Dependencies in discrete tomography

Rob Tijdeman, Milan, Italy, 20 April 2015

Universiteit Leiden

1. Switching components

Let A be a finite set in $\boldsymbol{Z}^2$.

A primitive direction is a relatively prime pair (a,b).
Let D be a finite set of directions.

We consider a function $f : A \rightarrow \mathbf{Z}$, a so-called *table*.

We call the sum of all values $f(x,y)$ along a line $bx - ay = t$ for some t in $\boldsymbol{Z}$ a *line sum* in the direction of (a,b).

We assume that f is unknown, but all the line sums in the directions of D are known. Which functions f are possible?

Equal line sums

Suppose we have $D = \{(1,0), (0,1), (1,1), (1,-1)\}$.
Then the following configurations have equal lines sums.

0	1	0	0
0	0	0	1
1	0	0	0
0	0	1	0

0	0	1	0
1	0	0	0
0	0	0	1
0	1	0	0

Subtract:

‘Switching element’

0	1	-1	0
-1	0	0	1
1	0	0	-1
0	-1	1	0

The line sums are zero in the four directions.

Questions

1. Which structures have zero line sums?
2. How much redundant information is there?
3. What is the structure of the redundant information?

Not in this lecture:

1. What solutions are possible (geometric approach)?
2. What is the most likely original?

2. Generating polynomials

Joint work with Lajos Hajdu (Debrecen, HU)
J. reine angew. Math. 534 (2001), 119-128.

1	0	0	0
1	0	1	0
1	1	1	1
0	0	1	0

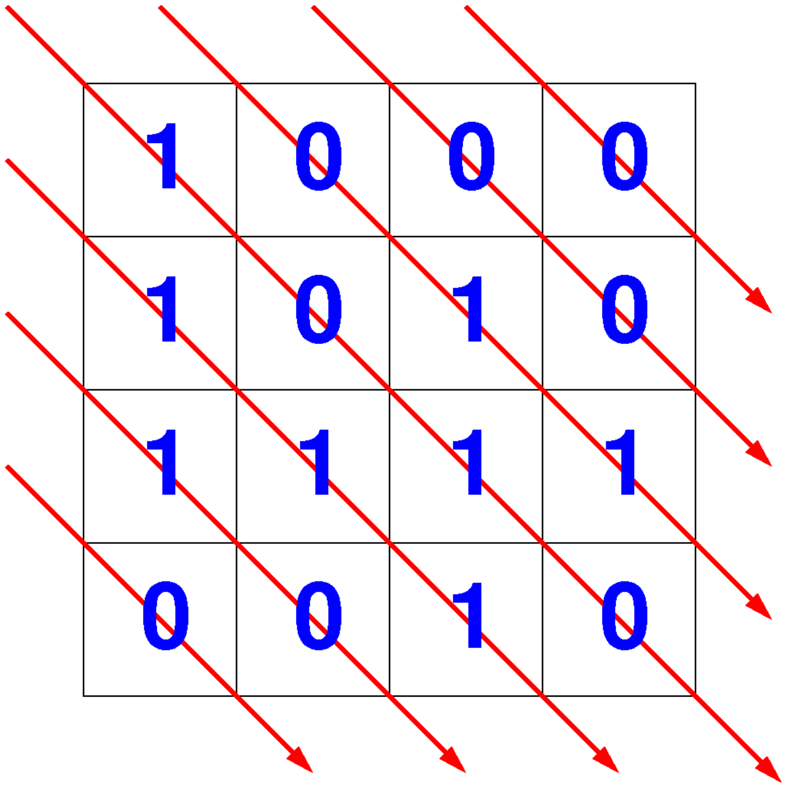
$$1 + y + x^2y + y^2 + xy^2 + x^2y^2 + x^3y^2 + x^2y^3$$

Row sums

1	0	0	0
1	0	1	0
1	1	1	1
0	0	1	0

$$1+y+x^2y+y^2+xy^2+x^2y^2+x^3y^2+x^2y^3 \equiv 1+2y+4y^2+y^3 \pmod{x-1}$$

Diagonal sums



$$1+y+x^2y+y^2+xy^2+x^2y^2+x^3y^2+x^2y^3 \equiv y^2+3y+2+2x \pmod{xy-1}$$

Column sums and anti-diagonal sums

Column sums:

$1 + y + x^2y + y^2 + xy^2 + x^2y^2 + x^3y^2 + x^2y^3$

$\cong \quad 3 + x + 3x^2 + x^3 \pmod{y-1}$

Anti-diagonal sums:

$1 + y + x^2y + y^2 + xy^2 + x^2y^2 + x^3y^2 + x^2y^3$

$\cong \quad 1 + x + x^2 + 2x^3 + x^4 + 2x^5 \pmod{x-y}$

Chinese remainder theorem for polynomials

f(x,y) has rest $1+2y+4y^2+y^{3,}$ when divided by x-1,

f(x,y) has rest $3+x+3x^2+x$, when divided by y-1,

f(x,y) has rest $y^2+3y+2+2x$, when divided by xy-1,

f(x,y) has rest $1+x+x^2+2x^{3+}x^4+2x5$, when divided by x-y.

Then the Chinese remainder theorem implies:

f(x,y) has unique solution modulo
the l.c.m. of x-1, y-1, xy-1, x-y.

Switching element

The lcm of x-1,y-1, x-y and xy-1 is

$(x-1)(y-1)(x-y)(xy-1) = x^3y^2 - x^3y + x^2 - x^2y^3 + xy^3 – x + y - y^2.$

Corresponding rectangle:

0	-1	1	0
1	0	0	-1
-1	0	0	1
0	1	-1	0

Number theory

This is the smallest nontrivial rectangle with zero line sums in all considered directions.

Every polynomial corresponding to a rectangle with zero line sums in the considered directions is divisible by the lcm of the polynomials corresponding to the directions.

Let (a_i, b_i) be the directions for $i = 1,\ldots,d$.
Suppose the lcm of the corresponding polynomials is their product. Then the degree of the product polynomial is $\Sigma_i |a_i|$ in x, $\Sigma_i |b_i|$ in y. The number of superfluous line sums equals $(\Sigma_i |a_i|)(\Sigma_i |b_i|) - \Sigma_i |a_i b_i|$.

Minimal switching hull

The smallest convex polygon covering the minimal switching component is obtained by ordering the directions (a_i,b_i) to increasing values of b_i/a_i and forming a chain of these vectors and their opposites.

For example,
Directions ordered (1,-1), (1,0), (1,1), (0,1).
Opposites (-1,1), (-1,0), (-1,-1), (0,-1).

This yields a polygon with sides the directions in D and their opposites so that in every direction of D there are two parallel sides of equal length.

3. Thesis of Birgit van Dalen (2007)

Let $A = \{(x,y) \text{ in } \mathbf{Z} \mid 0 \le x < m,\ 0 \le y < n\}$,
$f : A \rightarrow \mathbf{Z}$,
D: directions (a_i,b_i) for $i = 1,2,\ldots, k$.
Put $s(i,h) = \Sigma\, f(x,y)$ over (x,y) in A, $a_iy\text{-}b_ix=h$.

A linear dependency between line sums is of the form

$$\Sigma_i \Sigma_{h \text{ in } \mathbf{Z}}\ c_{ih}\, s(i,h) = 0. \quad (c_{ih} \text{ independent of } f)$$

A linear dependency between line sums is *global* if the coefficients are also independent of A.

Conjecture: There are $\Sigma_{i<j} \mid a_ib_j - a_jb_i \mid$ linear independent global dependencies between the line sums.

Birgit van Dalen –Arjen Stolk

Conjecture: There are $\Sigma_{i<j} \mid a_i b_j - a_j b_i \mid$ linear independent global dependencies between the line sums.

Birgit van Dalen proved the conjecture for $k = 2,3,4$ in the case that A is a 'rectangle'.

Arjen Stolk (2011) proved the full conjecture in his Ph.D.-thesis '*Discrete tomography for integer-valued functions* for A being a non-empty D-rounded convex grid set in $\mathbf{Z}^2$.

A is *convex* if it contains all integer points in its convex hull
A is *D-rounded* if there is a convex, closed and bounded set Z in $\mathbf{Z}^2$ such that $A = D + Z$.

