p_0, 2)$ and the Stokes problem with $p \in (3/2, 2)$ (with (3.1)). In the case $d = 2$ we consider the Stokes problem (with (3.2)). We can now apply a bootstrapping argument. Indeed, again by elliptic regularity we have

$$\|\vartheta\|_{W^{3,d+\delta}(\Omega)} \leq \frac{1}{\kappa} \left(\|g\|_{W^{1,d+\delta}(\Omega)} + \|\mathbf{u} \cdot \nabla\vartheta\|_{W^{1,d+\delta}(\Omega)} + \|\mathbf{u} \cdot \nabla\Theta_0\|_{W^{1,d+\delta}(\Omega)} \right). \tag{5.41}$$

Clearly, the only new terms to be controlled are the ones related to the spatial gradients. We have

$$\|\nabla(\mathbf{u} \cdot \nabla\vartheta)\|_{\mathbf{L}^{d+\delta}(\Omega)} \leq \|\nabla\mathbf{u}\|_{\mathbf{L}^{d+\delta}(\Omega)} \|\vartheta\|_{W^{1,\infty}(\Omega)} + \|\mathbf{u}\|_{\mathbf{L}^{\infty}(\Omega)} \|\vartheta\|_{W^{2,d+\delta}(\Omega)},$$

but, since, for $d = 3$, $3 + \delta < \overline{q}$ for $p \in (3/4, 2)$, and $p_0 > 3/4$ (the same goes in the Stokes case, since we have $p > 3/2$), due to the embeddings (5.36) and $\mathbf{W}^{1,\overline{q}}(\Omega) \hookrightarrow \mathbf{L}^{\infty}(\Omega)$, we infer that this term is bounded thanks to the regularity obtained in the first part of the proof. The same goes for the case $d = 2$, since we have the $\mathbf{W}^{1,\infty}(\Omega)-$control over $\mathbf{u}$ (see Remark 3.4). Similarly, we have

$$\|\nabla(\mathbf{u} \cdot \nabla\Theta_0)\|_{\mathbf{L}^{d+\delta}(\Omega)} \leq \|\nabla\mathbf{u}\|_{\mathbf{L}^{d+\delta}(\Omega)} \|\Theta_0\|_{W^{1,\infty}(\Omega)} + \|\mathbf{u}\|_{\mathbf{L}^{\infty}(\Omega)} \|\Theta_0\|_{W^{2,d+\delta}(\Omega)},$$

where $\|\Theta_0\|_{W^{1,\infty}(\Omega)} \leq C\|\Theta_0\|_{W^{2,d+\delta}(\Omega)} \leq C\|\theta\|_{W^{2-\frac{1}{d+\delta},d+\delta}(\Omega)}$ and thus also this term is bounded, concluding the proof, thanks to assumption (3.5) for $d = 3$ and (3.8) for $d = 2$.

5.3 Proofs of Section 4

5.3.1 Proof of Theorem 4.1

Let $\mathbf{u}^\# \in \mathbf{L}^4_{div}$ be given. Then, recalling that div $\mathbf{u}^\# = 0$, it is not difficult to prove that there is a unique $\vartheta^\# = \vartheta^\#(\mathbf{u}^\#) \in V$ which solves

$$\int_\Omega \kappa \nabla \vartheta^\# \cdot \nabla \phi + \int_\Omega u_j^\# \vartheta^\# \partial_{x_j} \phi = \langle g, \phi \rangle + \int_\Omega u_j^\# \Theta_0 \partial_{x_j} \phi \qquad \forall\, \phi \in V_0. \quad (5.42)$$

Consider now the problem of finding $\mathbf{u}^* \in \mathbf{V}^r_{div}$ such that

$$\begin{aligned} &\int_\Omega \left[\nu(\vartheta^\# + \Theta_0)\left(\tau_{ij}(x, \varepsilon(\mathbf{u}^*)) + \sigma|\varepsilon(\mathbf{u}^*))|^{r-2}\varepsilon_{ij}(\mathbf{u}^*)\right)\right] \varepsilon_{ij}(\mathbf{v})dx \\ &- \int_\Omega u_j^\# u_i^\# \partial_{x_j} v_i dx = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall\, \mathbf{v} \in \mathbf{V}^r_{div}. \end{aligned} \quad (5.43)$$

This problem has a unique solution. Thus we have constructed a nonlinear mapping $\mathcal{F} : \mathbf{L}^4_{div} \to \mathbf{L}^4_{div}$ defined as follows $\mathcal{F}(\mathbf{u}^\#) = \mathbf{u}^*$. It is easy to realize that $\mathcal{F}$ maps bounded sets of $\mathbf{L}^4_{div}$ into bounded sets of $\mathbf{V}^r_{div} \hookrightarrow\hookrightarrow \mathbf{L}^4_{div}$. Just take $\mathbf{v} = \mathbf{u}^*$ in (5.43). We now prove that $\mathcal{F}$ is also continuous so $\mathcal{F}$ is a compact operator. Indeed, let $\{\mathbf{u}_n\}$ be such that $\mathbf{u}_n \to \mathbf{u}^\#$ in $\mathbf{L}^4_{div}$. Then there exists $\{\vartheta_{n_h}\}$ which converges (weakly) in V_0 and strongly in H to $\vartheta^\#$. On the other hand, $\{\mathcal{F}(\mathbf{u}_{n_h})\} \subset \mathbf{L}^4_{div}$ is also bounded in $\mathbf{V}^r_{div}$. Using classical arguments (see, for instance, [49]), we can prove that there exists a subsequence $\{\mathbf{u}_{n_{h_m}}\}$ such that $\{\mathcal{F}(\mathbf{u}_{n_{h_m}})\}$ converges weakly in $\mathbf{V}^r_{div}$ and strongly in $\mathbf{L}^4_{div}$ to a solution $\tilde{\mathbf{u}}$ to (5.43) which coincides, due to uniqueness, with $\mathcal{F}(\mathbf{u}^\#)$. The class limit Λ of $\{\mathcal{F}(\mathbf{u}_n)\}$ is non-empty and bounded in $\mathbf{V}^r_{div}$. Arguing as above, it is easy to show that any $\mathbf{w} \in \Lambda$ is such that $\mathbf{w} = \mathcal{F}(\mathbf{u}^\#)$. Hence $\mathcal{F}$ is continuous from $\mathbf{L}^4_{div}$ to itself.

Observe now that if $\mathbf{w} \in \mathbf{L}^4_{div}$ is such that $\mathbf{w} = \lambda\mathcal{F}(\mathbf{w})$ for some $\lambda \in (0, 1)$ then there exists $C_1 > 0$ such that $\|\mathbf{w}\|_{\mathbf{L}^4_{div}} \le C_1$. Indeed, we have

$$\begin{aligned} &\int_\Omega \left[\nu(\vartheta(\mathbf{w} + \Theta_0)\left(\tau_{ij}(x, \varepsilon(\lambda^{-1}\mathbf{w})) + \sigma|\varepsilon(\lambda^{-1}\mathbf{w}))|^{r-2}\varepsilon_{ij}(\lambda^{-1}\mathbf{w})\right)\right] \varepsilon_{ij}(\mathbf{v})dx \\ &- \int_\Omega w_j w_i \partial_{x_j} v_i dx = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall\, \mathbf{v} \in \mathbf{V}^r_{div}. \end{aligned} \quad (5.44)$$

Then, taking $\mathbf{v} = \mathbf{w}$ and using Young's and Korn's inequalities, we deduce the existence of C_2 depending on d, Ω, σ, and $\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}$ such that

$$\|\mathbf{u}\|_{\mathbf{V}^r_{div}} \le C_2.$$

Thus we can apply Schaefer's fixed point theorem and conclude that $\mathcal{F}$ has a fixed point in $\mathbf{L}^4_{div}$. This is equivalent to say that the approximating problem (4.5)-(4.6) has a solution $(\mathbf{u}_\sigma, \vartheta_\sigma)$.

5.3.2 Proof of Theorems 3.4 and 4.2

We start from the proof of Theorem 4.2 to introduce the setting, since the proof of Theorem 3.4 exploits similar estimates and notations. Suppose $(\mathbf{u}^k, \vartheta^k) \in \mathbf{V}^r_{div} \times V_0$, $k = 1, 2$, satisfy the following system

$$\int_\Omega \left[\nu(\vartheta + \Theta_0) \left(\tau_{ij}(x, \varepsilon(\mathbf{u})) + \sigma|\varepsilon(\mathbf{u}))|^{r-2}\varepsilon_{ij}(\mathbf{u})\right)\right] \varepsilon_{ij}(\mathbf{v}) dx$$
$$- \int_\Omega u_j u_i \partial_{x_j} v_i dx = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall\, \mathbf{v} \in \mathbf{V}^r_{div}, \tag{5.45}$$
$$\int_\Omega \kappa \nabla\vartheta \cdot \nabla\phi + \int_\Omega u_j \vartheta \partial_{x_j}\phi = \langle g, \phi \rangle + \int_\Omega u_j \Theta_0 \partial_{x_j}\phi \qquad \forall\, \phi \in V_0. \tag{5.46}$$

Suppose $r = 2$ and $d = 3$, that is, the worst case. Then set $\tilde{\mathbf{u}} = \mathbf{u}^1 - \mathbf{u}^2$ and $\tilde{\vartheta} = \vartheta^1 - \vartheta^2$ and observe that $(\tilde{\mathbf{u}}, \tilde{\vartheta})$ satisfies

$$\int_\Omega \nu(\vartheta^1 + \Theta_0) \left[(\tau_{ij}(x, \varepsilon(\mathbf{u}^1)) - \tau_{ij}(x, \varepsilon(\mathbf{u}^2))\right] \varepsilon_{ij}(\mathbf{v}) dx$$
$$+ \sigma \int_\Omega \nu(\vartheta^1 + \Theta_0) \nabla\tilde{\mathbf{u}} \cdot \nabla\mathbf{v} dx$$
$$= - \int_\Omega \left[\nu(\vartheta^1 + \Theta_0) - \nu(\vartheta^2 + \Theta_0)\right] (\tau_{ij}(x, \varepsilon(\mathbf{u}^2)) \varepsilon_{ij}(\mathbf{v}) dx$$
$$- \sigma \int_\Omega \left[\nu(\vartheta^1 + \Theta_0) - \nu(\vartheta^2 + \Theta_0)\right] \nabla\mathbf{u}^2 \cdot \nabla\mathbf{v} dx$$
$$+ \int_\Omega (\tilde{u}_j u^1_i + u^2_j \tilde{u}_i) \partial_{x_j} v_i dx \qquad \forall\, \mathbf{v} \in \mathbf{V}^r_{div}, \tag{5.47}$$
$$\int_\Omega \kappa \nabla\tilde{\vartheta} \cdot \nabla\phi + \int_\Omega u^1_j \tilde{\vartheta} \partial_{x_j}\phi = - \int_\Omega \tilde{u}_j \vartheta^2 \partial_{x_j}\phi + \int_\Omega \tilde{u}_j \Theta_0 \partial_{x_j}\phi \qquad \forall\, \phi \in V_0. \tag{5.48}$$

Take now $\mathbf{v} = \tilde{\mathbf{u}}$ in (5.47) and $\phi = \tilde{\vartheta}$ in (5.48). Recalling that $\mathbf{u}_1$ is divergence free, this gives

$$\int_\Omega \nu(\vartheta^1 + \Theta_0) \left[(\tau_{ij}(x, \varepsilon(\mathbf{u}^1)) - \tau_{ij}(x, \varepsilon(\mathbf{u}^2))\right] \varepsilon_{ij}(\tilde{\mathbf{u}}) dx + \sigma \int_\Omega \nu(\vartheta^1 + \Theta_0) |\nabla\tilde{\mathbf{u}}|^2 dx$$
$$= - \int_\Omega \left[\nu(\vartheta^1 + \Theta_0) - \nu(\vartheta^2 + \Theta_0)\right] (\tau_{ij}(x, \varepsilon(\mathbf{u}^2)) \varepsilon_{ij}(\tilde{\mathbf{u}}) dx$$
$$- \sigma \int_\Omega \left[\nu(\vartheta^1 + \Theta_0) - \nu(\vartheta^2 + \Theta_0)\right] \nabla\mathbf{u}^2 \cdot \nabla\tilde{\mathbf{u}} dx$$
$$+ \int_\Omega (\tilde{u}_j u^1_i + u^2_j \tilde{u}_i) \partial_{x_j} \tilde{u}_i dx \qquad \forall\, \mathbf{v} \in \mathbf{V}^r_{div}, \tag{5.49}$$
$$\kappa \int_\Omega |\nabla\tilde{\vartheta}|^2 = - \int_\Omega \tilde{u}_j \vartheta^2 \partial_{x_j}\tilde{\vartheta} + \int_\Omega \tilde{u}_j \Theta_0 \partial_{x_j}\tilde{\vartheta} \qquad \forall\, \phi \in V_0. \tag{5.50}$$

Using (H1) and (H4), we deduce the inequalities

$$\sigma\nu_1 \int_\Omega |\nabla\tilde{\mathbf{u}}|^2 dx \leq I_1, \tag{5.51}$$

$$\kappa \int_\Omega |\nabla\tilde{\vartheta}|^2 \leq I_2, \tag{5.52}$$

where

$$\begin{aligned} I_1 = &- \int_\Omega \left[\nu(\vartheta^1+\Theta_0) - \nu(\vartheta^2+\Theta_0)\right] (\tau_{ij}(x,\varepsilon(\mathbf{u}^2))\varepsilon_{ij}(\tilde{\mathbf{u}})dx \\ &- \sigma \int_\Omega \left[\nu(\vartheta^1+\Theta_0) - \nu(\vartheta^2+\Theta_0)\right] \nabla\mathbf{u}^2 \cdot \nabla\tilde{\mathbf{u}}dx \\ &+ \int_\Omega (\tilde{u}_j u_i^1 + u_j^2 \tilde{u}_i)\partial_{x_j}\tilde{u}_i dx, \end{aligned} \tag{5.53}$$

$$I_2 = -\int_\Omega \tilde{u}_j(\vartheta^2 - \Theta_0)\partial_{x_j}\tilde{\vartheta}. \tag{5.54}$$

Observe now that, recalling (H1), thanks to Hölder's inequality we have

$$\begin{aligned} I_1 \leq\ & \|\nu'\|_{L^\infty(\mathbb{R})}\|\tilde{\vartheta}\|_{L^6(\Omega)} \left(\|\tau(\cdot,\varepsilon(\mathbf{u}^2))\|_{\mathbf{L}^3(\Omega)}\|\varepsilon(\tilde{\mathbf{u}})\|_{\mathbf{L}^2(\Omega)} + \sigma N\|\nabla\tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\right) \\ &+ (\|\mathbf{u}^1\|_{\mathbf{L}^4(\Omega)} + \|\mathbf{u}^2\|_{\mathbf{L}^4(\Omega)})\|\tilde{\mathbf{u}}\|_{\mathbf{L}^4(\Omega)}\|\nabla\tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{5.55}$$

Here we have $N = \|\nabla\mathbf{u}^2\|_{\mathbf{L}^3(\Omega)}$.

