PATTERN FORMATION: REACTION-DIFFUSION PDE SYSTEMS WITH STRONG COMPETITION

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Several phenomena can be described by a certain number of densities (of mass, population, probability, ...) distributed in a domain and subject to diffusion, reaction, and competitive interaction. When the competition is the prevailing aspect, one can reasonably expect pattern formation: the densities should not coexist in the same region, and they should tend to segregate, hence determining a partition of the domain.

From an analytic point of view, this amounts to study semilinear elliptic systems of $k \geq 2$ stationary equations:

$$\underset{\substack{\uparrow\\ \text{diffusion}}}{-\Delta u_i} = \underset{\substack{\uparrow\\ \text{reaction}}}{f_i(x, u_i)} + \underset{\substack{\uparrow\\ \text{interaction}}}{g_i(x, u_1, \dots, u_k)}, \qquad i = 1, \dots, k. \tag{1}$$

where $u_1 = u_1(x), \dots, u_k = u_k(x)$ are the unknown densities, defined either in $\mathbb{R}^N$ or in a bounded domain with suitable boundary conditions. Also transport terms, or more general diffusions, may be included.

Systems like (1) appear when searching for:

- equilibrium solutions to the corresponding evolution equations:

$$\partial_t u_i - \Delta u_i = \dots, \qquad \partial_{tt} u_i - \Delta u_i = \dots$$

- travelling wave solutions to diffusion equations: substituting

$$U_i(x, t) = u_i(x - ct), \quad c \in \mathbb{R}^N,$$

  into

$$\partial_t U_i - \Delta U_i = h(U_1, \dots, U_k),$$

  we obtain

$$-\Delta u_i - c \cdot \nabla u_i = h(u_1, \dots, u_k),$$

- rotating wave solutions to diffusion equations: substituting

$$U_i(x, t) = u_i\left(e^{i\omega t} x\right), \quad \omega \in \mathbb{R}, \quad x \in \mathbb{R}^2 \cong \mathbb{C},$$

  into

$$\partial_t U_i - \Delta U_i = h(U_1, \dots, U_k),$$

  we obtain

$$-\Delta u_i + \omega x^\perp \cdot \nabla u_i = h(u_1, \dots, u_k),$$

- solitary wave solutions to Schrödinger equations: substituting
$$\Phi_i(x,t) = e^{\mathrm{i}\lambda_i t} u_i(x), \quad \lambda \in \mathbb{R},$$
into
$$\mathrm{i}\partial_t \Phi_i + \Delta\Phi_i = h(|\Phi_1|, \dots, |\Phi_k|)\Phi_i,$$
we obtain
$$-\Delta u_i + \lambda_i u_i = -h(u_1, \dots, u_k)u_i.$$

Generally speaking, the main goals when dealing with (1) are:

- existence/multiplicity of non-trivial solutions (i.e. solutions with each component non-vanishing),
- asymptotic behavior of such solutions in particular regimes (strong interaction, semiclassical approximation, ...), pattern formation.

A lot of studies have been performed (and a lot of interesting problems are still open) in the regime of strong competition, that is when the system writes
$$-\Delta u_i \;=\; f_i(x, u_i) \;+\; \beta \underbrace{g_i(x, u_1, \dots, u_k)}_{\uparrow \atop \text{competitive interaction}}, \qquad i = 1, \dots, k,$$
and the competition parameter β goes to $+\infty$.

We say that a family of solutions $\{\mathbf{u}_\beta\}_\beta$ segregates if
$$u_{i,\beta} \to u_i, \qquad u_i \cdot u_j \equiv 0, \qquad \text{as } \beta \to +\infty.$$
If segregation occurs, two main questions have to be addressed:

- in which sense the convergence happens;
- what are the properties of the limiting states.

Moreover, different kind of competition may result in different properties of the limiting states.

A first example comes from population dynamics: for instance, a common model for two populations diffusing in some region is given by the so-called Lotka-Volterra interaction:
$$\begin{cases} -\Delta u = f(x,u) + \beta_1 uv & \text{in } \Omega \\ -\Delta v = g(x,v) + \beta_2 uv & \text{in } \Omega \end{cases}$$
Here:

- if $\beta_1 < 0 < \beta_2$ we have the usual prey-predator system,
- if $\beta_1, \beta_2 < 0$ there is competition between species (strong competition and segregation in case they both diverge).

The properties of the segregated limit profiles depend on the parameters involved (Figs. 1, 2).

Another well-studied model comes from the theory of Bose-Einstein condensation. The Gross-Pitaevskii (cubic) model for a binary mixture of condensates writes
$$\begin{cases} -\Delta u + [V(x) + \lambda_1]u = \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^3 \\ -\Delta v + [V(x) + \lambda_2]v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^3, \end{cases}$$
where the potential V models the magnetic trap which confine the bosonic gases. Here

- $\beta > 0$ entails attractive interaction (synchronization),
- $\beta < 0$ provides repulsive interaction (phase separation).

