Generalized conics' theory and its applications in geometric tomography

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Abstract. Generalized conics are subsets in the space all of whose points have the same average distance from a given set of points (focal set). We would like to present some results about

the algebraic properties of generalized conics with respect to the taxicab distance

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x^i - y^i| \tag{1}$$

in the coordinate space $\mathbb{R}^n$,

the minimizer of the function measuring the average taxicab distance,

applications in geometric tomography (reconstruction of compact connected hv-convex planar sets given by their coordinate X-rays, the problem of unicity)

Algebraic properties I. Suppose that the conic $\mathcal{C}_m$ is defined by

$$c = f_m(\mathbf{x}), \text{ where } c \in \mathbb{R} \text{ and } f_m(\mathbf{x}) := \frac{1}{m}\sum_{i=1}^{m} d_1(\mathbf{x}, \mathbf{x}_i), \tag{2}$$

i.e. we have finitely many focal points $\mathbf{x}_1$, ..., $\mathbf{x}_m \in \mathbb{R}^n$ and the average distance is given by the arithmetic mean of taxicab distances from the focuses.

Theorem 1 There exists a polynomial $p \in \mathbb{Q}(\mathbf{x}_1, \ldots, \mathbf{x}_m)[\mathbf{x}, x^{n+1}]$ over the extension of the rationals with the coordinates of the focuses such that $p(\mathbf{x}, f_m(\mathbf{x})) = 0$.

$$p(\mathbf{x}, x^{n+1}) := \prod_{\omega \in \mathcal{M}_{m\times n}(\pm 1)} \left(m x^{n+1} - \sum_{i=1}^{m}\sum_{j=1}^{n} \omega_j^i |x^j - x_i^j| \right), \tag{3}$$

where ω runs through the matrices of type $m \times n$ with ± 1. The right hand side is an even function of the variable $\alpha_j^i = |x^j - x_i^j|$ $\Rightarrow$ the Taylor expansion at the origin contains only even powers. Under the choice $\omega_j^i = 1$ it follows that $p(\mathbf{x}, f_m(\mathbf{x})) = 0$. For the same result with respect to the average Euclidean distance see Nie et. al. [6].

Conics with infinitely many focal points [7], [8]. The *generalized conic function* f_K^p associated to a compact set $K \subset \mathbb{R}^n$ is the mapping

$$f_K^p\colon \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f_K^p(\mathbf{x}) := \int_K d_p(\mathbf{x},\mathbf{y})\, d\mathbf{y}, \tag{4}$$

where d_p is the distance function coming from the p - norm

$$d_p(\mathbf{x},\mathbf{y}) = \sqrt[p]{(x^1-y^1)^p + \ldots + (x^n-y^n)^p} \quad (p \geq 1).$$

f_K^p is a convex function satisfying a kind of growth condition in case of positive Lebesgue measure of K: $\lambda_n(K) \neq 0$. The *generalized* p*-conic domain* $\mathcal{C}_K^p$ with focal set K is defined by

$$\frac{1}{\lambda_n(K)} f_K(\mathbf{x}) \leq c. \tag{5}$$

It is a convex compact subset in $\mathbb{R}^n$. Inequality (5) says that $\mathcal{C}_K^p$ is a "ball" with "center" $K \subset \mathbb{R}^n$ with respect to the average distance. The following question is natural: is the center K uniquely determined by the average distance? Generalized p-conics represent a class of subsets with affirmative answer in the following sense.

Theorem 1 *Let C be a generalized p-conic and suppose that C^* is a compact set with the same area as C. If the generalized p-conic functions associated to C and C^* coincide then $C \approx C^*$, i.e. C is equal to C^* except on a set of measure zero.*

Proof Let C be defined by the inequality $f_K^p(x,y) \leq c$ and suppose that C^* is a compact set with

$$\lambda_n(C^*) = \lambda_n(C) \quad \text{and} \quad f_C^p = f_{C^*}^p.$$

By the Fubini theorem

$$\int_C f_K^p = \int_K f_C^p = \int_K f_{C^*}^p = \int_{C^*} f_K^p \tag{6}$$

and thus

$$\int_{C\setminus C^*} f_K^p = \int_C f_K^p - \int_{C\cap C^*} f_K^p \overset{(6)}{=} \int_{C^*} f_K^p - \int_{C\cap C^*} f_K^p = \int_{C^*\setminus C} f_K^p. \tag{7}$$

The constant c is working as an upper bound for f_K^p on $C \setminus C^*$ but c is a (strict) lower bound for f_K^p on $C^* \setminus C$. Therefore $0 = \lambda_n(C \setminus C^*) = \lambda_n(C^* \setminus C)$.

Corollary 1 *Let C and C^* be generalized p-conics. If the generalized p-conic functions associated to C and C^* coincide then $C = C^*$.*

From the tomographic point of view we also have the following corollary.

Corollary 2 *Generalized 1-conics are determined by their X-rays parallel to the coordinate hyperplanes among compact sets.*

To prove Corollary 2 we have to pay a special attention to the case $p = 1$. The function

$$f_K := f_K^1, \quad f_K(\mathbf{x}) := \int_K d_1(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$$

is strongly related to the parallel X-rays as follows:

$$D_i D_i f_K(\mathbf{x}) =_{\text{a.e}} 2X_i K(x^i) \quad (i = 1, \ldots, n),$$

where $X_i K(t) := \lambda_{n-1}(t =_i K)$ and $t =_i K := \{\mathbf{x} \in K \mid t = x^i\}$. On the other hand

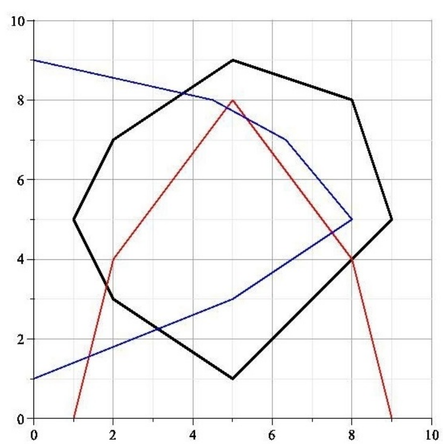
$$f_K(\mathbf{x}) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} |x^i - t| X_i K(t)\, dt. \tag{8}$$

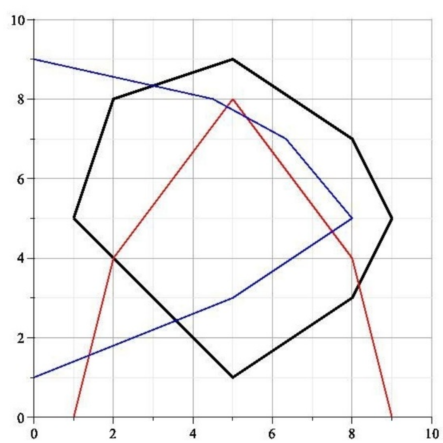
Theorem 2 [7, 10] *$f_K = f_L$ iff the X-rays parallel to the coordinate hyperplanes of K and L coincide almost everywhere.*

Consider the special case of $n = 2$; the X-ray functions X_1K and X_2K measure the vertical and the horizontal slices, respectively:

The coordinate X-rays in the plane are special cases of the parallel X-rays into a given direction; see Gardner [2].

Algebraic properties II If K is a convex polygon in the plane then the coordinate X-rays are piecewise linear functions.





Substituting piecewise linear functions into the special case

$$f_K(\mathbf{x}) = \int\limits_{-\infty}^{\infty} |x^1 - t| X_1 K(t)\, dt + \int\limits_{-\infty}^{\infty} |x^2 - t| X_2 K(t)\, dt$$

of formula (8) it follows that f_K is a piecewise polynomial function of degree at most 3:

$$\mathbf{x} = (x^1, x^2) \in [t_i^1, t_{i+1}^1] \times [t_j^2, t_{j+1}^2] \;\Rightarrow\; f_K(\mathbf{x}) = p_i(x^1) + q_j(x^2),$$

where $i = 0, \ldots, k,\ j = 0, \ldots l,$

$t_1^1, \ldots, t_k^1,\ t_1^2, \ldots, t_l^2$ are the coordinates of the vertices

$t_0^1 := -\infty,\ t_{k+1}^1 := \infty$ and $t_0^2 := -\infty,\ t_{l+1}^2 := \infty,$

p_i and q_j are polynomials of degree at most 3. The product

$$p(\mathbf{x}, x^3) := \prod_{i=0}^{k} \prod_{j=0}^{l} \Big(x^3 - p_i(x^1) + q_j(x^2)\Big)$$

gives a polynomial over the extension of the rationals with the coordinates of the vertices such that $p(\mathbf{x}, f_K(\mathbf{x})) = 0$.

The minimizer of the generalized 1-conic functions Let $K \subset \mathbb{R}^n$ be a compact subset with $\lambda_n(K) \neq 0$; recall the generalized conic function

$$f_K\colon \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f_K(\mathbf{x}) := \int_K d_1(\mathbf{x}, \mathbf{y})\, d\mathbf{y}.$$

Since f_K is a differentiable convex function it is enough to check the first order conditions to find the minimizer:

$$D_i f_K(\mathbf{x}) = \lambda_n(K \leq x^i) - \lambda_n(x^i \leq K) \quad (i = 1, \ldots, n),$$

where

$$K \leq x^i := \{\mathbf{z} \in K \mid z^i \leq x^i\}, \quad x^i \leq K := \{\mathbf{z} \in K \mid x^i \leq z^i\},$$

Theorem 3 *The point $\mathbf{x}_* \in \mathbb{R}^n$ is the minimizer of f_K if and only if each coordinate hyperplane at $\mathbf{x}_*$ divides K into two parts of equal measure.*

Since f_K is a differential convex function with Lipschitzian gradient we can use the gradient method to find the minimzer. It can be also formulated in terms of a stochastic algorithm as follows [1].

Using a starting point $\mathbf{x}_0 \in K$ let P_k be a sequence of K-valued independent uniformly distributed random variables. Consider the recursion

$$X_{k+1} = X_k - t_{k+1} Q_{k+1}, \tag{9}$$

where $X_0 := \mathbf{x}_0$

$$Q_{k+1} := (\text{sgn } (X_k^1 - P_{k+1}^1), \ldots, \text{sgn } (X_k^n - P_{k+1}^n))$$

and the step size is a decreasing sequence of positive real numbers t_k satisfying the following conditions:

$$\sum_{k=1}^{\infty} t_k = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} t_k^2 < \infty.$$

Then we have the following conditional probability provided that $\lambda_n(K) = 1$:

$$P(Q_{k+1} = (1, \ldots, 1) | X_k) = \lambda_n((K < X_k^1) \cap \ldots \cap (K < X_k^n))$$

because $Q_{k+1} = (1, \ldots, 1)$ means that X_k is greater than P_{k+1} with respect to the partial ordering

$$\mathbf{x} \prec \mathbf{y} \;\Leftrightarrow\; x^1 < y^1, \ldots, x^n < y^n.$$

