MOX–Report No. 06/2012

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Anisotropic recovery-based a posteriori error estimators for advection-diffusion-reaction problems

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January 23, 2012

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Keywords: advection-diffusion-reaction problems; anisotropic mesh adaptation; Zienkiewicz-Zhu a posteriori error estimator; goal-oriented analysis.

AMS Subject Classification: 65N15, 65N30, 65N50, 65K10

Abstract

We combine the good properties of recovery-based error estimators with the richness of information typical of an anisotropic a posteriori analysis. This merging yields error estimators which are general purpose yet simple and easy to implement, and automatically incorporate detailed geometric information about the computational mesh. This allows us to devise an effective anisotropic mesh adaptation procedure suited to control the discretization error both in the energy norm and in a goal-oriented framework. The advection-diffusion-reaction problem is considered as a computational paradigm.

1 Introduction

Advection-diffusion-reaction (ADR) problems can be interesting *per se* (e.g., pollution transport in air or rivers, population dynamics in biology) or can be employed as downscaled models for studying more complex problems in computational fluid dynamics (e.g., the Navier-Stokes equations for modeling viscous flows around bodies). A joint effect of geometry, advective field pattern, and boundary conditions may sometimes render ADR problems hard to be numerically solved unless *ad hoc* numerical schemes or computational meshes are employed.

The objective of this work is to propose practical a posteriori error estimators for driving an anisotropic adaptation of the mesh, i.e., where not only the size but also the shape and the orientation of the elements are controlled so as to match the directional features of the solution. It is in fact well known that anisotropic mesh adaptation is cost-effective in dealing with a broad range of problems ([2, 3, 4, 8]). In particular, we stick to recovery-based estimators, relying on the ideas proposed by O.C. Zienkiewicz and J.Z. Zhu in [9]. After devising a simple recovery technique, different from the standard one, we introduce an estimator for controlling the H^1-seminorm of the discretization error [5, 6, 7]. This estimator automatically includes the anisotropic features (size, aspect ratio, and orientation) of the triangulation in contrast to the standard Zienkiewicz-Zhu estimator.

The strong interest in approximating goal quantities for practical applications led us to extend the theory in [5, 6, 7] to a goal-oriented setting, showing that recovery-based and goal-oriented are compatible approaches which can be merged in an effective and practical way.
As a reference ADR problem used to introduce the new anisotropic estimator we employ the standard one completed with homogeneous Dirichlet boundary conditions, i.e., find $u \in V = H_0^1(\Omega)$, such that

$$a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \, d\mathbf{x} + \int_\Omega \boldsymbol{\beta} \cdot \nabla u \, v \, d\mathbf{x} + \int_\Omega \sigma \, u \, v \, d\mathbf{x} = \int_\Omega f v \, d\mathbf{x} \quad \forall v \in V, \quad (1)$$

where Ω is a polygonal domain in $\mathbb{R}^2$, $\mu > 0$ is the diffusion coefficient, $\boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^2$ is the advective field, $\sigma \in L^\infty(\Omega)$ is the reactive coefficient, and where standard notation are adopted for the Lebesgue and Sobolev spaces and their norms. To guarantee the well-posedness of (1) we add the assumption $\sigma - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \geq 0$.
The structure of a recovery-based estimator allows us a straightforward extension of the a posteriori analysis below to other types of boundary conditions.

2 Zienkiewicz-Zhu like anisotropic error estimators

The good properties of the recovery-based error estimators (independence of the problem, computational easiness, effectiveness) justify their broad use, mostly because they work pretty well in practice in many engineering applications. On the other hand, the presence of strong directional features in such applications requires ad hoc meshes to sharply detect the phenomena of interest. To meet this demand, we have proposed in [5, 6, 7] a suitable enrichment of the standard recovery-based estimators, which explicitly takes into account the intrinsic directionalities of the problem. For this purpose, let us first lay down the anisotropic background.

