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Abstract

In this work, we propose an hN adaptive algorithm for optimal control problems whose space discretization is based on the spectral element method. We focus on a special class of problems described by quadratic cost functional and linear advection-diffusion state equation. This model has been employed to forecast and suitably reduce, by an optimization process, the pollutant concentration in air on a specific area of interest. Moreover, a local adaptive algorithm is employed to achieve a desired accuracy on the cost functional by reducing the computational cost of the problem.

Introduction

In environmental science different pollution models have been developed for analysing air pollution problem in towns or in specific areas of relevant interest, see e.g. [6]. These models are used to examine emissions from pollution sources, to forecast current and future air quality with respect to the air standard values. This analysis can drive the choice of the location of new industrial plants as well as the daily management of the emission sources. In this work we deal with optimal control problems for an advection-diffusion air pollution model, see e.g.

[5]. If the forecasted pollution is larger than a maximum level allowed, by optimization we aim at determining the reduction rates of some reducible emission sources, such that the pollution is held within the limits while keeping the reductions to a minimum. Such environmental applications often demand the use of different scales in the model description, in particular higher grid resolution is necessary in the vicinity of the area of observation. For these problems a uniform mesh could be unappropriate, especially when high order methods are used. To increase the grid resolution near the area of interest a local *a posteriori* error estimate on the cost functional is provided and employed in a local hN adaptive algorithm, by which an automatic distribution of the elements and polynomial degrees is designed, see e.g. [2, 11, 13].
This paper is organized as follows. In Section 1 the optimal control problem for air pollution control, the spectral element discretization and the optimization process are presented. In Section 2 a local *a posteriori* error estimate on the cost functional is provided and an hN adaptive algorithm is proposed to identify those areas where more resolution is needed. In Section 3 numerical tests are presented to show the efficiency of the proposed algorithm.

1 The optimal control problem for air pollution

The concentration of pollutants in air depends on the source activity, the type of fuel, and the atmospheric stability class that characterizes the specific meteorological situation under which the pollutant is emitted. In the following formulation we will observe the pollutant Sulphure Oxides (SO_2) in a plane parallel to soil at height H of emission. For urban-scale and short-time forecasting we assume negligible chemical reactions and stationary frame (that means source term and meteorological variables are time independent). Under these hypotheses the pollutant concentration can be described by a linear advection-diffusion equation. This simple model, although lacking several complex features of the real life situation, can provide a good deal of qualitative information. We describe it as follows.
Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a given domain, the pollutant concentration y satisfies the state equation:

$$\begin{cases} \mathcal{A}y \equiv -\mathrm{div}(\mu \nabla y) + \mathbf{b} \cdot \nabla y = \chi_{\Omega_c} u + f & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_D, \\ \dfrac{\partial y}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_N. \end{cases} \tag{1}$$

Here μ is the molecular viscosity, $\mathbf{b}$ the advection term that describes the wind field, $\Omega_c \subset \Omega$ a given control domain, χ_{Ω_c} the characteristic function of Ω_c, u is the control function defined on Ω_c, f is a given source term (possibly zero), Γ_D and Γ_N provide a partition of $\partial\Omega$ such that $\Gamma_D = \{\mathbf{x} \in \partial\Omega : \mathbf{b} \cdot \mathbf{n} < 0\}$, $\mathbf{n}$ being the outward unit normal with respect to Ω, $\Gamma_N = \partial\Omega \backslash \Gamma_D$.
Let V be the Hilbert space defined on Ω where we search the solution y of problem (1). In our situation, $V = H^1_{\Gamma_d}(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$ (see [15]) then the optimization process consists of determining the function $u \in U$, a Hilbert space of the control functions defined on Ω_c (in our case $U =$

$L^2(\Omega_c)$) such that the couple $(y,u) \in V \times U$ ($y = y(u)$ being the state function corresponding to the control u) minimizes a given functional $J : V \times U \to \mathbb{R}$ that represents the goal of the process. Different goals characterize different optimal control problems for air pollution. Our interest is to reduce the pollution concentration under a given desired threshold function z_d (connected to quality value for environment ecosystem and public health) by keeping however the control function near a desired emission-rate function, say u_d (related to the industrial productivity level). For these reasons we choose the following control functional J:

$$J(y,u) = \int_{\Omega_o} (y - z_d)^2 \, d\Omega + \frac{\alpha}{2} \int_{\Omega_c} (u - u_d)^2 \, d\Omega, \tag{2}$$

where $\alpha \geq 0$ is a penalization coefficient, Ω_o the observation domain, i.e. a sub-domain of Ω where we aim at keeping the pollutant concentration under the threshold.
To summarize, the optimal control problem associated to (1-2) reads as follows: look for $(y,u) \in V \times U$, state and control variable, respectively, such that:

$$\begin{cases} \min\limits_{u \in U} J(y,u) \\ \\ \text{where } y = y(u) \text{ satisfies problem (1).} \end{cases} \tag{3}$$

