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Arnold, D.N.; Boffi, D.; Bonizzoni,F.

MOX, Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox@mate.polimi.it http://mox.polimi.it

Tensor product finite element differential forms and their approximation properties *

Douglas N. Arnold[♯] Daniele Boffi[†] Francesca Bonizzoni[‡]

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[♯] School of Mathematics,
University of Minnesota,
Minneapolis, MN 55455, USA
[†] Dipartimento di Matematica "F. Casorati",
Università di Pavia, 27100 Pavia, Italy
[‡] MOX – Dipartimento di Matematica "F. Brioschi",
Politecnico di Milano, 20133 Milano, Italy
arnold@umn.edu
daniele.boffi@unipv.it
francesca1.bonizzoni@mail.polimi.it

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Abstract

We discuss the tensor product construction for complexes of differential forms and show how it can be applied to define shape functions and degrees of freedom for finite element differential forms on cubes in n dimensions. These may be extended to curvilinear cubic elements, obtained as images of a reference cube under diffeomorphisms, by using the pullback transformation for differential forms to map the shape functions and degrees of freedom from the reference cube to the image finite element. This construction recovers and unifies several known finite element approximations in two and three dimensions. In this context, we study the approximation properties of the resulting finite element spaces in two particular cases: when the maps from the reference cube are affine, and when they are multilinear. In the former case the rate of convergence depends only on the degree of polynomials contained in the reference space of shape functions. In the latter case, the rate of approximation is degraded, with the loss more severe for differential forms of higher form degree.

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1 Introduction

Finite element simulations are often performed using meshes of quadrilaterals in two dimensions or of hexahedra in three. In simple geometries the meshes may consist entirely of squares, rectangles, or parallelograms, or their three-dimensional analogues, leading to simple data structures. However, for more general geometries, a larger class of quadrilateral or hexahedral elements is needed, a common choice being elements which are the images of a square or cube under an invertible multilinear map. In such cases, the shape functions for the finite element space are defined on the square or cubic reference element and then transformed to the deformed physical element. In [5] it was shown in the case of scalar (H^1) finite elements in two dimensions that, depending on the choice of reference shape function space, this procedure may result in a loss of accuracy in comparison with the accuracy achieved on meshes of squares. Thus, for example, the serendipity finite element space achieves only about one half the rate of approximation when applied on general quadrilateral meshes, as compared to what it achieves on meshes of squares. The results of [5] were extended to three dimensions in [15]. The case of vector ($H(\mathrm{div})$) finite elements in two dimensions was studied in [6]. In that case the transformation of the shape functions must be done through the Piola transform and it turns out the same issue arises, but even more strongly, the requirement on the reference shape functions needed to ensure optimal order approximation being more stringent. Some results were obtained for $H(\mathrm{curl})$ and $H(\mathrm{div})$ finite elements in 3-D in [14].

The setting of finite element exterior calculus (see [1, 7, 8]) provides a unified framework for the study of this problem. In this paper we discuss the construction of finite element subspaces of the domain $H\Lambda^k$ of the exterior derivative acting on differential k-forms, $0 \le k \le n$, in any number n of dimensions. This includes the case of scalar H^1 finite elements ($k = 0$), L^2 finite elements (in which the Jacobian determinant enters the transformation, $k = n$), and, in three-dimensions, finite elements in $H(\mathrm{curl})$ ($k = 1$) and $H(\mathrm{div})$ ($k = 2$).

The paper begins with a brief review of the relevant concepts from differential forms on domains in Euclidean space. It then describes the tensor product construction for differential forms, and complexes of differential forms. These will be used to construct reference shape functions and degrees of freedom. First, in Section 4, we discuss the construction of finite element spaces of differential forms and the use of reference domains and mappings. Combining these constructions, in Section 5, we define the $\mathcal{Q}_r^-\Lambda^k$ finite element spaces, which may be seen to be the most natural finite element subspace of $H\Lambda^k$ using mapped cubical meshes. In Section 6 we obtain conditions for $O(h^{r+1})$ approximation in L^2 for spaces of differential k-forms on mapped cubic meshes. More precisely, in the case of multilinear mappings Theorem 6.1 shows that a sufficient condition for $O(h^{r+1})$ approximation in L^2 is that the reference finite element space contains $\mathcal{Q}_{r+k}^-\Lambda^k$. Again we see loss of accuracy when the mappings are multilinear, in comparison to affine (for which $\mathcal{P}_r\Lambda^k$ guarantees approximation of order $r + 1$), with the effect becoming more severe for larger k. In the final section we determine the extent of the loss of accuracy for the $\mathcal{Q}_r^- \Lambda^k$ spaces. We discuss several examples of finite element spaces, including standard $\mathcal{Q}_r$ spaces ($k = 0$), $\mathcal{P}_r\Lambda^n$ spaces, and serendipity spaces $\mathcal{S}_r\Lambda^k$ which have been recently described in [3].

2 Preliminaries on differential forms

We begin with a brief review of some basic notations, definitions, and properties of differential forms defined on a domain S in $\mathbb{R}^n$. A differential k-form on S is a function $S \to \operatorname{Alt}^k \mathbb{R}^n$, where $\operatorname{Alt}^k \mathbb{R}^n$ denotes the space of alternating k-linear forms mapping $(\mathbb{R}^n)^k = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$. By convention $\operatorname{Alt}^0 \mathbb{R}^n = \mathbb{R}$, so differential 0-forms are simply real-valued functions. The space $\operatorname{Alt}^k \mathbb{R}^n$ has dimension $\binom{n}{k}$ for $0 \le k \le n$ and vanishes for other values of k. A basis is formed by the basic alternators $dx^\sigma := dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}$, for $\sigma = (\sigma_1, \dots, \sigma_k)$ belonging to the set $\Sigma(k,n)$ of increasing maps from $\{1, \dots, k\}$ to $\{1, \dots, n\}$. Here $dx^i \in \operatorname{Alt}^1 \mathbb{R}^n = (\mathbb{R}^n)^*$ denotes the functional $dx^i(v) = v^i$ for $v = (v^1, \dots, v^n) \in \mathbb{R}^n$ and the wedge product of alternating forms is the skew part of the tensor product:

$$f \wedge g = \binom{k+l}{k} \operatorname{skw}(f \otimes g), \quad f \in \operatorname{Alt}^k \mathbb{R}^n,\ g \in \operatorname{Alt}^l \mathbb{R}^n.$$

Thus a general differential k-form on S can be expressed uniquely as

$$f = \sum_{\sigma \in \Sigma(k,n)} f_\sigma \, dx^\sigma = \sum_{1 \le \sigma_1 \le \cdots \le \sigma_k \le n} f_{\sigma_1 \cdots \sigma_n} \, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}, \tag{1}$$

for some coefficient functions f_σ on S. In particular, a differential n-form can be expressed as $f = g\,dx$ where the volume form $dx = dx^1 \wedge \cdots \wedge dx^n$ and the coefficient g is a function on S. As long as the coefficient is integrable, the integral

$$\int_S f = \int_S g \, dx$$

of the differential form is defined and has the value the notation suggests.

