Nonlocal, nonlinear elliptic and parabolic equations.

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Nonlocal diffusion operators are being widely studied since the recent, pioneering works of Caffarelli, Silvestre, Vázquez (see, e.g. [1], [2], [3], [4], [5], [11], [10]). Besides linear equations, in the literature some generalizations of the porous medium equation have been introduced and investigated. More precisely, in a model, the pressure is a fractional power of the Laplacian, see e.g. [3]. The associated equation reads

$$\partial_t u - \operatorname{div}(u\nabla p) = 0\,, \quad x \in \mathbb{R}^n, t > 0\,,$$

where the pressure p is given by

$$p = (-\Delta)^{-s}u,\ 0 < s < 1.$$

In another model, the generator is the fractional Laplacian. In this case the associated nonlocal nonlinear equation is

$$\partial_t u = -(-\Delta)^s(u^m)\,, \quad x \in \mathbb{R}^n, t > 0$$

(see, e.g., [4], [5]). We have studied a *weighted* version of the previous equation, corresponding to spatially inhomogeneous media. To be specific, in [6] and [7] equation

$$\rho\,\partial_t u = -(-\Delta)^s(u^m)\,, \quad x \in \mathbb{R}^n, t > 0$$

has been considered, where $\rho = \rho(x)$ is a positive weight. In dependence of the behaviour of the function $\rho(x)$, existence, uniqueness and asymptotic behaviour of solutions have been studied. Observe that the proofs also use some abstract results, concerning the fractional Laplacian in L^p spaces, contained in [8]. Moreover, the question of uniqueness in suitable weighted L^p spaces, in the linear case, i.e. $m = 1$, has been investigated in [9].

Crucial issues are still open. We mention in particular uniqueness for bounded distributional solutions. As a second step, also unbounded initial data and unbounded distributional solutions could be considered. Furthermore, extensions to more general integro-differential operators are also planned.

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