A classical way to solve the problem (3) is to introduce the Lagrangian functional $\mathcal{L}$ by which we transform the optimal control problem as the search for the saddle-point of $\mathcal{L}$. More precisely, let us define $\mathcal{L} : V \times V \times U \to \mathbb{R}$ as

$$\mathcal{L}(y,p,u) = J(y,u) + a(y,p) - b(u,p) - < f, p >_{V' \times V}, \tag{4}$$

where $p \in V$ is the Lagrange multiplier, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the bilinear form associated with the linear elliptic operator $\mathcal{A}$ in (1), $b(\cdot,\cdot) : U \times V \to \mathbb{R}$ is given by $b(u,p) =< \chi_{\Omega_c} u, p >_{V' \times V}$, $< \cdot, \cdot >_{V' \times V}$ being the duality among V and its dual space V'.
If $\mathbf{x} = (y,p,u)$ is a regular point and it is a local optimal solution of the problem (3), then

$$\nabla \mathcal{L}(\mathbf{x})[\phi, \varphi, \psi] = 0, \quad \forall (\phi, \varphi, \psi) \in V \times V \times U, \tag{5}$$

where the operator ∇ is the Fréchet gradient, see e.g. [14]. Upon taking the derivatives with respect to each variable, this yields the Karush-Kuhn-Tucker (KKT) optimality system:

$$\begin{cases} \nabla_p \mathcal{L}(\mathbf{x})[\phi] \equiv a(y,\phi) - b(u,\phi) - < f, \phi >_{V' \times V} = 0 & \forall \phi \in V, \quad \text{(state equation)} \\ \\ \nabla_y \mathcal{L}(\mathbf{x})[\varphi] \equiv a(\varphi,p) - < y - z_d, \varphi >_{V' \times V} = 0 & \forall \varphi \in V, \quad \text{(adjoint equation)} \\ \\ \nabla_u \mathcal{L}(\mathbf{x})[\psi] \equiv < \alpha u + \chi_{\Omega_c} p, \psi >_{U' \times U} = 0 & \forall \psi \in U, \quad \text{(optimality conditions).} \end{cases} \tag{6}$$

Various gradient-based optimization algorithms can be used to minimize $\mathcal{L}$ based on its gradient information, see e.g. [12, 14]. In this paper we have employed line search methods with BFGS updating with respect to the control function u. Namely, given an initial control u the corresponding state y is given by the solution of the state equation in (6). To apply the stopping criterium $\|\nabla_u \mathcal{L}\| \leq tol_{iter}$ (tol_{iter} being a prescribed tolerance) we need information on the Lagrange multiplier p. To obtain p we need to solve the adjoint problem $\nabla_y \mathcal{L}(\mathbf{x})[\varphi] = 0 \; \forall \varphi \in V$, for given u, y. If the stopping criterium is satisfied we stop, otherwise we update the control variable u by line search method and start a new iteration. The optimization process can be summarized in the following Algorithm 1.

Algorithm 1 Optimization algorithm

Let tol_{iter} be a prescribed tolerance, we choose an initial control u^0

1. solve the state problem to obtain the state variable y and the corresponding functional evaluation $J(y, u)$;
2. solve the adjoint problem to obtain the adjoint variable p;
3. evaluate $J'(u) = \nabla_u \mathcal{L}(\mathbf{x})$ according to the optimality condition in the KKT system;
4. **if** $\|\nabla_u \mathcal{L}(\mathbf{x})\| \geq tol_{iter}$ **then**

   update the control variable u through the line search and start again from point 1
5. **else**

   the current $\mathbf{x} = (y, p, u)$ is the optimal solution.
6. **end if**

Remark 1.1 *Let (y, u) be the exact solution of problem (3) and (y^j, u^j) the solution of the $j-th$ iterative step of Algorithm 1. The iteration error $\epsilon_{iter}^{(j)} = J(y, u) - J(y^j, u^j)$ has the following expression:*

$$\epsilon_{iter}^{(j)} = \frac{1}{2}(\nabla_u \mathcal{L}(\mathbf{x}^j), u - u^j). \tag{7}$$

moreover, if a steepest-descent iterative method with constant relaxation parameter τ is used, $|\epsilon_{iter}^{(j)}|$ can be estimated as:

$$|\epsilon_{iter}^{(j)}| \simeq \frac{1}{2}\tau \left\|\nabla_u \mathcal{L}(\mathbf{x}^j)\right\|^2. \tag{8}$$

(For the proof, see [5]). This motivates the choice of the stopping criterium in Algorithm 1.

1.1 Numerical discretization

As described in Algorithm 1, for each step of the iterative method we solve both the state and the adjoint equations. For their numerical discretization, a Galerkin spectral element method has been employed. More precisely, we proceed as follows.
We assume that Ω is a two dimensional domain and that $\mathcal{T}$ be a shape regular, not-necessarily matching partition of Ω into K spectral elements such that $\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k$. Let $\overrightarrow{\delta} = \{\delta_k\}_{k=1}^{K}$ be the vector of discretization parameters, $\delta_k = (h_k, N_k)$, h_k being the diameter of Ω_k and N_k the degree of the polynomial approximation in Ω_k. Moreover, let V_δ, U_δ be the spectral discretization spaces of V, U, respectively, see e.g. [3]. The corresponding discrete KKT system is then given by:
we search $\mathbf{x}_\delta = (y_\delta, p_\delta, u_\delta) \in V_\delta \times V_\delta \times U_\delta$ such that:

$$\begin{cases} \nabla_p \mathcal{L}(\mathbf{x}_\delta)[\phi] \equiv a(y_\delta, \phi) - b(u_\delta, \phi) - < f, \phi >_{V'_\delta \times V_\delta} = 0 & \forall \phi \in V_\delta, \\ \nabla_y \mathcal{L}(\mathbf{x}_\delta)[\varphi] \equiv a(\varphi, p_\delta) - < y_\delta - z_d, \varphi >_{V'_\delta \times V_\delta} = 0 & \forall \varphi \in V_\delta, \\ \nabla_u \mathcal{L}(\mathbf{x}_\delta)[\psi] \equiv < \alpha u_\delta + \chi_{\Omega_c} p_\delta, \psi >_{U'_\delta \times U_\delta} = 0 & \forall \psi \in U_\delta. \end{cases} \tag{9}$$