If $\mathcal{F}(S)$ is some space of real-valued functions on S, then we denote by $\mathcal{F}\Lambda^k(S)$ the space of differential k-forms with coefficients in $\mathcal{F}(S)$. This space is naturally isomorphic to $\mathcal{F}(S) \otimes \operatorname{Alt}^k \mathbb{R}^n$. Examples are the spaces $C^\infty\Lambda^k(S)$ of smooth k-forms, $L^2\Lambda^k(S)$ of L^2 k-forms, $H^r\Lambda^k(S)$ of k-forms with coefficients in a Sobolev space, and $\mathcal{P}_r\Lambda^k(S)$ of forms with polynomial coefficients of degree at most r. The space $L^2\Lambda^k(S)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{L^2\Lambda^k(S)} = \sum_{\sigma \in \Sigma(k,n)} \langle f_\sigma, g_\sigma \rangle_{L^2(S)},$$

and similarly for the space $H^r\Lambda^k(S)$, using the usual Sobolev inner-product

$$\langle u, v \rangle_{H^r(S)} = \sum_{|\alpha| \le r} \langle D^\alpha u, D^\alpha v \rangle_{L^2(S)}, \quad u, v \in H^r(S),$$

where the sum is over multi-indices α of order at most r.

The exterior derivative of the k-form (1) is the $(k+1)$-form

$$df = \sum_{\sigma \in \Sigma(k,n)} \sum_{j=1}^{n} \frac{\partial f_\sigma}{\partial x^j} \, dx^j \wedge dx^\sigma,$$

assuming that the indicated partial derivatives exist. We may view the exterior derivative as an unbounded operator $L^2\Lambda^k(S) \to L^2\Lambda^{k+1}(S)$ with domain

$$H\Lambda^k(S) := \{\, f \in L^2\Lambda^k(S) \,|\, df \in L^2\Lambda^{k+1}(S) \,\},$$

which is a Hilbert space when equipped with the graph norm

$$\langle f, g \rangle_{H\Lambda^k(S)} = \langle f, g \rangle_{L^2\Lambda^k(S)} + \langle df, dg \rangle_{L^2\Lambda^{k+1}(S)}.$$

The complex

$$0 \to H\Lambda^0(S) \xrightarrow{d} H\Lambda^1(S) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^m(S) \to 0 \tag{2}$$

is the L^2 de Rham complex on S.

A very important construction is the pullback of a differential form under a mapping. Let $\hat{S}$ be a domain in $\mathbb{R}^n$ and $F = (F^1, \dots, F^n)$ a C^1 mapping of $\hat{S}$ into a domain S in some $\mathbb{R}^m$. Given a differential k-form v on S, its pullback F^*v is a differential k-form on $\hat{S}$. If

$$v = \sum_{1 \le i_1 < \cdots < i_k \le m} v_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

then

$$F^*v = \sum_{1 \le i_1 < \cdots < i_k \le m} \sum_{j_1, \dots, j_k = 1}^{n} (v_{i_1 \cdots i_k} \circ F) \frac{\partial F^{i_1}}{\partial \hat{x}^{j_1}} \cdots \frac{\partial F^{i_k}}{\partial \hat{x}^{j_k}} \, d\hat{x}^{j_1} \wedge \cdots \wedge d\hat{x}^{j_k}. \tag{3}$$

The pullback operation satisfies $(G \circ F)^* = F^* \circ G^*$, and, if F is a diffeomorphism, $(F^*)^{-1} = (F^{-1})^*$. The pullback preserves the wedge product and the exterior derivative in the sense that

$$F^*(v \wedge w) = F^*v \wedge F^*w, \quad dF^*v = F^*dv.$$

If F is a diffeomorphism of $\hat{S}$ onto S which is orientation preserving (i.e., its Jacobian determinant is positive), then the pullback preserves the integral as well:

$$\int_{\hat{S}} F^*v = \int_S v, \quad v \in L^1\Lambda^n(S).$$

An important application of the pullback is to define the trace of a differential form on a lower dimensional subset. If S is a domain in $\mathbb{R}^n$ and f is a subset of the closure $\bar{S}$ and also an open subset of a hyperplane of dimension $\le n$ in

$\mathbb{R}^n$, then the pullback of the inclusion map $f \hookrightarrow S$ defines the trace operator tr_f taking k-forms on $\bar{S}$ to k-forms on f.

Before continuing, we recall that vector proxies exist for differential 1-forms and $(n-1)$-forms on a domain in $\mathbb{R}^n$. That is, we may identify the 1-form $\sum_i v_i\, dx^i$ with the vector field $(v_1, \ldots, v_n) : S \to \mathbb{R}^n$, and similarly we may identify the $(n-1)$-form

$$\sum_i (-1)^i v_i\, dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n$$

with the same vector field. A 0-form is a scalar function, and an n-form can be identified with its coefficient which is a scalar function. Under these identifications, the de Rham complex (2) on a three-dimensional domain becomes

$$0 \to H^1(S) \xrightarrow{\text{grad}} H(\text{curl}, S) \xrightarrow{\text{curl}} H(\text{div}, S) \xrightarrow{\text{div}} L^2(S) \to 0.$$

The pullback of a scalar function f, viewed as a 0-form, is just the composition: $\hat{v}(\hat{x}) = v\big(F(\hat{x})\big)$, $\hat{x} \in \hat{S}$. If we identify scalar functions with n-forms, then the pullback is $\hat{v}(\hat{x}) = \det[DF(\hat{x})]v\big(F(\hat{x})\big)$ where $DF(\hat{x})$ is the Jacobian matrix of F at $\hat{x}$. For a vector field v, viewed as a 1-form or an $(n-1)$-form, the pullback operation corresponds to

$$\hat{v}(\hat{x}) = [DF(\hat{x})]^T v\big(F(\hat{x})\big), \quad \hat{v}(\hat{x}) = \text{adj}[DF(\hat{x})] v\big(F(\hat{x})\big),$$

respectively. The adjugate matrix, $\text{adj}\, A$, is the transposed cofactor matrix, which is equal to $(\det A)A^{-1}$ in case A is invertible. The latter formula, representing the pullback of an $(n-1)$-form, is called the Piola transform.

Next we consider how Sobolev norms transform under pullback. If F is a diffeomorphism of $\hat{S}$ onto S smooth up to the boundary, then each Sobolev norm of the pullback F^*v can be bounded in terms of the corresponding Sobolev norm of v and bounds on the partial derivatives of F and F^{-1}. Specifically, we have the following theorem.