In a similar way

$$P(Q_{k+1} = (1, -1, 1, \dots, 1)|X_k) =$$

$$\lambda_n((K < X_k^1) \cap (X_k^2 < K) \cap (K < X_k^3), \dots \cap (K < X_k^n))$$

and so on. Then

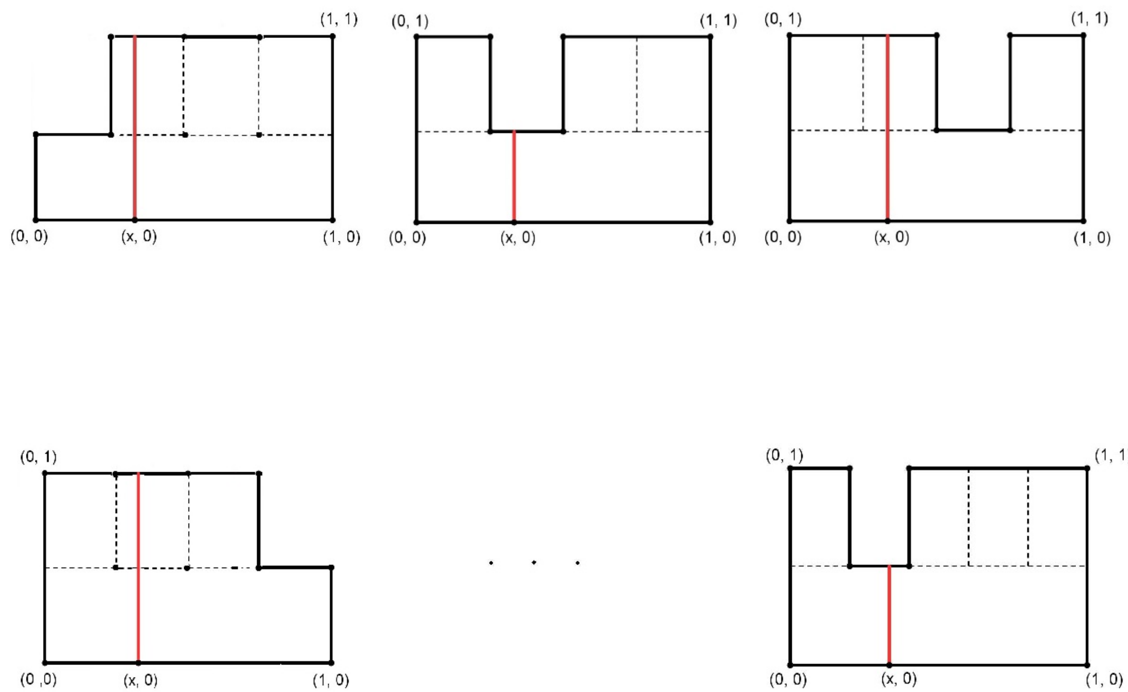
$$\mathbb{E}(Q_{k+1}|X_k) = \text{grad } f_K(X_k) \quad \text{and} \quad \mathbb{E}(X_{k+1}) = \mathbf{x}_0 - \sum_{i=1}^{k+1} t_i \mathbb{E} \text{ grad } f_K(X_k),$$

$$X_k \overset{\text{a.s.}}{\to} \mathbf{x}_* \quad \text{and} \quad \lim_{k\to\infty} \mathbb{E}\|X_k - \mathbf{x}_*\|^m = 0$$

for any positive integer $m \in \mathbb{N}$; see [1].

Continuity properties and reconstruction. In what follows we restrict ourselves to the coordinate plane $\mathbb{R}^2$. The reconstruction of planar sets by their coordinate X-rays is originally motivated by Gardner's unicity problem [2]: Characterize those convex bodies that can be determined by two X-rays. The following figure shows that X-rays can have deviant behavior under the Hausdorff convergence of the sets:

the measure of the vertical slice changes the values 1 and 2 with an increasing lenght of period at "a great amount" of elements of the supporting interval.



The generalized conic functions are more regular objects in some sense. This makes them to be a natural starting point of the reconstruction.

Definition 1 *The Hausdorff convergence* $L_n \to K$ *is called regular iff*

$$\lim_{n\to\infty} \lambda_2(L_n) = \lambda_2(K).$$

It is X-regular iff $\lim_{n\to\infty} \lambda_2(I_n) = \lambda_2(K)$, *where* $I_n := \cap_{n=i}^{\infty} L_i$.

Theorem 4 [8], [9] *If* $L_n \to K$ *with respect to the Hausdorff metric then*

$$\limsup_{n\to\infty} f_{L_n}(\mathbf{x}) \leq f_K(\mathbf{x}).$$

If the Hausdorff convergence $L_n \to K$ *is regular then*

$$\lim_{n\to\infty} f_{L_n}(\mathbf{x}) = f_K(\mathbf{x})$$

and the convergence $f_{L_n} \to f_K$ *is uniform over any compact subset in* $\mathbb{R}^2$. *If the Hausdorff convergence* $L_n \to K$ *is X-regular then it is regular,*

$$\lim_{n\to\infty} X_1 L_n(s) =_{a.e} X_1 K(s) \text{ and } \lim_{n\to\infty} X_2 L_n(t) =_{a.e} X_2 K(t).$$

Under the hypothesis of the Hausdorff convergence the regularity is equivalent to the convergence in symmetric difference.

Theorem 5 Bianchi et al. [5] *The sequence L_n converges in Hausdorff distance to K if and only if*

$$\lim_{n\to\infty} \lambda_2((L_n)_\delta \triangle K_\delta) = 0 \quad \textit{for each} \quad \delta > 0.$$

Example 1 Bianchi et al. [5] *If each L_n is obtained from a compact set L via finitely many Steiner symmetrizations and Euclidean isometries then the Hausdorff convergence $L_n \to K$ is regular.*

Example 2 [9] *If L_n is a sequence of compact connected hv-convex sets tending to the limit K with respect to the Hausdorff metric, then the convergence is regular.*

Example 3 [8] *Any outer Hausdorff approximation $K \subset L_n \to K$ is X-regular.*

Example 4 *The Hausdorff convergence of compact convex subsets L_n to K with non-empty interior is X-regular.*

In the sense of Example 4, the Hausdorff convergence in the class of compact convex sets (with nonempty) interior implies the X-regularity and the reconstruction can be based on direct comparisons of X-rays; Gardner and Kiderlen [3] (four directions, compact convex planar bodies).

In the sense of Example 2, the Hausdorff convergence in the class of compact connected hv-convex sets implies the regularity and the reconstruction can be based on direct comparisons of generalized conic functions. More precisely we have the following result:

Theorem 6 *Consider the collection of compact connected hv-convex sets contained in the axis parallel bounding box $B \subset \mathbb{R}^2$ and let K be one of them; for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_B |f_L(\mathbf{x}) - f_K(\mathbf{x})| \, d\mathbf{x} < \delta$$

implies that $H(L, K^*) < \varepsilon$, where $f_K = f_{K^*}$, i.e. K and K^* have the same coordinate X-rays almost everywhere.

An algorithm for the reconstruction [10]

Input: $n \in \mathbb{N}$ and X_1K, X_2K, the coordinate X-rays of a non-empty compact connected hv-convex set $K \subset \mathbb{R}^2$ with $K = \mathrm{cl}\,(\mathrm{int}\ (K))$.

STEP 1: Let B and the function f_K associated to K be given by the formulas

$$B = \mathrm{supp}\ (X_1K) \times \mathrm{supp}\ (X_2K) = [a,b] \times [c,d] \tag{10}$$

and

$$f_K(\mathbf{x}) = \int_{-\infty}^{\infty} |x_1 - t|\, X_1K(t)\, ds + \int_{-\infty}^{\infty} |x_2 - s|\, X_2K(s)\, ds. \tag{11}$$

Remark 1 Condition $K = \mathrm{cl}\,(\mathrm{int}\ (K))$ implies that the Cartesian product of the supports of the coordinate X-rays gives a box containing K, i.e. the vertical and horizontal ears are cutted.

STEP 2: Let $s_i \in [a,b]$ and $t_i \in [c,d]$ be equally spaced points as follows:

$$s_i = a + i\frac{b-a}{n}, \quad t_i = d - i\frac{d-c}{n} \quad (i = 0, \ldots, n)$$

STEP 3: $B_{ij}^n = [s_{i-1}, s_i] \times [t_j, t_{j-1}]$, where $i, j = 1, \ldots, n$.

STEP 4: The control grid $G_K^n := \left\{ \mathbf{y}_{ij} \in B_K | i, j = 1, \ldots, n \right\}$ consists of the centers of the subrectangles B_{ij}^n.

STEP 5: $L \in \mathcal{H}_n \Leftrightarrow L$ is a compact connected hv-convex set consisting of B_{ij}^n's and

$$f_L(\mathbf{y}_{ij}) \geq f_K(\mathbf{y}_{ij}) \quad \text{for any} \quad i, j = 1, \ldots, n. \tag{12}$$

STEP 6: Choose L_n from $\mathcal{H}_n$ that minimizes

$$\sum_{i,j=1}^{n} \frac{\left| f_{L_n}(\mathbf{y}_{ij}) - f_K(\mathbf{y}_{ij}) \right|}{n^2}$$

Output: L_n.

This procedure can be formulated in terms of a linear 0 - 1 programming as follows. Any element L in the feasible set can be represented as a $0-1$ (interval) matrix by the variables

$$x_{kl} = \begin{cases} 1 & \text{if } B^n_{kl} \subset L \\ \\ 0 & \text{otherwise} \end{cases} \quad (k, l = 1, \ldots, n) \quad \text{and} \quad \bar{x}_{kl} = 1 - x_{kl}.$$

Constraints:

$$x_{k1} + \ldots + x_{kn} \geq 1 \text{ and } x_{1l} + \ldots + x_{nl} \geq 1,$$

$k = 1, \ldots, n$ and $l = 1, \ldots, n$. Connectedness: equations like

$$x_{kj} x_{k+1j-1} = 1, \quad x_{kj} x_{k+1j} = 1 \quad \text{or} \quad x_{kj} x_{k+1j+1} = 1 \tag{13}$$

allow us to step left-and-down, down or right-and-down. Their sum provides the connectedness. The convexity into the horizontal direction means that the implication

$$x_{kl} = 0 \Rightarrow$$

$$((x_{k1} = 0) \wedge \ldots \wedge (x_{kl} = 0)) \vee ((x_{kl} = 0) \wedge \ldots \wedge (x_{kn} = 0))$$

must be true. In an equivalent way

$$\overline{x}_{kl} = 1 \Rightarrow$$

$$((\overline{x}_{k1} = 1) \wedge \ldots \wedge (\overline{x}_{kl} = 1)) \vee ((\overline{x}_{kl} = 1) \wedge \ldots \wedge (\overline{x}_{kn} = 1))\,,$$

i.e.

$$\overline{x}_{k1} \cdot \ldots \cdot \overline{x}_{kl} + \overline{x}_{kl} \cdot \ldots \cdot \overline{x}_{kn} \geq \overline{x}_{kl}.$$

Using

$$f_L(\mathbf{y}_{ij}) = \sum_{k,l=1}^{n} x_{kl} f_{B_{kl}^n}(\mathbf{y}_{ij}) \tag{14}$$

we can also formulate the last condition

$$f_L(\mathbf{y}_{ij}) \geq f_K(\mathbf{y}_{ij}) \quad \text{for any} \quad i,j = 1,\ldots,n$$

of **STEP 5** in terms of the variables x_{kl}. These inequalities imply that the objective function in **STEP 6** is linear. The linearization of the constraints is based on Li and Sun [4].

To make the algorithm more effective we can use greedy versions [10] based on deleting the subrectangle which causes the extremal average decreasing at the control points or the version adapted to finitely many and/or noisy measurements of the coordinate X-rays [11]. These ideas follow Gardner and Kiderlen's original work [3] but we use the generalized conic function ($\Leftrightarrow$ two X-rays) and the procedure is working for compact connected hv-convex planar sets.