Then we apply Algorithm 1 to (9) to obtain the numerical approximation of problem (3).
In what follows, we will provide an estimate of the discretization error associated with the spectral discretization method. The focus of the local *a posteriori* discretization error is to have an appropriate error measure of the error associated with the discretization method. This error will guide a suitable adaptive algorithm according to which the discretization parameters, that is either the mesh size or the polynomial degree, will be modified.

2 A posteriori error estimate on the cost functional

A classical error measure is represented by the energy norm error estimates associated with the optimal control problem, that means the sum of errors on state, adjoint and control variables. However, in many applications, this indicator is not suitable to provide useful information on the error of the quantity of physical interest, represented in our case by the cost functional. A different approach is based on the Dual Weighted Residual (DWR) method, that allows to bound the error on the cost functional by a suitable combination of the state, adjoint and control errors. Such a technique, proposed in [2, 10] in the context of finite element approximation for partial differential equations, has been extensively used for h-adaptive finite element strategies, see e.g. [1, 5, 11]. In this work we apply the DWR approach in the context of spectral element methods. Moreover, an hN-adaptive strategy is provided to select which parameters of the spectral discretization, either the diameter size h_k or the polynomial degree N_k, must be refined when an element Ω_k is marked for refinement, see e.g. [4, 7, 8, 9].

We start by introducing the following notation:

$$R^y(\mathbf{x}_\delta) = \chi_{\Omega_c} u_\delta + f - A y_\delta \ \text{ on } \Omega, \quad r^y(\mathbf{x}_\delta) = \begin{cases} \frac{1}{2}[\![\mu \frac{\partial y_\delta}{\partial \overrightarrow{\mathbf{n}}_k}]\!] & \text{on } \partial\Omega_k \backslash \partial\Omega, \\ \mu \frac{\partial y_\delta}{\partial \mathbf{n}} & \text{on } \Gamma_N, \\ 0 & \text{on } \Omega_k \cup \Gamma_D, \end{cases}$$

$$R^u(\mathbf{x}_\delta) = \alpha u_\delta + \chi_{\Omega_c} p_\delta \ \text{ on } \Omega_c,$$

$$R^p(\mathbf{x}_\delta) = y_\delta - z_d - A^* p_\delta \ \text{ on } \Omega, \qquad r^p(\mathbf{x}_\delta) = \begin{cases} \frac{1}{2}[\![\frac{\mu \partial p_\delta}{\partial \overrightarrow{\mathbf{n}}_k}]\!] & \text{on } \partial\Omega_k \backslash \partial\Omega, \\ \mu \frac{\partial p_\delta}{\partial \mathbf{n}} & \text{on } \Gamma_N, \\ 0 & \text{on } \Omega_k \cup \Gamma_D, \end{cases} \tag{10}$$

for $k = 1, \ldots, K$, $\mathbf{n}$ being the outward unit normal with respect to Ω and $\overrightarrow{\mathbf{n}}_k$ that with respect to Ω_k. Namely, R^y, R^p (r^y, r^p, respectively) are the interior (edge, respectively) residual associated with the strong form of the state and adjoint problems. A (A^*, respectively) is the discrete operator associated with the continuous operator $\mathcal{A}$ ($\mathcal{A}^*$, respectively). The symbol $[\![\cdot]\!]$ indicates jump. From now, we will suppose that the partition $\mathcal{T}$ and the polynomial approximation orders satisfy the following *local bounded variation* property.

Assumption 2.1 *(local bounded variation) There exist $\varrho_1, \varrho_2 > 0$ such that for each couple of neighboring elements Ω_{k_1}, Ω_{k_2} :*

- *the element sizes h_{k_1}, h_{k_2} satisfy $\frac{1}{\varrho_1} h_{k_1} \leq h_{k_2} \leq \varrho_1 h_{k_2}$,*
- *the polynomial degrees N_{k_1}, N_{k_2} satisfy $\frac{1}{\varrho_2} (N_{k_1} + 1) \leq N_{k_2} + 1 \leq \varrho_2 (N_{k_2} + 1)$.*

Then, the following estimate for the discretization error holds.