On the other hand, owing to (H3), we can find $C_3 = C_3(\tau_2) > 0$ such that

$$\begin{aligned} \|\tau(\cdot,\varepsilon(\mathbf{u}^2))\|_{\mathbf{L}^3(\Omega)} &\leq C(\tau_1)\left(1 + \|\varepsilon(\mathbf{u}^2)\|^{p-1}_{\mathbf{L}^{3p-3}(\Omega)}\right) \\ &\leq C_3(\tau_1)\left(1 + \|\varepsilon(\mathbf{u}^2)\|^{p-1}_{\mathbf{L}^2(\Omega)}\right), \end{aligned} \tag{5.56}$$

for $p \in (1, 5/3]$, but we know that there exists a positive constant C_4 depending on τ_1 and $\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}$ such that

$$\sqrt{\nu_1}\|\varepsilon(\mathbf{u}^k)\|_{\mathbf{L}^p(\Omega)} + \sqrt{\nu_1\sigma}\|\nabla\mathbf{u}^k\|_{\mathbf{L}^2(\Omega)} \leq C_4, \quad k = 1, 2. \tag{5.57}$$

Indeed, we can take $\mathbf{v} = \mathbf{u}^k$ in (5.45) written for $(\mathbf{u}^k, \vartheta^k)$ and using (H1), (H3), (H5), and Young's inequality. Thus, using Poincaré's and Korn's inequalities and the continuous embedding $\mathbf{V}^2_{div} \hookrightarrow \mathbf{L}^4(\Omega)^3$, we get

$$\begin{aligned} I_1 \leq\ & \|\nu'\|_{L^\infty(\mathbb{R})}\|\tilde{\vartheta}\|_{L^6(\Omega)} \\ &\times \left(C_6\left(1 + \frac{1}{(\sqrt{\nu_1\sigma})^{p-1}}\right)\|\varepsilon(\tilde{\mathbf{u}})\|_{\mathbf{L}^2(\Omega)} + \sigma N\|\nabla\tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\right) \\ &+ \frac{C_7}{\sqrt{\nu_1\sigma}}\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned} \tag{5.58}$$

where C_6 and C_7 also depend on the Poincaré constant and on the embedding constant of the embedding quoted above (therefore they depend on Ω).

Regarding I_2, we have

$$I_2 \leq \|\tilde{\mathbf{u}}\|_{\mathbf{L}^3_{div}}(\|\vartheta^2\|_{L^6(\Omega)} + \|\Theta_0\|_{L^6(\Omega)})\|\nabla\tilde{\vartheta}\|. \tag{5.59}$$

Taking $\phi = \vartheta^2$ in (5.46) written for $(\mathbf{u}^2, \vartheta^2)$ we get

$$\kappa\|\nabla\vartheta^2\|^2 \leq \|g\|_{V'}\|\vartheta^2\|_{H^1(\Omega)} + \|\mathbf{u}^2\|_{\mathbf{L}^3_{div}}\|\Theta_0\|_{L^6(\Omega)}\|\nabla\vartheta^2\|. \tag{5.60}$$

Thus, using Poincaré's, we can find $C_8 = C_8(\Omega) > 0$ such that

$$\kappa\|\nabla\vartheta^2\|^2 \leq \left(C_8\|g\|_{V'} + \|\mathbf{u}^2\|_{\mathbf{L}^3_{div}}\|\Theta_0\|_{L^6(\Omega)}\right)\|\nabla\vartheta^2\|$$

and Young's inequality give

$$\frac{\kappa}{2}\|\nabla\vartheta^2\|^2 \leq \frac{1}{2\kappa}\left(C_7\|g\|_{V'} + \|\mathbf{u}^2\|_{\mathbf{L}^3_{div}}\|\Theta_0\|_{L^6(\Omega)}\right)^2.$$

Thus, recalling (5.57), the continuous embedding $\mathbf{V}^2_{div} \hookrightarrow \mathbf{L}^3(\Omega)$, Korn's and Poincaré's inequalities, we can find a constant $C_9 > 0$ depending on Ω, τ_1, τ_2, and $\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}$ such that

$$\|\vartheta^2\|_{V_0} \leq \frac{C_9}{\kappa}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right).$$

Hence, from (5.59) we infer

$$I_2 \leq \frac{C_{10}}{\kappa}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)\|\tilde{\mathbf{u}}\|_{\mathbf{L}^3_{div}}\|\nabla\tilde{\vartheta}\|, \tag{5.61}$$

where $C_{10} > 0$ has the same dependencies as C_9 does. Then, using Young's inequality, from (5.52) we obtain

$$\frac{\kappa}{2}\|\nabla\tilde{\vartheta}\|^2_{H^3} \leq \frac{C_{10}^2}{2\kappa^2}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)^2\|\tilde{\mathbf{u}}\|^2_{\mathbf{L}^3_{div}}. \tag{5.62}$$

Therefore, on account of (5.61), from (5.58) Korn's and Poincaré's inequalities and Sobolev embeddings, we deduce

$$\begin{aligned} I_1 \leq C_{11}\frac{\|\nu'\|_{L^\infty(\mathbb{R})}}{\kappa^{3/2}}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)\left(1 + \frac{1}{(\sqrt{\nu_1\sigma})^{p-1}} + \sigma N\right)\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \\ + \frac{C_7}{\sqrt{\nu_1\sigma}}\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{5.63}$$

Here $C_{11} > 0$ has the same dependencies as the other constants.

Combining (5.51) with (5.63) we get

$$\left[\sigma\nu_1 - C_{11}\frac{\|\nu'\|_{L^\infty(\mathbb{R})}}{\kappa^{3/2}}\left(\|g\|_{V'}+\frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)\left(1+\frac{1}{(\sqrt{\nu_1\sigma})^{p-1}}+\sigma N\right)+\frac{C_7}{\sqrt{\nu_1\sigma}}\right]$$
$$\times\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}\le 0,$$

which eventually yields uniqueness provided that (4.15) holds with $M_1 = C_{10}$ and $M_2 = C_7$.

Let us now consider the proof of Theorem 3.4. This corresponds to the case when τ is defined as in (3.2), $\sigma = 0$ and there is no convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. First notice that the weak formulation for the fluid difference is again (5.47)-(5.48) (without the convective term in (5.47)), since the test functions $\phi \in V_0$ are sufficiently regular also in the case of $d = 3$ (differently from what observed in Remark 2.1). Indeed, this is possible due to the embedding $\mathbf{W}^{1,p}(\Omega)\hookrightarrow\mathbf{L}^3(\Omega)$ for $p \ge 3/2$.

In this situation we cannot exploit (5.51), but we exploit a more refined estimate of the first term in (5.49). We can suppose generically the dimension $d = 2, 3$. In particular, by means of Lemma 6.1, exploiting Hölder's inequality with a negative exponent, we get

$$\int_\Omega \nu(\vartheta^1+\Theta_0)\left[(\tau_{ij}(x,\varepsilon(\mathbf{u}^1)) - \tau_{ij}(x,\varepsilon(\mathbf{u}^2))\right]\varepsilon_{ij}(\tilde{\mathbf{u}})dx$$
$$\ge C\nu_1\left(1+\|\mathbf{u}^1\|_{\mathbf{W}^{1,p}(\Omega)}+\|\mathbf{u}^2\|_{\mathbf{W}^{1,p}(\Omega)}\right)^{p-2}\|\tilde{\mathbf{u}}\|^2_{\mathbf{L}^p(\Omega)}\ge \nu_1 C_{12}\|\tilde{\mathbf{u}}\|^2_{\mathbf{L}^p(\Omega)},$$

thanks to the fact that, by standard inequalities, for $k = 1, 2$,

$$\|\mathbf{u}^k\|_{\mathbf{W}^{1,p}(\Omega)}\le C\|\mathbf{f}\|_{\mathbf{L}^{p'}(\Omega)}. \tag{5.64}$$

We can estimate the remaining terms in I_1 by means of Hölder's inequality, Sobolev embeddings and (H3),

$$\begin{aligned} I_1 &= -\int_\Omega\left[\nu(\vartheta^1+\Theta_0)-\nu(\vartheta^2+\Theta_0)\right]\tau_{ij}(x,\varepsilon(\mathbf{u}^2))\varepsilon_{ij}(\tilde{\mathbf{u}})dx \\ &\le C\|\tau(\cdot,\varepsilon(\mathbf{u}^2))\|_{\mathbf{L}^\alpha(\Omega)}\|\varepsilon(\tilde{\mathbf{u}})\|_{\mathbf{L}^p(\Omega)}\|\vartheta\|_{\mathbf{L}^\beta(\Omega)} \\ &\le \frac{\nu_1 C_{12}}{2}\|\tilde{\mathbf{u}}\|^2_{\mathbf{L}^p(\Omega)^d}+C_{13}\left(1+\|\mathbf{u}^2\|^{2p-2}_{\mathbf{W}^{1,\alpha(p-1)}(\Omega)}\right)\|\nabla\vartheta\|^2, \end{aligned} \tag{5.65}$$

where, for $d = 2$, $\alpha = \dfrac{q}{p-1}$ and $\beta = (1-1/\alpha-1/p)^{-1}$, with q defined in the assumptions, whereas, for $d = 3$, we set $\alpha = \dfrac{6p}{5p-6}$ and $\beta = 6$. Concerning the temperature, we have

$$I_2 \le \|\tilde{\mathbf{u}}\|_{\mathbf{L}^\gamma_{div}}(\|\vartheta^2\|_{L^t(\Omega)}+\|\Theta_0\|_{L^t(\Omega)})\|\nabla\tilde{\vartheta}\|, \tag{5.66}$$

where $\gamma = \dfrac{2p}{2-p}$, $t = \dfrac{2\gamma}{\gamma - 2}$ for $d = 2$ and $\gamma = 3$, $t = 6$ for $d = 3$. Moreover, taking again $\phi = \vartheta^2$ in (5.46) written for $(\mathbf{u}^2, \vartheta^2)$ we get

$$\kappa \|\nabla \vartheta^2\|^2 \leq \|g\|_{V'} \|\vartheta^2\|_{H^1(\Omega)} + \|\mathbf{u}^2\|_{\mathbf{L}^s_{div}} \|\Theta_0\|_{L^t(\Omega)} \|\nabla \vartheta^2\|, \tag{5.67}$$

for the same exponents γ, t. Indeed, with this choice of γ we have, for $d = 2$, the embedding $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^\gamma(\Omega)$ for any $p \in (1,2)$ and thus also $\|\mathbf{u}^2\|_{\mathbf{L}^\gamma(\Omega)} \leq C$ as it is needed in this estimate. Notice that also for $d = 3$, when $p \geq 3/2$ we get $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$ and thus $\|\mathbf{u}^2\|_{\mathbf{L}^3(\Omega)} \leq C$. Exploiting then the embedding $H^1(\Omega) \hookrightarrow L^t(\Omega)$ for any $t \in [2, \infty)$ if $d = 2$, for $t = 6$ if $d = 3$, we get, similarly to (5.62),

$$\frac{\kappa}{2} \|\nabla \tilde{\vartheta}\|^2 \leq \frac{C_{10}^2}{2\kappa^2} \left(\|g\|_{V'} + \|\Theta_0\|_V\right)^2 \|\tilde{\mathbf{u}}\|^2_{\mathbf{L}^\gamma_{div}}, \tag{5.68}$$

Therefore, for $p \in (1,2)$ in the case $d = 2$ and for $p \in [3/2, 2)$ when $d = 3$, we get

$$\frac{\kappa}{2} \|\nabla \tilde{\vartheta}\|^2 \leq \frac{C_{14}}{\kappa^2} \left(\|g\|_{V'} + \|\Theta_0\|_V\right)^2 \|\tilde{\mathbf{u}}\|^2_{\mathbf{W}^{1,p}(\Omega)}. \tag{5.69}$$

Putting together these results we obtain in the end

$$\|\tilde{\mathbf{u}}\|^2_{\mathbf{W}^{1,p}(\Omega)} \left(\frac{\nu_1 C_{12}}{2} - \frac{C_{13} C_{14}}{\kappa^3} \left(\|g\|_{V'} + \|\Theta_0\|_V\right)^2 \left(1 + \|\mathbf{u}^2\|^{2p-2}_{\mathbf{W}^{1,\alpha(p-1)}(\Omega)}\right) \right) \leq 0,$$

from which we deduce uniqueness provided that (3.9) holds with $M_3 = C_{12}$ and $M_4 = C_{13} C_{14}$. The proof is concluded.

