A weighted wavelet scheme for region of interest tomography

Reconstructing piecewise constant functions from tomography data

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joint work with Ronny Ramlau (JKU) and Todd Quinto (Tufts)

Reconstructing piecewise constant functions from tomography data

Reconstruction problem

– given (indirect) information of an object...
  what can we find out about the object? (also "inverse problem")

Piecewise constant functions

– let $\Omega_n \subset \mathbb{R}^2$ and $c_n \in \mathbb{R}$, then $f(x) = \sum_{n=1}^{N} c_n \chi_{\Omega_n}$

Tomography data

– line integrals of an object

Overview

1. The reconstruction or inverse problem
2. Region of interest tomography
3. Wavelets for region of interest tomography
4. Filtered backprojection for region of interest tomography

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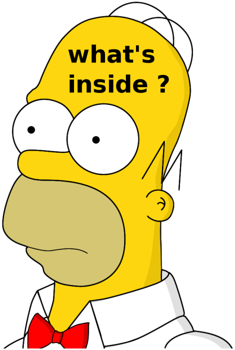
An important question



Ideas for answers:

- ask the creator
- open head (quite destructive)
- ...
- ...
- deduce from behavior
- non-invasive head scan

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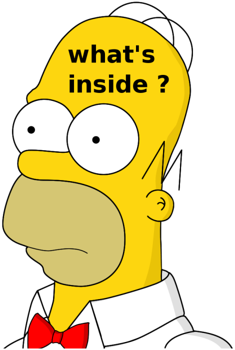

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Non-invasive means indirect information.
We can only observe the effect caused by the inside.

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Forward, direct and *backward, indirect, inverse*

Assume one knows what is inside, say f;
and one also knows the process, say K;
(e.g., sums along certain lines)
Then, the expected outcome can be computed, say g.

Forward or direct problem

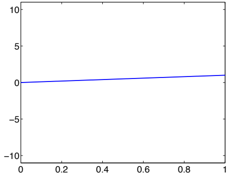
Given f and $K : X \to Y$, compute $g = Kf$. *(we consider the linear case)*

Backward or indirect or inverse problem

Given $K : X \to Y$ and $g \in Y$, compute f with $Kf = g$.

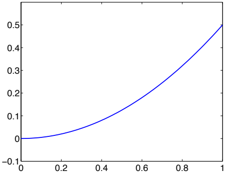
As soon as there is an inverse to K this should not be too hard!

Example: Integration and differentiation



$$x \xrightarrow{\text{integrate}} \frac{1}{2}x^2$$

$$\xleftarrow{\text{differentiate}}$$



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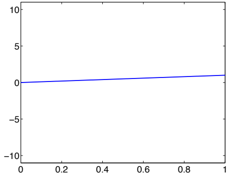

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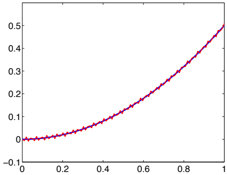

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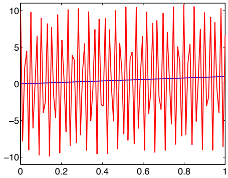

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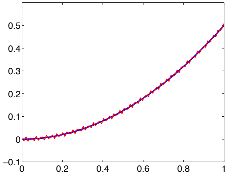

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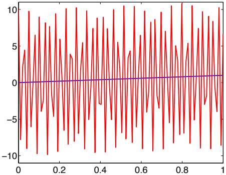

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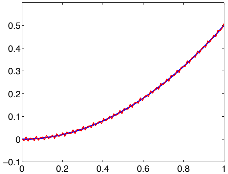

- This is a very academic and artificial example.
- It shows a typical behavior:

g^δ close to g (with $g = Kf$) leads to $f^\delta := K^{-1}g^\delta$ not close at all to $f = K^{-1}g$

I$\ell\ell$-posed inverse problems and regularization

Given K and g^δ that approximates $g = Kf$, e.g., $\|g - g^\delta\| \leq \delta$.

ill-posed inverse problems: issues with

- existence and uniqueness of solutions and
- stability: $g^\delta \to g$ BUT $f^\delta := K^{-1}g^\delta \not\to f = K^{-1}g$

regularization: replace K^{-1} by R_α with $\alpha = \alpha(\delta) > 0$ such that

- approximation: $\lim_{\alpha\to 0} R_\alpha g = K^{-1}g$
- stability: $g^\delta \to g$ implies $f_\alpha^\delta := R_\alpha g^\delta \to f = K^{-1}g$

The parameter α is called *regularization parameter*. It controls the compromise between approximation and stability.