2.1 The anisotropic setting

Let $\mathcal{T}_h = \{K\}$ be a conforming partition of Ω consisting of triangles. According to the anisotropic framework in [1], the size, shape, and orientation of each element

K of $\mathcal{T}_h$ are described by means of the affine map $T_K : \widehat{K} \to K$ between the reference triangle $\widehat{K}$ and the generic element $K \in \mathcal{T}_h$. In particular, we pick $\widehat{K}$ as the equilateral triangle centered at the origin, with coordinates $(-\sqrt{3}/2, -1/2)$, $(\sqrt{3}/2, -1/2)$, $(0, 1)$ and edge length $\sqrt{3}$. The map T_K writes out as $\mathbf{x} = T_K(\widehat{\mathbf{x}}) = M_K \widehat{\mathbf{x}} + \mathbf{t}_K$, where $M_K \in \mathbb{R}^{2\times 2}$ is the Jacobian and $\mathbf{t}_K \in \mathbb{R}^2$ is the shift vector. To get the anisotropic information associated with K out of T_K, we factorize M_K via the polar decomposition as $M_K = B_K Z_K$, where $B_K \in \mathbb{R}^{2\times 2}$ is symmetric positive definite, and $Z_K \in \mathbb{R}^{2\times 2}$ is orthogonal. Then B_K is spectrally decomposed as $B_K = R_K^T \Lambda_K R_K$, with $R_K^T = [\mathbf{r}_{1,K}, \mathbf{r}_{2,K}]$ and $\Lambda_K = \mathrm{diag}(\lambda_{1,K}, \lambda_{2,K})$ the eigenvector and eigenvalue matrix, respectively. The map T_K stretches the unit circle circumscribing $\widehat{K}$ into an ellipse circumscribing K: the unit vectors $\{\mathbf{r}_{i,K}\}$ provide us with the corresponding principal directions, whereas the eigenvalues $\{\lambda_{i,K}\}$ are the length of the ellipse semi-axes. Without loss of generality, we assume $\lambda_{1,K} \geq \lambda_{2,K} > 0$ so that the aspect ratio, $s_K = \lambda_{1,K}/\lambda_{2,K}$ is always greater than or equal to one, for any $K \in \mathcal{T}_h$, equality holding when K is equilateral.

2.2 An error estimator for the H^1-seminorm

In [5] we propose an a posteriori error estimator for the H^1-seminorm of the discretization error $e_h = u - u_h$, where u_h is the Galerkin affine finite element approximation to (1). The actual estimator reads

$$\eta_{H^1}^2 = \sum_{K\in\mathcal{T}_h} \big[\eta_{K,H^1}\big]^2, \quad \big[\eta_{K,H^1}\big]^2 = \frac{1}{\lambda_{1,K}\lambda_{2,K}} \sum_{i=1}^{2} \lambda_{i,K}^2 \left(\mathbf{r}_{i,K}^T\, G_K(\mathbf{E}_K(u_h))\mathbf{r}_{i,K}\right), \tag{2}$$

where $G_K(\cdot)$ is the symmetric positive semidefinite matrix with entries

$$[G_K(\mathbf{w})]_{i,j} = \sum_{T\in\Delta_K} \int_T w_i\, w_j\, d\mathbf{x}, \quad \text{with } i,j = 1,2, \tag{3}$$

with $\Delta_K = \{T \in \mathcal{T}_h : T \cap K \neq \emptyset\}$, and where $\mathbf{E}_K(u_h) = P_{\Delta_K}(\nabla u_h) - \nabla u_h|_{\Delta_K}$ is the approximation, over Δ_K, to the error on the gradient via a suitable recovered gradient $P_{\Delta_K}(\nabla u_h)$ ([9]). In particular, in [5, 6, 7] we employ as recovery procedure the area-weighted average over the patch Δ_K of the gradients of the discrete solution. Estimator (2) exhibits the standard recovery-based structure in the term $\mathbf{E}_K(u_h)$, while the anisotropic contribution is represented by the weighted projection of the isotropic estimator onto the anisotropic directions $\mathbf{r}_{i,K}$.

2.3 A goal-oriented error estimator

The strong interest in engineering applications prompted us in [7] to generalize estimator (2) to a goal-oriented approach. That approach is however constrained to the Poisson problem and to a special choice of the functional of interest $J : V \to \mathbb{R}$. Here we propose a more general approach suited to dealing with problem