5.3.3 Proof of Theorem 4.3

First of all we prove the following

Lemma 5.1. *Let the assumptions of Theorem 4.2 hold, together with* (4.16). *Then a solution ϑ_σ given by Theorem 4.1, for any $\sigma > 0$, belongs to $L^\infty(\Omega)$.*

Proof. We recall that by Theorem 4.1 we have $\mathbf{u}_\sigma \in \mathbf{V}^2_{div}$ and $\vartheta_\sigma \in V_0$. We concentrate on (4.3). By a classical elliptic regularity result, we get

$$\|\vartheta_\sigma\|_{H^2(\Omega)} \leq \frac{1}{\kappa} \left(\|g\| + \|\mathbf{u}_\sigma \cdot \nabla \vartheta_\sigma\| + \|\mathbf{u}_\sigma \cdot \nabla \Theta_0\|\right),$$

but, by Hölder and Sobolev-Gagliardo-Nirenberg's inequalities, using also Sobolev embeddings, we have

$$\|\mathbf{u}_\sigma \cdot \nabla \vartheta_\sigma\| \leq \|\mathbf{u}_\sigma\|_{\mathbf{L}^6(\Omega)} \|\nabla \vartheta_\sigma\|_{\mathbf{L}^3(\Omega)} \leq C \|\nabla \mathbf{u}_\sigma\| \|\nabla \vartheta_\sigma\|^{1/2} \|\vartheta_\sigma\|^{1/2}_{H^2(\Omega)}.$$

Analogously, recalling the properties of the lift operator, i.e., $\|\Theta_0\|_{H^2(\Omega)} \leq C\|\theta\|_{H^{3/2}(\partial\Omega)}$, we find

$$\|\mathbf{u}_\sigma \cdot \nabla\Theta_0\| \leq \|\mathbf{u}_\sigma\|_{\mathbf{L}^6(\Omega)}\|\nabla\Theta_0\|_{\mathbf{L}^3(\Omega)}$$
$$\leq C\|\nabla\mathbf{u}_\sigma\|\|\Theta_0\|_{H^2(\Omega)} \leq C\|\nabla\mathbf{u}_\sigma\|\|\theta\|_{H^{3/2}(\partial\Omega)}.$$

Therefore, by Young's inequality, we immediately deduce

$$\|\vartheta_\sigma\|_{H^2(\Omega)} \leq \frac{C}{\kappa^2}\|\nabla\mathbf{u}_\sigma\|^2\|\nabla\vartheta_\sigma\| + \frac{1}{2}\|\vartheta_\sigma\|_{H^2(\Omega)} + \frac{C}{\kappa}\|\nabla\mathbf{u}_\sigma\|\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|,$$

i.e.,

$$\|\vartheta_\sigma\|_{H^2(\Omega)} \leq \frac{C}{\kappa^2}\|\nabla\mathbf{u}_\sigma\|^2\|\nabla\vartheta_\sigma\| + \frac{C}{\kappa}\|\nabla\mathbf{u}_\sigma\|\|\theta\|_{H^{3/2}(\partial\Omega)}, \tag{5.70}$$

meaning that $\vartheta_\sigma \in H^2(\Omega)$ (and thus also $\vartheta_\sigma + \Theta_0 \in H^2(\Omega)$). In particular, this implies that ϑ_σ is bounded in $L^\infty(\Omega)$. □

We now prove the Theorem. Let us consider the same setting as in the proof of Theorem 4.2 (with $r = 2$ and $d = 3$) and take $\mathbf{v} = \mathbf{A}^{-1}\tilde{\mathbf{u}}$ in (5.47), and $\phi = \tilde{\vartheta}$ in (5.48). Recalling that $\mathbf{u}_1$ is divergence free, this gives

$$\begin{aligned}
&\int_\Omega \nu(\vartheta^1 + \Theta_0)\left[(\tau_{ij}(x, \varepsilon(\mathbf{u}^1)) - \tau_{ij}(x, \varepsilon(\mathbf{u}^2))\right]\varepsilon_{ij}(\mathbf{A}^{-1}\tilde{\mathbf{u}})dx \\
&+ \sigma\int_\Omega \nu(\vartheta^1 + \Theta_0)\varepsilon_{ij}(\tilde{\mathbf{u}})\varepsilon_{ij}(\mathbf{A}^{-1}\tilde{\mathbf{u}})dx \\
&= -\int_\Omega \left[\nu(\vartheta^1 + \Theta_0) - \nu(\vartheta^2 + \Theta_0)\right](\tau_{ij}(x, \varepsilon(\mathbf{u}^2))\varepsilon_{ij}(\mathbf{A}^{-1}\tilde{\mathbf{u}})dx \\
&- \sigma\int_\Omega \left[\nu(\vartheta^1 + \Theta_0) - \nu(\vartheta^2 + \Theta_0)\right]\varepsilon_{ij}(\mathbf{u}^2)\varepsilon_{ij}(\mathbf{A}^{-1}\tilde{\mathbf{u}})dx \\
&+ \int_\Omega (\tilde{u}_j u_i^1 + u_j^2\tilde{u}_i)\partial_{x_j}(\mathbf{A}^{-1}\tilde{\mathbf{u}})_i dx \qquad \forall\, \mathbf{v} \in \mathbf{V}_{div}^r,
\end{aligned} \tag{5.71}$$

$$\kappa\int_\Omega |\nabla\tilde{\vartheta}|^2 = -\int_\Omega \tilde{u}_j\vartheta^2\partial_{x_j}\tilde{\vartheta} + \int_\Omega \tilde{u}_j\Theta_0\partial_{x_j}\tilde{\vartheta} \qquad \forall\, \phi \in V_0. \tag{5.72}$$

From now on, for the sake of simplicity, we will use the notation $D\mathbf{u}$ to indicate the symmetrized strain tensor. We observe that

$$\sigma\int_\Omega \nu(\vartheta^1 + \Theta_0)\varepsilon_{ij}(\tilde{\mathbf{u}})\varepsilon_{ij}(\mathbf{A}^{-1}\tilde{\mathbf{u}})dx = \sigma(\nu(\vartheta^1 + \Theta_0)D\tilde{\mathbf{u}}, D\mathbf{A}^{-1}\tilde{\mathbf{u}})$$
$$= \sigma(\nu(\vartheta^1 + \Theta_0)\nabla\tilde{\mathbf{u}}, D\mathbf{A}^{-1}\tilde{\mathbf{u}})$$

and

$$\sigma(\nu(\vartheta^1 + \Theta_0)\nabla\tilde{\mathbf{u}}, D\mathbf{A}^{-1}\tilde{\mathbf{u}}) = -\sigma(\mathrm{div}(\nu(\vartheta^1 + \Theta_0)D\mathbf{A}^{-1}\tilde{\mathbf{u}}), \tilde{\mathbf{u}})$$

$$= -\frac{\sigma}{2}(\nu(\vartheta^1+\Theta_0)\Delta\mathbf{A}^{-1}\tilde{\mathbf{u}},\tilde{\mathbf{u}}) - \sigma(D\mathbf{A}^{-1}\tilde{\mathbf{u}}\nabla\Theta_1\nu'(\vartheta^1+\Theta_0),\tilde{\mathbf{u}}).$$

Now by the classical properties of $\mathbf{A}$ (see, e.g., [38, Appendix B], we know that, for a suitable $\tilde{p}\in H^1(\Omega)$,

$$-\Delta\mathbf{A}^{-1}\tilde{\mathbf{u}} + \nabla\tilde{p} = \tilde{\mathbf{u}} \qquad \text{a.e. in } \Omega,$$

and

$$\|\mathbf{A}^{-1}\tilde{\mathbf{u}}\|_{\mathbf{H}^2(\Omega)} \le C\|\tilde{\mathbf{u}}\|, \qquad \|\tilde{p}\|_{H^1(\Omega)} \le C\|\tilde{\mathbf{u}}\|.$$

Thus, by (H1),

$$\begin{aligned}
&-\frac{1}{2}(\nu(\vartheta^1+\Theta_0)\Delta\mathbf{A}^{-1}\tilde{\mathbf{u}},\tilde{\mathbf{u}}) = \frac{1}{2}(\nu(\vartheta^1+\Theta_0)\tilde{\mathbf{u}},\tilde{\mathbf{u}}) - \frac{1}{2}(\nu(\vartheta^1+\Theta_0)\nabla\tilde{p},\tilde{\mathbf{u}})\\
&\ge \frac{\nu_1}{2}\|\tilde{\mathbf{u}}\|^2 - \frac{1}{2}(\nu(\vartheta^1+\Theta_0)\nabla\tilde{p},\tilde{\mathbf{u}}).
\end{aligned}$$

Using also (H4), we deduce the inequalities

$$\frac{\sigma\nu_1}{2}\int_\Omega |\tilde{\mathbf{u}}|^2 dx \le I_1, \tag{5.73}$$

$$\kappa\int_\Omega |\nabla\tilde{\vartheta}|^2 \le I_2, \tag{5.74}$$

where

$$\begin{aligned}
I_1 = &-\int_\Omega \left[\nu(\vartheta^1+\Theta_0)-\nu(\vartheta^2+\Theta_0)\right](\tau_{ij}(x,\varepsilon(\mathbf{u}^2))\varepsilon_{ij}(\mathbf{A}^{-1}\tilde{\mathbf{u}})dx\\
&-\sigma\int_\Omega \left[\nu(\vartheta^1+\Theta_0)-\nu(\vartheta^2+\Theta_0)\right]\varepsilon_{ij}(\mathbf{u}^2)\varepsilon_{ij}(\mathbf{A}^{-1}\tilde{\mathbf{u}})dx\\
&+\int_\Omega (\tilde{u}_j u_i^1 + u_j^2\tilde{u}_i)\partial_{x_j}(\mathbf{A}^{-1}\tilde{\mathbf{u}})_i dx\\
&+\frac{\sigma}{2}(\nu(\vartheta^1+\Theta_0)\nabla p,\tilde{\mathbf{u}}) + \sigma(D\mathbf{A}^{-1}\tilde{\mathbf{u}}\nabla\Theta_1\nu'(\vartheta^1+\Theta_0),\tilde{\mathbf{u}}),
\end{aligned} \tag{5.75}$$

$$I_2 = -\int_\Omega \tilde{u}_j(\vartheta^2-\Theta_0)\partial_{x_j}\tilde{\vartheta}. \tag{5.76}$$

Observe now that, recalling (H1), thanks to Hölder's and Korn's inequalities, the properties of $\mathbf{A}$ and Sobolev embedding $\mathbf{V}^2_{div}\hookrightarrow\mathbf{L}^q(\Omega)$, $q\in[2,6]$, we have

$$\begin{aligned}
&I_1 \le \|\nu'\|_{L^\infty(\mathbb{R})}\|\tilde{\vartheta}\|_{L^6(\Omega)}\\
&\times\left(\|(\tau(\cdot,\varepsilon(\mathbf{u}^2))\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{A}^{-1}\tilde{\mathbf{u}}\|_{\mathbf{L}^3(\Omega)} + \sigma\|\nabla\mathbf{u}^2\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{A}^{-1}\tilde{\mathbf{u}}\|_{\mathbf{L}^3(\Omega)}\right)\\
&+(\|\mathbf{u}^1\|_{\mathbf{L}^4(\Omega)}+\|\mathbf{u}^2\|_{\mathbf{L}^4(\Omega)})\|\tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{A}^{-1}\tilde{\mathbf{u}}\|_{\mathbf{L}^4(\Omega)}\\
&+\sigma C\left(\|\nu\|_{L^\infty(\mathbb{R})}+\|\nu'\|_{L^\infty(\mathbb{R})}\|\nabla\Theta_1\|_{\mathbf{L}^4(\Omega)}\right)\|\tilde{\mathbf{u}}\|^2\\
&\le C_1\|\nu'\|_{L^\infty(\mathbb{R})}\|\tilde{\vartheta}\|_{L^6(\Omega)}\left(\|\tau(\cdot,\varepsilon(\mathbf{u}^2))\|_{\mathbf{L}^3(\Omega)}\|\tilde{\mathbf{u}}\| + \sigma\|\nabla\mathbf{u}^2\|_{\mathbf{L}^2(\Omega)}\|\tilde{\mathbf{u}}\|\right)
\end{aligned}$$

$$+ (\|\mathbf{u}^1\|_{\mathbf{L}^4(\Omega)} + \|\mathbf{u}^2\|_{\mathbf{L}^4(\Omega)})\|\tilde{\mathbf{u}}\|^2$$
$$+ \sigma C_2 \left(\|\nu\|_{L^\infty(\mathbb{R})} + \|\nu'\|_{L^\infty(\mathbb{R})}\|\nabla\Theta_1\|_{\mathbf{L}^4(\Omega)}\right)\|\tilde{\mathbf{u}}\|^2.$$