Example: Tikhonov

- Standard Tikhonov: solving $Kf = g$ is replaced by

$$\text{minimize } \|Kf - g^\delta\|_2^2 + \alpha\|f\|_2^2;$$

This is called a *variational formulation*.

☺ parameter choice rule exists, one even has a convergence rate

- Generalizations: solving $Kf = g$ is replaced by

$$\text{minimize } \|Kf - g^\delta\|_X^p + \alpha\mathcal{P}(f)$$

The penalty term $\mathcal{P}$

  - assures the stable dependence on the data,
  - describes properties of the solution like sparsity or smoothness;

$$\|f\|_2^2 \quad \text{or} \quad \mathrm{TV}(f) \quad \text{or} \quad \|f\|_{H^s} \quad \text{or} \quad \ldots$$

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Tomography, Radon and ill-posedness

- In computerized tomography (CT) one wants an image/some information from inside a body given measurements from the outside.
- The data is modeled as the Radon transform of the body density f

$$Rf(s,\omega) = \int_{\mathbb{R}} f(s\omega^{\perp} + t\omega)\mathrm{d}t \qquad (s,\omega) \in \mathbb{R} \times [0,\pi)$$

- Wanted:

$$\text{get } f \text{ from } g = Rf$$

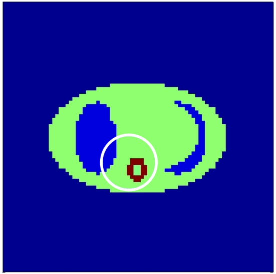
☞ There is an inverse Radon transform.

☠ R smoothes of order $1/2$ in Sobolev scales

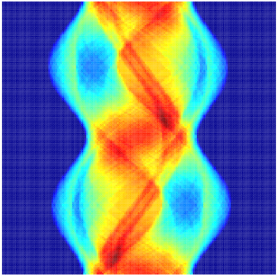
  - The inversion is instable/ill-posed.
  - Partial data (region of interest) makes it worse.

Region of interest tomography or **Interior problem**

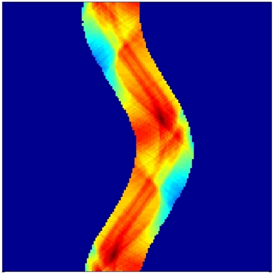
- information is wanted/available only about a region of interest of an object not about the whole object
- image only that region; save time and money, minimize exposure
- increases the ill-posedness/instability



original with ROI



full data



ROI data

Region of interest tomography – the big question

Let $\Omega \subset D \subset \mathbb{R}^2$ be the region of interest; and $f : D \to \mathbb{R}$ is wanted.

Radon data/line integrals of f are given

1. over all lines passing through Ω (continuous data)
2. over some lines (densely sampled) passing through Ω (discrete data)

Can one get back f (or maybe just $f_{|\Omega}$) from continuous ROI data?

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In general: no ...... For piecewise constant functions: yes.

K., Quinto, Ramlau. *A weighted wavelet method for region of interest tomography* (2015).

Related work (theoretical results and TV approach):

Han, Yu, Wang (2009); and Ye, Yu, Wang (2009); and Yu, Jang, Jian, Wang (2009).

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Wavelet representation

- A (1d-)wavelet ψ is a small wave, a function with zero mean.
- It generates a family of wavelets via $\psi_{jk}(t) := 2^{-j/2}\psi\left(\frac{t-k\cdot 2^j}{2^j}\right)$, $j,k \in \mathbb{Z}$
  - j determines the size of the wavelet:
    - for $j > 0$ we get a slow/long wave; general trend or approximation
    - for $j < 0$ we get a fast/short wave; details
  - k determines the location of the wavelet.
- For some wavelets the ψ_{jk} are an orthonormal system; even a basis:

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{jk} \rangle \psi_{jk}$$

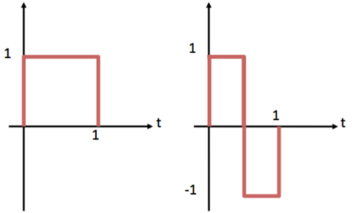
- For some wavelets there exists a second function, the so-called *scaling function* or *father wavelet*.

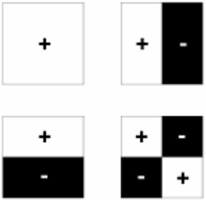
Wavelets in 2d – example

1d-Haar scaling function and wavelet.