(1) and where J can be any functional in the dual space V'. The dual problem associated with (1) is: find $z \in V$, such that

$$a(v, z) = J(v) \quad \forall v \in V. \tag{4}$$

Combining (1) with (4) and using the Galerkin orthogonality, we get the error representation

$$J(u - u_h) = a(u - u_h, z - z_h), \tag{5}$$

with z_h the Galerkin affine finite element approximation to (4). In a recovery-based spirit, (5) suggests the quantity

$$\int_{\Delta_K} \mu\, \mathbf{E}_K(u_h) \cdot \mathbf{E}_K(z_h)\, d\mathbf{x} + \int_{\Delta_K} \boldsymbol{\beta} \cdot \mathbf{E}_K(u_h) F_K(z_h)\, d\mathbf{x} + \int_{\Delta_K} \sigma\, F_K(u_h) F_K(z_h)\, d\mathbf{x} \tag{6}$$

as a first attempt to estimate $J(e_h)\big|_K$, where the explicit definition of $a(\cdot,\cdot)$ is used and suitable recovered quantities replace the exact fields. In particular, $F_K(u_h) = \big(R(u_h) - u_h\big)\big|_{\Delta_K}$, where $R(u_h)$ is the affine field recovered via the arithmetic average over the patch $\Delta_N = \{T \in \mathcal{T}_h : T \ni N\}$ of u_h at the centroids of $T \in \Delta_N$, with N the generic node of $\mathcal{T}_h$.

The next step is to convert (6) into an anisotropic source of information. The strategy that we pursue casts the generic term of (1) in the reference framework ($\widehat{\Delta}_K = T_K^{-1}(\Delta_K)$) and then carries it back to the physical framework, employing the spectral properties of T_K. This leads for free to a structure similar to the one in (2), i.e., with built-in anisotropic quantities. Let us exemplify this procedure starting from the diffusive term. We employ the relations $\widehat{\nabla}\widehat{u} = M_K^T \nabla u$, $|\widehat{\Delta}_K| = |\Delta_K|/(\lambda_{1,K}\lambda_{2,K})$, and $\widehat{u} = u \circ T_K$ (and similarly for v), and the decompositions of the Jacobian M_K in Sect. 2.1, to get

$$\begin{aligned}
\int_{\widehat{\Delta}_K} \widehat{\mu} \widehat{\nabla}\widehat{u} \cdot \widehat{\nabla}\widehat{v}\, d\widehat{\mathbf{x}} &= \frac{1}{\lambda_{1,K}\lambda_{2,K}} \int_{\Delta_K} \mu \Lambda_K R_K(\nabla u) \cdot \Lambda_K R_K \nabla v\, d\mathbf{x} \\
&= \int_{\Delta_K} \mu \big[s_K (\mathbf{r}_{1,K} \cdot \nabla u)(\mathbf{r}_{1,K} \cdot \nabla v) + s_K^{-1} (\mathbf{r}_{2,K} \cdot \nabla u)(\mathbf{r}_{2,K} \cdot \nabla v) \big]\, d\mathbf{x} \\
&= s_K \mathbf{r}_{1,K}^T G_{K,\mu}(\nabla u, \nabla v) \mathbf{r}_{1,K} + s_K^{-1} \mathbf{r}_{2,K}^T G_{K,\mu}(\nabla u, \nabla v) \mathbf{r}_{2,K},
\end{aligned} \tag{7}$$

where $G_{K,\mu}$ is the matrix with entries $[G_{K,\mu}(\mathbf{t}, \mathbf{w})]_{ij} = \int_{\Delta_K} \mu \mathbf{t}_i \mathbf{w}_j\, d\mathbf{x}$, $i, j = 1, 2$, for $\mathbf{t}, \mathbf{w} : \Omega \to \mathbb{R}^2$. It is guaranteed the consistency with the isotropic case ($\lambda_{1,K} = \lambda_{2,K}$). In an analogous manner, the advective term becomes

$$\begin{aligned}
&(\lambda_{1,K}\lambda_{2,K})^{1/2} \int_{\widehat{\Delta}_K} \widehat{\boldsymbol{\beta}} \cdot \widehat{\nabla}\widehat{u}\; \widehat{v}\, d\widehat{\mathbf{x}} = (\lambda_{1,K}\lambda_{2,K})^{-1/2} \int_{\Delta_K} \boldsymbol{\beta}^T Z_K^T R_K^T \Lambda_K R_K \nabla u\; v\, d\mathbf{x} \\
&= (\lambda_{1,K}\lambda_{2,K})^{-1/2} \int_{\Delta_K} (Z_K \boldsymbol{\beta})^T \big[\lambda_{1,K} (\mathbf{r}_{1,K} \cdot \nabla u) \mathbf{r}_{1,K} + \lambda_{2,K} (\mathbf{r}_{2,K} \cdot \nabla u) \mathbf{r}_{2,K} \big]\, v\, d\mathbf{x} \\
&= s_K^{1/2} \mathbf{r}_{1,K}^T G_{K,\boldsymbol{\beta}}(\nabla u, v) \mathbf{r}_{1,K} + s_K^{-1/2} \mathbf{r}_{2,K}^T G_{K,\boldsymbol{\beta}}(\nabla u, v) \mathbf{r}_{2,K},
\end{aligned} \tag{8}$$