On the other hand, owing to (H3), since $p \in (1,2)$, we can find $C_3 = C_3(\tau_2) > 0$ such that

$$\|\tau(\cdot, \varepsilon(\mathbf{u}^2))\|_{\mathbf{L}^2(\Omega)} \leq C\left(1 + \|\varepsilon(\mathbf{u}^2)\|^{p-1}_{\mathbf{L}^{2p-2}(\Omega)}\right) \leq C_3\left(1 + \|\varepsilon(\mathbf{u}^2)\|^{p-1}_{\mathbf{L}^2(\Omega)}\right), \tag{5.77}$$

but we know that there exists a positive constant C_4 depending on τ_1 and $\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}$ such that

$$\sqrt{\nu_1}\|\varepsilon(\mathbf{u}^k)\|_{\mathbf{L}^p(\Omega)} + \sqrt{\nu_1\sigma}\|\nabla\mathbf{u}^k\|_{\mathbf{L}^2(\Omega)} \leq C_4, \quad k = 1, 2. \tag{5.78}$$

Indeed, we can take $\mathbf{v} = \mathbf{u}^k$ in (5.45) written for $(\mathbf{u}^k, \vartheta^k)$ and using (H1), (H3), (H5), and Young's inequality we get this result, meaning that (5.77) is bounded (note that this is possible due to the $\mathbf{L}^2$ control on the gradient of $\tilde{\mathbf{u}}$, being $\sigma > 0$). Thus, using Poincaré's inequality and the continuous embedding $\mathbf{V}^2_{div} \hookrightarrow \mathbf{L}^4(\Omega)$, we get

$$I_1 \leq C_6\|\nu'\|_{L^\infty(\mathbb{R})}\|\tilde{\vartheta}\|_{L^6(\Omega)}\left(\left(1 + \frac{1}{(\sqrt{\nu_1\sigma})^{p-1}}\right)\|\tilde{\mathbf{u}}\| + \frac{\sigma}{\sqrt{\nu_1\sigma}}\|\tilde{\mathbf{u}}\|\right)$$
$$+ \frac{C_7}{\sqrt{\nu_1\sigma}}\|\tilde{\mathbf{u}}\|^2 + \sigma C_2\left(\|\nu\|_{L^\infty(\mathbb{R})} + \|\nu'\|_{L^\infty(\mathbb{R})}\|\nabla\Theta_1\|_{\mathbf{L}^4(\Omega)}\right)\|\tilde{\mathbf{u}}\|^2,$$

where C_2, C_6 and C_7 also depend on the Poincaré's constant and on the embedding constants of the embeddings quoted above (therefore they depend on Ω). Clearly they also depend on $\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}$.

Regarding I_2, we have

$$I_2 \leq \|\tilde{\mathbf{u}}\|(\|\vartheta^2\|_{L^\infty(\Omega)} + \|\Theta_0\|_{L^\infty(\Omega)})\|\nabla\tilde{\vartheta}\|. \tag{5.79}$$

Considering (5.70) written for $(\mathbf{u}^2, \vartheta^2)$ we get

$$\|\vartheta^2\|_{H^2(\Omega)} \leq \frac{C_8}{\kappa^2}\|\nabla\mathbf{u}^2\|^2\|\nabla\vartheta^2\| + \frac{C_9}{\kappa}\|\nabla\mathbf{u}^2\|\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|, \tag{5.80}$$

Thus, by (5.78), we can find $C_{10} > 0$ such that

$$\|\vartheta^2\|_{H^2(\Omega)} \leq C_{10}(\frac{1}{\kappa^2\nu_1\sigma}\|\nabla\vartheta^2\| + \frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|). \tag{5.81}$$

Moreover, taking $\phi = \vartheta^2$ in (5.46) written for $(\mathbf{u}^2, \vartheta^2)$ we get

$$\kappa\|\nabla\vartheta^2\|^2 \leq \|g\|_{V'}\|\vartheta^2\|_{H^1(\Omega)} + \|\mathbf{u}^2\|_{\mathbf{L}^3_{div}}\|\Theta_0\|_{L^6(\Omega)}\|\nabla\vartheta^2\|.$$

Thus, using Poincaré's inequality,

$$\kappa\|\nabla\vartheta^2\|^2 \le \left(C\|g\|_{V'} + \|\mathbf{u}^2\|_{\mathbf{L}^3_{div}}\|\Theta_0\|_{L^6(\Omega)}\right)\|\nabla\vartheta^2\|$$

and Young's inequality gives

$$\frac{\kappa}{2}\|\nabla\vartheta^2\|^2 \le \frac{1}{2\kappa}\left(C\|g\|_{V'} + \|\mathbf{u}^2\|_{\mathbf{L}^3_{div}}\|\Theta_0\|_{L^6(\Omega)}\right)^2.$$

Thus, recalling (5.78), we can find a constant $C_{11} > 0$ depending on Ω, τ_1, τ_2, and $\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}$ such that

$$\|\vartheta^2\|_{V_0} \le \frac{C_{11}}{\kappa}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right).$$

Hence, from (5.81) we infer

$$\|\vartheta^2\|_{H^2(\Omega)} \le C_{12}\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right) + \frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right). \tag{5.82}$$

Analogously, being $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we deduce

$$\|\vartheta^2\|_{L^\infty(\Omega)} \le C_{13}\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right) + \frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right). \tag{5.83}$$

Therefore, on account of (5.82), recalling that $\|\Theta_0\|_{L^\infty(\Omega)} \le C\|\theta\|_{H^{3/2}(\partial\Omega)}$,

$$I_2 \le C_{14}\|\tilde{\mathbf{u}}\|\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right) + \frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right)\|\nabla\tilde{\vartheta}\|, \tag{5.84}$$

which means, by Young's inequality,

$$\|\nabla\tilde{\vartheta}\|^2_{H^3} \le \frac{C_{15}}{\kappa^2}\|\tilde{\mathbf{u}}\|^2\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'} + \frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right) + \frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right)^2 \tag{5.85}$$

Combining (5.85) with (5.58) we get, since $\Theta^1 = \vartheta^1 + \Theta_0$, applying Young's and Poincaré's inequalities (recalling that (5.82) is still valid for ϑ^1) and Sobolev embeddings,

$$\begin{aligned}
I_1 &\le C_{16}\|\nu'\|^2_{L^\infty(\mathbb{R})}\|\tilde{\vartheta}\|^2_{L^6(\Omega)}\left(1 + \frac{1}{(\sqrt{\nu_1\sigma})^{p-1}} + \frac{\sigma}{\sqrt{\nu_1\sigma}}\right)^2 \\
&+ C_{17}\Bigg(1 + \frac{1}{\sqrt{\nu_1\sigma}} + \sigma\big(\|\nu\|_{L^\infty(\mathbb{R})} \\
&+\|\nu'\|_{L^\infty(\mathbb{R})}\left(\|\vartheta_1\|_{H^2(\Omega)} + \|\theta\|_{H^{3/2}(\partial\Omega)}\right)\big)\Bigg)\|\tilde{\mathbf{u}}\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le C_{16}\|\nu'\|^2_{L^\infty(\mathbb{R})}\|\tilde{\vartheta}\|^2_{L^6(\Omega)}\left(1+\frac{1}{(\sqrt{\nu_1\sigma})^{p-1}}+\frac{\sigma}{\sqrt{\nu_1\sigma}}\right)^2\\
&+C_{17}\left(1+\frac{1}{\sqrt{\nu_1\sigma}}+\sigma\left(\|\nu\|_{L^\infty(\mathbb{R})}\right.\right.\\
&+\|\nu'\|_{L^\infty(\mathbb{R})}\left(\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'}+\frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)+\frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)}\right.\right.\\
&\left.\left.\left.\left.+\frac{1}{\kappa}\|g\|\right)+\|\theta\|_{H^{3/2}(\partial\Omega)}\right)\right)\right)\|\tilde{\mathbf{u}}\|^2\\
&\le \frac{C_{18}}{\kappa^2}\|\nu'\|^2_{L^\infty(\mathbb{R})}\left(1+\frac{1}{(\sqrt{\nu_1\sigma})^{p-1}}+\frac{\sigma}{\sqrt{\nu_1\sigma}}\right)^2\\
&\times\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'}+\frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)+\frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)}+\frac{1}{\kappa}\|g\|\right)^2\|\tilde{\mathbf{u}}\|^2\\
&+C_{17}\left(1+\frac{1}{\sqrt{\nu_1\sigma}}+\sigma\left(\|\nu\|_{L^\infty(\mathbb{R})}\right.\right.\\
&+\|\nu'\|_{L^\infty(\mathbb{R})}\left(\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'}+\frac{\|\Theta_0\|_V}{\sqrt{\nu_1\sigma}}\right)+\frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)}\right.\right.\\
&\left.\left.\left.\left.+\frac{1}{\kappa}\|g\|\right)+\|\theta\|_{H^{3/2}(\partial\Omega)}\right)\right)\right)\|\tilde{\mathbf{u}}\|^2. \qquad (5.86)
\end{aligned}
$$

In conclusion, by (5.73), recalling that $\|\Theta_0\|_V\le\tilde{C}\|\theta\|_{H^{1/2}(\partial\Omega)}$,

$$
\begin{aligned}
&\left[-\frac{\sigma\nu_1}{2}+\frac{C_{18}}{\kappa^2}\|\nu'\|^2_{L^\infty(\mathbb{R})}\left(1+\frac{1}{(\sqrt{\nu_1\sigma})^{p-1}}+\frac{\sigma}{\sqrt{\nu_1\sigma}}\right)^2\right.\\
&\times\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'}+\frac{\tilde{C}\|\theta\|_{H^{1/2}(\partial\Omega)}}{\sqrt{\nu_1\sigma}}\right)+\frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)}+\frac{1}{\kappa}\|g\|\right)^2\\
&+C_{17}\left(1+\frac{1}{\sqrt{\nu_1\sigma}}+\sigma\left(\|\nu\|_{L^\infty(\mathbb{R})}\right.\right.\\
&+\|\nu'\|_{L^\infty(\mathbb{R})}\left(\left(\frac{1}{\kappa^3\nu_1\sigma}\left(\|g\|_{V'}+\frac{\tilde{C}\|\theta\|_{H^{1/2}(\partial\Omega)}}{\sqrt{\nu_1\sigma}}\right)+\frac{1}{\kappa\sqrt{\nu_1\sigma}}\|\theta\|_{H^{3/2}(\partial\Omega)}\right.\right.\\
&\left.\left.\left.\left.\left.+\frac{1}{\kappa}\|g\|\right)+\|\theta\|_{H^{3/2}(\partial\Omega)}\right)\right)\right)\right]\|\tilde{\mathbf{u}}\|^2\ge 0, \qquad (5.87)
\end{aligned}
$$

which eventually yields uniqueness provided that (4.17) holds with $M_1=C_{18}$, $M_2=\tilde{C}$ and $M_3=C_{17}$.

6 Finite element approximation: the Carreau law case

6.1 Preliminaries

In this section, we consider the finite element approximation of a specific instance of the boundary value problem (2.1)-(2.4), again under the same assumptions **(H1)-(H7)** namely we drop the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and we focus on the relevant case of non-Newtonian fluids governed by the Carreau law, i.e. $\tau(\varepsilon(\mathbf{u})) = \eta(|\varepsilon(\mathbf{u})|^2)\varepsilon(\mathbf{u})$ with

$$\eta(z) := \eta_\infty + (\eta_0 - \eta_\infty)(1 + \lambda z)^{(p-2)/2} \tag{6.1}$$

$$\eta_0 > \eta_\infty \geq 0, \lambda > 0 \text{ and } p \in (1, 2). \tag{6.2}$$

For the sequel, it is instrumental to recall the following result (cf. [10, Lemmas 3.1 and 3.2]) where $|\cdot|$ denotes the euclidean matrix norm, i.e. for $\mathbf{K} \in \mathbb{R}^{n\times n}$ real matrix, $|\mathbf{K}|^2 = \sum_{i,j=1}^{n} (K_{i,j})^2$.