4. Elimination process

Let *A* and *D* be given.
Suppose there is a line along *D* which meets *A* in only one point *a*. Then the value *f*(*a*) is known and the lines meeting only *a* and the point *a* itself can be taken apart. We adjust the values of the remaining line sums for the loss of *f*(*a*).

If there is more than one line from D going only through *a* then they create a local dependency. All the line sums of lines passing only through *a* have to have the same line sums.

We continue this elimination process until every line from *D* through a point of *A* passes through another point of *A*.

Elimination process (2)

Every line from D through a point of the final A^* passes through another point of A^*.

Then the sides of the convex hull of the set A^* are in the directions of D. Moreover, every direction occurs twice.

A^* need not be convex itself, but we bring back the inner points and the lines from D through them.

It follows that either the final set A^* is empty (in which case we know all the dependencies) or it contains a switching element.

Example of a-symmetric switching hull

Directions: (1,0), (1,-2), (3,1). Final set *A*.

$$
\begin{array}{ccccccccccccc}
 & & & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \\
\underline{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \underline{1} \\
 & \underline{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \underline{1} & & & \\
 & & 1 & 1 & 1 & 1 & \underline{1} & & & & & & \\
 & & \underline{1} & \underline{1} & & & & & & & & &
\end{array}
$$

If the parallel sides have equal lengths, then the set is *D*-rounded and the dependencies have a basis of global dependencies according to Arjen Stolk.
(There are no local dependencies anymore.)

5. Dimension of dependencies

Let B be the incidence matrix of A.
If the number of points of A is p and the number of lines from D along A equals l, then B has l rows and p columns.

Put $R = \{Bx : x \text{ in } \mathbf{Z}^p\}$ and $N = \{x \text{ in } \mathbf{Z}^p : Bx = 0\}$.
Write r and n for their ranks. Then $r + n = p$.

Put $S = \{B^T x : x \text{ in } \mathbf{Z}^l\}$ and $V = \{x \text{ in } \mathbf{Z}^l : B^T x = 0\}$.
Write r and v for their ranks. Then $r + v = l$.

Thus $v = l – p + n$. This formula can be used to compute the dimension of the dependencies.

Dimensions of dependencies (2)

We have $v = l - p + n$. Suppose A is convex. Then

l is the number of lines from D going through a point of A,
p is the number of points in A,
n is the number of switching elements fitting in A,
v is the dimension of the linear dependencies.

In many cases l, p and n can be calculated.
This provides the value of v.

Symmetric switching hulls

Let the convex A be surrounded by successively
r_1 times (a_1,b_1), …, r_k times $(a_k.b_k)$, r_1 times $(-a_1,-b_1)$, …,
r_k times $(-a_k.-b_k)$. Put $\det_{ij} = |\, a_i b_j - a_j b_i \,|$. Then

$l = k + \Sigma_{i<j} (r_i + r_j)\ \det_{ij}$,

$p = 1 + \Sigma_i r_i + \Sigma_{i<j} r_i\, r_j \det_{ij}$,

$r = - k + 1 + \Sigma_i r_i + \Sigma_{i<j} (r_i-1)(r_j-1) \det_{ij}$,

It follows that
$v = \Sigma_{i<j} \det_{ij} = \Sigma_{i<j} |\, a_i b_j - a_j b_i \,|$.

This is exactly the dimension of the global dependencies!

Conclusion

The considered switching hull is a *D*-rounded convex grid set. According to Stolk there is a basis of global dependencies.

I think I see a similar way to prove that
$v = \Sigma_{i<j} |a_i b_j - a_j b_i|$ in case of switching hulls where parallel edges do not have the same length, but I have to work it still out.
This together would mean that for given *A* and *D* we first can apply the elimination process and then apply this result. Hence we can say exactly what the dimension is of the global dependencies and what are the local dependencies.

6. Explicit global dependencies

A second goal is to make the basis of global dependencies explicit. In 2001 Hajdu and I computed a basis of the seven global dependencies in case of the directions (1,0), (0,1),(1,1)(1,-1).

