Automatic alignment for Tomographic reconstruction

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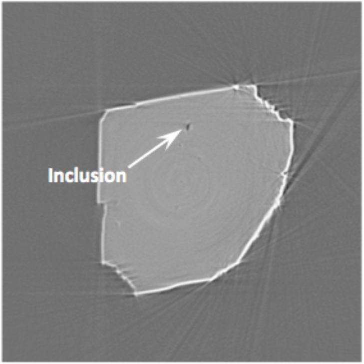
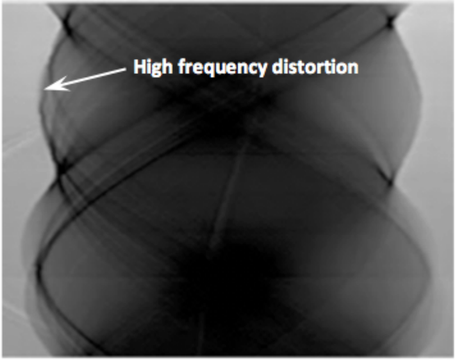

Inclusion


High frequency distortion

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Mathematical formulation & approach

Express the reconstruction problem as an optimization problem:

$$\min_{\mathbf{x}} \|W\mathbf{x} - \mathbf{p}\|_2^2,$$

where

- $\mathbf{x} \in \mathbb{R}^n$ - image
- $\mathbf{p} \in \mathbb{R}^{lm}$ - projection data
- $W \in \mathbb{R}^{lm \times n}$ - projection matrix
- n - number of pixels (voxels)
- l - number of angles
- m - number of detector pixels

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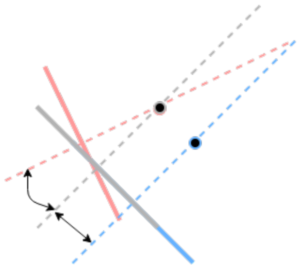

For example, in a 2D parallel beam geometry, $\mathbf{a}$ consists of an angle and offset perturbation for each projection angle.

Bi-level optimization problem

$$\min_{\mathbf{x},\mathbf{a}} \|W(\mathbf{a})\mathbf{x} - \mathbf{p}\|_2^2,$$

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$$\min_{\mathbf{x},\mathbf{a}} \sum_{i=1}^{I} \|W_i(\mathbf{a}_i)\mathbf{x} - \mathbf{p}_i\|_2^2,$$

- W_i - projection matrix for the i^{th} angle
- $\mathbf{a}_i$ - alignment parameters for the i^{th} angle,
- $\mathbf{p}_i$ - projection data for the i^{th} angle.

Can we retrieve both $\mathbf{x}$ and $\mathbf{a}$?

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- If W has full row-rank, we can fit the data for *any* $\mathbf{a}$
- How well we can retrieve $\mathbf{a}$ depends on $\mathbf{x}$
- Solution is at best unique up to global shifts and rotations

How do we solve the bi-level optimization problem?

$$\min_{\mathbf{x},\mathbf{a}} \sum_{i=1}^{I} \| W_i(\mathbf{a}_i)\mathbf{x} - \mathbf{p}_i \|_2^2,$$

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Exploit structure:

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- for fixed $\mathbf{x}$, separates in I independent low-dimensional problems for $\mathbf{a}_i$

Intermezzo: Variable projection

Intermezzo: Variable projection [Golub & Pereyra, 1973]

$$\min_{\mathbf{x},\mathbf{y}} \|A(\mathbf{y})\mathbf{x} - \mathbf{b}\|_2^2,$$

- eliminate $\mathbf{x}$ explicitly as $\mathbf{x} = A(\mathbf{y})^\dagger \mathbf{b}$,
- formulate a non-linear least-squares problem:

$$\min_{\mathbf{y}} \left\| \left(A(\mathbf{y})A(\mathbf{y})^\dagger - I \right) \mathbf{b} \right\|_2^2,$$

- this approach is superior to solving the joint problem with a GN method.

This idea can be generalized to solve problems of the form

$$\min_{\mathbf{x},\mathbf{y}} f(\mathbf{x},\mathbf{y}),$$

by solving for $\overline{\mathbf{x}}$ s.t. $\nabla_{\mathbf{x}} f(\overline{\mathbf{x}},\mathbf{y}) = 0$ and substituting this back:

$$\min_{y} \overline{f}(\mathbf{y}),$$

with $\overline{f}(\mathbf{y}) = f(\overline{\mathbf{x}}(\mathbf{y}),\mathbf{y})$.