Lemma 6.1. *Let η obey the Carreau law* (6.1) *with $p \in (1, 2)$. Then there exists positive constants C_i, $i = 1, \ldots, 3$ such that for all symmetric matrices $\mathbf{K}, \mathbf{L} \in \mathbb{R}^2$ there holds*

$$\begin{aligned} |\eta(|\mathbf{K}|^2)\mathbf{K} - \eta(|\mathbf{L}|^2)\mathbf{L}| &\leq C_1|\mathbf{K} - \mathbf{L}| \qquad \text{if } \eta_\infty \neq 0 \\ |\eta(|\mathbf{K}|^2)\mathbf{K} - \eta(|\mathbf{L}|^2)\mathbf{L}| &\leq C_3|\mathbf{K} - \mathbf{L}|^{p-1} \qquad \text{if } \eta_\infty = 0. \end{aligned}$$

Moreover it holds

$$\begin{aligned} &\sum_{i,j=1}^{2} \left(\eta(|\mathbf{K}|^2)K_{i,j} - \eta(|\mathbf{L}|^2)L_{i,j}\right)(K_{i,j} - L_{i,j}) \\ &\geq \{\eta_\infty + C_2[1 + |\mathbf{K}| + |\mathbf{L}|]^{p-2}\}|\mathbf{K} - \mathbf{L}|^2. \end{aligned}$$

We first recall the variational formulation of the continuous problem. Let us choose:

$$V = H^1(\Omega), \quad V_0 = H^1_0(\Omega), \quad \mathbf{V} = \mathbf{W}^{1,s}(\Omega), \quad \mathbf{V}_0 = \mathbf{W}^{1,s}_0(\Omega),$$

$$Q = L^{s'}(\Omega), \quad Q_* = L^{s'}_0(\Omega) = \left\{ q \in Q | \int_\Omega q = 0 \right\},$$

where $1/s' + 1/s = 1$ with $s = p$ if $\eta_\infty = 0$ and $s = 2$ otherwise. We set $\|\cdot\|_{\mathbf{V}} := |\cdot|_{\mathbf{W}^{1,s}(\Omega)}$ and $\|\cdot\|_Q = \|\cdot\|_{Q_*} := \|\cdot\|_{L^{s'}(\Omega)}$. Moreover, let us introduce the following forms

$$a_1(\vartheta, \mathbf{u}; \mathbf{v}) = \int_\Omega 2\nu(\vartheta)\eta(|\varepsilon(\mathbf{u})|^2)\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}), \qquad a_2(\vartheta, \varrho) = \int_\Omega \kappa \nabla\vartheta \cdot \nabla\varrho,$$

$$b(\mathbf{v}, p) = -\int_\Omega p \nabla \cdot \mathbf{v}, \qquad c(\mathbf{u}, \vartheta, \varrho) = \int_\Omega (\mathbf{u} \cdot \nabla)\vartheta\varrho,$$

$$f(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \qquad g(\varrho) = \int_\Omega g\varrho.$$

Since the velocity space is not divergence free, we rewrite the variational formulation of a weak solution given in Definition 2.1 as follows: let $p \in (1, 2)$ if $d = 2$ and $p \in (3/2, 2)$ for $d = 3$. Find $(\mathbf{u}, \pi, \vartheta) \in \mathbf{V}_0 \times Q_* \times V_0$ such that

$$a_1(\vartheta + \Theta_0, \mathbf{u}; \mathbf{v}) + b(\mathbf{v}, \pi) = f(\mathbf{v}), \tag{6.3}$$

$$b(\mathbf{u}, q) = 0, \tag{6.4}$$

$$a_2(\vartheta, \varrho) + c(\mathbf{u}; \vartheta, \varrho) = g(\rho) - a_2(\Theta_0, \varrho) - c(\mathbf{u}, \Theta_0, \varrho), \tag{6.5}$$

$\forall(\mathbf{v}, q, \varrho) \in \mathbf{V}_0 \times Q_* \times V_0$.

Remark 6.1. *The weak formulation is well posed since, also in the case* $\eta_\infty = 0$, *differently from the general case of Remark 2.1, we are considering a range of* p*'s for* $d = 2, 3$ *ensuring the embedding* $\boldsymbol{V}_0 \hookrightarrow \boldsymbol{L}^3(\Omega)$. *Therefore* $\varrho \in V_0$ *is allowed as a test function in* (6.5).

We collect well known results that will be employed in the following. First, the bilinear form $b(\mathbf{v}, q)$ is continuous, that is there exists $M > 0$ such that

$$b(\mathbf{v}, q) \leq M\|\mathbf{v}\|_{\mathbf{V}}\|q\|_{Q_*} \quad \forall(\mathbf{v}, q) \in \mathbf{V} \times Q_* \tag{6.6}$$

and it satisfies the following compatibility condition ([22, 39]): there exists $c = c(\Omega)$ such that

$$c\|q\|_{Q_*} \leq \sup_{\mathbf{v}\in\mathbf{V}_0} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}}} \quad \forall q \in Q_*. \tag{6.7}$$

Second, the trilinear form $c(\mathbf{u}, \theta, \varrho)$ satisfies the antisymmetry property when $\nabla \cdot \mathbf{u} = 0$. Indeed

$$\begin{aligned} &c(\mathbf{u}, \theta, \varrho) = -c(\mathbf{u}, \varrho, \theta), \\ &c(\mathbf{u}, \theta, \theta) = 0. \end{aligned} \tag{6.8}$$

Moreover, for all $(\mathbf{u}, \theta, \varrho) \in \mathbf{V}_0 \times V_0 \times V_0$, using the generalized Hölder inequality and the Sobolev embedding,

$$c(\mathbf{u}, \theta, \varrho) \leq \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}\|\nabla\theta\|\|\nabla\varrho\| \tag{6.9}$$

where $p \in (1,2)$ for $d=2$ and $p \in [3/2,2)$ for $d=3$. Indeed, for $d=2$, for any $\iota > 2$,

$$c(\mathbf{u},\theta,\varrho) \leq \|\mathbf{u}\|_{\mathbf{L}^\iota(\Omega)}\|\varrho\|_{L^{\frac{2\iota}{\iota-2}}(\Omega)}\|\nabla\theta\| \leq \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}\|\nabla\theta\|\|\nabla\varrho\|,$$

due to the embedding $H^1(\Omega) \hookrightarrow L^\iota(\Omega), \forall \iota \in [2,\infty)$. Therefore, being ι arbitrary, it is enough to use $\iota = \dfrac{2p}{2-p} > 2$ to obtain the embedding: for any $p \in (1,2)$ the inequality is true. For $d=3$, we have at most

$$c(\mathbf{u},\theta,\varrho) \leq \|\mathbf{u}\|_{\mathbf{L}^3(\Omega)}\|\varrho\|_{L^6(\Omega)}\|\nabla\theta\| \leq \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}\|\nabla\theta\|\|\nabla\varrho\|,$$

due to the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Therefore, we need $p \in [3/2,2)$.

6.2 Definition of the discrete problem

Let $\Omega \subset \mathbb{R}^d$, $d=2,3$ be a polygonal/polyedral domain. We introduce (see, e.g,. [21, 26]) a regular family of triangulations $\mathcal{T}_h$ of Ω where h denotes the maximum of the diameters of the elements $K \in \mathcal{T}_h$.

Let $\mathbf{V}_h \subset \mathbf{V}$, $Q_h \subset Q_*$ and $V_h \subset V$ be finite dimensional spaces and we set $\mathbf{V}_{0,h} = \mathbf{V}_h \cap \mathbf{V}_0$, $V_{0,h} = V_h \cap V_0$. We assume that the following approximability properties hold:

(P1) $\inf_{s_h \in V_h} \|\nabla(\theta - s_h)\| \to 0 \quad$ as $h \to 0 \quad \forall \theta \in V_h$,

(P2) $\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{\mathbf{V}} \to 0 \quad$ as $h \to 0 \quad \forall \mathbf{v} \in \mathbf{V}_h$,

(P3) $\inf_{q_h \in Q_h} \|\pi - q_h\|_{Q_*} \to 0 \quad$ as $h \to 0 \quad \forall \pi \in Q_h$.

Moreover, we assume (see, e.g., [22, 39] for examples in the context of finite elements) that the discrete spaces $\mathbf{V}_h, Q_h$ are chosen so that there exists $\beta > 0$ independent of h such that

$$\beta\|q_h\|_{Q_*} \leq \sup_{\mathbf{v}_h \in \mathbf{V}_{0,h}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \quad \forall q_h \in Q_h. \tag{6.10}$$

We introduce the following discrete form

$$c_h(\mathbf{u}_h, \vartheta_h, \varrho_h) = \frac{1}{2}\left(c(\mathbf{u}_h, \vartheta_h, \varrho_h) - c(\mathbf{u}_h, \varrho_h, \vartheta_h)\right) \tag{6.11}$$

to recover the antisymmetry property of its continuous counterpart (cf. (6.8)). Similarly to the continuous case, it holds

$$c_h(\mathbf{u}_h, \theta_h, \varrho_h) \leq \|\mathbf{u}_h\|_{\mathbf{W}^{1,p}(\Omega)}\|\nabla\theta_h\|\|\nabla\varrho_h\| \tag{6.12}$$

where $p \in (1,2)$ for $d = 2$ and $p \in [3/2, 2)$ for $d = 3$. Let assume for simplicity $\Theta_0 \in V_h$, then the finite dimensional approximation of problem (6.3)-(6.5), reads: find $(\mathbf{u}_h, \pi_h, \vartheta_h) \in \mathbf{V}_{0,h} \times Q_h \times V_{0,h}$ such that

$$a_1(\vartheta_h + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) + b(\mathbf{v}_h, \pi_h) = \mathbf{f}(\mathbf{v}_h) \tag{6.13}$$

$$b(\mathbf{u}_h, q_h) = 0 \tag{6.14}$$

$$a_2(\vartheta_h, \varrho_h) + c_h(\vartheta_h, \varrho_h; \mathbf{u}_h) = g(\varrho_h) - a_2(\Theta_0, \varrho_h) - c_h(\mathbf{u}_h, \Theta_0, \varrho_h) \tag{6.15}$$

$\forall(\mathbf{v}_h, q_h, \varrho_h) \in \mathbf{V}_{0,h} \times Q_h \times V_{0,h}$. Let us first remark that in the case of $\eta_\infty = 0$, which is the most interesting case, the existence of a solution to (6.13)-(6.15) follows by adapting the proof of Theorem 4.1, employing the pair $\mathbf{V}_h^{div} \times V_h$, $\mathbf{V}_h^{div} = \{\mathbf{v}_h \in \mathbf{V}_{0,h} : b(\mathbf{v}_h, q_h) = 0 \ \forall q_h \in Q_h\}$, so that (6.13)-(6.15) is equivalent to the weak formulation of Definition 2.1 on the spaces $\mathbf{V}_h^{div} \times V_h$. Conditional uniqueness is then a consequence of Theorem 3.4, exploiting again the pair $\mathbf{V}_h^{div} \times V_h$. Analogously, in the case $\eta_\infty > 0$, the existence of a solution to (6.13)-(6.15) follows by adapting the proof of Theorem 4.1 with the pair $\mathbf{V}_h^{div} \times V_h$, whereas conditional uniqueness is a consequence of Theorem 4.2 with $\sigma = \eta_\infty$ and $r = 2$ (only for $p \in (1, 5/3]$ in the case $d = 3$), adapted to the space $\mathbf{V}_h^{div} \times V_h$. Notice that Theorem 4.3 cannot be applied, since the solution $\mathbf{u}_h$ does not enjoy zero divergence and moreover we cannot ensure $\mathbf{A}^{-1}\mathbf{u}_h \in \mathbf{V}_h$.

6.3 A priori error analysis: Main results

Let us now state and prove the main results of this section, namely a priori error estimates for the discrete solution of (6.13)-(6.15). First we state the most relevant result, which corresponds to the case when $\eta_\infty = 0$.

Theorem 6.2. *Let $\eta_\infty = 0$, $p \in (1,2)$ for $d = 2$ and $p \in (3/2, 2)$ for $d = 3$. Let $(\mathbf{u}, \pi, \vartheta)$ be a solution of (6.3)-(6.5). Let $(\mathbf{u}_h, \pi_h, \vartheta_h)$ be a solution of (6.13)-(6.15) where we assume $\mathbf{u}_h \in \boldsymbol{W}^{1,(p-1)p_2}(\Omega)$, with $p_2 = q/(p-1)$ with $q > p$ when $d = 2$ and $p_2 = 6p/(5p-6)$ when $d = 3$. In addition, we suppose that the following smallness condition on $\mathbf{u}_h$ holds*

(A1) $$\kappa - 2\left[\frac{\tilde{C}_2}{\kappa}\|g\| + \tilde{C}_3\,(1 + C_f)\,\|\theta\|_{H^{1/2}(\Gamma)}\right] D_6 \|\varepsilon(\mathbf{u}_h)\|^{p-1}_{\boldsymbol{L}^{(p-1)p_2}(\Omega)} > 0,$$

with $C_f = (1 + \|\mathbf{f}\|_{\boldsymbol{W}^{1,p'}(\Omega)})^{1/(p-1)}$ and also $\tilde{C}_2, \tilde{C}_3, D_6 > 0$ possibly depending on the data of the problem. Then there exists $h_0 > 0$ such that, for any $h \le h_0$, the following inequalities hold

$$\|\nabla(\vartheta - \vartheta_h)\| \lesssim \min_{(\mathbf{v}_h, q_h, s_h) \in \mathbf{V}_h \times Q_h \times V_h} \Big(\|\nabla(\vartheta - s_h)\| + \|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{p-1}_{\boldsymbol{L}^p(\Omega)}$$

$$
+ \|\pi - q_h\|_{L^{p'}(\Omega)}\Big) \tag{6.16}
$$

$$
\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|_{\boldsymbol{L}^p(\Omega)} \lesssim \min_{(\mathbf{v}_h,q_h,s_h)\in\mathbf{V}_h\times Q_h\times V_h} \Big(\|\varepsilon(\mathbf{u}-\mathbf{v}_h)\|^{p-1}_{\boldsymbol{L}^p(\Omega)} + \|\pi - q_h\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\nabla(\vartheta - s_h)\|\big), \tag{6.17}
$$

$$
\|\pi - \pi_h\|_{L^{p'}(\Omega)} \lesssim \min_{(\mathbf{v}_h,q_h,s_h)\in\mathbf{V}_h\times Q_h\times V_h} \Big(\|\pi - q_h\|^{p-1}_{L^{p'}(\Omega)} + \|\varepsilon(\mathbf{u}-\mathbf{v}_h)\|^{(p-1)^2}_{\boldsymbol{L}^p(\Omega)} + \|\nabla(\vartheta - \vartheta_h)\|^{p-1}\Big), \tag{6.18}
$$

where $\boldsymbol{u}$ *is the solution to the continuous problem, and the hidden constants are independent of* h.

Remark 6.2. *Notice that in force of Theorems 3.2 and 3.3 we have that any solution* $\boldsymbol{u}$ *to the corresponding continuous problem is sufficiently regular to perform the error estimates, if we assume the additional hypotheses* (3.4) *(for* $d = 3$*) and* (3.7) *(for* $d = 2$*). To be precise, Theorem 3.2 concerns a slightly different law for* τ*, i.e.,* (3.1)*, but we believe that the regularity results are similar if not the same. Therefore, we expect that also* $\boldsymbol{u}_h$*, shares an analogous regularity (in case of a sufficiently regular domain* Ω*), which is required in Theorem 6.2.*

In conclusion, we state another result for the simpler case $\eta_\infty > 0$.