An algorithmic answer to the problem of unicity The "plain enumeration" is based on the following estimation: let K and L be compact connected hv-convex sets contained in the box B; then

$$|f_L(\mathbf{x}) - f_K(\mathbf{x})| \leq 2kH(K,L)\left(\frac{k}{2} + 2H(K,L)\right), \tag{15}$$

where k is the perimeter of the box [10]. Using the minimal covering

$$L_n^* := \bigcup_{B_{kl}^n \cap K \neq \emptyset} B_{kl}^n$$

of K we have that

$$|f_{L_n^*}(\mathbf{x}) - f_K(\mathbf{x})| \leq \frac{k^3(n+2)}{n^2}. \tag{16}$$

If

$$\mathcal{O}_n = \{L_n^1, L_n^2, \ldots, L_n^{m_n}\}$$

is the set of elements of the feasible set such that inequality (16) is satisfied then $\mathcal{O}_n$ contains the minimal covering of any K^* for which $f_K = f_{K^*}$, i.e. K and K^* have the same coordinate X-rays almost everywhere (in case of convex sets the phrase *almost everywhere* can be omitted). Therefore K is uniquely determined by its coordinate X-rays iff

$$\lim_{n\to\infty} \text{diam } \mathcal{O}_n = 0,$$

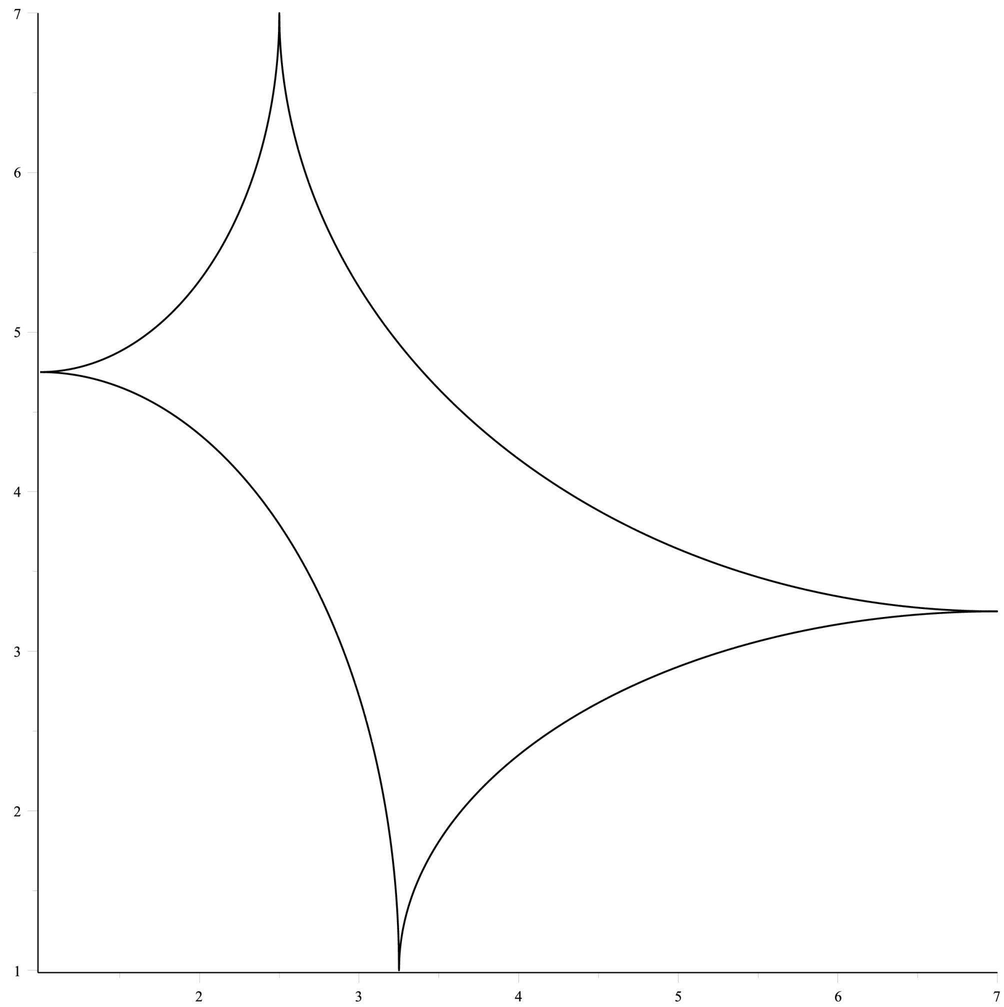
i.e. $\mathcal{O}_n$ collapses to a single element as $n \to \infty$. As a special consequence if K is a compact connected hv-convex set determined by the coordinate X-rays then the (set-valued) inverse Φ^{-1} of the mapping $\Phi\colon L \to \Phi(L) := f_L$ is continuous at f_K. For the class of $\mathcal{K}_0^2$ (nonempty compact convex bodies) Gardner [2] proved that the sets that are determined by the coordinate X-rays form a dense subset. Therefore we can also formulate the converse statement.

Theorem 7 The body $K \in \mathcal{K}_0^2$ is determined by the coordinate X-rays if and only if the (set-valued) inverse Φ^{-1} of the mapping

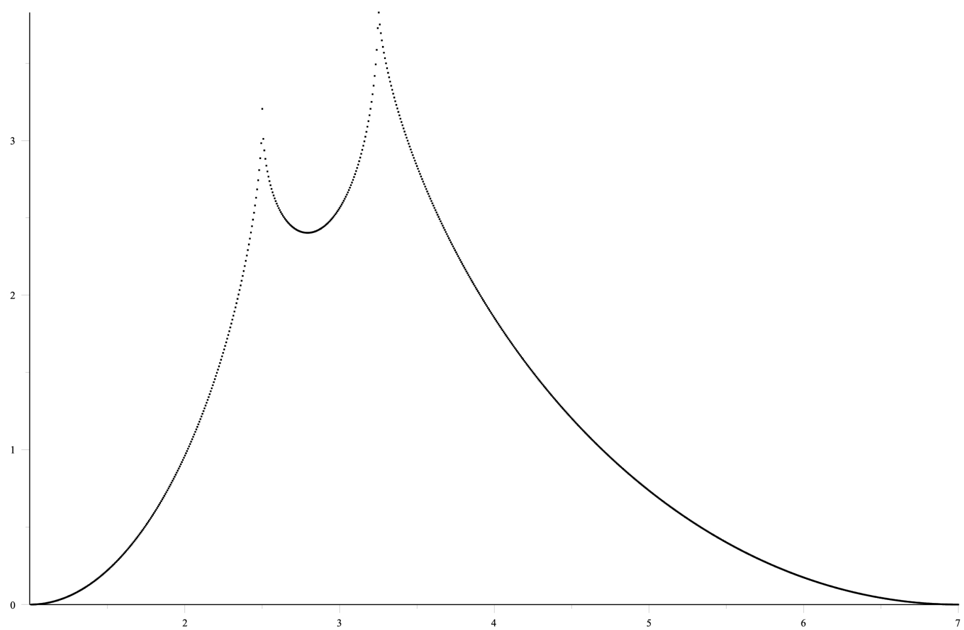
$$\Phi\colon L \to \Phi(L) := f_L$$

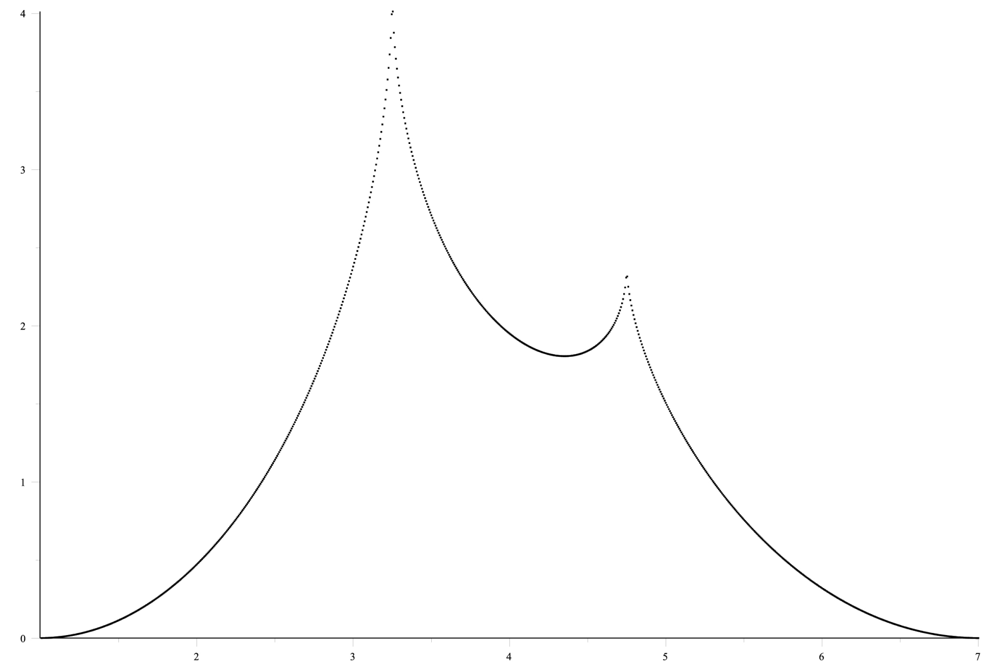
is continuous at f_K.

The set we are looking for.