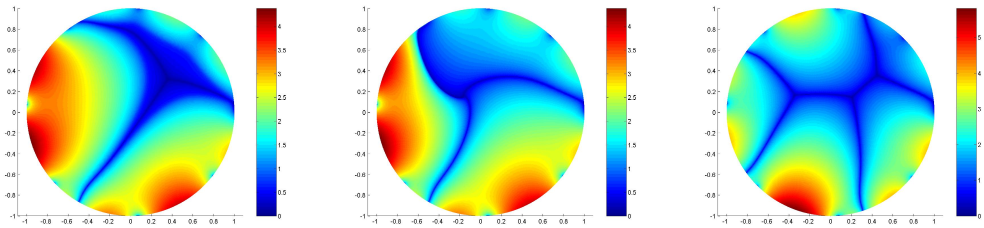

FIGURE 1. Strong competition with 3 (symmetric/asymmetric) and 5 densities.

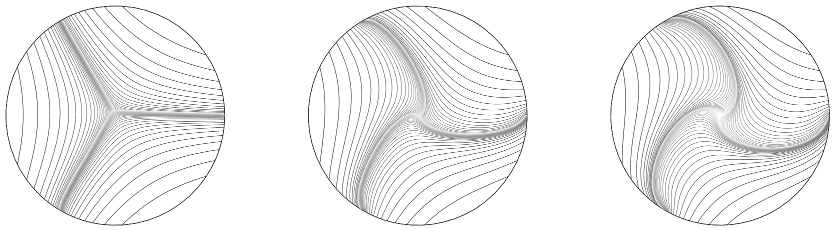

FIGURE 2. If the strong competition is asymmetric, spiraling interfaces emerge.

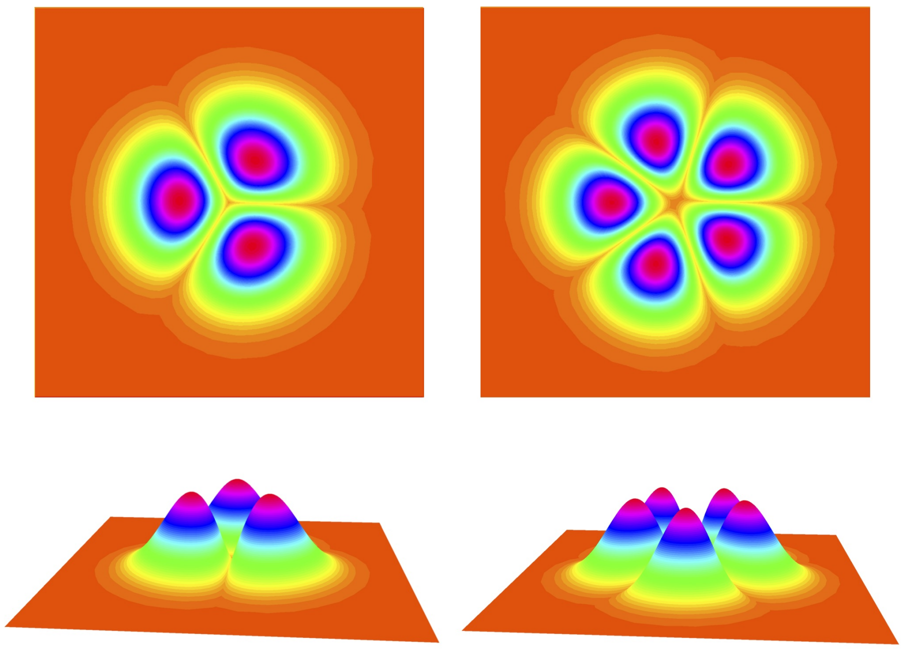

FIGURE 3. Phase separation in condensates with 3 or 5 spin states.

Again, strong competition implies segregation (Fig. 3) .

The toolbox for the analytic study of this kind of problems is variagated. A list of keywords follows.

Nonlinear analysis techniques:

- variational methods (the solutions correspond to critical points of suitable functionals (energy, action))
  - Mountain Pass Theorem,
  - Nehari manifold,
  - Constrained optimization, natural constraints,
- topological methods (the solutions correspond to zeroes of suitable maps between functional spaces);
  - Inversion Theorem, Implicit Function Theorem,
  - Fixed Point Theorems,
  - Bifurcation theory, topological degree.

Free boundary techniques, regularity issues:

- blow-up analysis,
- Alt-Caffarelli-Friedman monotonicity formula,
- Almgren frequency formula,
- Liouville-type theorems.

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