Theorem 6.3. *Let* $\eta_\infty > 0$, $p \in (1,2)$ *for* $d = 2$ *and* $p \in (3/2, 2)$ *for* $d = 3$*. Let* $(\mathbf{u}, \pi, \vartheta)$ *be a solution of* (6.3)-(6.5)*. Let* $(\mathbf{u}_h, \pi_h, \vartheta_h)$ *be a solution of* (6.13)-(6.15) *where we assume* $\mathbf{u}_h \in \boldsymbol{W}^{1,p_2}(\Omega)$*, with* $p_2 > p$ *when* $d = 2$ *and* $p_2 = 6p/(5p-6)$ *when* $d = 3$*. In addition, we suppose that the following smallness condition on* $\mathbf{u}_h$ *holds*

(B1) $\kappa - 2\left[\frac{\tilde{C}_2}{\kappa}\|g\| + \tilde{C}_3\,(1 + C_f)\,\|\theta\|_{H^{1/2}(\Gamma)}\right] D_6 \|\varepsilon(\mathbf{u}_h)\|_{\boldsymbol{L}^q(\Omega)} > 0,$

with $C_f = C/(\nu_1\eta_\infty)\|\mathbf{f}\|$ *and also* $\tilde{C}_2, \tilde{C}_3, D_6 > 0$ *possibly depending on the data of the problem. Then there exists* $h_0 > 0$ *such that, for any* $h \leq h_0$*, the following inequalities hold*

$$
\|\nabla(\vartheta - \vartheta_h)\| \lesssim \min_{(\mathbf{v}_h,q_h,s_h)\in\mathbf{V}_h\times Q_h\times V_h} \big(\|\nabla(\vartheta - s_h)\| + \|\varepsilon(\mathbf{u}-\mathbf{v}_h)\| + \|\pi - q_h\|\big) \tag{6.19}
$$

$$
\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\| \lesssim \min_{(\mathbf{v}_h,q_h,s_h)\in\mathbf{V}_h\times Q_h\times V_h} \big(\|\varepsilon(\mathbf{u}-\mathbf{v}_h)\| + \|\pi - q_h\| + \|\nabla(\vartheta - s_h)\|\big), \tag{6.20}
$$

$$
\|\pi - \pi_h\| \lesssim \min_{(\mathbf{v}_h,q_h,s_h)\in\mathbf{V}_h\times Q_h\times V_h} \big(\|\pi - q_h\| + \|\varepsilon(\mathbf{u}-\mathbf{v}_h)\| + \|\nabla(\vartheta - \vartheta_h)\|\big), \tag{6.21}
$$

*where **u** is the solution to the continuous problem, and the hidden constants are independent of h.*

Remark 6.3. *Notice that in the case $\eta_\infty > 0$ we get optimal approximation results, whereas when $\eta_\infty = 0$ we deduce error estimates that are coherent with the ones obtained, though in the simpler context of isothermal non-Newtonian fluids, in [19]. For further comments on these aspects we refer to the numerical results reported in Section 6.5.*

6.4 Proof of Theorems 6.2 and 6.3

For the sake of brevity, we consider both the cases $\eta_\infty > 0$ and $\eta_\infty = 0$. In particular, as before, we set $s = p$ if $\eta_\infty = 0$ and $s = 2$ if $\eta_\infty > 0$. We preliminarily collect some instrumental results that will be employed during the proof. Let us first recall that in view of Assumption (**H1**) we have that $\nu(\Theta)$ is a bounded continuous function defined on $(0, +\infty)$ satisfying the following properties:

$$\nu \in C^1(\mathbb{R}), \tag{6.22}$$

$$0 < \nu_1 \le \nu(\xi) \le \nu_2 \quad \text{for } \xi \in \mathbb{R}^+, \tag{6.23}$$

$$|\nu'(\xi)| \le \nu_3 \quad \text{for } \xi \in \mathbb{R}^+. \tag{6.24}$$

Moreover, let us observe that, exploiting Lemma 6.1 combined with the definition of $a_1(\cdot,\cdot;\cdot)$, (6.23) and the generalized (also to negative exponents) Hölder's inequality, we can prove the following inequalities holding $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}_0$:

Case $\eta_\infty > 0$

$$|a_1(\vartheta, \mathbf{u}; \mathbf{w}) - a_1(\vartheta, \mathbf{v}; \mathbf{w})| \le C_1 \nu_2 \|\varepsilon(\mathbf{u} - \mathbf{v})\| \|\varepsilon(\mathbf{w})\|, \tag{6.25}$$

$$a_1(\vartheta, \mathbf{u}; \mathbf{u} - \mathbf{v}) - a_1(\vartheta, \mathbf{v}; \mathbf{u} - \mathbf{v}) \ge \nu_1 \eta_\infty \|\varepsilon(\mathbf{u} - \mathbf{v})\|^2. \tag{6.26}$$

Case $\eta_\infty = 0$

$$|a_1(\vartheta, \mathbf{u}; \mathbf{w}) - a_1(\vartheta, \mathbf{v}; \mathbf{w})| \le C_3 \nu_2 \|\varepsilon(\mathbf{u} - \mathbf{v})\|_{\mathbf{L}^p}^{p-1} \|\varepsilon(\mathbf{w})\|_{\mathbf{L}^p}, \tag{6.27}$$

$$\begin{aligned} & a_1(\vartheta, \mathbf{u}; \mathbf{u} - \mathbf{v}) - a_1(\vartheta, \mathbf{v}; \mathbf{u} - \mathbf{v}) \\ & \ge C_2 \nu_1 \left(\int_\Omega 1 + |\varepsilon(\mathbf{u})|^p + |\varepsilon(\mathbf{v})|^p \right)^{\frac{p-2}{p}} \|\varepsilon(\mathbf{u} - \mathbf{v})\|_{\mathbf{L}^p}^2. \end{aligned} \tag{6.28}$$

Using the definition of $a_2(\cdot,\cdot)$, it is trivial to see that the following inequalities hold

$$a_2(\vartheta, \varrho) \le \kappa \|\nabla\vartheta\| \|\nabla\varrho\|, \quad a_2(\vartheta, \vartheta) \ge \kappa \|\nabla\vartheta\|^2 \quad \forall \vartheta, \varrho \in V_0. \tag{6.29}$$

Finally, we collect some stability estimates. We first observe that the following stability estimate for the discrete velocity holds

$$\|\nabla \mathbf{u}_h\|_{\mathbf{L}^s(\Omega)} \le C_f \tag{6.30}$$

where $C_f = (1 + \|\mathbf{f}\|_{\mathbf{W}^{1,p'}(\Omega)})^{1/(p-1)}$ when $\eta_\infty = 0$ and $C_f = C/(\nu_1\eta_\infty)\|\mathbf{f}\|$ when $\eta_\infty > 0$. Indeed, in the case $\eta_\infty > 0$, testing (6.13)-(6.14) with $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = \pi_h$ and employing the monotonicity of $a_1(\cdot,\cdot;\cdot)$ together with the properties of ν combined with the Korn inequality, we obtain (6.30) where C is the Korn constant. In the case $\eta_\infty = 0$, it is sufficient to proceed as in the proof of Theorem 5.1 (cf. (5.24)) to get the desired stability estimate.

Testing (6.15) with $\varrho_h = \theta_h$ and employing the coercivity and continuity properties of $a_2(\cdot,\cdot)$, the Poincaré inequality, the continuity and antisimmetry of $c_h(\cdot,\cdot,\cdot)$, the above stability result of the discrete velocity and the stability of the Dirichlet lifting Θ_0, we get the following stability estimate for the discrete temperature

$$\|\nabla \vartheta_h\| \le \frac{1}{\kappa}[\tilde{C}_2\|g\| + \kappa\tilde{C}_3\|\theta\|_{H^{1/2}(\Gamma)} + \tilde{C}_3 C_f\|\theta\|_{H^{1/2}(\Gamma)}]. \tag{6.31}$$

We remark that an analogous stability estimate holds for the continuous temperature as well.

We are now ready to prove (6.16)-(6.18). Let us consider (6.3) which, together with (6.13), yields

$$\begin{aligned} b(\mathbf{v}_h, \pi - \pi_h) &= a_1(\vartheta_h + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) - a_1(\vartheta + \Theta_0, \mathbf{u}; \mathbf{v}_h) \\ &= a_1(\vartheta_h + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) - a_1(\vartheta + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) \\ &\quad + a_1(\vartheta_h + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) - a_1(\vartheta_h + \Theta_0, \mathbf{u}; \mathbf{v}_h) \end{aligned}$$

$\forall \mathbf{v}_h \in \mathbf{V}_{h,0}$. By linearity we have

$$\begin{aligned} b(\mathbf{v}_h, q_h - \pi_h) &= a_1(\vartheta_h + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) - a_1(\vartheta + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) \\ &\quad + a_1(\vartheta_h + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) - a_1(\vartheta_h + \Theta_0, \mathbf{u}; \mathbf{v}_h) \\ &\quad + b(\mathbf{v}_h, q_h - \pi) \end{aligned} \tag{6.32}$$

$\forall q_h \in Q_h$. Now, considering the compatibility condition (6.10) together with (6.32), we obtain

$$\begin{aligned} \beta\|\pi_h - q_h\|_{L^{s'}(\Omega)} &\le \sup_{\mathbf{v}_h \in \mathbf{V}_{0,h}} \frac{b(\mathbf{v}_h, q_h - \pi_h)}{\|v_h\|_{\mathbf{V}}} \\ = \sup_{\mathbf{v}_h \in \mathbf{V}_{0,h}} &\Bigg[\frac{(a_1(\vartheta_h + \Theta_0, \mathbf{u}_h; \mathbf{v}_h) - a_1(\vartheta + \Theta_0, \mathbf{u}_h; \mathbf{v}_h)}{\|v_h\|_{\mathbf{V}}} \end{aligned}$$

$$
\begin{aligned}
&+ \left.\frac{a_1(\vartheta_h+\Theta_0,\mathbf{u}_h,\mathbf{v}_h)-a_1(\vartheta_h+\Theta_0,\mathbf{u};\mathbf{v}_h)+b(\mathbf{v}_h,q_h-\pi)}{\|v_h\|_{\mathbf{V}}}\right] \\
&\le C_s\nu_3\|\nabla(\vartheta-\vartheta_h)\|\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)} \qquad (6.33)\\
&+ C_s\nu_2\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)} + M\|\pi-q_h\|_{L^{s'}(\Omega)},
\end{aligned}
$$

where $C_s = C_1$ for $s = 2$, $C_s = C_3$ for $s \neq 2$ (cf. (6.25) and (6.27)). Moreover, p_2 is the one defined in the assumptions of Theorems 6.2 and 6.3. We note that in the last step we employed the continuity property (6.25) (or (6.27)) for the second term, and the continuity of $b(\cdot,\cdot)$ for the last term. For the first term, we employed the properties of ν together with Lemma 6.1 and the generalized Hölder's inequality to get

$$
\begin{aligned}
&a_1(\vartheta_h+\Theta_0,\mathbf{u}_h;\mathbf{v}_h)-a_1(\vartheta+\Theta_0,\mathbf{u}_h;\mathbf{v}_h)\\
&\le C_s\nu_3\|\theta-\theta_h\|_{L^\beta(\Omega)}\left(\int_\Omega|\varepsilon(\mathbf{u}_h)|^{(s-1)p_2}\right)^{1/p_2}\|\varepsilon(\mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)}\\
&\le C_s\nu_3\|\theta-\theta_h\|_{L^\beta(\Omega)}\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)}\|\varepsilon(\mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \qquad (6.34)
\end{aligned}
$$

where $1/\beta + 1/p_2 + 1/p = 1$ and $1/\beta = 1 - 1/p_2 - 1/p$, $p_2 = q/(p-1)$ with $q > p$ when $d = 2$ and $\beta = 6$, $p_2 = 6p/(5p-6)$ when $d = 3$.