where the entries of $G_{K,\boldsymbol{\beta}}$ are $[G_{K,\boldsymbol{\beta}}(\mathbf{t},w)]_{ij} = \int_{\Delta_K} (Z_K\boldsymbol{\beta})_i \mathbf{t}_j w \, d\mathbf{x}$, $i,j=1,2$, for $\mathbf{t}:\Omega\to\mathbb{R}^2$ and $w:\Omega\to\mathbb{R}$. The consistency with the isotropic case is recovered via the scaling factor $(\lambda_{1,K}\lambda_{2,K})^{1/2}$.
The reactive term does not provide any anisotropic contribution.

The right-hand sides in (7) and (8) yield the anisotropic counterpart of the first two terms in (6) after replacing ∇u with $\mathbf{E}_K(u_h)$, ∇v with $\mathbf{E}_K(z_h)$, and ∇u with $\mathbf{E}_K(u_h)$, v with $F_K(z_h)$, respectively. This suggests as a first anisotropic attempt to estimate $J(e_h)\big|_K$ the quantity

$$
\begin{aligned}
& s_K \mathbf{r}_{1,K}^T G_{K,\mu}(\mathbf{E}_K(u_h),\mathbf{E}_K(z_h))\mathbf{r}_{1,K} + s_K^{-1}\mathbf{r}_{2,K}^T G_{K,\mu}(\mathbf{E}_K(u_h),\mathbf{E}_K(z_h))\mathbf{r}_{2,K} \\
+ \; & s_K^{1/2}\mathbf{r}_{1,K}^T G_{K,\boldsymbol{\beta}}(\mathbf{E}_K(u_h),F_K(z_h))\mathbf{r}_{1,K} + s_K^{-1/2}\mathbf{r}_{2,K}^T G_{K,\boldsymbol{\beta}}(\mathbf{E}_K(u_h),F_K(z_h))\mathbf{r}_{2,K} \\
+ \; & \int_{\Delta_K} \sigma \, F_K(u_h) F_K(z_h)\, d\mathbf{x}.
\end{aligned} \tag{9}
$$

To make such an estimator effective, we have to introduce a suitable regularization since both $G_{K,\mu}$ and $G_{K,\boldsymbol{\beta}}$ are neither symmetric nor positive definite.
Since

$$
\begin{aligned}
\mathbf{r}_{i,K}^T G_{K,\mu}(\mathbf{t},\mathbf{w})\mathbf{r}_{i,K} &= \mathbf{r}_{i,K}^T G_{K,\mu}(\mathbf{w},\mathbf{t})\mathbf{r}_{i,K} = \mathbf{r}_{i,K}^T G_{K,\mu}^T(\mathbf{t},\mathbf{w})\mathbf{r}_{i,K}, \\
\mathbf{r}_{i,K}^T G_{K,\boldsymbol{\beta}}(\mathbf{t},w)\mathbf{r}_{i,K} &= \mathbf{r}_{i,K}^T G_{K,Z_K^T\mathbf{t}}(Z_k\boldsymbol{\beta},w)\mathbf{r}_{i,K} = \mathbf{r}_{i,K}^T G_{K,\boldsymbol{\beta}}^T(\mathbf{t},w)\mathbf{r}_{i,K},
\end{aligned}
$$