2d functions (wavelets) as products of 1d functions

- $\Psi_1(x,y) := \phi(x)\phi(y)$
- $\Psi_2(x,y) := \phi(x)\psi(y)$
- $\Psi_3(x,y) := \psi(x)\phi(y)$
- $\Psi_4(x,y) := \psi(x)\psi(y)$





Different amount of contribution

We have $f = \sum_{j,k} c_{jk}\psi_{jk}$ but also $f = f_{\text{ROI}} + f_{D\setminus\text{ROI}}$.

Different types of wavelets (position of the wavelet relative to the ROI).

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Different types of wavelets (position of the wavelet relative to the ROI).

1. empty overlap: $\operatorname{supp}\psi \cap \text{ROI} = \varnothing$;

   these wavelets do not contribute to f_{ROI}
2. one containes the other: $\operatorname{supp}\psi \subset$ ROI; or ROI $\subset \operatorname{supp}\psi$;

   strong contribution to f_{ROI}
3. neither 1 nor 2, i.e., nonempty overlap but not contained;

   some contribution to f_{ROI}

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Think about it: What would you do with type 1?

Weighted wavelets for ROI tomography – the functional

- Given (noisy) ROI data g^δ_{ROI} of f, i.e., $g_{\text{ROI}} = R_{\text{ROI}} f$
- compute $\vec{c} = (c_{jk}) \in \ell_2$ and by that $f = F^* \vec{c} = \sum c_{jk} \psi_{jk}$ as

$$\vec{c} = \arg\min \left\{ \| R_{\text{ROI}} F^* \vec{c} - g^\delta_{\text{ROI}} \|^2 + \alpha \sum_{j,k} \omega_{jk} |c_{jk}| \right\}$$

- with weights emphasizing the positioning relative to the ROI
  1. empty overlap: $\omega_{jk} = \omega_{\text{out}}$
  2. one containes the other: $\omega_{jk} = \omega_{\text{in}} = 1$
  3. neither 1 nor 2, i.e., $\omega_{jk} = \omega_{\text{rim}}$ interpolates ω_{out} and ω_{in}

$$\vec{c} = \arg\min \Big\{ \| R_{\text{ROI}} F^* \vec{c} - g^\delta_{\text{ROI}} \|^2 + \alpha \Big(\sum_{\text{in}} |c_{jk}| + \sum_{\text{rim}} w_{jk} |c_{jk}| + \omega_{\text{out}} \sum_{\text{out}} |c_{jk}| \Big) \Big\}$$

Influence of the weights

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We set $\omega_{\text{in}} = 1$.

- $\omega_{\text{out}} = 1$ . . . all wavelets have the same influence.
- $\omega_{\text{out}} << 1$, e.g., close to zero; remember: minimize the functional!

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We set $\omega_{\text{out}} > \omega_{\text{in}} = 1$, but still bounded, e.g., $\omega_{\text{out}} = 5$ or 10.

Regularization result

We can find conditions for convergence in

Daubechies, Defrise, De Mol. *An iterative thresholding algorithm for linear inverse problems with a sparsity constraint.* 2004.

Ramlau. *Regularization properties of Tikhonov regularization with sparsity constraints.* 2008.

All of the necessary conditions can be fulfilled; and we have a unique solution for piecewise constant functions from complete ROI tomography data.

Theorem (Convergence result)

The regularized solutions of the weighted wavelet method converge to the exact solution.

Related work

Small wavelets outside the ROI do not contribute to it – get rid of them!

$$f = \sum_{j=0}^{N} \sum_k c_{jk} \psi_{jk} = \sum_{j=0}^{N} \sum_k c_{jk} \psi_{jk}^{\text{ROI}} + \sum_{j=0}^{N-1} \sum_k c_{jk} \psi_{jk}^{\neg\text{ROI}}$$

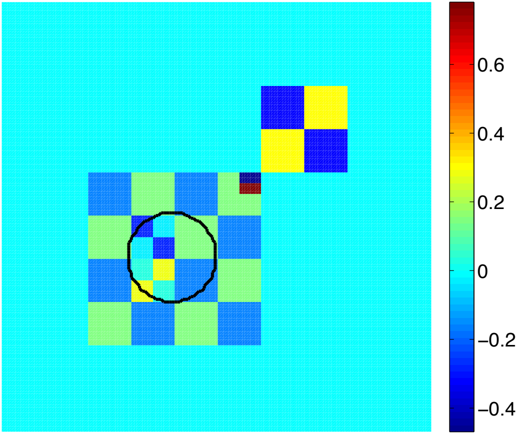
Deleting complete scales of wavelets is called linear shrinkage and it reduces the number of unknowns!

