MULTI-PHASE SHAPE OPTIMIZATION PROBLEMS

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This research project is focused on different kinds of shape optimization problems having as a common feature the presence of several "unknown phases". Some archetypal examples are:

- *Optimal partition problems*, in which one has to minimize a geometric or variational energy depending on the family of cells of the partition;
- *Classical packing problems*, in which one has to find the best arrangement of a family of balls in a given box (*cf.* Figure 1);
- *Multi-label image segmentation problems*, in which one has to reconstruct with some accuracy the different regions of an input image.

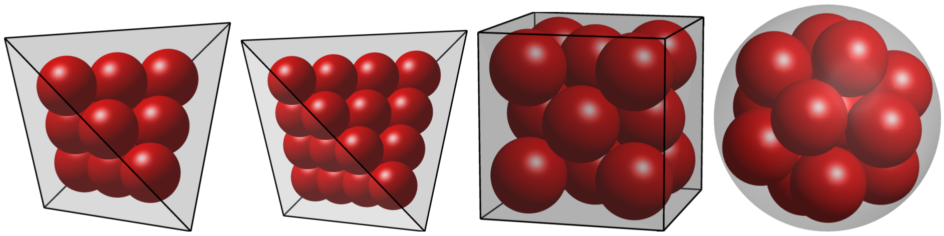

FIGURE 1. Sphere packing examples in 3D.

Depending on the situation under study, the target may be establishing the existence of a solution in a suitable class of domains, or the regularity of the boundaries of the cells of an optimal configuration (the so-called "free boundaries"), or some qualitative properties such as the asymptotical behaviour of solutions when the number of cells becomes very large. In this respect, in many cases a hexagonal configuration is predicted, as it occurs in natural wonders like the bees' honeycombs or the giant's causeway (*cf.* Figure 2).

FIGURE 2. The giant's causeway in Northern Ireland.

Since the field is in rapid evolution, the objective of the Thesis will be fixed in detail at the start-up of the PhD program.

To this day, some "hot topics" under study are:

- Existence and regularity of solutions for optimal partitions problems involving Robin Laplacian eigenvalues (see [7, 9, 10] for the same kind of problems with Dirichlet boundary conditions, and the recent work [8]).
- Approximation of optimal packing problems by means of Cheeger partitions, including possible extensions to packing of ellipsoids and implementation of numerical algorithms (in the spirit of [2]).
- Hexagonal behaviour of column joints occurring in the propagation of cracks in cooling lava (see the model discussed in [11], and [3, 4, 5] for some recent achievements of this kind).
- Multiphase monotonicity formulas (extending the main result proved in [6]) and applications to computer vision [1].

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