Thus, using the triangle inequality we obtain

$$
\begin{aligned}
\|\pi-\pi_h\|_{L^{s'}(\Omega)} &\le \|\pi-q_h\|_{L^{s'}(\Omega)}+\|\pi_h-q_h\|_{L^{s'}(\Omega)}\\
&\le (1+\frac{M}{\beta})\|\pi-q_h\|_{L^{s'}(\Omega)}+\frac{C_s\nu_2}{\beta}\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)}\\
&\quad +\frac{C_s\nu_3}{\beta}\|\nabla(\vartheta-\vartheta_h)\|\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)}. \qquad (6.35)
\end{aligned}
$$

Next, we estimate the error on the temperature. Taking the difference between (6.15) and (6.5) and choosing $\varrho = \varrho_h$ yield

$$
a_2(\vartheta_h-\vartheta,\varrho_h)+c_h(\mathbf{u}_h,\vartheta_h,\varrho_h)-c_h(\mathbf{u},\vartheta,\varrho_h)=c_h(\mathbf{u}-\mathbf{u}_h,\Theta_0,\varrho_h).
$$

Adding and subtracting the two terms $a_2(s_h,\varrho_h)$ and $c_h(\mathbf{u}_h,s_h,\varrho_h)$ we have, for all $s_h \in V_{0,h}$ and for all $\mathbf{v}_h \in \mathbf{V}_h$

$$
\begin{aligned}
a_2(\vartheta_h-s_h,\varrho_h)+c_h(\mathbf{u}_h,\vartheta_h-s_h,\varrho_h) &= a_2(\vartheta-s_h,\varrho_h)+c_h(\mathbf{u}_h,\vartheta-s_h,\varrho_h)\\
&+ c_h(\mathbf{u}-\mathbf{v}_h,\vartheta,\varrho_h)+c_h(\mathbf{v}_h-\mathbf{u}_h,\vartheta,\varrho_h)\\
&+ c_h(\mathbf{u}-\mathbf{v}_h,\Theta_0,\varrho_h)+c_h(\mathbf{v}_h-\mathbf{u}_h,\Theta_0,\varrho_h). \qquad (6.36)
\end{aligned}
$$

Next, taking $\varrho_h = \vartheta_h - s_h$ in (6.36), noting $c_h(\mathbf{u}_h,\vartheta_h-s_h,\vartheta_h-s_h)=0$, using the coercivity and continuity properties of $a_2(\cdot,\cdot)$, the continuity property (6.12) of $c_h(\cdot,\cdot,\cdot)$, the stability estimates for $\mathbf{u}_h$, ϑ and Θ_0 we have

$$\begin{aligned}\|\nabla(\vartheta_h - s_h)\| \quad &\leq \quad \Lambda_1\|\nabla(\vartheta - s_h)\| \\ &\qquad + \Lambda_2\left(\|\varepsilon(\mathbf{u}-\mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} + \|\varepsilon(\mathbf{v}_h-\mathbf{u}_h)\|_{\mathbf{L}^s(\Omega)}\right) \end{aligned} \tag{6.37}$$

where

$$\begin{aligned}\Lambda_1 &= 1 + C_f, \\ \Lambda_2 &= \frac{2}{\kappa}\left(\frac{\tilde{C}_2}{\kappa}\|g\| + \left(\tilde{C}_3 + \tilde{C}_3 C_f\right)\|\theta\|_{H^{1/2}(\Gamma)}\right). \end{aligned} \tag{6.38}$$

To take advantage of the above inequality, we now estimate $\|\varepsilon(\mathbf{u}_h - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)}$.

Consider the momentum equation and take the difference between (6.3) and (6.13). Choosing $\mathbf{u}_h - \mathbf{v}_h$ as test function we obtain

$$a_1(\vartheta+\Theta_0,\mathbf{u};\mathbf{u}_h-\mathbf{v}_h) - a_1(\vartheta_h+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h) + b(\mathbf{u}_h-\mathbf{v}_h,\pi-\pi_h) = 0. \tag{6.39}$$

Adding and subtracting $a_1(\vartheta+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h)$ we get

$$\begin{aligned}&a_1(\vartheta+\Theta_0,\mathbf{u};\mathbf{u}_h-\mathbf{v}_h) - a_1(\vartheta+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h) \\ &+ a_1(\vartheta+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h) - a_1(\vartheta_h+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h) \\ &+ b(\mathbf{u}_h-\mathbf{v}_h,\pi-\pi_h) = 0.\end{aligned} \tag{6.40}$$

Now, recalling

$$\begin{aligned}b(\mathbf{u}_h-\mathbf{v}_h,\pi-\pi_h) &= b(\mathbf{u}_h-\mathbf{u},\pi-q_h) + b(\mathbf{u}_h-\mathbf{u},q_h-\pi_h) \\ &\quad + b(\mathbf{u}-\mathbf{v}_h,\pi-\pi_h) = b(\mathbf{u}_h-\mathbf{u},\pi-q_h) + b(\mathbf{u}-\mathbf{v}_h,\pi-\pi_h),\end{aligned}$$

adding and subtracting $a_1(\vartheta+\Theta_0,\mathbf{v}_h;\mathbf{u}_h-\mathbf{v}_h)$ we get

$$\begin{aligned}&a_1(\vartheta+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h) - a_1(\vartheta+\Theta_0,\mathbf{v}_h;\mathbf{u}_h-\mathbf{v}_h) = \\ &a_1(\vartheta+\Theta_0,\mathbf{u};\mathbf{u}_h-\mathbf{v}_h) - a_1(\vartheta+\Theta_0,\mathbf{v}_h;\mathbf{u}_h-\mathbf{v}_h) \\ &+ a_1(\vartheta+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h) - a_1(\vartheta_h+\Theta_0,\mathbf{u}_h;\mathbf{u}_h-\mathbf{v}_h) \\ &+ b(\mathbf{u}_h-\mathbf{u},\pi-q_h) + b(\mathbf{u}-\mathbf{v}_h,\pi-\pi_h).\end{aligned} \tag{6.41}$$

From (6.41) we employ, depending on the value of η_∞, (6.25)-(6.26) (or (6.27)-(6.28)), combined with (5.64), (6.34) and (6.6). This yields

$$\begin{aligned}&C\nu_1\|\varepsilon(\mathbf{u}_h-\mathbf{v}_h)\|^2_{\mathbf{L}^s(\Omega)} \\ &\leq C_s\nu_2\|\varepsilon(\mathbf{u}-\mathbf{v}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)}\|\varepsilon(\mathbf{u}_h-\mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ &+ C_s\nu_3\|\nabla(\vartheta-\vartheta_h)\|\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)}\|\varepsilon(\mathbf{u}_h-\mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ &+ M\|\varepsilon(\mathbf{u}_h-\mathbf{u})\|_{\mathbf{L}^s(\Omega)}\|\pi-q_h\|_{L^{s'}(\Omega)} + M\|\varepsilon(\mathbf{u}-\mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)}\|\pi-\pi_h\|_{L^{s'}(\Omega)}\end{aligned} \tag{6.42}$$

where $C = \eta_\infty$ for $\eta_\infty > 0$ and $C = C_2\left(1 + \|\mathbf{u}_h\|_{1,p}^p + \|\mathbf{v}_h\|_{1,p}^p\right)^{(p-2)/p}$ for $\eta_\infty = 0$. Using (6.35) we have

$$\begin{aligned} C\nu_1\|\varepsilon(\mathbf{u}_h - \mathbf{v}_h)\|^2_{\mathbf{L}^s(\Omega)} \le\ & C_s\nu_2\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)}\|\varepsilon(\mathbf{u}_h - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ & + C_s\nu_3\|\nabla(\vartheta - \vartheta_h)\|\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)}\|\varepsilon(\mathbf{u}_h - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ & + M\|\varepsilon(\mathbf{u}_h - \mathbf{u})\|_{\mathbf{L}^s(\Omega)}\|\pi - q_h\|_{L^{s'}(\Omega)} \\ & + M(1 + \frac{M}{\beta})\|\pi - q_h\|_{L^{s'}(\Omega)}\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ & + M\frac{C_3\nu_2}{\beta}\|\varepsilon(\mathbf{u} - \mathbf{u}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)}\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ & + M\frac{\eta_0\nu_3}{\beta}\|\nabla(\vartheta - \vartheta_h)\|\|\mathbf{u}_h\|_{\mathbf{W}^{1,p_2}(\Omega)}\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)}. \end{aligned}$$

Then, applying the triangle inequality together with the generalized Young's inequality (with exponents $2/(s-1)$ and $(3-s)/2$) we have, with suitable positive constants $D_1, D_2, D_3, D_4, D_5, D_6$ independent of h and possibly dependent on problem data, the following inequalities

$$\begin{aligned} \|\varepsilon(\mathbf{u}_h - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \le\ & D_1\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)} + D_2\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{s/2}_{\mathbf{L}^s(\Omega)} \\ & + D_3\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{\frac{1}{3-s}}_{\mathbf{L}^s(\Omega)} + D_4\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ & + D_5\|\pi - q_h\|_{\mathbf{L}^{s'}(\Omega)} + D_6\|\nabla(\vartheta - \vartheta_h)\|\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)} \\ \le\ & D_7\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)} + D_5\|\pi - q_h\|_{L^{s'}(\Omega)} \\ & + D_6\|\nabla(\vartheta - \vartheta_h)\|\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)} \end{aligned} \tag{6.43}$$

where in the last step we assumed, for a sufficiently small h, the existence of $\mathbf{v}_h$ so that $e_h := \|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} < 1$ (cf. (P1)) and we retained the lowest power of e_h for $s \in (1, 2]$.

Now, using (6.37) and (6.43) we obtain

$$\begin{aligned} \|\nabla(\vartheta_h - s_h)\| \le\ & \Lambda_1\|\nabla(\vartheta - s_h)\| + \Lambda_2\left(\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} + \|\varepsilon(\mathbf{v}_h - \mathbf{u}_h)\|_{\mathbf{L}^s(\Omega)}\right) \\ \le\ & \Lambda_1\|\nabla(\vartheta - s_h)\| + \Lambda_2\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ & + \Lambda_2\left(D_5\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)} + D_2\|\pi - q_h\|_{L^{s'}(\Omega)}\right) \\ & + \Lambda_2 D_6\|\nabla(\vartheta - \vartheta_h)\|\|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)}. \end{aligned}$$

Employing the triangle inequality we then get

$$\begin{aligned} \|\nabla(\vartheta - \theta_h)\| \le\ & (1 + \Lambda_1)\|\nabla(\vartheta - s_h)\| + \Lambda_2\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|_{\mathbf{L}^s(\Omega)} \\ & + \Lambda_2\left(D_5\|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|^{s-1}_{\mathbf{L}^s(\Omega)} + D_2\|\pi - q_h\|_{L^{s'}(\Omega)}\right) \end{aligned}$$

$$+\Lambda_2 D_6 \|\nabla(\vartheta - \vartheta_h)\| \|\varepsilon(\mathbf{u}_h)\|^{s-1}_{\mathbf{L}^{(s-1)p_2}(\Omega)}. \tag{6.44}$$

Therefore, in the case $s = p$, i.e., $\eta_\infty = 0$, thanks to assumption (A1), we infer (6.16) and thus, by (6.43), (6.17). Finally, using both (6.16) and (6.17), from (6.35) we are led to (6.18). Analogously, in the case $s = 2$, i.e., $\eta_\infty > 0$, exploiting (6.44) and assuming (B1), we infer first (6.19), then (6.20) and in conclusion (6.21). The proof is finished.

6.5 Numerical Experiments

The aim of this section is twofold: (a) to corroborate the theoretical estimates of Theorems 6.2 and 6.3; (b) to explore the rôle of the regularizing parameter σ in the approximating problem (4.1)-(4.4).

Having in mind these goals, we perform the numerical tests on the two-dimensional unit square $\Omega = (0,1)^2$ by employing the following finite element spaces

$$\begin{aligned}
\mathbf{V}_h &= \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}) | \forall K \in \mathcal{T}_h,\ \mathbf{v}_h|_K \in \mathbb{P}_{r+1}(K)^2\},\\
Q_h &= \{q_h \in C(\overline{\Omega}) | \forall K \in \mathcal{T}_h,\ q_h|_K \in \mathbb{P}_r(K)\},\\
V_h &= \{\varrho_h \in C(\overline{\Omega}) | \forall K \in \mathcal{T}_h,\ \varrho_h|_K \in \mathbb{P}_{r+1}(K)\},
\end{aligned}$$

with $r \geq 1$. Note that the compatibility condition (6.10) is satisfied; see, e.g., [22, 39].

The discrete nonlinear problem (6.13)-(6.15) is solved by resorting to the following fixed point strategy.
Set $(\mathbf{u}_h^{(0)}, \vartheta_h^{(0)}) = (\mathbf{0}, 0)$, $k = 0$ and iterate:
Step 1. Given $(\mathbf{u}_h^{(k)}, \vartheta_h^{(k)})$ find $(\mathbf{u}_h^{(k+1)}, \pi_h^{(k+1)})$ such that for all $(\mathbf{v}_h, q_h)$ there holds

$$\begin{aligned}
\widetilde{a}_1(\vartheta_h^{(k)} + \Theta_0, \mathbf{u}_h^{(k)}; \mathbf{u}_h^{(k+1)}, \mathbf{v}_h) + b(\mathbf{v}_h, \pi_h^{(k+1)}) &= \mathbf{f}(\mathbf{v}_h)\\
b(\mathbf{u}_h^{(k+1)}, q_h) &= 0
\end{aligned}$$

where
$$\widetilde{a}_1(\vartheta, \mathbf{w}; \mathbf{u}, \mathbf{v}) = \int_\Omega 2\nu(\vartheta)\eta(|\varepsilon(\mathbf{w})|^2)\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}).$$

Step 2. Given $\mathbf{u}_h^{(k+1)}$ find $\vartheta_h^{(k+1)}$ such that for all ρ_h there holds

$$a_2(\vartheta_h^{(k+1)}, \varrho_h) + c_h(\mathbf{u}_h^{(k)}, \vartheta_h^{(k+1)}, \varrho_h) = g(\varrho_h) - a_2(\Theta_0, \varrho_h) - c_h(\mathbf{u}_h^{(k)}, \Theta_0, \varrho_h).$$

Step 3. $k + 1 \to k$

The iteration is stopped when

$$\|\mathbf{u}_h^{(k+1)} - \mathbf{u}_h^{(k)}\|_{\mathbf{L}^s(\Omega)} + \|\pi_h^{(k+1)} - \pi_h^{(k)}\|_{L^{s'}(\Omega)} + \|\vartheta_h^{(k+1)} - \vartheta_h^{(k)}\| < \texttt{tol}.$$

Numerical tests are performed with $\texttt{tol} = 10^{-10}$ using the high level C++ interface of FEniCS-DOLFIN (see [50] and [51]).