Theorem 2.2 *Assume the mesh $\mathcal{T}$, with elements Ω_k $k = 1, \ldots, K$ and polynomial distribution that satisfy the Assumption 2.1. Let (y, u) be the exact solution of the optimal control problem (3) and (y_δ, u_δ) the corresponding spectral element approximation, that is the solution of (9). The discretization error $\epsilon_{dis} = J(y, u) - J(y_\delta, u_\delta)$ can be bounded as follows:*

$$|\epsilon_{dis}| \leq \eta = C \sum_{k=1}^{K} \eta_k, \tag{11}$$

where

$$\eta_k = \rho_k^y \frac{h_k}{N_k} \|\nabla p_\delta\|_{L^2(\omega_k)} + \rho_k^p \frac{h_k}{N_k} \|\nabla y_\delta\|_{L^2(\omega_k)} + \rho_k^u \frac{h_k}{N_k} \|\nabla u_\delta\|_{L^2(\omega_k)}, \tag{12}$$

ω_k is the patch associated with Ω_k, and ρ_k^y, ρ_k^p, ρ_k^u are the residuals on Ω_k of state, adjoint and optimal conditions, respectively, defined as:

$$\begin{aligned} \rho_k^y &= \|R^y(\mathbf{x}_\delta)\|_{\Omega_k} + \big(\frac{h_k}{N_k}\big)^{-\frac{1}{2}} \|r^y(y_\delta)\|_{\partial\Omega_k}, \\ \rho_k^p &= \|R^p(\mathbf{x}_\delta)\|_{\Omega_k} + \big(\frac{h_k}{N_k}\big)^{-\frac{1}{2}} \|r^p(p_\delta)\|_{\partial\Omega_k}, \\ \rho_k^u &= \|R^u(\mathbf{x}_\delta)\|_{\Omega_k}. \end{aligned} \tag{13}$$

$R(\cdot,\cdot)$, $r(\cdot,\cdot)$ are the interior and edges residuals, respectively, defined in (10).

To obtain this results we have employed a dual weighted residual approach. Namely, for linear control problems (that is the state equation is linear and the functional is quadratic in the state variable) the error on the cost functional is given by:

$$J(y) - J(y_\delta) = \frac{1}{2}\Big[\rho(y_\delta, p - p_\delta) + \rho(p_\delta, y - y_\delta) + \rho(u_\delta, u - u_\delta)\Big], \tag{14}$$

where $\rho(y_\delta, \varphi)$, $\rho(p_\delta, \varphi)$, $\rho(u_\delta, \phi)$ are the residual of the state, adjoint and control functions, associated with the weak formulation, given by:

$$\begin{aligned} \rho(y_\delta, \varphi) &=< \chi_{\Omega_c} u_\delta + f, \varphi >_{V_\delta' \times V_\delta} - a(y_\delta, \varphi), \\ \rho(p_\delta, \varphi) &=< y_\delta - z_d, \varphi >_{V_\delta' \times V_\delta} - a(\varphi, p_\delta), \\ \rho(u_\delta, \phi) &=< \alpha u_\delta + \chi_{\Omega_c} p_\delta, \phi >_{U_\delta' \times U_\delta}, \end{aligned} \tag{15}$$

respectively. Integrating by parts, employing Hölder inequality and hN Clément interpolation estimates (see e.g. [15]), the estimate (11) can be proven. We refer to [7] for the complete proof and more details about the provided estimate.
By this estimate we can select which spectral elements need to be refined according to the error on the cost functional. After this marking phase, to complete the description of the adaptive algorithm we need to characterize how to choose the local discretization parameters h_k and N_k.

Suppose that after having computed a discrete solution of problem (3), the corresponding error is estimated according to (11). Now we update the mesh and solve the optimal control problem on the new mesh until the tolerance on the discretization error will be satisfied. At the step j of the iterative algorithm associated with the adaptive process we compute, for each element Ω_k, the energy norm $\eta_k^{j,E}$ associated with the optimal control problem and moreover we compute $\eta_k^{j,pred}$, a prediction of the energy error at the $(j+1) - th$ step of the iterative

algorithm. The latter will be employed, for comparison, in the next step of the algorithm. More precisely, we compute the local estimator $\eta_k^{j,E}$ ($\eta_k^{j,pred}$, respectively) associated with the optimal control problem as the sum of the three local contributions $\eta_{k;y}^{j,E}$, $\eta_{k;p}^{j,E}$ and $\eta_{k;u}^{j,E}$ ($\eta_{k;y}^{j,pred}$, $\eta_{k;p}^{j,pred}$ and $\eta_{k;u}^{j,pred}$, respectively). Here $\eta_{k;y}^{j,E}$ ($\eta_{k;p}^{j,E}$, $\eta_{k;u}^{j,E}$, respectively) are the state (the adjoint and the control, respectively) local *a posteriori* residual error indicator and $\eta_{k;y}^{j,pred}$ ($\eta_{k;p}^{j,pred}$, $\eta_{k;u}^{j,pred}$, respectively) are the corresponding predicted error indicators. We refer to [13] for more details about the construction of both the energy and predictable error estimates.
As mentioned before, at the step $j+1$ we can compare $\eta_k^{j+1,E}$ with $\eta_k^{j,pred}$ to yield information about the exactness of our prediction. If the prediction will be right, we will implicitly have information on the regularity r_k of the solution on Ω_k. Indeed, since the prediction estimate is constructed by using *a priori* error estimate, implicitly a suitable assumption on the regularity of the solution is made.
As usual in hN adaptive algorithms, where the polynomial approximation order N_k is less than or equal to r_k, a N-refinement is performed, otherwise an h-refinement is performed. We refer to [7] for more details about this algorithm. The adaptive process can be summarized in Algorithm 2.