$i=1,2$, we can replace in (9) the two matrices with their symmetric counterparts $G_{K,\mu}^{sym}(\cdot,\cdot) = (G_{K,\mu}(\cdot,\cdot)+G_{K,\mu}^T(\cdot,\cdot))/2$ and $G_{K,\boldsymbol{\beta}}^{sym}(\cdot,\cdot) = (G_{K,\boldsymbol{\beta}}(\cdot,\cdot)+G_{K,\boldsymbol{\beta}}^T(\cdot,\cdot))/2$.
Next, to ensure the positive definiteness, we replace the symmetrized matrices with the modulus matrices (e.g., if $G = V^T D V$, then $|G| = V^T|D|V$, with D and V the eigenvalues and eigenvectors matrices, respectively); this leads to the definitive estimator

$$
\begin{aligned}
& \eta_{K,J} = \\
& s_K \mathbf{r}_{1,K}^T \big|G_{K,\mu}^{sym}(\mathbf{E}_K(u_h),\mathbf{E}_K(z_h))\big|\mathbf{r}_{1,K} + s_K^{-1}\mathbf{r}_{2,K}^T \big|G_{K,\mu}^{sym}(\mathbf{E}_K(u_h),\mathbf{E}_K(z_h))\big|\mathbf{r}_{2,K} \\
& + s_K^{1/2}\mathbf{r}_{1,K}^T \big|G_{K,\boldsymbol{\beta}}^{sym}(\mathbf{E}_K(u_h),F_K(z_h))\big|\mathbf{r}_{1,K} + s_K^{-1/2}\mathbf{r}_{2,K}^T \big|G_{K,\boldsymbol{\beta}}^{sym}(\mathbf{E}_K(u_h),F_K(z_h))\big|\mathbf{r}_{2,K} \\
& + \Big| \int_{\Delta_K} \sigma \, F_K(u_h) F_K(z_h)\, d\mathbf{x} \Big|.
\end{aligned}
$$

An example of the benefits due to the regularization above is shown in Fig. 1, where we compare the contour lines associated with the absolute value of the quantity in (9) with those of $\eta_{K,J}$, for given indefinite matrices $G_{K,\mu}$, $G_{K,\boldsymbol{\beta}}$ associated with the test case in Sect. 3. Only in the second case does it exist a unique minimum.

Finally, the matrix Z_K in the definition of $G_{K,\boldsymbol{\beta}}(\cdot,\cdot)$ is, in practice, taken as the identity matrix. It represents a degree of freedom in the mesh generation associated with a rotation of K inside the ellipse given by $\{\lambda_{i,K},\mathbf{r}_{i,K}\}_{i=1,2}$. However this information is, usually, not required by a metric-based mesh generator.

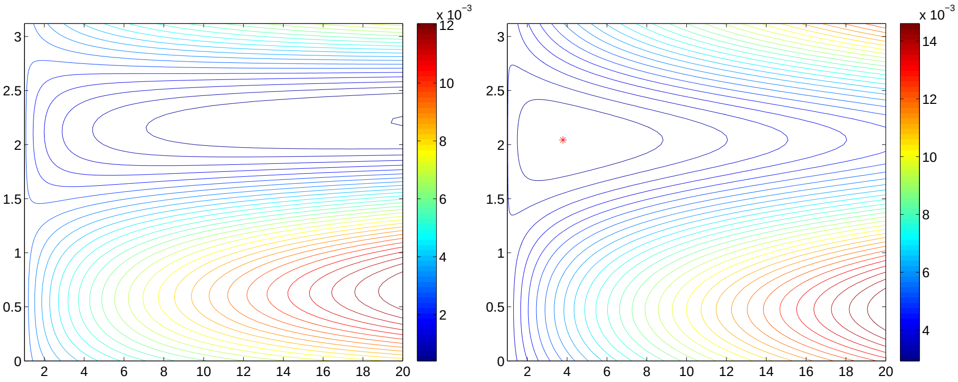

Figure 1: Contour lines of the absolute value of (9) (left) and of $\eta_{K,J}$ (right): the red star marks the minimum

3 Numerical assessment

We provide here the actual procedure employed to convert $\eta_{K,J}$ into a practical tool for driving the mesh adaptation.

