Nonlinear elliptic and parabolic equations on the Euclidean space and on manifolds.

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Nonlinear diffusion phenomena have been widely investigated in the past both because of their physical significance in applications and of their intrinsic mathematical interest. A key model differential equation describing some of such phenomena is the following:

$$u_t = \Delta(u^m) = m\,\nabla(u^{m-1}\nabla u),$$

which is a *degenerate* parabolic equation when $m > 1$, and is labeled *porous medium equation* in that case, whereas it is a singular parabolic equation when $m < 1$ and is known as *fast diffusion equation*. While the theory is fairly well posed in the Euclidean setting (see [14]), a number of relevant mathematical issues remain open, e.g. existence and uniqueness of solutions for non-uniformly elliptic generators, detailed asymptotics of solutions in inhomogeneous media, rate of convergence to "stationary states" in rescaled variables for general solutions. In particular asymptotic behaviour of solutions is sometimes related to the properties of solutions of associated *nonlinear, elliptic* equations obtained by separating variables, hence the study of such solutions is of particular relevance, see e.g. [4]. A number of the above topics, and several others, are actively being studied also in geometrically nontrivial examples whose model is the hyperbolic space. For example, we have recently studied existence and uniqueness of solutions with measure data [9] and with big initial data [8], smoothing effects [6] (i.e. quantitative versions of the fact that L^q data yield solutions that are bounded at all times $t > 0$), asymptotic behaviour [10, 7]. A number of properties related to the qualitative properties of the evolutions are also connected to the validity of suitable *functional inequalities*, of independent interest, see e.g. [3, 5], whereas surprising phenomena associated to the solvability of elliptic equations in sufficiently negatively curved manifolds, see e.g. [2], make striking differences with the Euclidean situation appear. Clearly, the study of elliptic and parabolic semilinear problems on Riemannian manifolds have also an independent interest. In particular, existence and nonexistence of solutions of elliptic and parabolic semilinear equations on Riemannian manifolds are related to the volume growth of geodesics balls. Some results in this direction have been obtained in [11], [12], [13], where both positive solutions and sign-changing stable solutions have been considered. Furthermore, existence and nonexistence of *patterns* in dependence of the Ricci curvature have been addressed in [1].

The proposed research activity may focus either on the Euclidean or on the manifold setting, depending on the attitude and background of the candidates. We stress that, however, the necessary geometrical expertise even in the manifold setting is, at least for several of the research issues, not a real problem and can easily be acquired in reasonable time if necessary. The research topics are very active and have lead to important breakthrough recently, so we expect that a successful candidate could quickly enter into leading research lines.

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