This is done for ROI tomography (also $N-2$, $N-3$ etc.) in

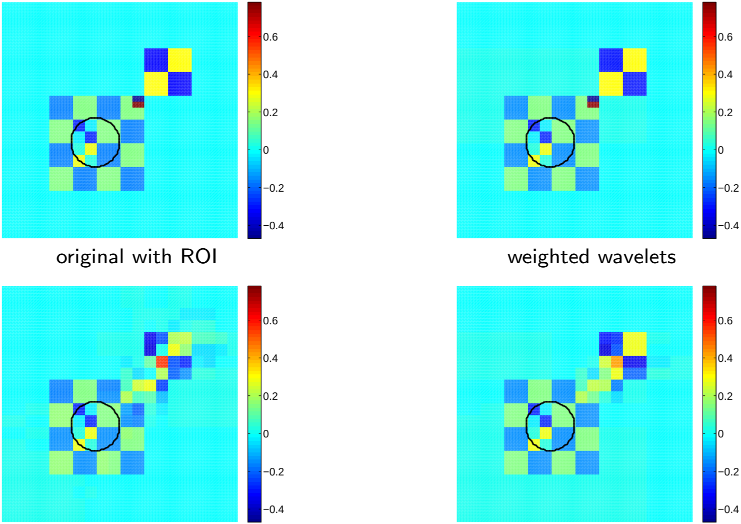
Niinimäki, Siltanen, Kohlemainen. *Bayesion multiresolution method for local tomography in dental X-ray imaging.* 2007.

A very sparse wavelet example

A very sparse and very academic example built from 8 wavelets:



A very sparse wavelet example – reconstruction



original with ROI

weighted wavelets

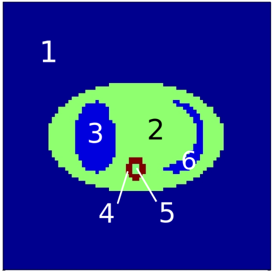
delete 2 outer levels; α_1

delete 2 outer levels; α_2

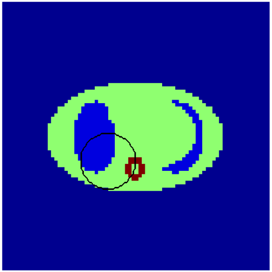
A very sparse wavelet example - remarks

- Allowing small wavelets outside the ROI results in perfect recovery; 8 nonzero wavelet coefficient.
- The reconstruction quality inside the ROI is also very good for the linear shrinkage.
- However, one looses the overall sparsity.

The MCAT torso phantom

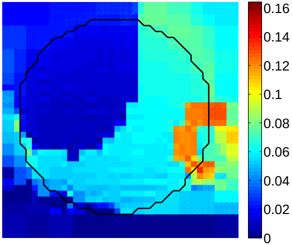


j	μ_j
1	0
2	0.0600
3	0.0100
4	0.1200
5	0.0600
6	0.0100

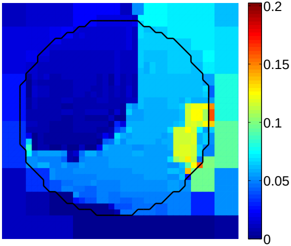


Torso phantom with numbered domains and ROI. Note the spine region!

Weighted wavelets versus linear shrinkage



weighted wavelets, $\partial_{\text{ROI}} = 10.34\%$



linear shrinkage, $\partial_{\text{ROI}} = 14.93\%$

More results, pictures, tables (also for noisy data) in

K., Quinto, Ramlau. *A weighted wavelet method for region of interest tomography* (2015).

Allowing small details outside the ROI can lead to a better reconstruction inside the ROI.

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Region of interest tomography – filtered backprojection

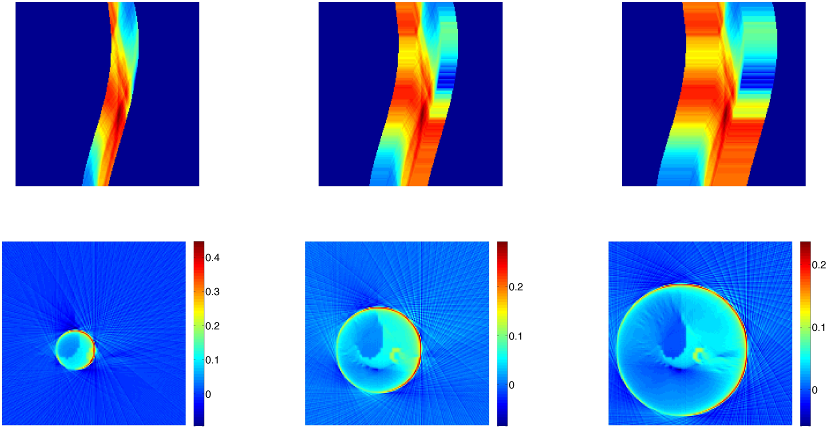
Please be aware! Most of this is this weekend's 'shot in the dark'.