3.1 The adaptive procedure

Following, e.g., [7], we first properly scale the matrices in $\eta_{K,J}$ with respect to $|\Delta_K|$, to factor out the patch size information. This yields

$$\eta_{K,J} = |\widehat{\Delta}_K|\lambda_{1,K}\lambda_{2,K}\Big[s_K \mathbf{r}_{1,K}^T \Big|\widetilde{G}_{K,\mu}^{sym}(\mathbf{E}_K(u_h),\mathbf{E}_K(z_h))\Big|\mathbf{r}_{1,K}$$
$$+s_K^{-1}\mathbf{r}_{2,K}^T\Big|\widetilde{G}_{K,\mu}^{sym}(\mathbf{E}_K(u_h),\mathbf{E}_K(z_h))\Big|\mathbf{r}_{2,K} + s_K^{1/2}\mathbf{r}_{1,K}^T\Big|\widetilde{G}_{K,\boldsymbol{\beta}}^{sym}(\mathbf{E}_K(u_h),F_K(z_h))\Big|\mathbf{r}_{1,K}$$
$$+s_K^{-1/2}\mathbf{r}_{2,K}^T\Big|\widetilde{G}_{K,\boldsymbol{\beta}}^{sym}(\mathbf{E}_K(u_h),F_K(z_h))\Big|\mathbf{r}_{2,K} + \frac{1}{|\Delta_K|}\Big|\int_{\Delta_K}\sigma\, F_K(u_h)F_K(z_h)\,d\mathbf{x}\Big|\Big],$$

where $\widetilde{G}_{K,\mu}^{sym}(\cdot,\cdot) = G_{K,\mu}^{sym}(\cdot,\cdot)/|\Delta_K|$, and likewise for $\widetilde{G}_{K,\boldsymbol{\beta}}^{sym}(\cdot,\cdot)$. Then, we minimize the expression in square brackets with respect to the pair $\{s_K, \mathbf{r}_{1,K}\}$ subject to the constraints $s_K \geq 1$ and $\mathbf{r}_{1,K}\cdot\mathbf{r}_{2,K} = 0$, with $\|\mathbf{r}_{1,K}\| = \|\mathbf{r}_{2,K}\| = 1$. For this purpose, we set $\mathbf{r}_{1,K} = [\cos\theta, \sin\theta]^T$ and $\mathbf{r}_{2,K} = [-\sin\theta, \cos\theta]^T$, for a certain $0 \leq \theta < \pi$, and let $F = F(s_K,\theta)$ be the quantity in brackets. For this minimization we use the Matlab function `fmincon`. Moreover, in a *predictive* fashion, the matrices and the area $|\Delta_K|$ are computed on the actual mesh, where both u_h and z_h are obtained. This yields the minimum $F^* = F(s_K^*, \theta^*)$ for the optimal values, $\{s_K^*, \theta^*\}$, and consequently $\mathbf{r}_{1,K}^* = [\cos\theta^*, \sin\theta^*]^T$.
To get the optimal values $\lambda_{1,K}^*$ and $\lambda_{2,K}^*$, we enforce the equidistribution of the error, i.e., $\eta_{K,J} = \mathtt{TOLL}/\#\mathcal{T}_h$, where $\#\mathcal{T}_h$ is the mesh cardinality and TOLL is the accuracy demanded on $J(e_h)$. This yields $\lambda_{1,K}^*\lambda_{2,K}^* = \mathtt{TOLL}/(\#\mathcal{T}_h|\widehat{\Delta}_K|F^*)$. To split the values $\lambda_{1,K}^*$, $\lambda_{2,K}^*$ we finally exploit the identity $s_K^* = \lambda_{1,K}^*/\lambda_{2,K}^*$.
The optimal metric is formed by the optimal values $\lambda_{1,K}^*, \lambda_{2,K}^*, \mathbf{r}_{1,K}^*$.

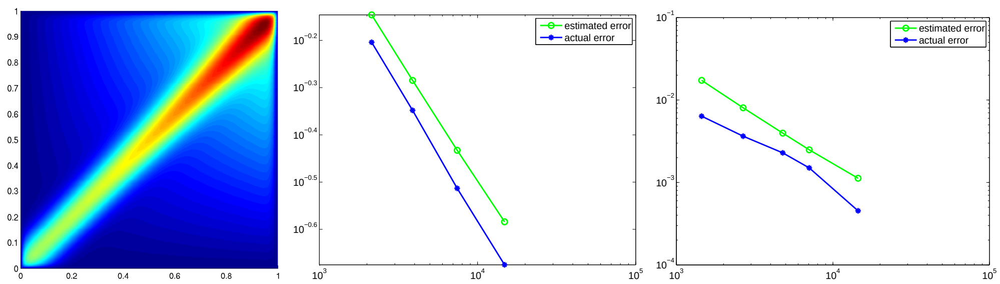

Figure 2: Solution u (left) and convergence history for the H^1-seminorm (center) and for $J = J_1$ (right)