Theorem [Aravkin & van Leeuwen, 2012]

Given a twice-differentiable function $f(\mathbf{x},\mathbf{y})$, define $\overline{f}(\mathbf{y}) = f(\overline{\mathbf{x}},\mathbf{y})$ with $\nabla_{\mathbf{x}} f(\overline{\mathbf{x}},\mathbf{y}) = 0$ then

$$\nabla\overline{f}(\mathbf{y}) = \nabla_{\mathbf{y}} f(\overline{\mathbf{x}},\mathbf{y}),$$

and

$$\nabla^2\overline{f} = \nabla^2_{\mathbf{y}} f - \nabla_{\mathbf{x},\mathbf{y}} f^T \left(\nabla^2_{\mathbf{x}} f\right)^{-1} \nabla_{\mathbf{x},\mathbf{y}} f.$$

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Corollary

If, in addition, we require that $\nabla^2_{\mathbf{x}} f(\overline{\mathbf{x}},\mathbf{y}) \succeq 0$, then a local minimum $\overline{\mathbf{y}}$ of $\overline{f}$, together with the corresponding $\overline{\mathbf{x}}$ is a local minimum of f.

A gradient-based method for minimizing $\overline{f}$ can be implemented as follows

Algorithm

$$\mathbf{x}^{(k+1)} = \text{argmin}_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}^{(k)})$$

$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} - \alpha_k \nabla_{\mathbf{y}} f(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k)})$$

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We are completely free to choose the most appropriate method to solve the inner and outer optimization problems.

Algorithms

Bi-level optimization problem

$$\min_{\mathbf{x},\mathbf{a}} \|W(\mathbf{a})\mathbf{x} - \mathbf{p}_i\|_2^2,$$

There are two natural choices:

- project out $\mathbf{a}$ (Align-then-Reconstruct),
- project out $\mathbf{x}$ (Reconstruct-then-Align).

Basic algorithm for A-R (modified)

$$\mathbf{a}_i^{(k+1)} = \underset{\mathbf{a}_i}{\text{argmin}} \, \|W_i(\mathbf{a}_i)\mathbf{x}^{(k)} - \mathbf{p}_i\|_2^2$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}$$

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- Use GN method to compute $\Delta\mathbf{x}^{(k)}$:

$$\Delta\mathbf{x}^{(k)} = W(\mathbf{a}^{(k+1)})^\dagger \left(\mathbf{p} - W(\mathbf{a}^{(k+1)})\mathbf{x}^{(k)}\right)$$

Algorithm for R-A

$$\mathbf{x}^{(k+1)} = W(\mathbf{a}^{(k)})^{\dagger}\mathbf{p}$$

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- Take $\Delta\mathbf{a}^{(k)}$ to be the (scaled) negative gradient:

  $\Delta\mathbf{a}^{(k)} \propto -J(\mathbf{x}^{(k+1)}, \mathbf{a}^{(k)})^T \left(W(\mathbf{a}^{(k)})\mathbf{x}^{(k+1)} - \mathbf{p}\right),$

  where $J(\mathbf{x}, \mathbf{a}) = \left(\frac{\partial W(\mathbf{a})\mathbf{x}}{\partial a}\right)$

Numerical examples

Simple linear projection model

$$\left(W_{h,\Delta\theta} f\right)(r,\theta) = \int_\Omega \mathrm{d}s\, f(s)\delta(r + h - n(\theta + \Delta\theta)\cdot s),$$

where $\Omega = [-1,1]^2$, $r \in [-1,1]$, $\theta \in [0,\pi)$, $n(\theta) = (\cos(\theta), \sin(\theta))^T$.