Theorem 2.1 *Let r be a non-negative integer and $M > 0$. There exists a constant C depending only on r, M, and the dimension n, such that*

$$\|F^*v\|_{H^r\Lambda^k(\hat{S})} \le C\|v\|_{H^r\Lambda^k(S)}, \quad v \in H^r\Lambda^k(S),$$

whenever $\hat{S}, S$ are domains in $\mathbb{R}^n$ and $F : \hat{S} \to S$ is a C^{r+1} diffeomorphism satisfying

$$\max_{1\le s\le r+1} |F|_{W^s_\infty(\hat{S})} \le M, \quad |F^{-1}|_{W^1_\infty(S)} \le M.$$

Proof. From (3),

$$\|F^*v\|_{H^r\Lambda^k(\hat{S})} = \sum_{1\le i_1<\cdots<i_k\le m} \sum_{|\alpha|\le r} \int_{\hat{S}} |D^\alpha[(v_{i_1\cdots i_k} \circ F)\frac{\partial F^{i_1}}{\partial \hat{x}^{j_1}} \cdots \frac{\partial F^{i_k}}{\partial \hat{x}^{j_k}}](\hat{x})|^2\, d\hat{x}.$$

Using the Leibniz rule and the chain rule, we can bound the integrand by

$$C \sum_{|\beta| \le r} |(D^\beta v_{i_1 \cdots i_k})\big(F(\hat{x})\big)|$$

where C depends only on $\max_{1 \le s \le r+1} |F|_{W^s_\infty(\hat{S})}$ and so can be bounded in terms of M. Changing variables to $x = F\hat{x}$ in the integral brings in a factor of the Jacobian determinant of F^{-1}, which is also bounded in terms of M, and so gives the result. $\square$

A simple case is when F is a dilation: $F(\hat{x}) = h\hat{x}$ for some $h > 0$. Then (3) becomes

$$(F^* v)(\hat{x}) = \sum_{\sigma \in \Sigma(k,n)} v_\sigma(h\hat{x}) h^k \, d\hat{x}^\sigma,$$

and therefore,

$$D^\alpha (F^* v)(\hat{x}) = \sum_{\sigma \in \Sigma(k,n)} (D^\alpha v_\sigma)(h\hat{x}) h^{r+k} \, d\hat{x}^\sigma,$$

where $r = |\alpha|$. We thus get the following theorem.

Theorem 2.2 *Let F be the dilation $F\hat{x} = hx$ for some $h > 0$, $\hat{S}$ a domain in $\mathbb{R}^n$, $S = F(\hat{S})$, and α a multi-index of order r. Then*

$$\|D^\alpha (F^* v)\|_{L^2\Lambda^k(\hat{S})} = h^{r+k-n/2} \|D^\alpha v\|_{L^2\Lambda^k(S)}, \quad v \in H^r\Lambda^k(S).$$

3 Tensor products of complexes of differential forms

In this section we discuss the tensor product operation on differential forms and complexes of differential forms. The tensor product of a differential k-form on some domain and a differential l-form on a second domain may be naturally realized as a differential $(k+l)$-form on the Cartesian product of the two domains. When this construction is combined with the standard construction of the tensor product of complexes, we are led to a realization of the tensor product of subcomplexes of the de Rham subcomplex on two domains as a subcomplex of the de Rham complex on the Cartesian product of the domains (see also [9, 10]).

We begin by identifying the tensor product of algebraic forms on Euclidean spaces. For $m, n \ge 1$, consider the Euclidean space $\mathbb{R}^{m+n}$ with coordinates denoted $(x_1, \ldots, x_m, y_1, \ldots, y_n)$. The projection $\pi_1 : \mathbb{R}^{m+n} \to \mathbb{R}^m$ on the first m coordinates defines, by pullback, an injection $\pi_1^* : \operatorname{Alt}^k \mathbb{R}^m \to \operatorname{Alt}^k \mathbb{R}^{m+n}$, with the embedding of $\pi_2^* : \operatorname{Alt}^l \mathbb{R}^n \to \operatorname{Alt}^l \mathbb{R}^{m+n}$ defined similarly. Therefore we may define a bilinear map

$$\operatorname{Alt}^k \mathbb{R}^m \times \operatorname{Alt}^l \mathbb{R}^n \to \operatorname{Alt}^{k+l} \mathbb{R}^{m+n}, \quad (\mu, \nu) \mapsto \pi_1^*\mu \wedge \pi_2^*\nu,$$

or, equivalently, a linear map

$$\operatorname{Alt}^k \mathbb{R}^m \otimes \operatorname{Alt}^l \mathbb{R}^n \to \operatorname{Alt}^{k+l} \mathbb{R}^{m+n}, \quad \mu \otimes \nu \mapsto \pi_1^*\mu \wedge \pi_2^*\nu.$$

This map is an injection. Indeed, a basis for $\mathrm{Alt}^k \mathbb{R}^m \otimes \mathrm{Alt}^l \mathbb{R}^n$ consists of the tensors $dx^\sigma \otimes dy^\tau$ with $\sigma \in \Sigma(k,m)$ and $\tau \in \Sigma(l,n)$, which simply maps to $dx^\sigma \wedge dy^\tau$, an element of the standard basis of $\mathrm{Alt}^{k+l} \mathbb{R}^{m+n}$. In view of this injection, we may view the tensor product of $\mu \otimes \nu$ of a k-form μ on $\mathbb{R}^m$ and an l-form on $\mathbb{R}^n$ as a $(k+l)$-form on $\mathbb{R}^{m+n}$ (namely, we identify it with $\pi_1^*\mu \wedge \pi_2^*\nu$).

Next we turn to differential forms defined on domains in Euclidean space. Let u be a differential k-form on a domain $S \subset \mathbb{R}^m$ and v a differential l-form on $T \subset \mathbb{R}^n$. We may identify the tensor product $u \otimes v$ with the differential $(k+l)$-form $\pi_S^* u \wedge \pi_T^* v$ on $S \times T$ where $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$ are the canonical projections. In coordinates, this identification is

$$(\sum_\sigma f_\sigma \, dx^\sigma) \otimes (\sum_\tau g_\tau \, dy^\tau) = \sum_{\sigma,\tau} f_\sigma \otimes g_\tau \, dx^\sigma \wedge dy^\tau.$$

Note that the exterior derivative of the tensor product is

$$\begin{aligned} d_{S\times T}(\pi_S^* u \wedge \pi_T^* v) &= d_{S\times T}(\pi_S^* u) \wedge \pi_T^* v + (-1)^k \pi_S^* u \wedge d_{S\times T}(\pi_T^* v) \\ &= \pi_S^*(d_S u) \wedge \pi_T^* v + (-1)^k \pi_S^* u \wedge \pi_T^*(d_T v). \end{aligned} \tag{4}$$

Having defined the tensor product of differential forms, we next turn to the tensor product of complexes of differential forms. A subcomplex of the L^2 de Rham complex (2),

$$0 \to V^0 \xrightarrow{d} V^1 \xrightarrow{d} \cdots \xrightarrow{d} V^m \to 0, \tag{5}$$

is called a de Rham subcomplex on S. This means that, for each k, $V^k \subset H\Lambda^k(S)$ and d maps V^k into V^{k+1}. Suppose we are given such a de Rham subcomplex of S and also a de Rham subcomplex,

$$0 \to W^0 \xrightarrow{d} W^1 \xrightarrow{d} \cdots \xrightarrow{d} W^n \to 0, \tag{6}$$

on T. The tensor product of the complexes (5) and (6) is the complex

$$0 \to (V \otimes W)^0 \xrightarrow{d} (V \otimes W)^1 \xrightarrow{d} \cdots \xrightarrow{d} (V \otimes W)^{m+n} \to 0, \tag{7}$$

where

$$(V \otimes W)^k := \bigoplus_{i+j=k} (V^i \otimes W^j), \quad k = 0, \ldots, m+n. \tag{8}$$

and the differential $(V \otimes W)^k \to (V \otimes W)^{k+1}$ is defined by

$$d(u \otimes v) = d_S^i u \otimes v + (-1)^i u \otimes d_T^j v, \quad u \in V^i, v \in W^j.$$

In view of the identification of the tensor product of differential forms with differential forms on the Cartesian product, the space $(V \otimes W)^k$ in (8) consists of differential k-forms on $S \times T$, and, in view of (4), the differential is the restriction of the exterior derivative on $H\Lambda^k(S \times T)$. Thus the tensor product complex (7) is a subcomplex of the de Rham complex on $S \times T$.