Algorithm 2 Adaptive algorithm

Let tol_{dis} be a prescribed tolerance, $\mathbf{x}_\delta$ a discrete solution of problem (3)

while $\eta_k > tol_{dis}$ **do**

if $\left(\eta_k^{j,E}\right)^2 > \left(\eta_k^{j,pred}\right)^2$ **then**

h refinement, the element Ω_k is split into four sub-elements obtained by joining the mid-points of each edge

else

N refinement, $N_k = N_k + 1$

end if

a discrete solution $\mathbf{x}_\delta$ of the problem (3) is computed on the new mesh

end while

The complete process for the solution of the optimal control problem and the hN adaptive strategy can be summarized in the following steps:

1. we solve the optimization process, following Algorithm 1,
2. we estimate the discretization error on the cost functional according to (11),
3. if the discretization error estimates are not satisfied, then

- mark elements for refinement and choose between h or N refinement, according to Algorithm 2,
- return to the optimization process at point 1 ,

otherwise STOP.

3 Numerical results

In this section we present some numerical tests in which the proposed complete optimization-adaptive algorithm is employed.

3.1 Distributed control and observation with $\Omega_c \subset \Omega$ and $\Omega_o \subset \Omega$

According to the problem (1), we set $\Omega =]-1,1[^2$, the observation and the control domains $\Omega_o = \Omega_c =]0.5,1[^2$. We start with the conforming mesh reported in Figure 1 (left), with $K = 9$ spectral elements, and $N_k = N = 2\,\forall k$. In this first test case according to the notation in (1), for the state equation we set $\mu = 2$, $\mathbf{b} = [1,0]$, $f = -12(x_1 + x_2)$, $\Gamma_D = \partial\Omega$, $\Gamma_N = \emptyset$, the desired function $z_d = (x_1 + x_2)^3$. Figure 2 shows the final state function (left) and the desired function z_d in the full domain (right). Moreover, in Table 1 we report the history of the complete algorithm (optimization plus adaptivity). We separate the number of iterations associated with the optimization algorithm, it_{iter}, from that of the iterations associated with the adaptive process, it_{dis}. In particular, we allow at most $it_{dis}^{max} = 2$ iterations. Indeed, by changing the approximation of the functional J, the optimal control computed on the old mesh could be very different from the one on the new mesh. For each step of the complete process, according to the notations introduced in Section 1 and Section 2, we report the iteration error estimate ϵ_{iter} and the discretization error estimate ϵ_{dis}. Moreover, in the same table for each iteration of the adaptive process the number of elements refined in h ($\sharp$**h-ref**) or N ($\sharp$**N-ref**) are displayed . In Figure 1 (right) we report the final mesh. We can observe that the adaptive process driven by the dual weighted indicator fits elements in those zones which are more relevant for the control problem. Whereas, an adaptive process driven by the energy error indicator would fit the elements in a quite uniform way in all the domain. We highlight that, in the approach based on the energy error indicator, the mesh is refined according to the residual error in the state and adjoint process separately, while the dual weighted residual approach weighs the residual of the state equation with that of the adjoint problem (and vice versa).

3.2 Air pollution control

Our second numerical test is inspired by the previously discussed environmental application. An example of computational domain could be a territory with a city. In this case, the city represents a subarea, in which we want to control the emissions from industrial chimneys in order to keep the pollutant concentration over the critical subarea below a prescribed threshold. Particularly, we refer to

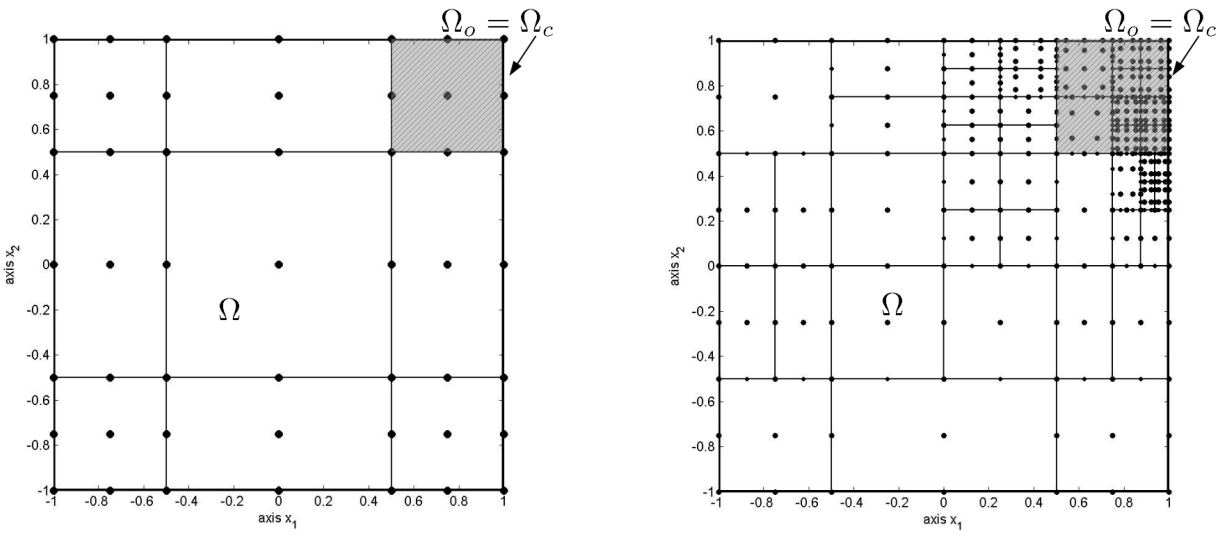

Figure 1: Initial (left) and final (right) mesh and corresponding degrees of freedom.