Σ_h s((1,0),h) = Σ_h s((0,1),h) = Σ_h s((1,1),h) = Σ_h s((1,-1),h),
$\Sigma_{h \text{ is odd}}$ s((1,1),h) = $\Sigma_{h \text{ is odd}}$ s((1,-1),h)

Σ_h h s((1,1),h) = Σ_h h s((1,0),h) + Σ_h h s((0,1),h),
Σ_h h s((1,-1),h) = Σ_h h s((1,0),h) - Σ_h h s((0,1),h),

Σ_h h^2 s((1,1),h) +Σ_h h^2 s((1,-1),h) =
2 Σ_h h^2 s((1,0),h) + 2 Σ_h h^2 s((0,1),h).

Conjecture of Birgit van Dalen

Birgit van Dalen (2007) made the following conjecture on the occurring degrees.

Define $G_m = \Sigma_{|D| = m} \gcd_{i,j \text{ in } S} \mid a_i b_j - a_j b_i \mid$,

where the sum runs over the sets S of cardinality h.
Thus G_2 is the total number of global dependencies.

Then the number of global dependencies of power t equals

$$\Sigma_j \ (-1)^{j} \ (t+j \text{ choose } j) \ G_{t+j+2} \ .$$

She proved that this is correct for $|D| \leq 4$.

Maximum degree

Birgit van Dalen (2007) showed

Let D be a set of $k \geq 2$ dependencies. Then there is a dependency of the form

$$\Sigma_{j \text{ in } D} \Sigma_{h \text{ in } \mathbf{Z}} h^{k-2} s(j,h) = 0 \quad (*)$$

but there is no dependency of the form

$$\Sigma_{j \text{ in } D} \Sigma_{h \text{ in } \mathbf{Z}} h^{r} s(j,h) = 0$$

with $r > k$-2.

The coefficients in (*) are uniquely determined (up to a nonzero factor) and all nonzero.

Explicit coefficients

Hajdu and I found a simple explicit expression for the coefficients with an elegant proof.

Denote the determinant of the $(m+1)$ by m matrix with i th column vector $\boldsymbol{x}_i$ by $\det(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m)_j$.
Put $\boldsymbol{a}^T = (a_1,\ldots,a_k)$ and $\boldsymbol{b}^T = (b_1,\ldots,b_k)$.
Set

$$E_j = \det(\boldsymbol{a}^{k-2}, \boldsymbol{a}^{k-3}\boldsymbol{b}, \boldsymbol{a}^{k-4}\boldsymbol{b}^2, \ldots, \boldsymbol{b}^{k-2}).$$

Then

$$\Sigma_j \ (-1)^j E_j \ \Sigma_{h \text{ in } \mathbf{Z}} h^{k-2} s(j,h) = 0.$$

Sketch of the proof

Obviously we have

$$\det(\boldsymbol{a}^h \boldsymbol{b}^{k-2-h}, \boldsymbol{a}^{k-2}, \boldsymbol{a}^{k-3}\boldsymbol{b}, \boldsymbol{a}^{k-4}\boldsymbol{b}^2, \ldots, \boldsymbol{b}^{k-2}) = 0.$$

for $0 \leq h \leq k$-2.

By developing to the first column we obtain for each h

$$\Sigma_j (-1)^{j-1} a_j^h b_j^{k-2-h} \det(\boldsymbol{a}^{k-2}, \boldsymbol{a}^{k-3}\boldsymbol{b}, \boldsymbol{a}^{k-4}\boldsymbol{b}^2, \ldots, \boldsymbol{b}^{k-2})_j = 0.$$

Hence, for every (x,y) in $\mathbf{Z}^2$

$$\Sigma_{x,y \text{ in } \mathbf{Z}} f(x,y) \, \Sigma_j (-1)^{j-1} (b_j x - a_j y)^{k-2} E_j = 0.$$

It follows that

$$\Sigma_j \Sigma_t \Sigma_{x,y} f(x,y) (-1)^{j-1} t^{k-2} E_j = 0,$$

where the third summation is over x,y with $b_j x - a_j y = t.$

Therefore

$$\Sigma_j \Sigma_t (-1)^{j-1} t^{k-2} E_j \, s(j,t) = 0.$$

Literature

See:

www.math.unideb.hu/~hajdul/

www.math.leidenuniv.nl/~tijdeman/
up to 2010,
on arXiv from 2010 on.