4 Finite elements

Let Ω be a bounded domain in $\mathbb{R}^n$, $0 \le k \le n$. As in [11], a finite element space of k-forms on Ω is assembled from several ingredients: a *triangulation* $\mathcal{T}$ of Ω, whose elements we call *finite elements*, and, for each finite element K, a space $V(K)$ of *shape functions* on K, and a set $\Xi(K)$ of *degrees of freedom*. We now describe these ingredients more precisely.

For the triangulation, we allow the finite elements to be curvilinear polytopes. This means that each K is the image $F_K(\hat{K})$ of an n-dimension polytope $\hat{K}$ (so a closed polygon in two dimensions and a closed polyhedron in three dimensions) under a smooth invertible map F_K of $\hat{K}$ into $\mathbb{R}^n$. The faces of K are defined as the images of the faces of $\hat{K}$, and the requirement that $\mathcal{T}$ be a triangulation of Ω means that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}} K$ and that the intersection of any two elements of $\mathcal{T}$ is either empty or is a common face of both of some dimension.

The space $V(K)$ of shape functions is a finite-dimensional space of k-forms on K. The degrees of freedom are a unisolvent set of functionals on $V(K)$, or, otherwise put, $\Xi(K)$ is a basis for the dual space $V(K)^*$. Further, each degree of freedom is associated to a specific face of K, and when two distinct elements K_1 and K_2 intersect in a common face f, the degrees of freedom of K_1 and K_2 associated to the face f are in 1-to-1 correspondence.

The finite element functions are then defined as the differential forms on Ω which belong to the shape functions spaces piecewise, and for which corresponding degrees of freedom are single-valued. That is, to define a finite element function u we specify, for all elements K, shape functions $u_K \in V(K)$ satisfying $\xi_1(u_{K_1}) = \xi_2(u_{K_2})$ whenever K_1 and K_2 share a common face and $\xi_1 \in \Xi_1(K)$ and $\xi_2 \in \Xi_2(K)$ are corresponding degrees of freedom associated to the face. Then u is defined almost everywhere by setting its restriction to the interior of each element K to be u_K. The finite element space $V(\mathcal{T})$ is defined to be the space of all such finite element functions.

The degrees of freedom associated to faces of dimension $< n$ determine the inter-element continuity imposed on the finite element functions. Specifying the continuity in this way leads to finite element spaces which can be efficiently implemented. We note that the finite element space is unchanged if we use a different set of degrees of freedom, as long as the span of the degrees of freedom associated to each face is unchanged. Thus we shall usually specify these spans, rather than a particular choice of basis for them.

When a finite element K is presented as $F_K(\hat{K})$ using a *reference element* $\hat{K} \subset \mathbb{R}^n$ and a diffeomorphic mapping $F_K : \hat{K} \to K$ (as for curvilinear polytopes), it is often convenient to specify the shape functions and degrees of freedom in terms of the reference element and the mapping. That is, we specify a finite dimensional space $V(\hat{K})$ of differential k-forms on $\hat{K}$, the *reference shape functions*, and define $V(K) = (F_K^{-1})^* V(\hat{K})$, the pullback under the diffeomorphism $F_K^{-1} : K \to \hat{K}$. Similarly, given a space $\Xi(\hat{K})$ of degrees of freedom for $V(\hat{K})$ on $\hat{K}$ and a reference degree of freedom $\hat{\xi} \in \Xi(\hat{K})$ associated to a face $\hat{f}$

of $\hat{K}$, we define ξ by $\xi(v) = \hat{\xi}(F_K^* v)$ for $v \in V(K)$ and associate ξ to the face $f = F_K(\hat{f})$ of K. In this way we determine the degrees of freedom $\Xi(K)$. In the present paper we shall be concerned with the case where each element of the mesh is presented as the image of the unit cube $\hat{K} = [0,1]^n$ under a multilinear map (i.e., each component of F_K is a polynomial of degree at most one in each of the n variables). A special case is when F_K is affine, in which case the elements K are arbitrary parallelotopes (the generalization of parallelograms in two dimensions and parallelepipeds in three dimensions). In the general case, the elements are arbitrary convex quadrilaterals in two dimensions, but in more dimensions they may be truly curvilinear: the edges are straight, but the faces of dimension two need not be planar.

We need some measure of the shape regularity of an element K. Since the shape regularity should depend on the shape, but not the size, of the element, we require it to be invariant under dilation. Therefore, let $h_K = \mathrm{diam}(K)$ and set $\bar{K} = h_K^{-1} K$, which is of unit diameter. Assuming only that the element is a Lipschitz domain, the Bramble–Hilbert lemma ensures that for $r \geq 0$ integer there exists a constant C such that

$$\inf_{p \in \mathcal{P}_r(\bar{K})} \|u - p\|_{L^2(\bar{K})} \leq C |u|_{H^{r+1}(\bar{K})}, \quad u \in H^{r+1}(\bar{K}). \tag{9}$$

We define $C(K, r)$ to be the least such constant, and take this as our shape regularity measure. If the domain K is convex then $C(K, r)$ can be bounded in terms only of r and the dimension n; see [18]. If the domain is star-shaped with respect to a ball of diameter at least δh_K for some $\delta > 0$, then $C(K, r)$ can be bounded in terms of r, n, and δ; see [13].

The next theorem will play a fundamental role in proving approximation properties of finite element differential forms.

Theorem 4.1 *Let K be a bounded Lipschitz domain in $\mathbb{R}^n$ with diameter h_K and let r be a non-negative integer. Then there exists a constant C depending only on r, n, and the shape constant $C(K, r)$, such that*

$$\inf_{p \in \mathcal{P}_r \Lambda^k(K)} \|u - p\|_{L^2 \Lambda^k(K)} \leq C h_K^{r+1} |u|_{H^{r+1} \Lambda^k(K)}, \quad u \in H^{r+1} \Lambda^k(K),$$

for $0 \leq k \leq n$.

Proof. Let $\bar{K} = h_K^{-1} K$, let $F : \bar{K} \to K$ be the dilation, and set $\bar{u} = F^* u$. Applying (9) to each component of $F^* u$, we obtain $p \in \mathcal{P}_r \Lambda^k(\bar{K})$ with

$$\|F^* u - p\|_{L^2 \Lambda^k(\bar{K})} \leq C |F^* u|_{H^{r+1} \Lambda^k(\bar{K})}. \tag{10}$$

Set $q = (F^{-1})^* p$. Since F^{-1} is a dilation, $q \in \mathcal{P}_r \Lambda^k(K)$, and clearly $F^*(u - q) = F^* u - p$. Combining (10) and Theorem 2.2 gives the desired estimate. $\square$

5 Tensor product finite element differential forms on cubes

In this section, we describe the construction of specific finite element spaces of differential forms. We suppose that the triangulation consists of curvilinear cubes, so that each element K of the triangulation $\mathcal{T}$ is specified by a smooth diffeomorphism F_K taking the unit cube $\hat{K} = [0,1]^n$ onto $K \subset \mathbb{R}^n$. As described above, the shape functions and degrees of freedom on K can then be determined by specifying a space $V(\hat{K})$ of reference shape functions and a set $\Xi(\hat{K})$ of reference degrees of freedom. Now we apply the tensor product construction of Section 3 to construct the reference shape functions and degrees of freedom.