3.2 The "arrow" test case

We consider (1), choosing $\mu = \sigma = 10^{-2}$, $\boldsymbol{\beta} = [1, 1]^T$ on $\Omega = (0, 1)^2$. The exact solution is tailor-made so that it exhibits one internal layer along the SW-NE diagonal and two boundary layers along the top and right sides of Ω (see Fig. 2 (left)):

$$u(x, y) = \big[\alpha(x, y) + \rho(x)\,\rho(y)\big]\,\delta(x)\delta(y),$$

with $\alpha(x, y) = e^{-(y-x)^2/0.01}$, $\rho(\zeta) = \zeta - \big(e^{(\zeta-1)/\varepsilon} - e^{-1/\varepsilon}\big)/\big(1 - e^{-1/\varepsilon}\big)$, $\delta(\zeta) = 1 - e^{-\zeta/\varepsilon} + e^{-1/\varepsilon} - e^{-(1-\zeta)/\varepsilon}$, and $\varepsilon = 10^{-2}$. The source term is computed as $f = -\mu\,\Delta u + \boldsymbol{\beta} \cdot \nabla u + \sigma\, u$.

On this test case we assess the performance of both the estimators defined by $\eta_{K,\,H^1}$ and $\eta_{K,J}$. Let us start from the H^1-seminorm control. In Fig. 3 (left) we show the adapted grid for the tolerance $\mathtt{TOLL} = 10^{-2}/2$: it consists of 3887 elements which perfectly capture all the internal and boundary layers. The maximum stretching factor over the mesh elements is $s_K^{\max} = 129.1$. The convergence history for this estimator is summarized in Fig. 2 (center) as a function of $\#\mathcal{T}_h$: the rate of convergence turns out to be about 1/2, accordingly to the a priori analysis.

Moving to the goal-oriented setting, we consider two different goal-functionals: we control the mean value of u on Ω via J_1 and the energy norm $a(u, u)$ via J_2. The grids associated with these two choices are quite different (see Fig. 3, (center) and (right)): in the case of J_1 we can appreciate the strong influence of the dual problem through the boundary layers on the left and bottom sides of Ω; this is not the case for J_2 since the control of the energy norm leads to identifying z with u. Moreover, the directions of the anisotropic features on the top and right sides are skew and parallel to these layers for $J = J_1$ and $J = J_2$, respectively. These differences confirm the sensitivity of the adapted mesh to the goal functional. The maximum stretching factor and the cardinality of $\mathcal{T}_h$ are $s_K^{\max} = 29.9$, $\#\mathcal{T}_h = 7061$, and $s_K^{\max} = 38.4$, $\#\mathcal{T}_h = 2103$ in the two cases, for $\mathtt{TOLL} = 10^{-2}/4$ and $\mathtt{TOLL} = 10^{-1}/2$, respectively.
Figure 2 (right) displays the convergence history for the functional error $J_1(e_h)$ which exhibits an $O(1/\#\mathcal{T}_h)$ order of convergence.

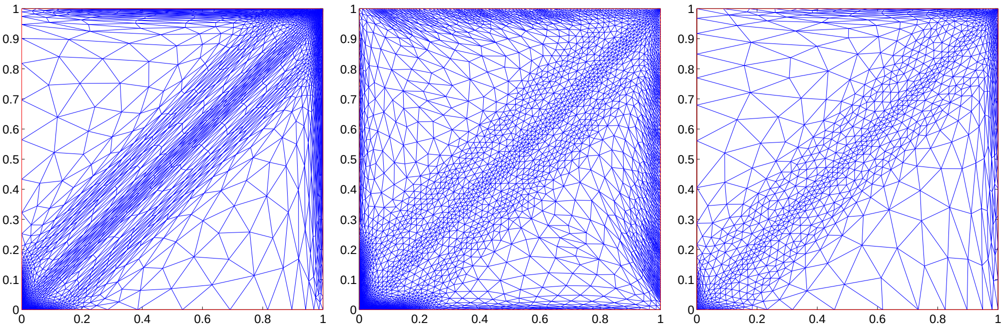

Figure 3: Adapted grids for the H^1-seminorm (left), $J = J_1$ (center) and $J = J_2$ (right) control

Prompted us by the above promising results, we are now extending the approach proposed in this paper to the more challenging shallow water system.

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