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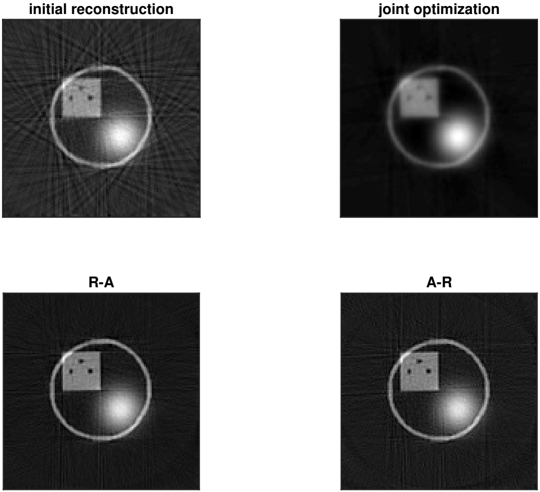
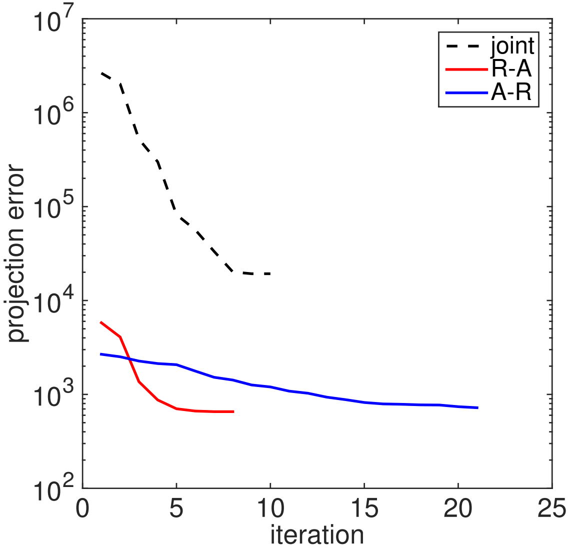
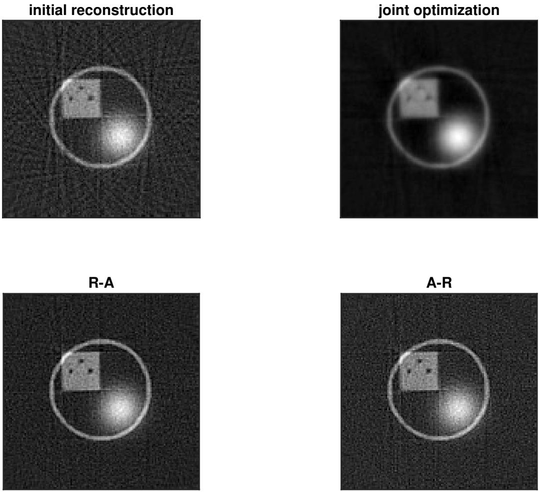

initial reconstruction
joint optimization
R-A
A-R


joint
R-A
A-R
projection error
iteration
10^7
10^6
10^5
10^4
10^3
10^2
0
5
10
15
20
25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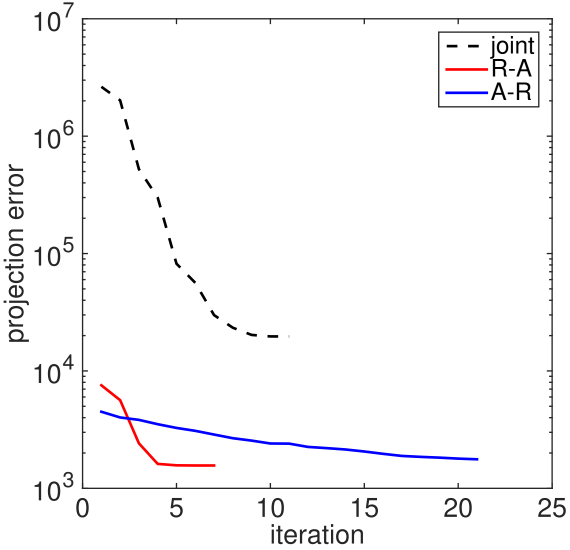
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Possible issues:

- Gradient w.r.t. $\mathbf{a}$ might not be available; use appropriate derivative-free method [Nocedal & Wright, 2006]
- Renegade null-space elements may cause problems in A-R approach

Conclusions

- Unified mathematical framework to derive algorithms for automatic alignment

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- Decouple alignment and reconstruction problems by *variational projection*

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- Decouple alignment and reconstruction problems by *variational projection*
- Regularization (e.g., TV) can be included
- Further analysis needed to understand exactly why R-A approach appears to work better

Thank you!

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