Fix an integer $r \geq 1$. With $I = [0,1]$ the unit interval, let $\mathcal{P}_r\Lambda^0(I) = \mathcal{P}_r(I)$ denote the space of polynomial functions (0-forms) on I of degree at most r, and let $\mathcal{P}_{r-1}\Lambda^1(I) = \mathcal{P}_{r-1}(I)\,dx$ denote the space of polynomial 1-forms of degree at most $r-1$. Connecting these two spaces with the exterior derivative, we obtain a subcomplex of the de Rham complex on the unit interval:

$$\mathcal{P}_r(I) \xrightarrow{d} \mathcal{P}_{r-1}(I)\,dx. \tag{11}$$

Taking the tensor product of this complex with itself, we obtain a subcomplex of the de Rham complex on I^2, whose spaces we denote by $\mathcal{Q}_r^-\Lambda^k = \mathcal{Q}_r^-\Lambda^k(I^2)$:

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^2. \tag{12}$$

Specifically, writing $\mathcal{P}_{r,s} = \mathcal{P}_{r,s}(I^2)$ for $\mathcal{P}_r(I) \otimes \mathcal{P}_s(I)$, we have on I^2,

$$\mathcal{Q}_r^-\Lambda^0 = \mathcal{P}_{r,r}, \quad \mathcal{Q}_r^-\Lambda^1 = \mathcal{P}_{r-1,r}\,dx \oplus \mathcal{P}_{r,r-1}\,dy, \quad \mathcal{Q}_r^-\Lambda^2 = \mathcal{P}_{r-1,r-1}\,dx \wedge dy.$$

The first space $\mathcal{Q}_r^-\Lambda^0$ is the tensor product polynomial space traditionally denoted $\mathcal{Q}_r$, and the last space is $\mathcal{Q}_{r-1}\,dx \wedge dy$.

To get spaces on the 3-D cube, we may further take the tensor product of the 2-D complex (12) with the 1-D complex (11), or, equivalently, take the tensor product of three copies of the 1-D complex, to obtain a de Rham subcomplex on the unit cube I^3:

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^2 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^3.$$

Here $\mathcal{Q}_r^-\Lambda^0 = \mathcal{Q}_r$, $\mathcal{Q}_r^-\Lambda^3 = \mathcal{Q}_{r-1}\,dx \wedge dy \wedge dz$, and

$$\mathcal{Q}_r^-\Lambda^1 = \mathcal{P}_{r-1,r,r}\,dx \oplus \mathcal{P}_{r,r-1,r}\,dy \oplus \mathcal{P}_{r,r,r-1}\,dz,$$
$$\mathcal{Q}_r^-\Lambda^2 = \mathcal{P}_{r,r-1,r-1}\,dy \wedge dz \oplus \mathcal{P}_{r-1,r,r-1}\,dx \wedge dz \oplus \mathcal{P}_{r-1,r-1,r}\,dx \wedge dy.$$

The extension of this construction to higher dimensions is clear, yielding spaces $\mathcal{Q}_r^-\Lambda^k(I^n)$ for $0 \leq k \leq n$, $n \geq 1$, spanned by quantities $p\,dx^\sigma$, where $\sigma \in \Sigma(k,n)$

and p is a polynomial of degree at most r in all variables and of degree at most $r-1$ in the variables x^{σ_i}. More precisely,

$$\mathcal{Q}_r^- \Lambda^k(I^n) = \bigoplus_{\sigma\in\Sigma(k,n)} \left[\bigotimes_{i=1}^n \mathcal{P}_{r-\delta_{i,\sigma}}(I)\right] dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},$$

where

$$\delta_{i,\sigma} = \begin{cases} 1, & i\in\{\sigma_1,\dots,\sigma_k\}, \\ 0, & \text{otherwise.}\end{cases}$$

In the case $k=0$, this space is understood to be

$$\mathcal{Q}_r^- \Lambda^0(I^n) = \bigotimes_{i=1}^n \mathcal{P}_r(I),$$

i.e., the space conventionally referred to as $\mathcal{Q}_r(I^n)$. This definition makes sense also if $r=0$, so $\mathcal{Q}_0^-\Lambda^0(I^n) = \mathcal{Q}_0(I^n) = \mathbb{R}$ is the space of constant functions. For $k>0$, $\mathcal{Q}_0^-\Lambda^k(I^n) = 0$. We also interpret this in the case $n=0$, so I^n is a single point. Then $\mathcal{Q}_r^-\Lambda^0(I^0)$ is understood to be the space $\mathbb{R}$ of constants. It is easy to check that

$$\dim \mathcal{Q}_r^-\Lambda^k(I^n) = \binom{n}{k}(r+1)^{n-k}r^k,$$

in all cases $0\le k\le n$, $r\ge 0$.

Let us now characterize the degrees of freedom of $\mathcal{Q}_r^-\Lambda^k(I^n)$. The space $\mathcal{P}_r\Lambda^0(I) = \mathcal{P}_r(I)$ has dimension $r+1$. It has one degree of freedom associated to each of the two vertices $p=0$ and $p=1$, namely the evaluation functional $v\mapsto v(p)$. The remaining $r-1$ degrees of freedom are associated to I itself, and are given by

$$v\mapsto \int_I v(x)q(x)\,dx, \quad q\in\mathcal{P}_{r-2}(I).$$

The space $\mathcal{P}_{r-1}\Lambda^1(I) = \mathcal{P}_{r-1}(I)$ has dimension r and its degrees of freedom are given by

$$v\mapsto \int_I v(x)q(x)\,dx, \quad q\in\mathcal{P}_{r-1}(I).$$

The tensor product construction then yields degrees of freedom for $\mathcal{Q}_r^-\Lambda^k(I^n)$. Namely, each degree of freedom for $\mathcal{Q}_r^-\Lambda^k(I^n)$ is of the form $\xi_1\otimes\ldots\otimes\xi_n$, and is associated to the Cartesian product face $f_1\times\ldots\times f_n$, where $\xi_j\in\Xi(I)$ is associated to the face f_j of the interval I $\forall\, j$. Hence, a set of unisolvent degrees of freedom for $\mathcal{Q}_r^-\Lambda^k(I^n)$ ($r\ge 1$, $0\le k\le n$) is given by

$$v\mapsto \int_f \operatorname{tr}_f v(x)\wedge q(x), \quad q\in\mathcal{Q}_{r-1}^-\Lambda^{d-k}(f), \tag{13}$$

for each face f of I^n of degree $d\ge k$.

Let us count that we have supplied the correct number of degrees of freedom. Since the number of faces of dimension d of I^n is $2^{n-d}\binom{n}{d}$, the total number of degrees of freedom is

$$\sum_{d=k}^{n} 2^{n-d}\binom{n}{d}\binom{d}{k} r^k (r-1)^{d-k}.$$

Substituting $\binom{n}{d}\binom{d}{k} = \binom{n}{k}\binom{n-k}{n-d}$ and then changing the summation index from d to $m = n - d$ we get:

$$\begin{aligned}\sum_{d=k}^{n} 2^{n-d}\binom{n}{k}\binom{n-k}{n-d} r^k (r-1)^{d-k} &= \binom{n}{k} r^k \sum_{m=0}^{n-k}\binom{n-k}{m} 2^m (r-1)^{n-k-m} \\ &= \binom{n}{k} r^k (r-1+2)^{n-k} = \dim \mathcal{Q}_r^- \Lambda^k(I^n),\end{aligned}$$

as desired.