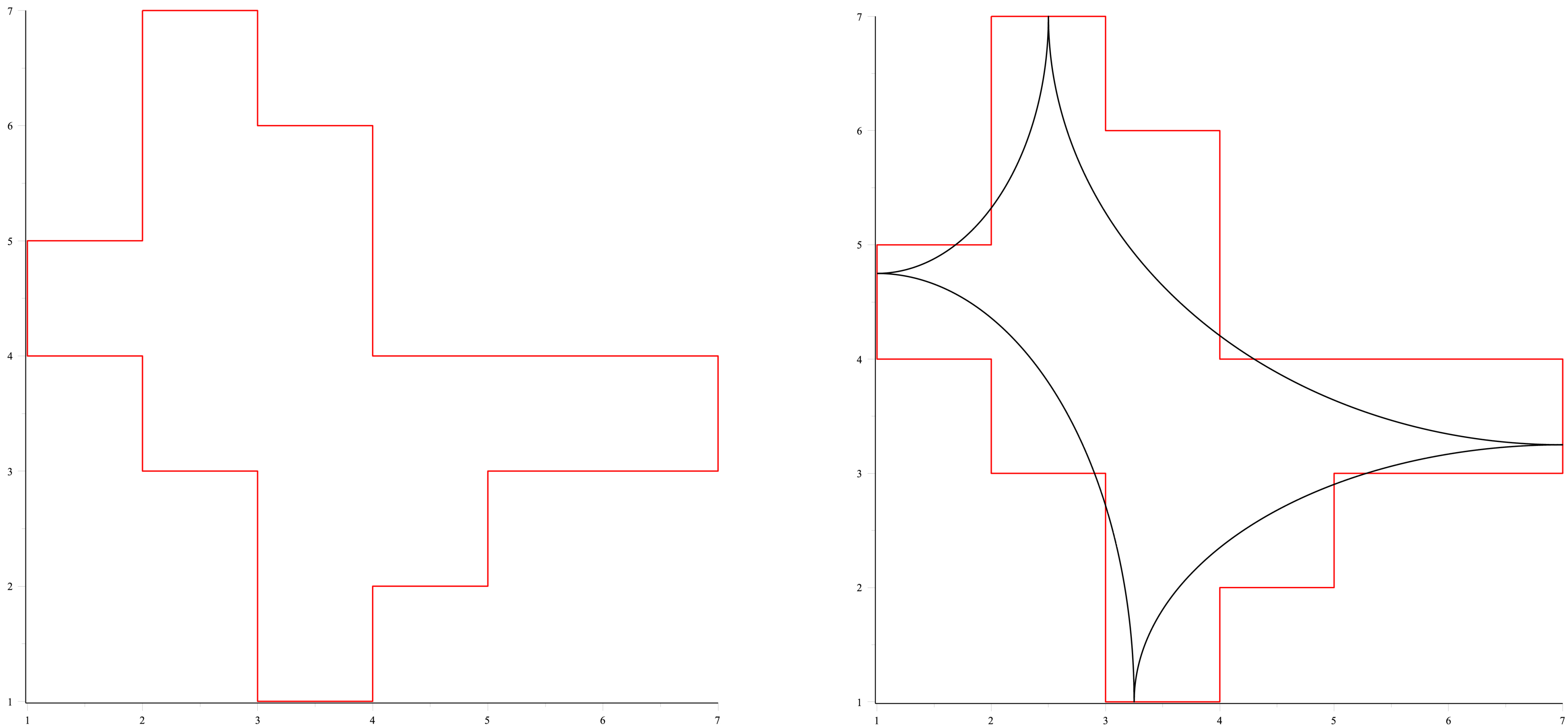


Input: the coordinate X-rays

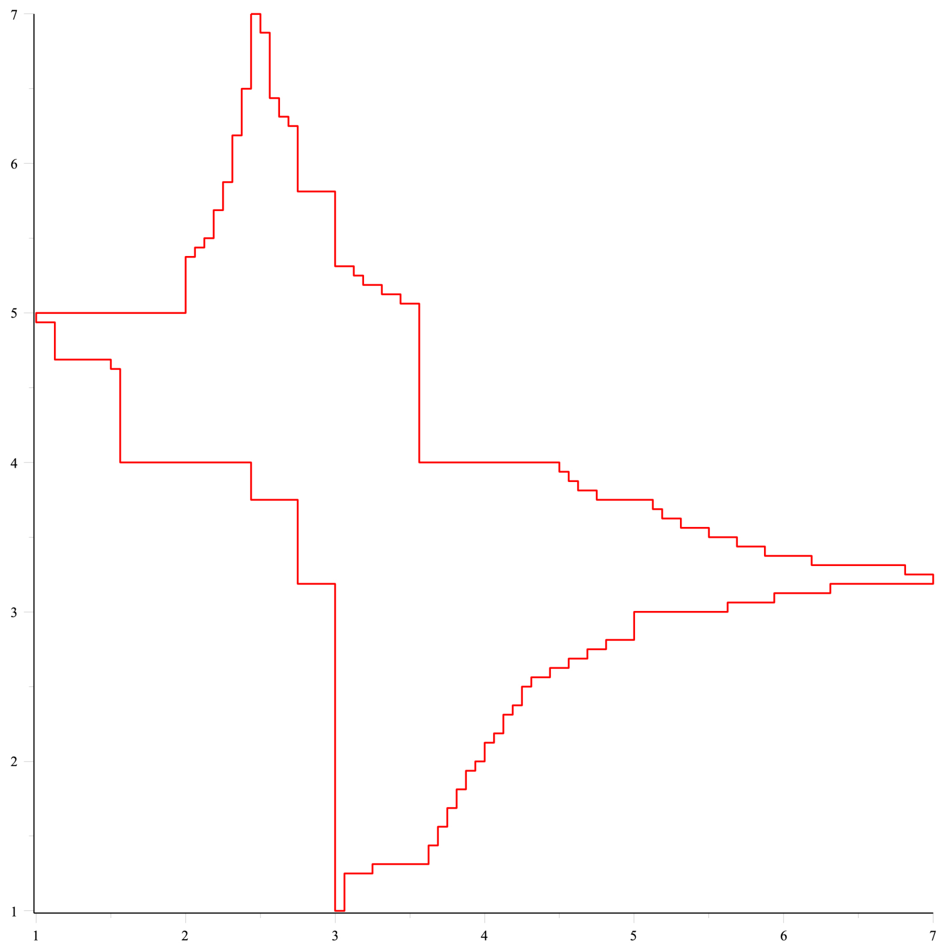


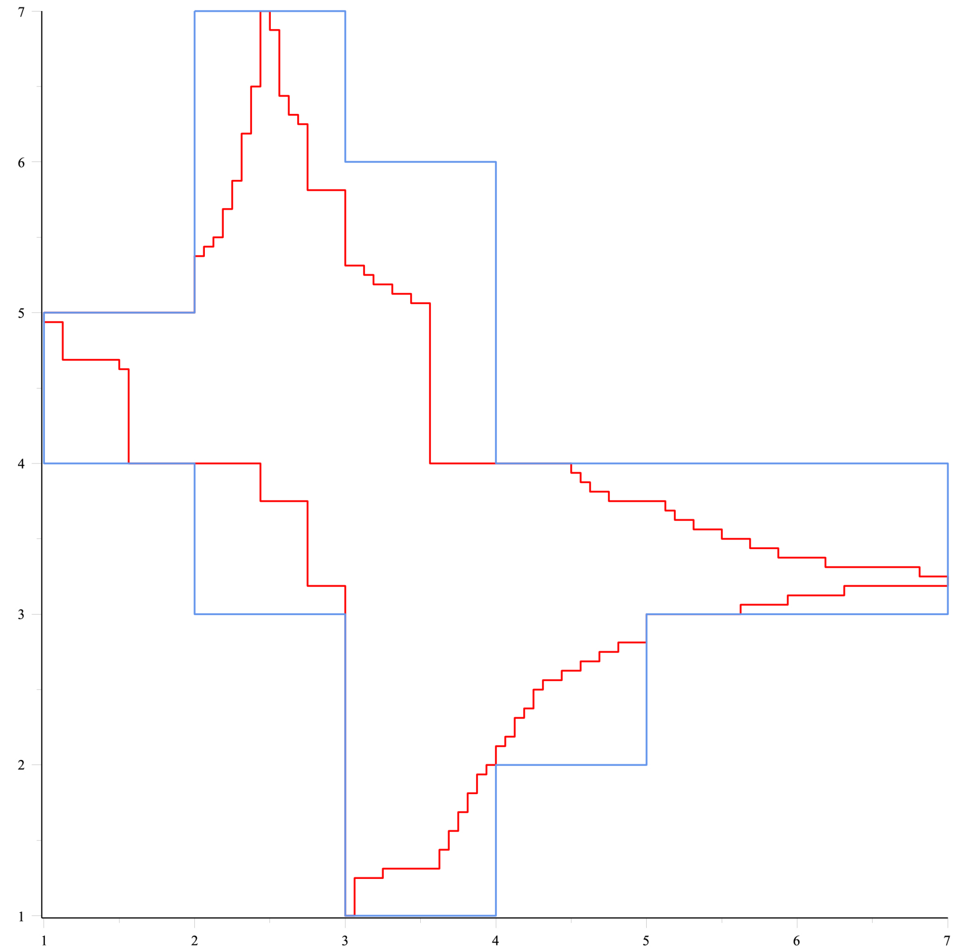


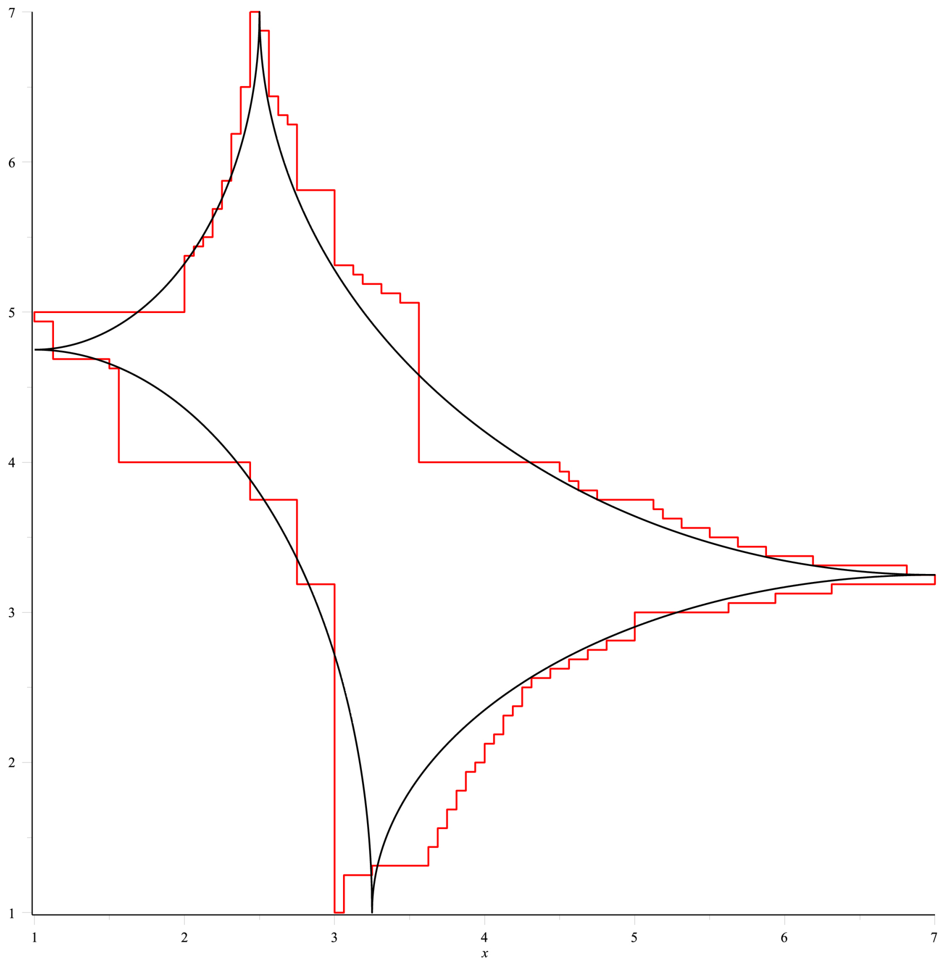
The optimal solution



The greedy version







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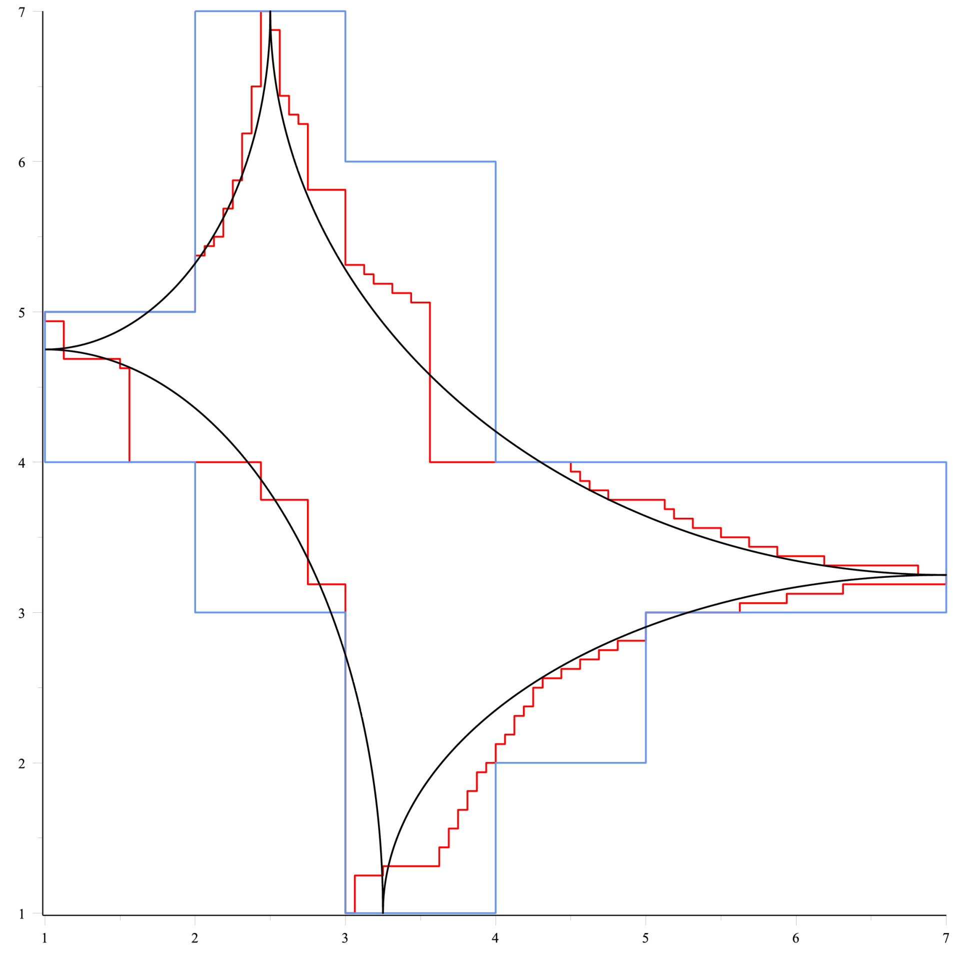