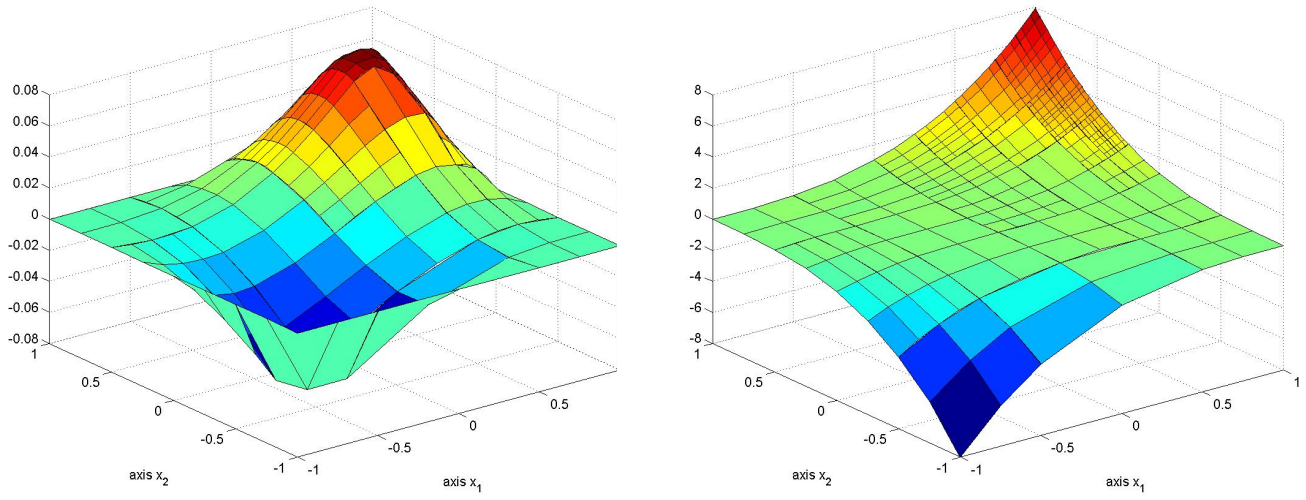

Figure 2: Final state function (left) and desired function (right).

pollution control problems, in which the observation domain $\Omega_o \subset \Omega$, and the control domain is given by $\Omega_c = \Omega_{c_1} \cup \Omega_{c_2}$, Ω_{c_1}, Ω_{c_2} being two sub-domains each one representing a chimney of the industrial plant. We want to define the daily management of such chimneys according to a given desired value $z_d = 100\mu g/m^3$ and a desired emission rate $u_d = 800m/s$. As discussed in Section 1, the goal consists in minimizing the pollutant concentration over a certain area by regulating the emissions rate. The data that we have employed are:

- effective height $H = 100\ m$;
- desired function $z_d = 100\mu g/m^3$;

it_{iter}	it_{dis}	ϵ_{iter}	ϵ_{dis}	♯**h-ref.**	♯**N-ref.**
1	0	2.88946e-006	0.0688201	0	0
1	1	0.001516330	0.00515035	2	0
1	2	0.000735936	0.00131672	3	1
2	0	7.78693e-006	0.0117715	0	0
2	1	0.000375459	0.00266594	3	1
2	2	0.000357549	0.00172755	4	3
3	0	2.58373e-006	0.00157696	0	0

Table 1: The error estimates at each optimization and adaptive step and the number of elements refined in h, N at each adaptive step with respect to the hN-refinement Algorithm 2.

- penalization coefficient $\alpha = 0.1$;
- desired emission rate for each chimney $u_d = 800 g/s$;
- the computational domain $\Omega =]0, 15[^2$ (units are in Km);
- $\Gamma_D = \Gamma_4$ and $\Gamma_N = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (according to the numbering side as in Figure 3, left) ;
- the control domain $\Omega_c = \Omega_{c_1} \cup \Omega_{c_2}$, with $\Omega_{c_1} =]2.5, 3[\times]2.5, 3[$, $\Omega_{c_2} =]2.5, 3[\times]7, 7.5[$;
- observation domain $\Omega_o =]8, 12[\times]2.5, 7[$;
- diffusion coefficient $\mu = \frac{\sigma_x^2}{2r}\|\mathbf{b}\|$, being r the radial coordinate, and σ_x suitable diffusivity coefficient according to the neutral stability class (see e.g. [7]);
- transport field $\mathbf{b} = (2.5, 0)\ m/s$;
- source term $f = 0\ g/s$.