Figure 1, taken from [2], shows degree of freedom diagrams for the $\mathcal{Q}_r^- \Lambda^k$ spaces in two and three dimensions. In such diagrams, the number of symbols drawn in the interior of a face is equal to the number of degrees of freedom associated to the face.

Now suppose that a mesh $\mathcal{T}$ of curvilinear cubes $K = F_K(\hat{K})$ is given. We assume that whenever two elements K_1 and K_2 meet in a common face, say $f = F_{K_1}(\hat{f}_1) = F_{K_2}(\hat{f}_2)$, then the map $F_{K_2}^{-1} \circ F_{K_1}|_{\hat{f}_1}$ mapping $\hat{f}_1$ onto $\hat{f}_2$ is linear. The assembled finite element space $\mathcal{Q}_r^- \Lambda^k(\mathcal{T})$ consists of functions u whose restriction u_K to K belongs to $(F_K^{-1})^* \mathcal{Q}_r^- \Lambda^k(I^n)$ and for which the corresponding degrees of freedom are single-valued on faces. The choice of degrees of freedom implies that if K_1 and K_2 share a face f of dimension $\geq k$, then the traces $\operatorname{tr}_f u_{K_1}$ and $\operatorname{tr}_f u_{K_2}$ coincide. This is exactly the condition needed for u to belong to the space $H\Lambda^k(\Omega)$, i.e., for the exterior derivative of u to belong to $L^2\Lambda^{k+1}(\Omega)$ (see [7, Lemma 5.1]). From the commutativity of the pullback with the exterior derivative, we see that $d\mathcal{Q}_r^- \Lambda^k(\mathcal{T}) \subset \mathcal{Q}_r^- \Lambda^{k+1}(\mathcal{T})$, and so we obtain a subcomplex of the L^2 de Rham complex on Ω:

$$\mathcal{Q}_r^- \Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{Q}_r^- \Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Q}_r^- \Lambda^n(\mathcal{T}).$$

The assembled finite element spaces $\mathcal{Q}_r^- \Lambda^k(\mathcal{T})$ are well-known, especially when the maps F_K are composed of dilations and translations, so the mesh consists of cubes. The space $\mathcal{Q}_r^- \Lambda^0(\mathcal{T})$ is the usual $\mathcal{Q}_r$ approximation of $H^1(\Omega)$ and the space $\mathcal{Q}_r^- \Lambda^n(\mathcal{T})$ is the discontinuous $\mathcal{Q}_{r-1}$ approximation of $L^2(\Omega)$. In three dimensions, if we identify $H\Lambda^1$ and $H\Lambda^2$ with $H(\mathrm{curl})$ and $H(\mathrm{div})$, respectively, then $\mathcal{Q}_r^- \Lambda^1(\mathcal{T})$ and $\mathcal{Q}_r^- \Lambda^2(\mathcal{T})$ are the Nédélec edge element and face element spaces of the first kind, respectively [17].

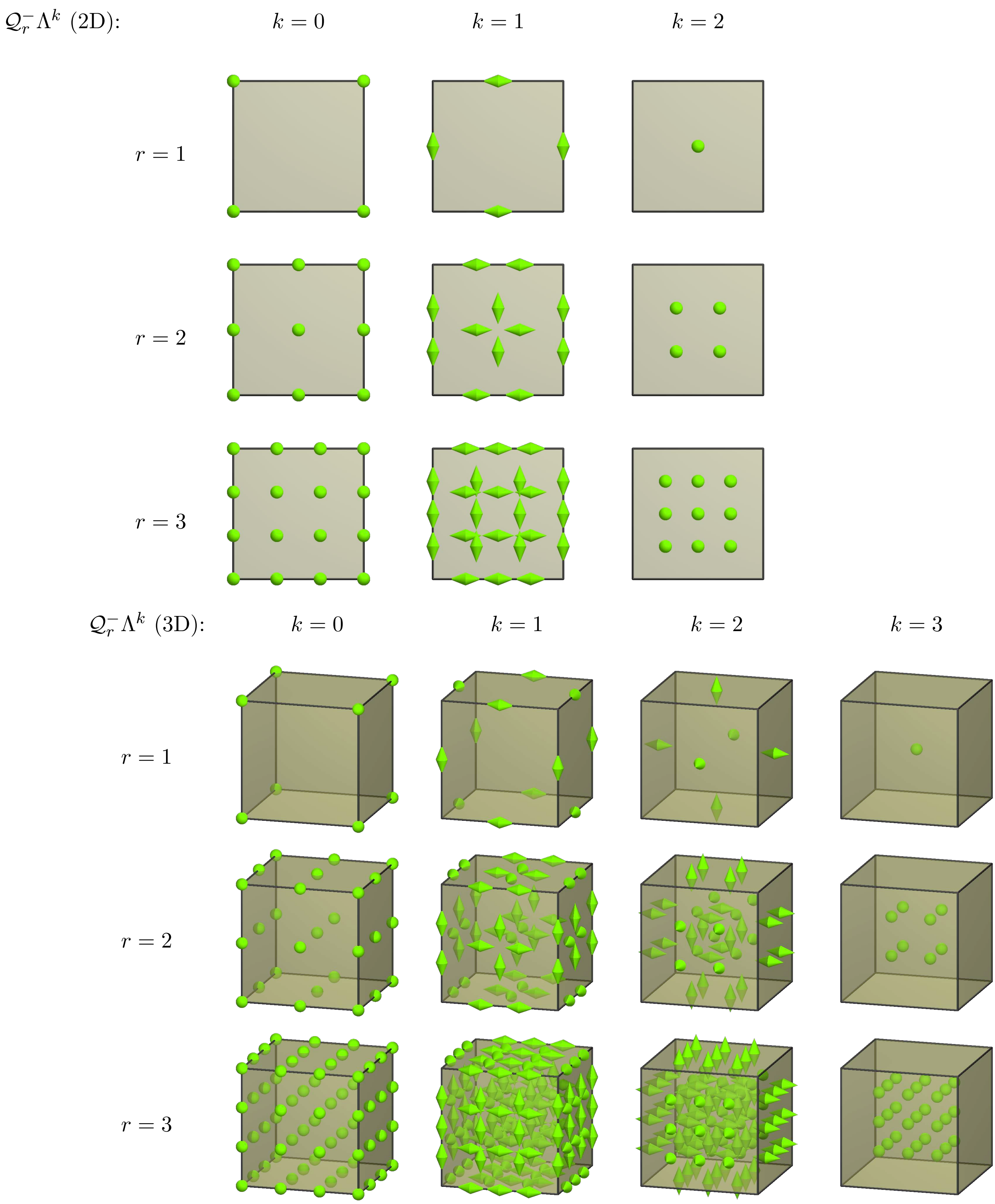

Figure 1: Degree of freedom diagrams for $\mathcal{Q}_r^-\Lambda^k$ spaces in two and three dimensions.