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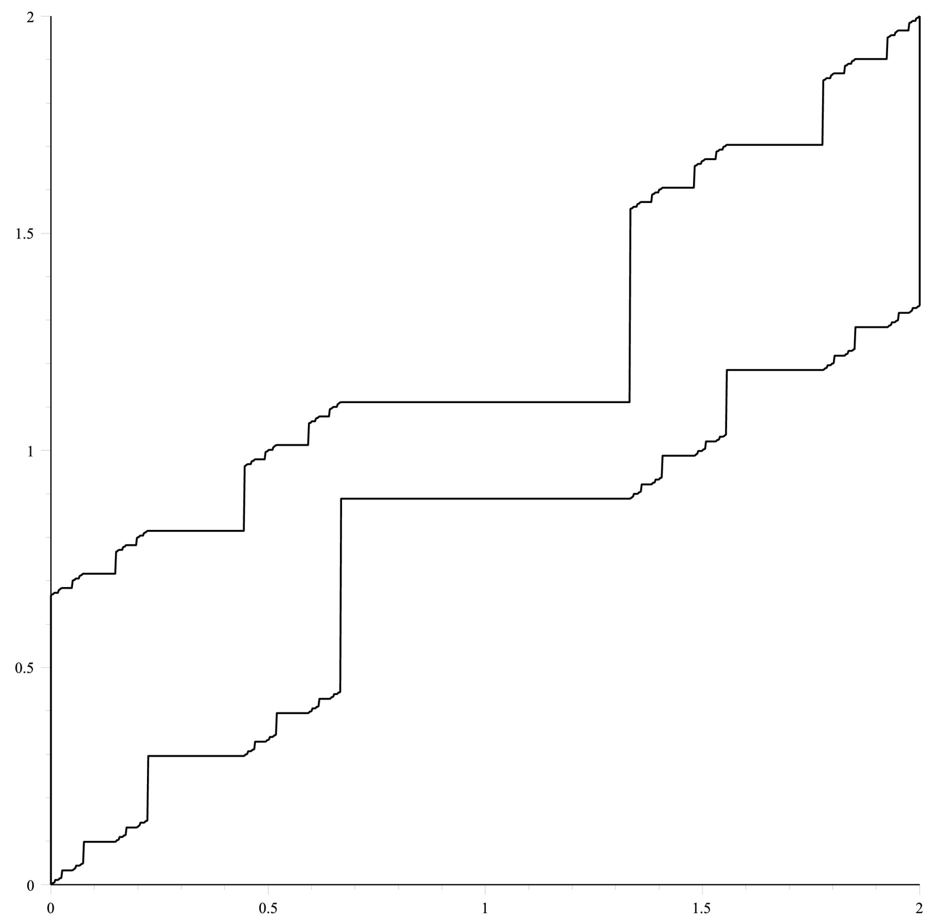
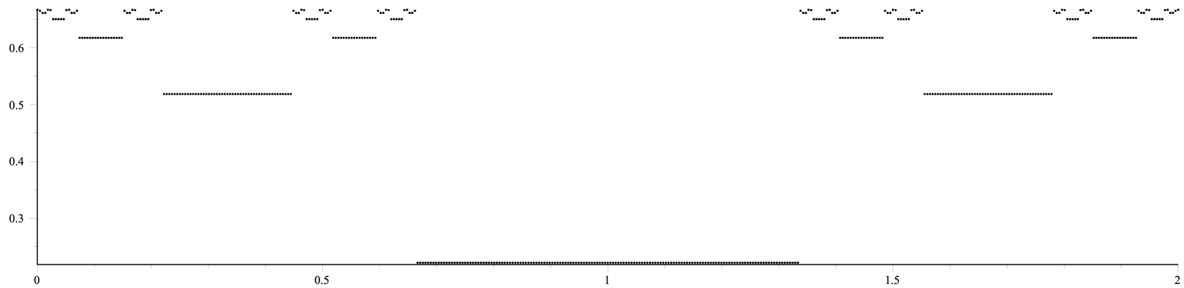
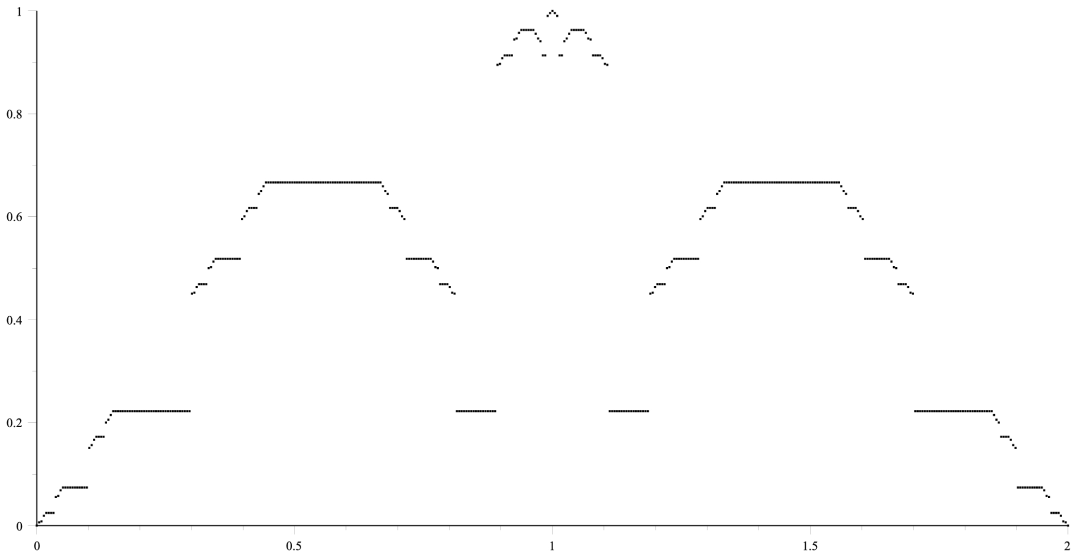
The set we are looking for

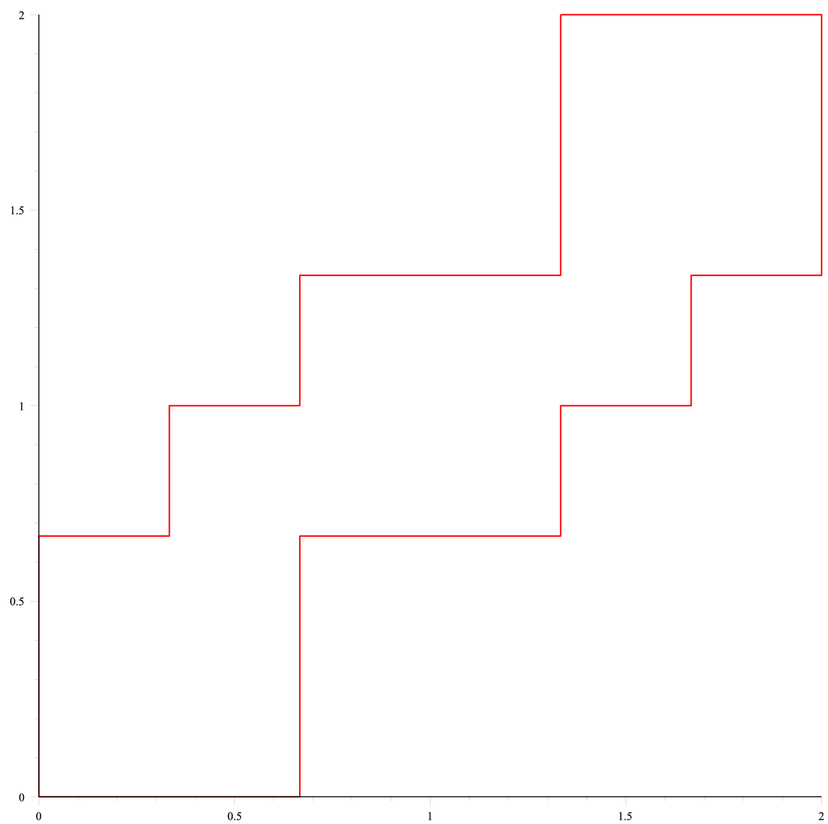


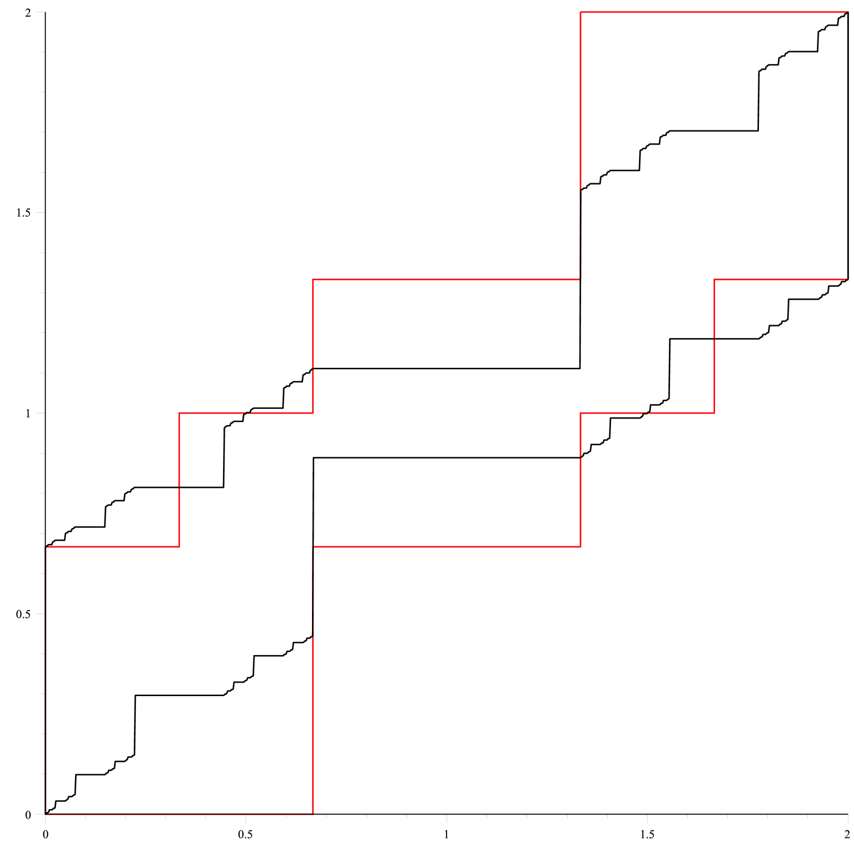
Input: the coordinate X-rays (finitely many measurements)