We have employed the initial mesh reported in Figure 3 (left), with $K = 30$ spectral elements, and $N_k = N = 2\ \forall k$. In the same Figure 3 (right) we plot the corresponding state function. We can observe that the pollutant concentration in the observation area is higher than the desired threshold. To reduce such a concentration, we apply the optimization-adaptive algorithm as before, with $tol_{iter} = tol_{dis} = 1e-3$. After two iterations of the complete algorithm, we obtain the final distribution of the state function reported in Figure 4 (right), which terms out to be reduced below the desired value. In Figure 4 (left) we report the final mesh. The refinement has been made according to the functional definition, that means with special attention to the elements near the observation and control areas. h refinement is predominant, some N refinements have been done in the area between observation and control domains.

To conclude we repeat this numerical test, but we change the position of the observation area $\Omega_o =]8, 12[\times]7.5, 10[$, see Figure 5 (left). We use the same

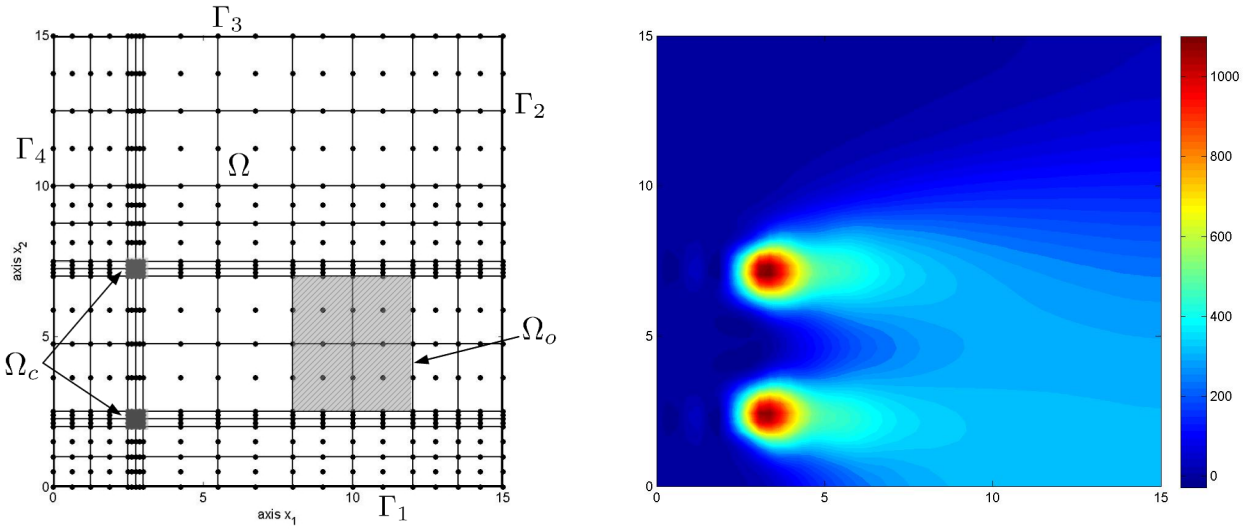

Figure 3: Neutral class: Initial mesh (left) and corresponding state function (right).

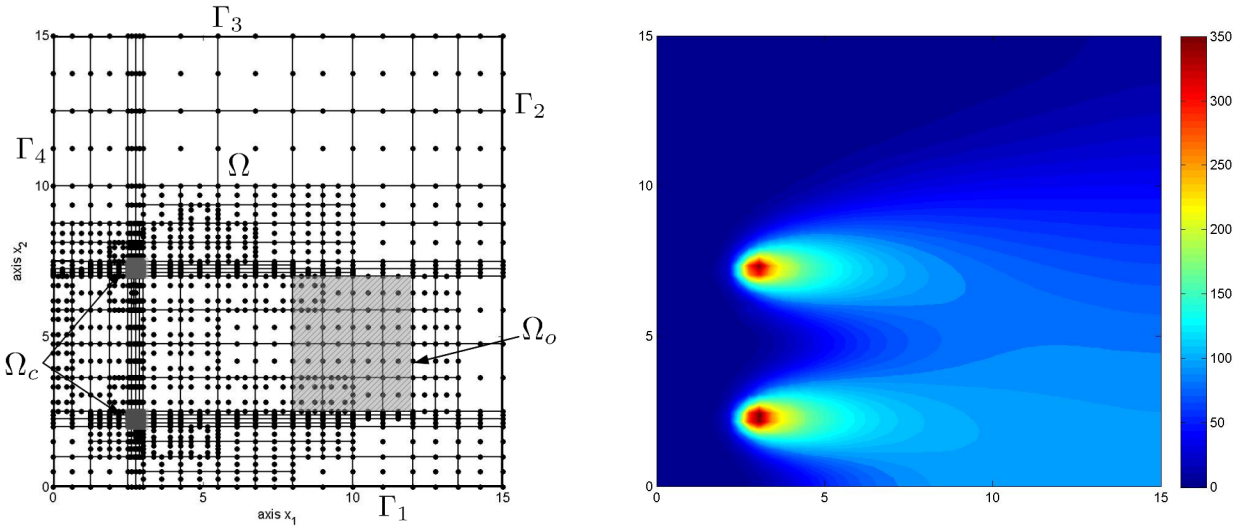

Figure 4: Neutral class: Final mesh (left) and corresponding state function (right).

initial state function because it just depends on the control domain, see Figure 5 (right). The final pollution distribution is reported in Figure 6 (right). Also in this case the refinement, see Figure 6 (left), is observed near the observation and control areas, with special attention to the control domain nearest to the observation area, as this can greatly influence the pollutant concentration on the city.