6 Approximation properties on curvilinear cubes

We now return to the situation of Section 4 and consider the approximation properties afforded by finite element spaces of differential k-forms on meshes of curvilinear cubes. Thus we suppose given:

1. A space of reference shape k-forms $V(\hat{K})$ and a set of reference degrees of freedom $\Xi(\hat{K})$ on the unit cube $\hat{K}$;
2. A mesh $\mathcal{T}$ of the domain Ω;
3. For each $K \in \mathcal{T}$ a diffeomorphism F_K of $\hat{K}$ onto K.

As described in Section 4, these determine a space of k-form shape functions $V(K)$ on each element K and an assembled finite element space $V(\mathcal{T})$ consisting of k-forms on Ω which belong piecewise to the $V(K)$. If it happens that $V(K)$ contains the full polynomial space $\mathcal{P}_r\Lambda^k(K)$, then Theorem 4.1 gives us the approximation result

$$\inf_{v \in V(K)} \|u - v\|_{L^2\Lambda^k(K)} \leq C h_K^{r+1} |u|_{H^{r+1}\Lambda^k(K)}, \tag{14}$$

with a constant C depending only on r, n, and the shape constant $C(K, r)$.

We now show how to extend estimate (14) to the global space $V(\mathcal{T})$. More precisely, under the same condition that for all $K \in \mathcal{T}$ the space $V(K)$ contains the full polynomial space $\mathcal{P}_r\Lambda^k(K)$, we want to show that

$$\inf_{v \in V(\mathcal{T})} \|u - v\|_{L^2\Lambda^k} \leq C h^{r+1} |u|_{H^{r+1}\Lambda^k}, \tag{15}$$

where h is as usual the maximum of the h_K's and the constant C depends only on r, n, and the shape constants $C(K, r)$. This result can be obtained by extending to this setting the classical construction of the Clément interpolant presented in [12]. An analogous extension has been presented in [7, Sec. 5.4] in the case of simplicial meshes. A Clément-like operator $\pi_h : L^2\Lambda^k \to V(\mathcal{T})$ can be defined as in (13), where $\operatorname{tr}_f v(x)$ is replaced by the trace of the L^2 projection of v onto $\mathcal{P}_r\Lambda^k(S)$, S being the union of elements in $\mathcal{T}$ containing the face f. Following [12], crucial conditions in order to get (15) are that global polynomial forms are preserved by π_h, that is $\pi_h w = w$ whenever w belongs to $\mathcal{P}_r\Lambda^k(\Omega)$, and suitable scaling estimates which in our case are consequences of Theorems 2.1 and 2.2.

Thus we are led to ask what conditions on the reference shape functions $V(\hat{K})$ and the mappings F_K ensure that $V(K) = (F_K^{-1})^* V(\hat{K})$ contains $\mathcal{P}_r\Lambda^k(K)$. The following result, which is straightforward to prove at this point, answers this question for affine and multilinear maps.

Theorem 6.1 *Suppose that either*

1. F_K is an affine diffeomorphism and that $\mathcal{P}_r\Lambda^k(\hat{K}) \subset V(\hat{K})$; or

2. F_K is a multilinear diffeomorphism and $\mathcal{Q}^-_{r+k}\Lambda^k(\hat{K}) \subset V(\hat{K})$

Then $V(K)$ contains $\mathcal{P}_r\Lambda^k(K)$ and so (14) *and* (15) *hold.*

Proof. Let us write F for F_K. The requirement that $V(K) = (F^{-1})^*V(\hat{K})$ contains $\mathcal{P}_r\Lambda^k(K)$ is equivalent to requiring that $F^*\big(\mathcal{P}_r\Lambda^k(K)\big) \subset V(\hat{K})$. If F is affine, then $F^*\big(\mathcal{P}_r\Lambda^k(K)\big) \subset \mathcal{P}_r\Lambda^k(\hat{K})$, as is clear from (3). The sufficiency of the first condition follows.

For F multilinear, we wish to show that $F^*\big(\mathcal{P}_r\Lambda^k(K)\big) \subset \mathcal{Q}^-_r\Lambda^k(\hat{K})$, for which it suffices to show that $F^*(p\,dx^\sigma) \in \mathcal{Q}^-_{r+k}\Lambda^k(\hat{K})$ if $p \in \mathcal{P}_r(K)$ and $\sigma \in \Sigma(k,n)$. From (3) it suffices to show that

$$(p \circ F)\frac{\partial F^{i_1}}{\partial \hat{x}^{j_1}} \cdots \frac{\partial F^{i_k}}{\partial \hat{x}^{j_k}}\, d\hat{x}^{j_1} \wedge \cdots \wedge d\hat{x}^{j_k} \in \mathcal{Q}^-_{r+k}\Lambda^k(\hat{K}). \tag{16}$$

Now $p \in \mathcal{P}_r$ and F multilinear imply that $p \circ F \in \mathcal{Q}_r(K)$. Moreover, $\partial F^{i_l}/\partial \hat{x}^{j_l}$ is multilinear, but also independent of $\hat{x}^{j_l}$. Therefore the product, $(p{\circ}F)\partial F^{i_1}/\partial\hat{x}^{j_1} \cdots \partial F^{i_k}/\partial\hat{x}^{j_k}$, is of degree at most $r+k$ in all variables, but of degree at most $r+k-1$ in the variables $\hat{x}^{j_l}$. Referring to the description of the spaces $\mathcal{Q}^-_r$ spaces derived in Section 5, we verify (16). □

Note that the requirement on the reference space $V(\hat{K})$ to obtain $O(h^{r+1})$ convergence is much more stringent when the maps F_K are only assumed to be multilinear, than in the case when they are restricted to be affine. Instead of just requiring that $V(\hat{K})$ contain the polynomial space $\mathcal{P}_r\Lambda^k(\hat{K})$, it must contain the much larger space $\mathcal{Q}^-_{r+k}\Lambda^k(\hat{K})$. For 0-forms, the requirement reduces to inclusion of the space $\mathcal{Q}_r(\hat{K})$. This result was obtained previously in [5] in two dimensions, and in [15], in three dimensions. The requirement becomes even more stringent as the form degree, k, is increased.

7 Application to specific finite element spaces

Theorem 6.1 gives conditions on the space $V(\hat{K})$ of reference shape functions which ensure a desired rate of L^2 approximation accuracy for the assembled finite element space $V(\mathcal{T})$. In this section we consider several choices for $V(\hat{K})$ and determine the resulting implied rates of approximation. According to the theorem, the result will be different for parallelotope meshes, in which each element is an affine image of the cube $\hat{K}$, and for the more general situation of curvilinear cubic meshes in which element is a multilinear image of $\hat{K}$ (which we shall simply refer to as curvilinear meshes in the remainder the section). Specifically, if s is the largest integer such that

$$\mathcal{P}_{s-1}\Lambda^k(\hat{K}) \subset V(\hat{K}), \tag{17}$$

then the theorem implies a rate of s (that is, L^2 error bounded by $O(h^s)$) on parallelotope meshes, while if s is the largest integer for which the more stringent condition

$$\mathcal{Q}^-_{s+k-1}\Lambda^k(\hat{K}) \subset V(\hat{K}), \tag{18}$$

holds, then the theorem implies a rate of s more generally on curvilinear meshes.