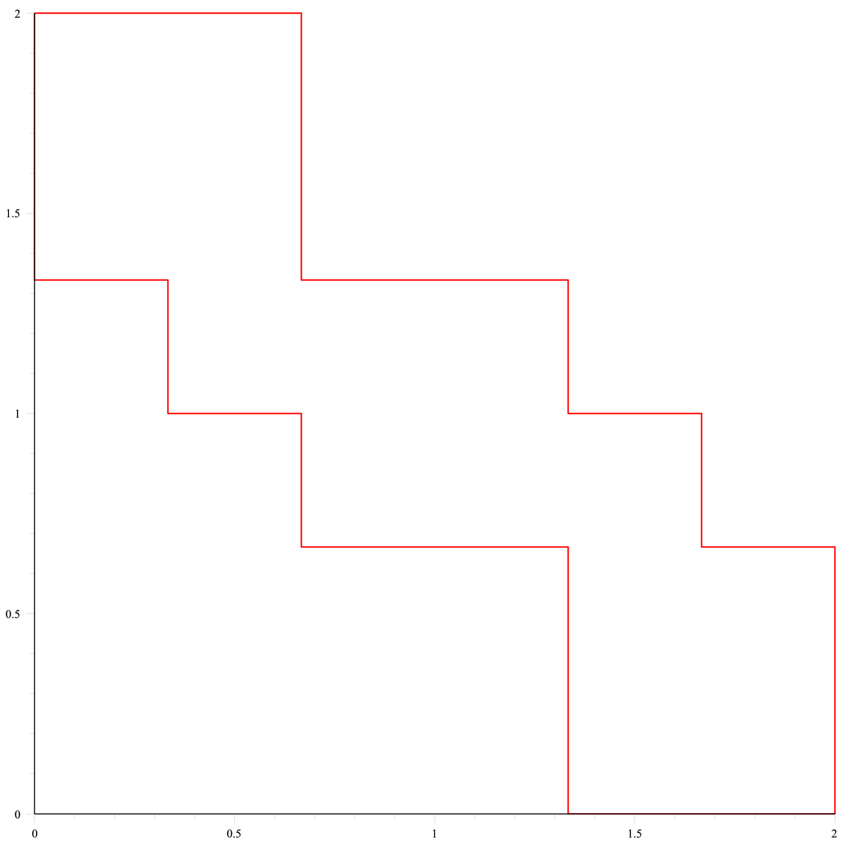


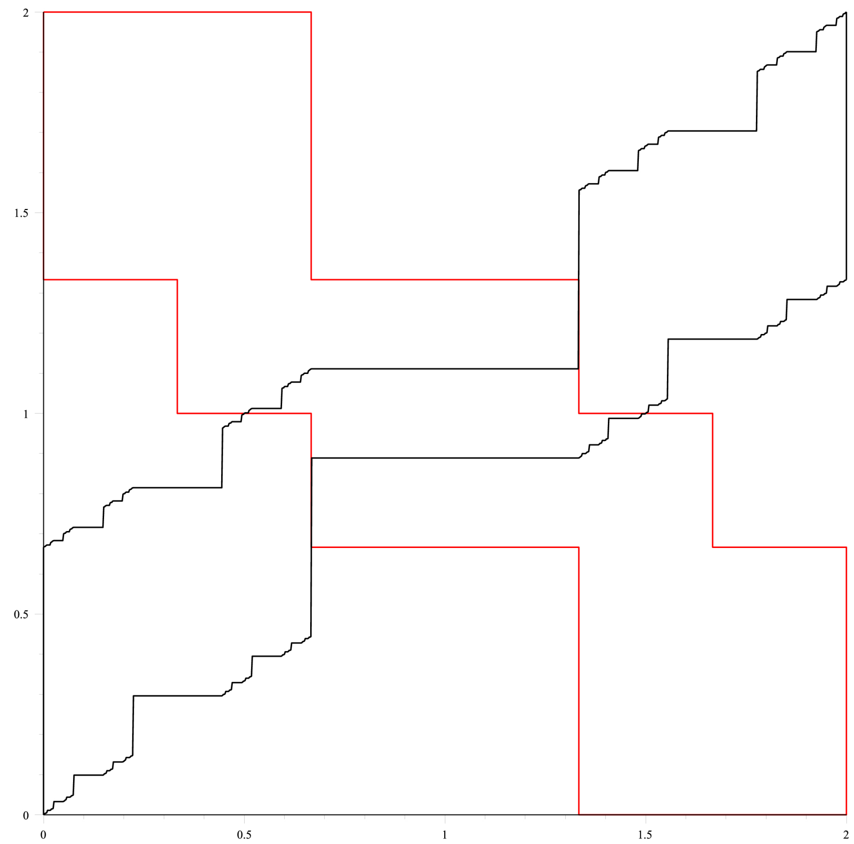
The optimal solution I

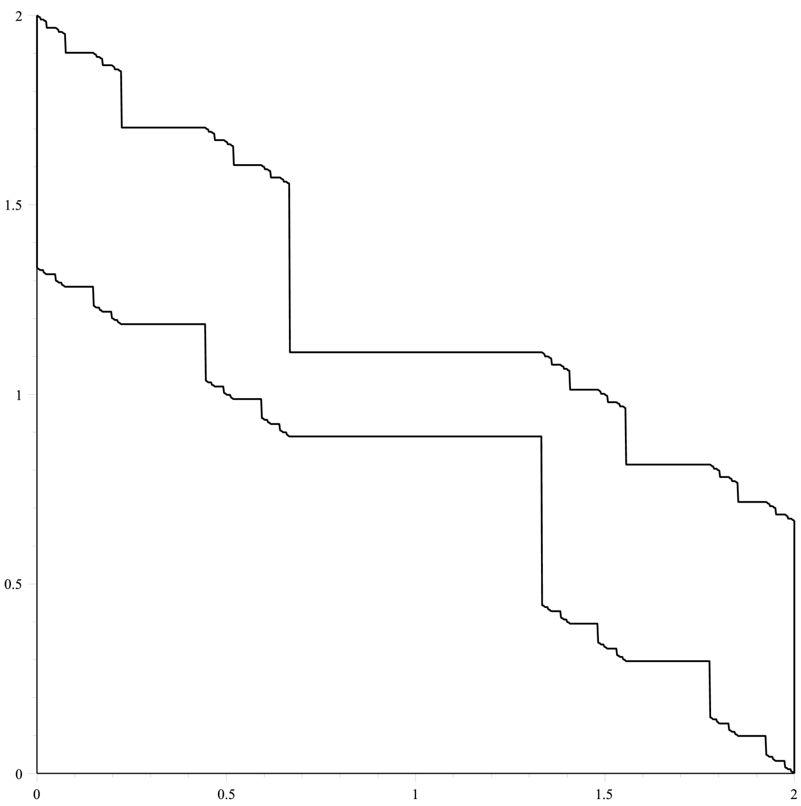


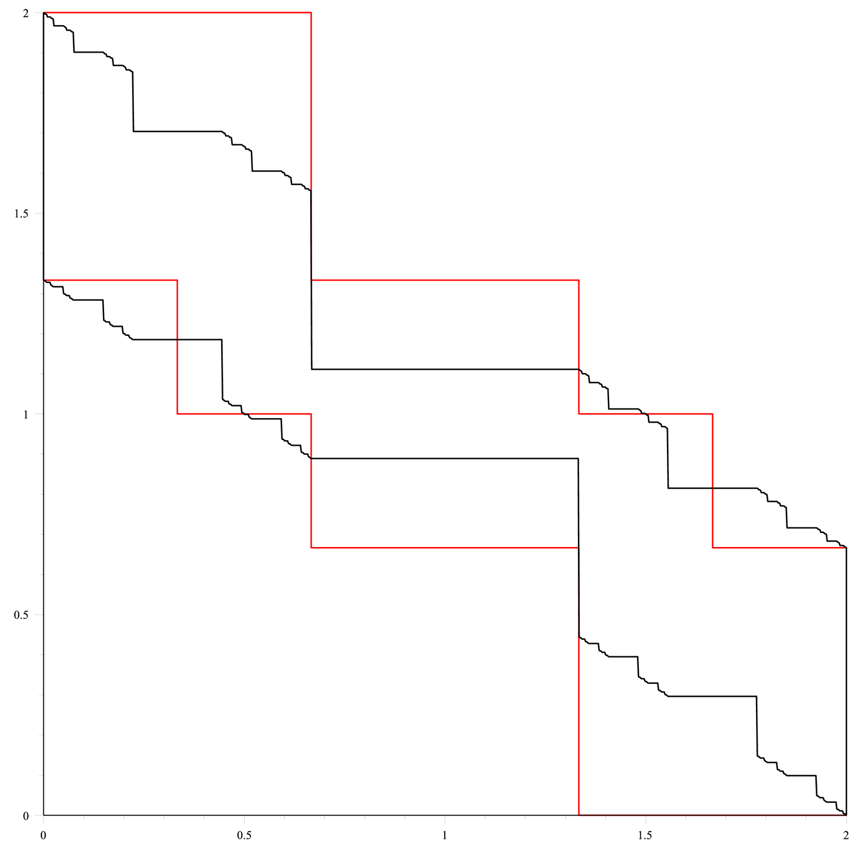


The optimal solution II

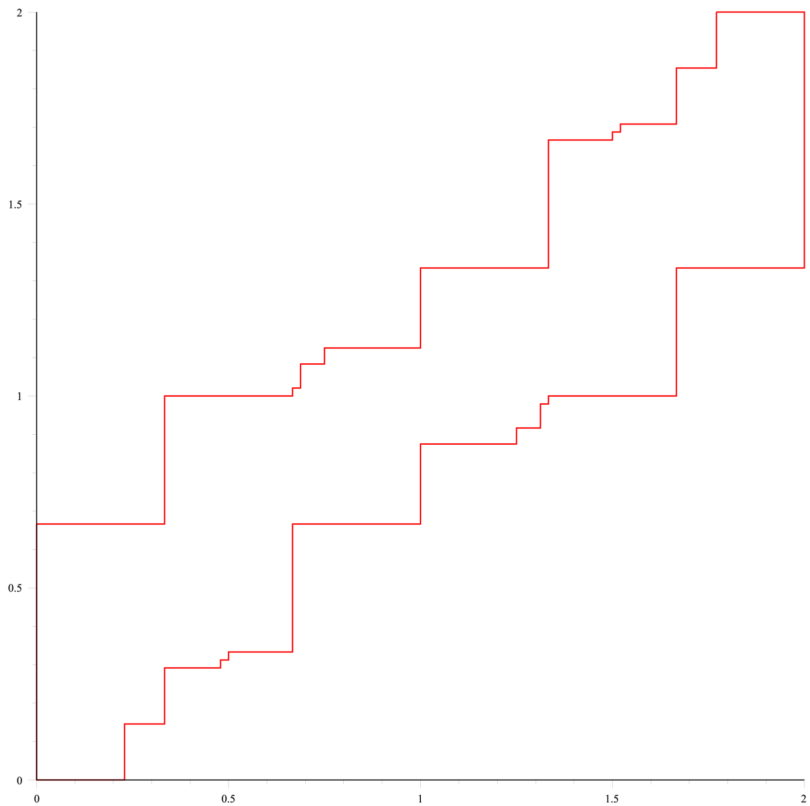


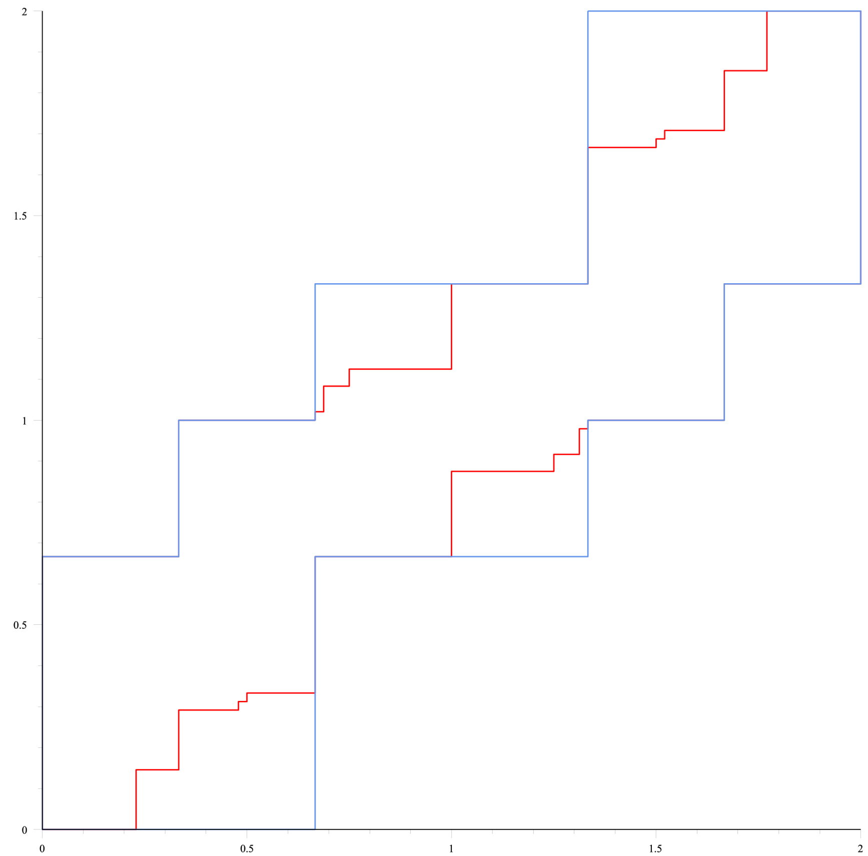


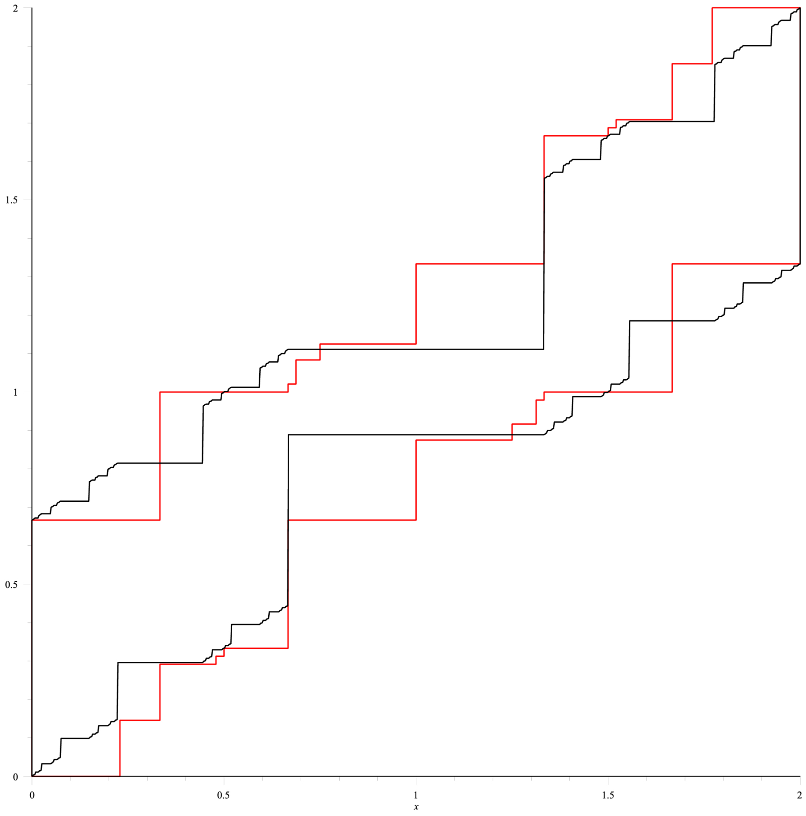

2
1.5
1
0.5
0
0
0.5
1
1.5
2


2
1.5
1
0.5
0
0
0.5
1
1.5
2

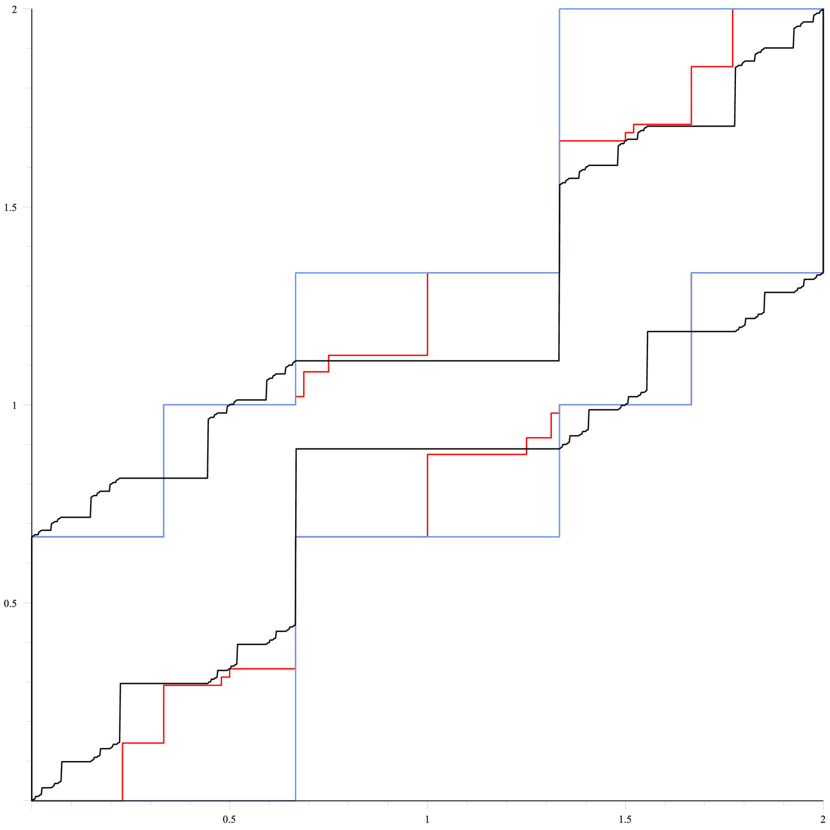