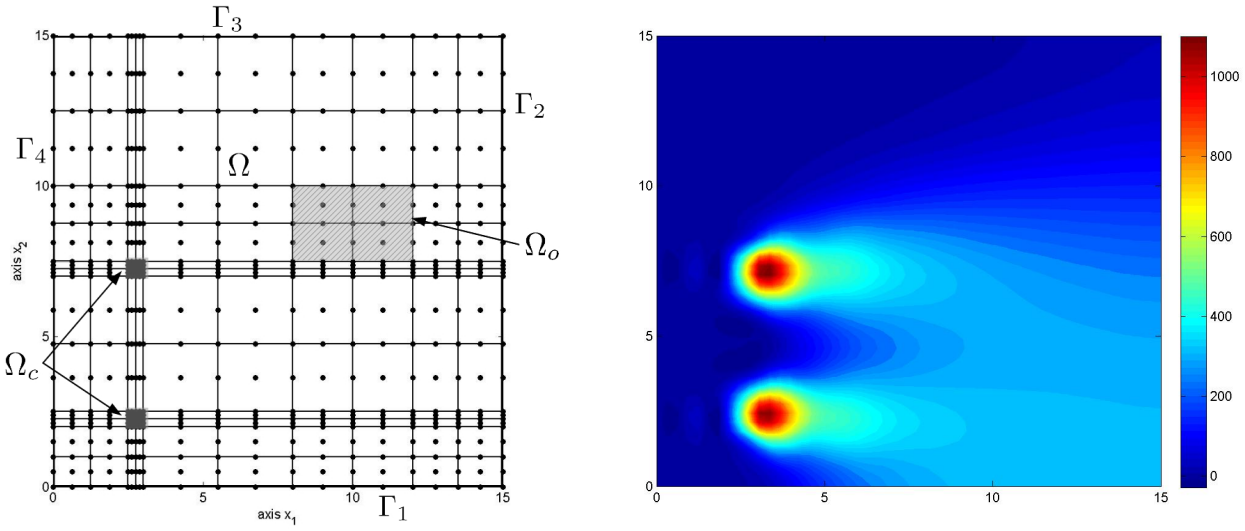

Figure 5: Neutral class: Initial mesh (left) and corresponding state function (right).

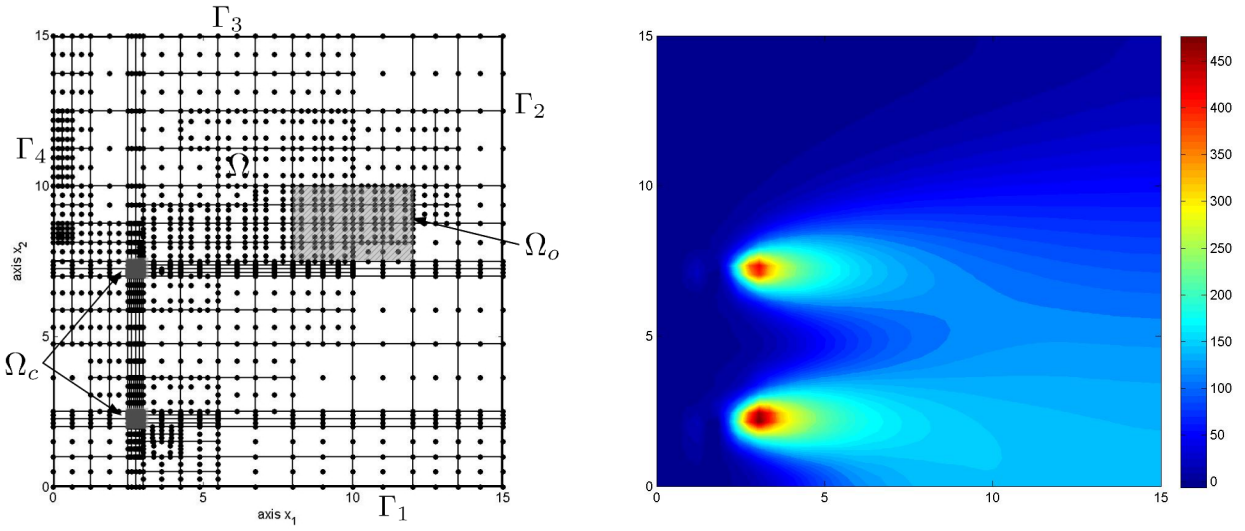

Figure 6: Neutral class: Final mesh (left) and corresponding state function (right).

Conclusion

In this work we have proposed a general algorithm for the solution of an optimal control problem. Motivated by the high accuracy that high order methods offer, we have presented a spectral element discretization of a quadratic optimal control problem. We have derived *a posteriori* error estimates on the cost functional, by using the approach of dual weighted residual methods. Such an estimate has been employed in an hN adaptive algorithm. The main advantage of the proposed algorithm is the automatic design of the discretization space associated with the

problem at hand. The method proposed in this work, applied to air pollution control, can be easily extended to different problems. Numerical results have been carried out, and even if h refinement is predominant for the environmental problems, the refined areas reflect the definition of the cost functional, namely more degrees of freedom have been located near the observation and control areas to better capture and describe the physical phenomena.

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