7.1 The space $\mathcal{Q}_r$

First we consider the case of 0-forms (H^1 finite elements) with $V(\hat{K}) = \mathcal{Q}^-_r\Lambda^0(\hat{K})$, which is the usual $\mathcal{Q}_r(\hat{K})$ space, consisting of polynomials of degree at most r in each variable. Then (17) holds for $s = r+1$, but not larger. Therefore we find the L^2 approximation rate to be $r+1$ on parallelotope meshes, as is well-known. Since $k = 0$, (18) also holds for $s = r+1$, and so on curvilinear meshes we obtain the same rate $r+1$ of approximation. Thus, in this case, the generalization from parallelotope to curvilinear meshes entails no loss of accuracy.

7.2 The space $\mathcal{Q}^-_r\Lambda^n$

The case of n-forms with $V(\hat{K}) = \mathcal{Q}^-_r\Lambda^n(\hat{K})$ is quite different. If we identify n-forms with scalar-valued functions, this is the space $\mathcal{Q}_{r-1}(\hat{K})$. Hence (17) holds with $s = r$, and so we achieve approximation order $O(h^r)$ on parallelotope meshes. However, for $k = n$, (18) holds only if $s \le r-n+1$, and so Theorem 6.1 only gives a rate of $r-n+1$ on curvilinear meshes in this case, and *no convergence at all* if $r \le n-1$. Thus curvilinear meshes entail a loss of one order of accuracy compared to parallelotope meshes in two dimensions, a loss of two orders in three dimensions, etc. This may seem surprising, since if we identify an n-form $v\,dx^1 \wedge \cdots \wedge dx^n$ with the function v, which is a 0-form, the space $\mathcal{Q}^-_r\Lambda^n(\hat{K})$ corresponds to the space $Q_{r-1}(\hat{K}) = \mathcal{Q}^-_{r-1}\Lambda^0(\hat{K})$, which achieves the same rate r on both classes of meshes. The reason is that the transformation of an n-form on $\hat{K}$ to an n-form on K is very different than the transform of a 0-form. The latter is simply $\hat{v} = v \circ F$ while the former is

$$\hat{v}\,d\hat{x}^1 \wedge \cdots \wedge \hat{x}^n = (v \circ F)(\det F)\,dx^1 \wedge \cdots \wedge dx^n.$$

7.3 The spaces $\mathcal{Q}^-_r\Lambda^k$

Next we consider the case $V(\hat{K}) = \mathcal{Q}^-_r\Lambda^k(\hat{K})$ for k strictly between 0 and n. Then (17) holds with $s = r$ and (18) holds with $s = r-k+1$ (but no larger). Thus, the rates of approximation achieved are r on parallelotope meshes and $r-k+1$ for curvilinear meshes. In particular, if $r < k$, then the space provides no approximation in the curvilinear case. For example, in three dimensions the space $\mathcal{Q}^-_1\Lambda^2(K)$, which, in conventional finite element language is the trilinearly mapped Raviart-Thomas-Nedelec space, affords no approximation in L^2. This fact was noted and discussed in [16].

7.4 The space $\mathcal{P}_r\Lambda^n$

Since the space $H\Lambda^n(\Omega)$ does not require inter-element continuity, the shape functions $V(\hat{K}) = \mathcal{P}_r\Lambda^n(\hat{K})$ may be chosen as reference shape functions (with all degrees of freedom in the interior of the element). In this case, (17) clearly holds for $s = r + 1$, giving the expected $O(h^{r+1})$ convergence on parallelotope meshes. But (18) holds if and only if $\mathcal{Q}_{s+n-2}(\hat{K}) \subset \mathcal{P}_r(\hat{K})$, which holds if and only if $n(s+n-2) \leq r$. In two dimensions this condition becomes $s \leq r/2$, and so we only obtain the rate of $\lfloor r/2 \rfloor$, and no approximation at all for $r = 0$ or 1. In three dimensions, the corresponding rate is $O\lfloor r/3 \rfloor - 1$, requiring $r \geq 6$ for first order convergence on curvilinear meshes. For general n, the rate is $\lfloor r/n \rfloor - n + 2$.

7.5 The serendipity space $\mathcal{S}_r$

The serendipity space $\mathcal{S}_r$, $r \geq 1$, is a finite element subspace of H^1 (i.e., a space of 0-forms). In two dimensions and, for small r, in three dimensions, the space has been used for many decades. In 2011, a uniform definition was given for all dimensions n and all degrees $r \geq 1$ [3]. According to this definition, the shape function space $\mathcal{S}_r(\hat{K})$ consists of all polynomials of *superlinear degree* less than or equal to r, i.e., for which each monomial has degree at most r ignoring those variables which enter the monomial linearly (example: the monomial x^2yz^3 has superlinear degree 5). From this definition, it is easy to see that $\mathcal{P}_r(\hat{K}) \subset \mathcal{S}_r(\hat{K})$ but $\mathcal{P}_{r+1}(\hat{K}) \not\subset \mathcal{S}_r(\hat{K})$. Thus, from (17), the rate of L^2 approximation of $\mathcal{S}_r$ on parallelotope meshes is $r + 1$. Now $\mathcal{Q}_1(\hat{K}) \subset \mathcal{S}_r(\hat{K})$ for all $r \geq 1$, but for $s \geq 2$, $\mathcal{Q}_s(\hat{K})$ contains the element $(x_1 \cdots x_n)^s$ of superlinear degree ns. It follows that (18) holds if and only if $s \leq \max(2, \lfloor r/n \rfloor + 1)$, which gives the L^2 rate of convergence of the serendipity elements on curvilinear meshes. This was shown in two dimensions in [5].

7.6 The space $\mathcal{S}_r\Lambda^k$

In [4], Arnold and Awanou defined shape functions and degrees of freedom for a finite element space $\mathcal{S}_r\Lambda^k$ on cubic meshes in n-dimensions for all form degrees k between 0 and n, and polynomial degrees $r \geq 1$. The shape function space $\mathcal{S}_r\Lambda^k(\hat{K})$ they defined contains $\mathcal{P}_r\Lambda^k(\hat{K})$, so the assembled finite element space affords a rate of approximation $r+1$ on parallelotope meshes. In the case $k = n$, $\mathcal{S}_r\Lambda^n(\hat{K})$ in fact coincides with $\mathcal{P}_r\Lambda^n(\hat{K})$, so, as discussed above, the rate is reduced to $\lfloor r/n \rfloor - n + 2$ on curvilinear meshes. In the case $k = 0$, $\mathcal{S}_r\Lambda^0$ coincides with the serendipity space $\mathcal{S}_r$, and so the rate is reduced to $\max(2, \lfloor r/n \rfloor + 1)$ on curvilinear meshes in that case. In $n = 2$ dimensions, this leaves the space $\mathcal{S}_r\Lambda^1$, which is the BDM space on squares, for which the reference shape functions are $\mathcal{P}_r\Lambda^1(\hat{K})$ together with the span of the two forms $d(x^{r+1}y)$ and $d(xy^{r+1})$. To check condition (18) we note that $\mathcal{Q}_s^-\Lambda^1(\hat{K})$ contains $x^{s-1}y^s\,dx$, which is contained in $\mathcal{S}_r\Lambda^1(\hat{K})$ only if $2s - 1 \leq r$. Thus the rate of approximation of $\mathcal{S}_r\Lambda^1$ on curvilinear meshes in two dimension is $\lfloor (r+1)/2 \rfloor$.

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Dipartimento di Matematica "F. Brioschi",
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

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