The greedy version





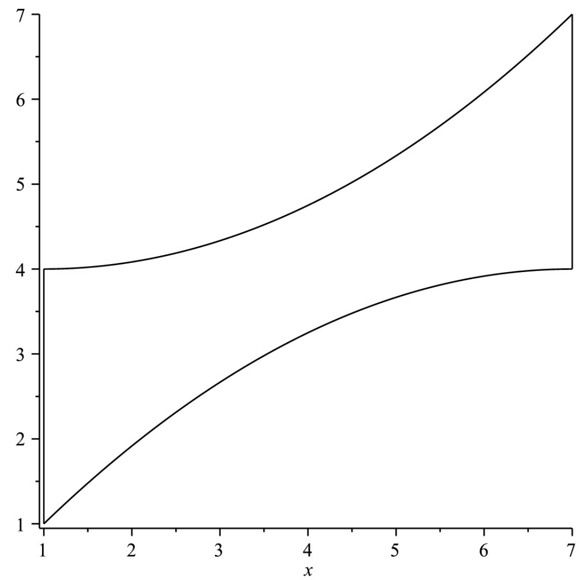


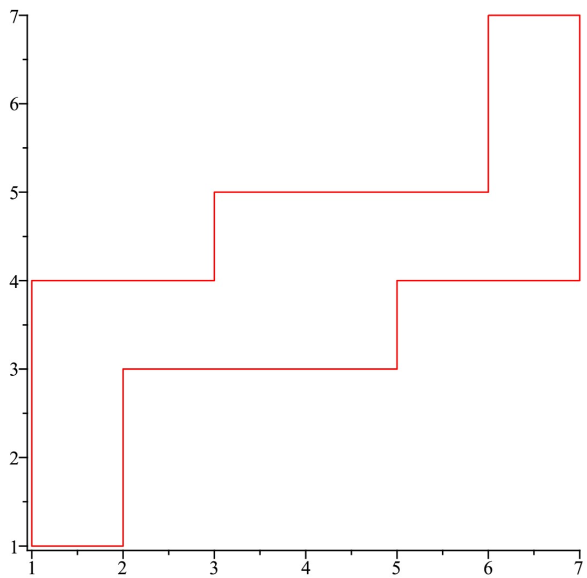
2
1.5
1
0.5
0
0
0.5
1
1.5
2
x

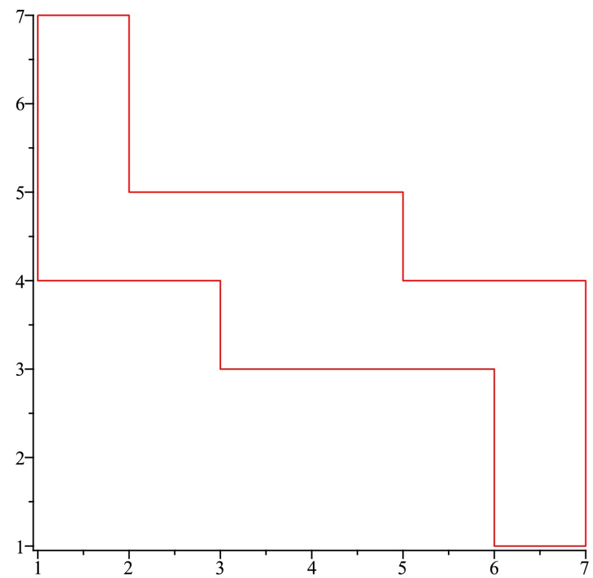


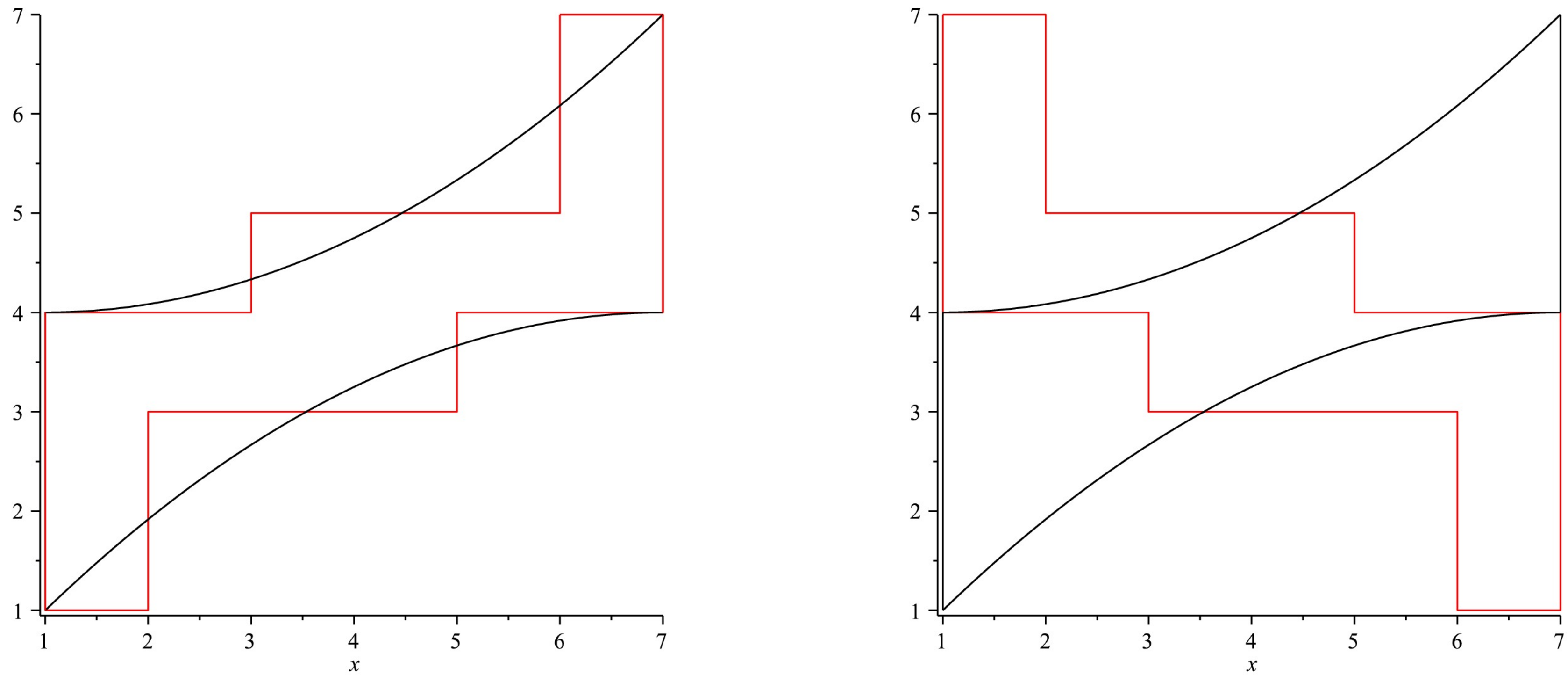
2
1.5
1
0.5
0.5
1
1.5
2

The set we are looking for and the optimal solutions

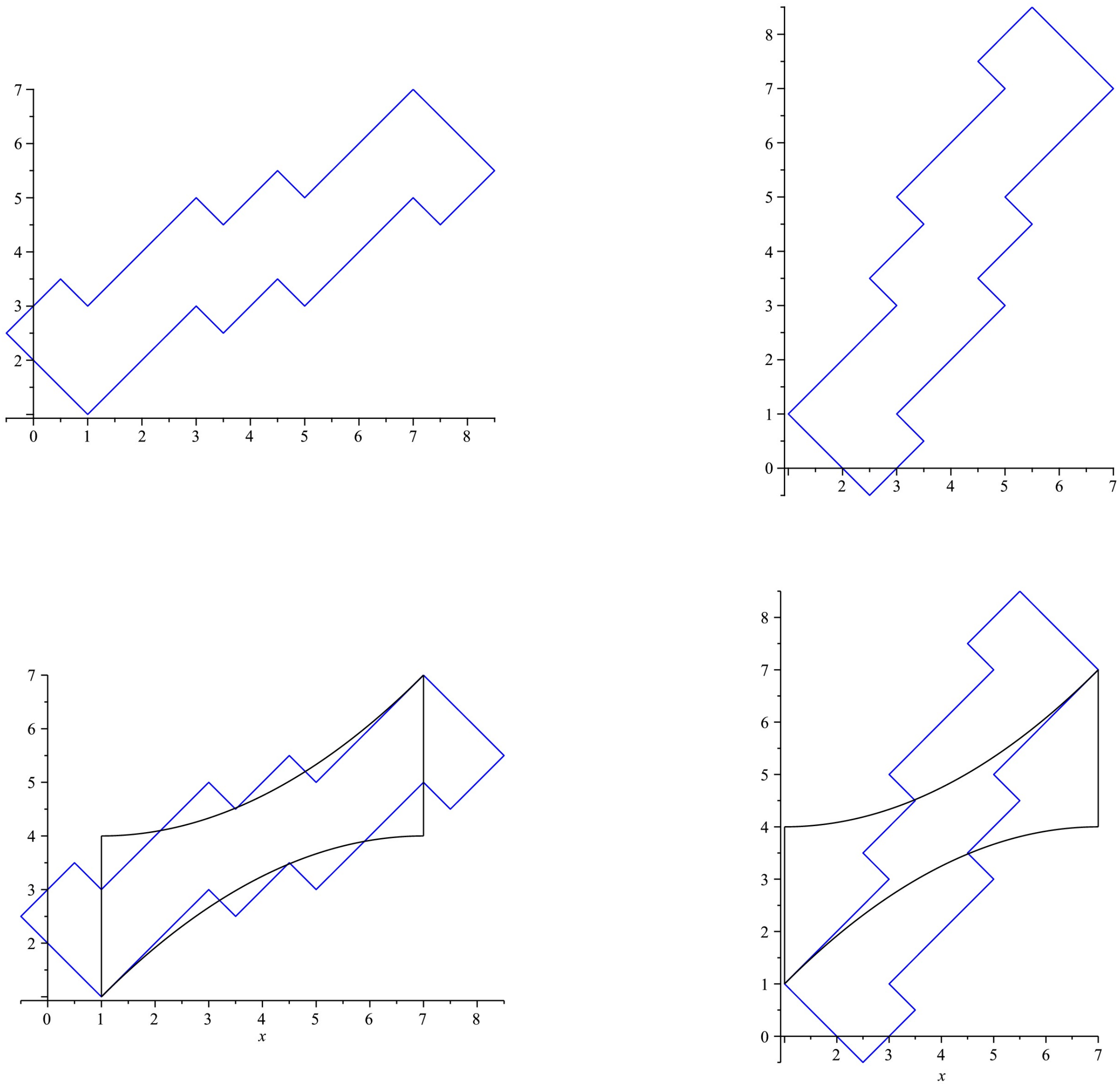




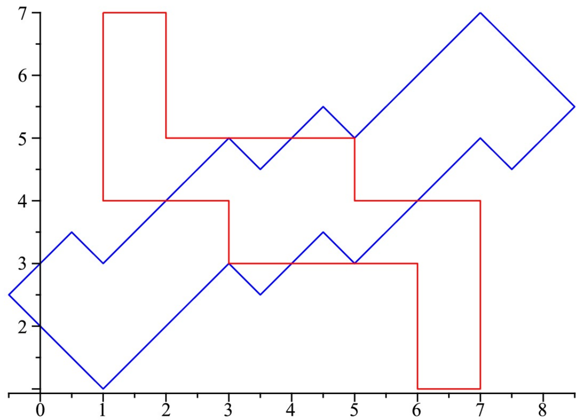


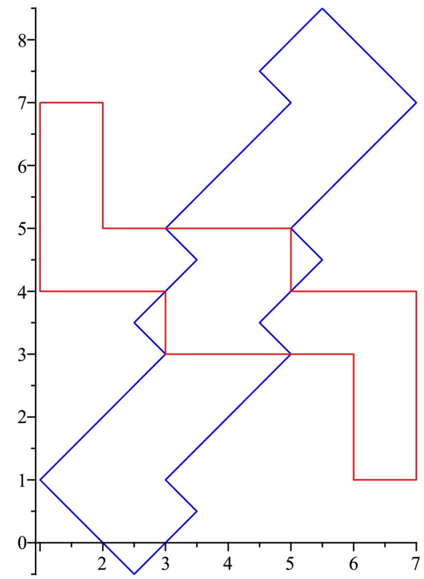

7
6
5
4
3
2
1
1
2