6.5.1 Test 1

We consider the finite element approximation of the non-isothermal non-Newtonian flow problem governed by the Carreau law with $\nu(\xi) = e^{-\xi}$, $\xi \in \mathbb{R}^+$, cf. (6.13)-(6.15). The source term $\mathbf{f}$ is manufactured so that the exact solution is given by

$$u_x(x,y) = 5y\sin(x^2+y^2) + 4y\sin(x^2-y^2), \quad (6.45)$$

$$u_y(x,y) = -5x\sin(x^2+y^2) + 4x\sin(x^2-y^2), \quad (6.46)$$

$$\pi(x,y) = \sin(x+y), \quad (6.47)$$

$$\theta(x,y) = \cos(xy), \quad (6.48)$$

where $\mathbf{u} = (u_x, u_y)$. Dirichlet boundary conditions for velocity and temperature given by the exact solution are imposed on the domain boundary.

We first consider the problem with the Carreau law parameters in (6.1) defined as $\eta_\infty = 0.5, \eta_0 = 2, \lambda = 1$ for different values of $p = 2, 1.6, 1.2$.

The convergence results in terms of the $\mathbf{L}^2$ and $\mathbf{H}^1$ velocity errors, L^2 pressure error and H^1 temperature error, obtained using quadratic ($r = 2$) finite elements for velocity and temperature and linear ($r = 1$) finite elements for pressure, are reported in Figure 1 (left).

The corresponding results when cubic ($r = 3$) finite elements for velocity and temperature and quadratic ($r = 2$) finite elements for pressure are adopted, as displayed in Figure 1 (right). In both cases, optimal convergence rates are achieved (with a slight superconvergence for the pressure error). This is in agreement with the approximation results of Theorem 6.3 combined with standard interpolation error estimates (see, e.g., [21]).

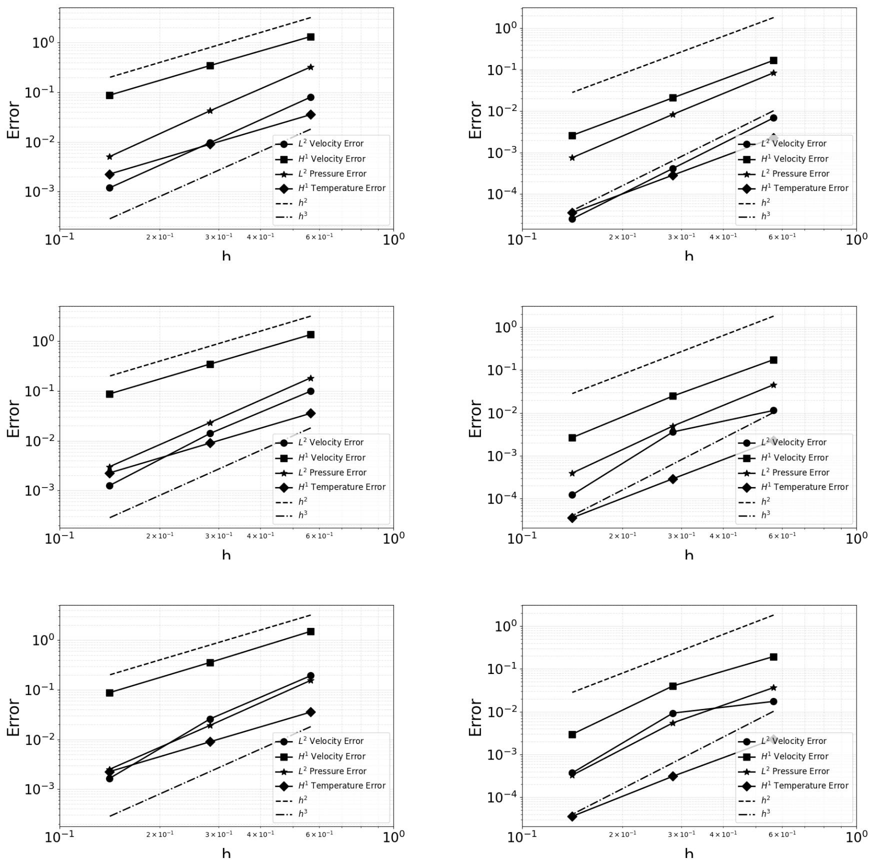

Figure 1: Test 1. Convergence test for the Carreau model with $\eta_\infty = 0.5, \eta_0 = 2, \lambda = 1$ and different values of $p = 2, 1.6, 1.2$ (from top to bottom) using $\mathcal{P}_2/\mathcal{P}_1/\mathcal{P}_2$ (left) and $\mathcal{P}_3/\mathcal{P}_2/\mathcal{P}_3$ (right) finite elements

6.5.2 Test 2

In this second test case, we investigate the rôle of the regularizing parameter σ in the approximating problem (4.1)-(4.4) by solving the same manufactured solution problem introduced in the first test case, but considering the Carreau model with $\eta_\infty = 0, \eta_0 = 2, \lambda = 1$ and different values of p. Consistently with the value of η_∞, the velocity error is computed in $\mathbf{W}^{1,p}$, the pressure error in $L^{p'}$, while the temperature error is computed in H^1.

The convergence results obtained using $\mathcal{P}_2/\mathcal{P}_1/\mathcal{P}_2$ finite elements are displayed in Figures 2, 3 and 4 for $p = 2, 1.6, 1.2$, respectively. In each plot the slope of the dotted reference line is computed by employing the values of the corresponding error obtained for $\sigma = 0$ and the two smallest values of h. As expected, for $\sigma \neq 0$ the error between the exact solution of (6.3)-(6.5) with $\eta_\infty = 0$ and the approximation of the solution to the regularized problem (4.1)-(4.4) exhibits an asymptotic plateau for h tending to zero, where the value of the plateau decreases as σ tends to zero. When the regularization parameter σ is set to zero, from Theorem 6.2 and standard interpolation error estimates we expect the velocity and the temperature errors to behave like $h^{2(p-1)}$, while the pressure error as $h^{2(p-1)^2}$. The obtained convergence results are coherent with the theoretical estimates in Theorem 6.2, in particular the rate of convergence of the pressure error decreases with p. However, it seems that the asymptotic regime is not yet reached; a similar numerical behavior, in the context of isothermal non-Newtonian fluids has been also observed, e.g., in [19].

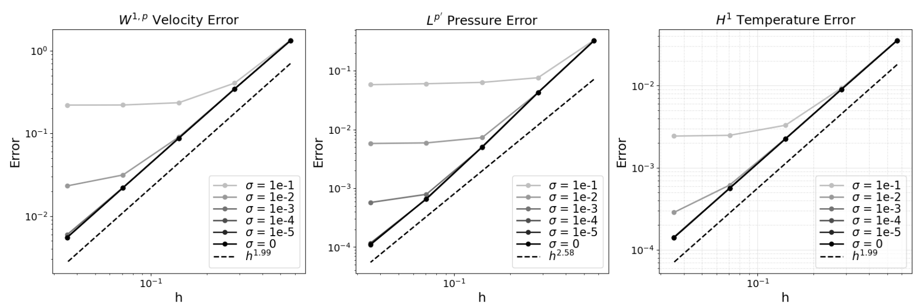

Figure 2: Test 2. Convergence test for the Carreau model with $\eta_\infty = 0, \eta_0 = 2.\lambda = 1, p = 2$ and different values of the regularization parameter σ using $\mathcal{P}_2/\mathcal{P}_1/\mathcal{P}_2$ finite elements.

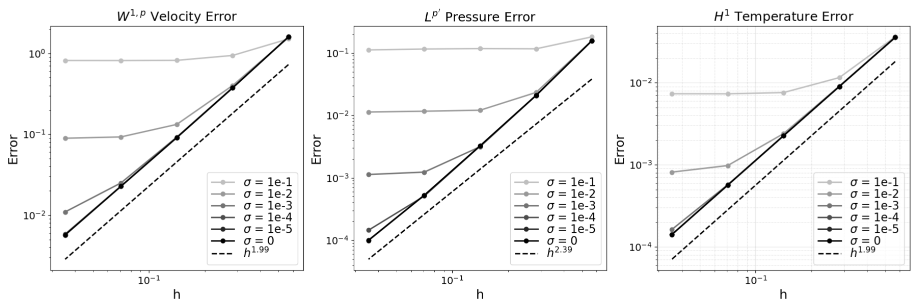

Figure 3: Test 2. Convergence test for the Carreau model with $\eta_\infty = 0, \eta_0 = 2, \lambda = 1, p = 1.6$ *and different values of the regularization parameter* σ *using* $\mathcal{P}_2/\mathcal{P}_1/\mathcal{P}_2$ *finite elements.*

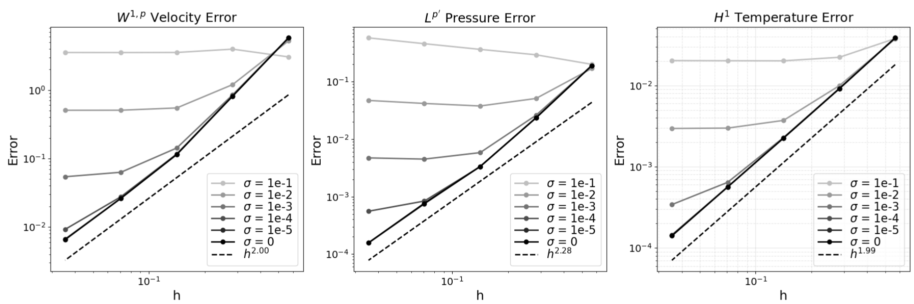

Figure 4: Test 2. Convergence test for the Carreau model with $\eta_\infty = 0, \eta_0 = 2, \lambda = 1, p = 1.2$ *and different values of the regularization parameter* σ *using* $\mathcal{P}_2/\mathcal{P}_1/\mathcal{P}_2$ *finite elements.*

Similar comments apply when $\mathcal{P}_3/\mathcal{P}_2/\mathcal{P}_3$ finite elements are employed. In this case we expect the velocity and the temperature errors to behave like $h^{3(p-1)}$, while the pressure error to converge as $h^{3(p-1)^2}$. The corresponding results are reported in Figures 5, 6 and 7 for $p = 2, 1.6, 1.2$, respectively.

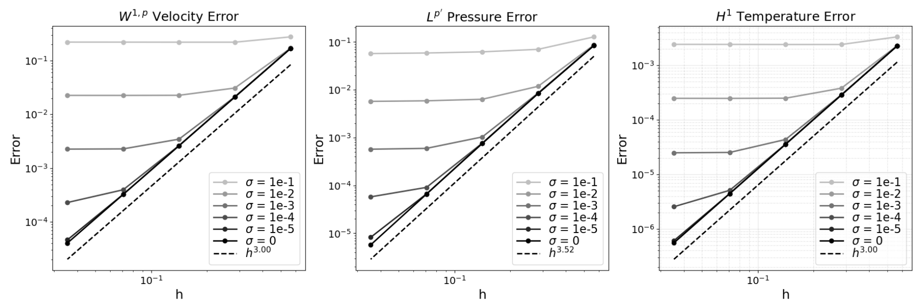

Figure 5: Test 2. Convergence test for the Carreau model with $\eta_\infty = 0, \eta_0 = 2, \lambda = 1, p = 2$ *and different values of the regularization parameter* σ *using* $\mathcal{P}_3/\mathcal{P}_2/\mathcal{P}_3$ *finite elements.*

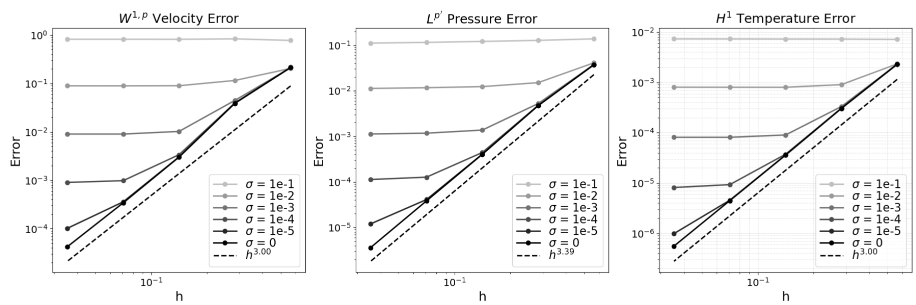

Figure 6: Test 2. Convergence test for the Carreau model with $\eta_\infty = 0, \eta_0 = 2, \lambda = 1, p = 1.6$ *and different values of the regularization parameter* σ *using* $\mathcal{P}_3/\mathcal{P}_2/\mathcal{P}_3$ *finite elements.*

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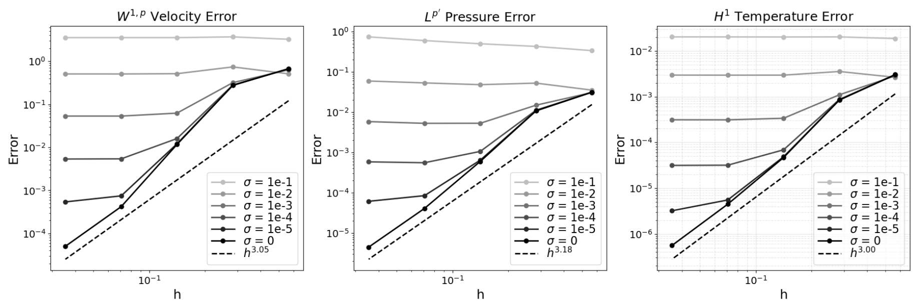

Figure 7: Test 2. Convergence test for the Carreau model with $\eta_\infty = 0, \eta_0 = 2, \lambda = 1, p = 1.2$ *and different values of the regularization parameter* σ *using* $\mathcal{P}_3/\mathcal{P}_2/\mathcal{P}_3$ *finite elements.*

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