3
4
5
6
7
x
7
6
5
4
3
2
1
1
2
3
4
5
6
7
x

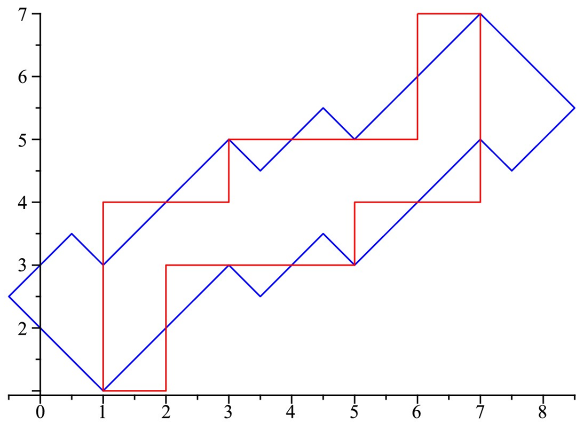
One more pair of directions - the optimal solutions

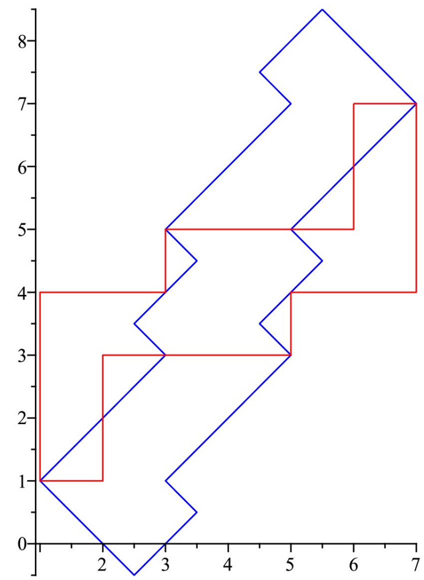


Comparison

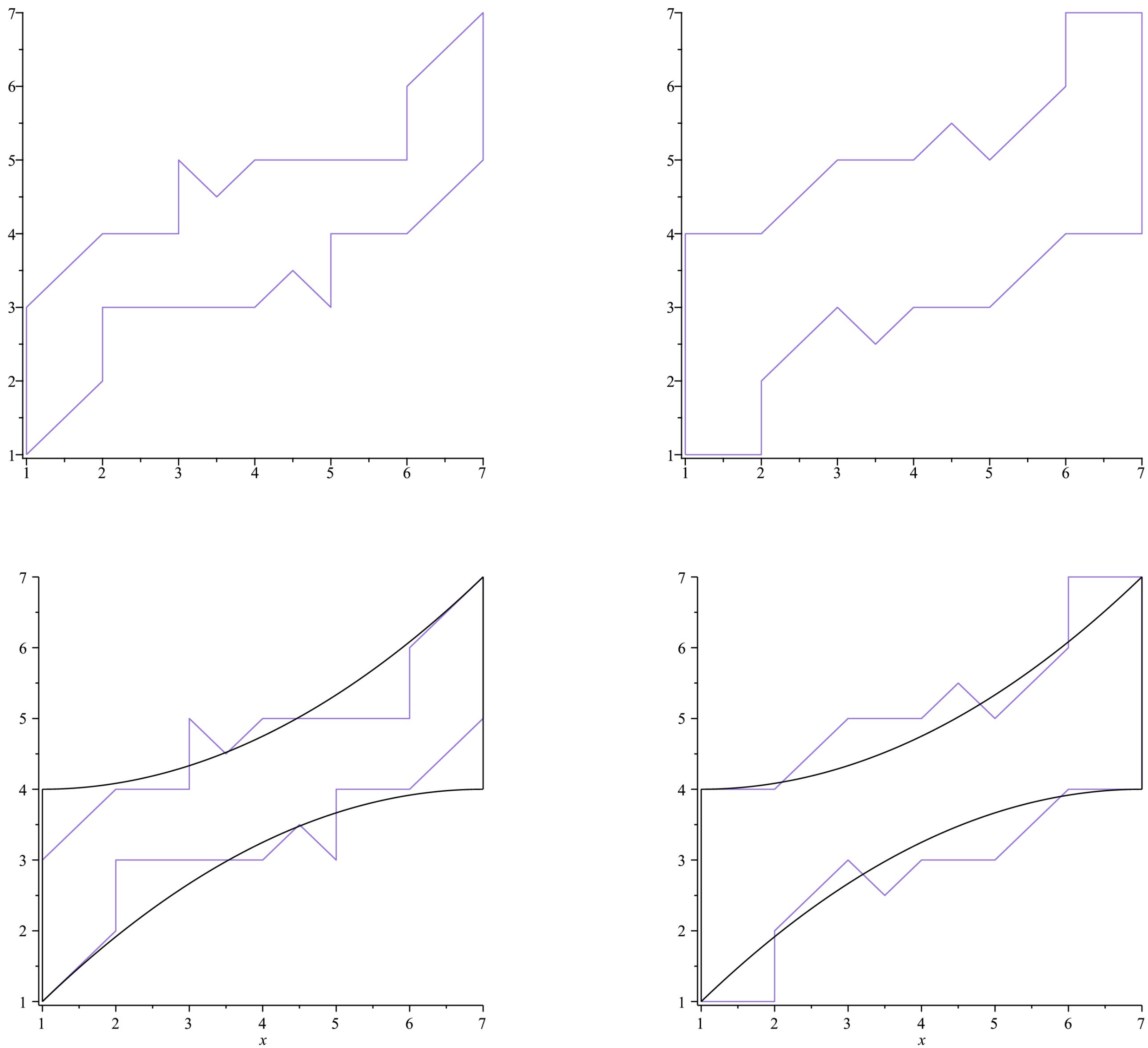

7
6
5
4
3
2
0 1 2 3 4 5 6 7 8


8
7
6
5
4
3
2
1
0
2 3 4 5 6 7


7
6
5
4
3
2
0 1 2 3 4 5 6 7 8


8
7
6
5
4
3
2
1
0
2 3 4 5 6 7

The intersection and the union



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