From some discussions, mainly last workshop in Leiden (Federica Marone & ROI discussion group), i got the following ideas what is used

- Consider the ROI data as a complete data set.
- Use FBP to reconstruct an object of the size of the region of interest.
- Data padding (get rid of the ring artefact);
  - introduce more data, i.e., pad the given sinogram, e.g., constantly with the first/last measurement;
  - again consider the padded data as complete data set and reconstruct a region of the appropriate size.
- Rely on the structure not on the values.

FBP of ROI data

This is again the MCAT phantom.



no padding | constant padding l/r 20 | constant padding l/r 40

FBP of ROI data

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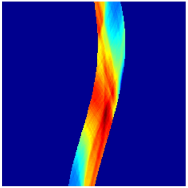
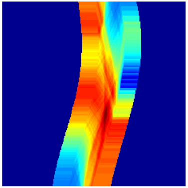
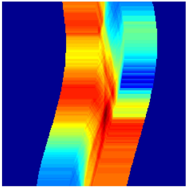
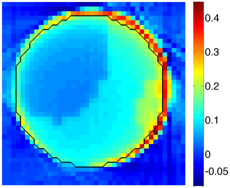
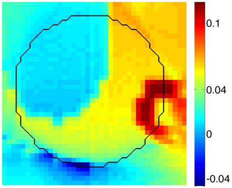
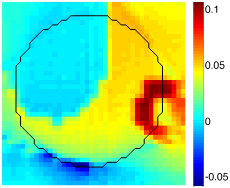






no padding



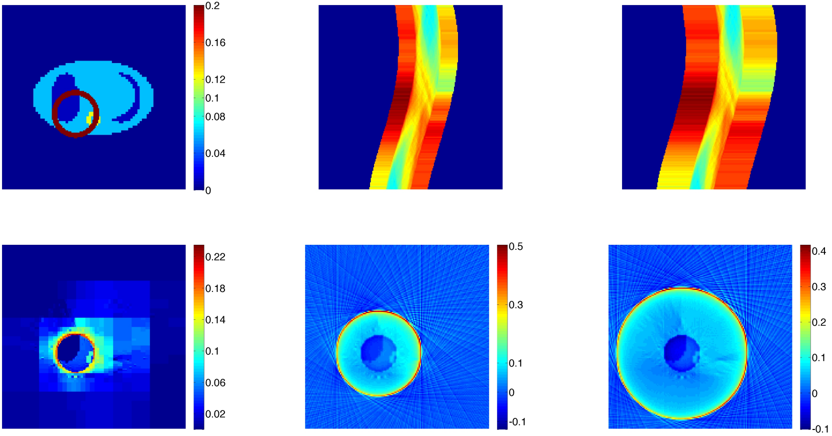
constant padding l/r 20



constant padding l/r 40

FBP of ROI data

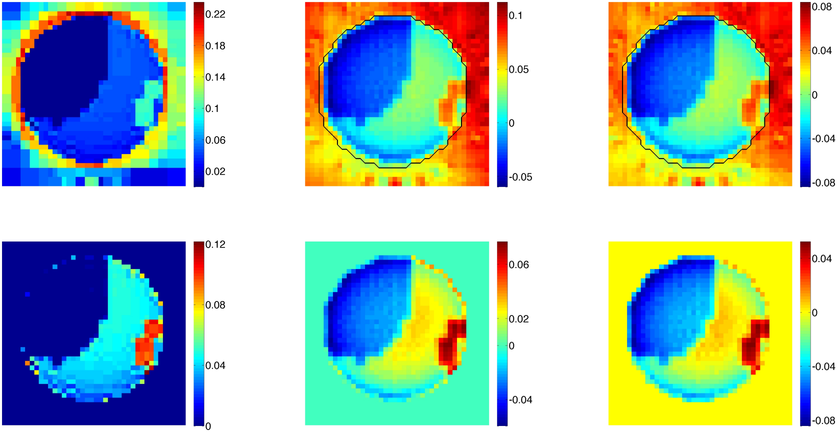
MCAT phantom with a circle of high attenuation directly outside the ROI.



weighted wavelet | constant padding l/r 20 | constant padding l/r 40

FBP of ROI data

Zoom into reco around ROI (first row) and only reco of ROI (2nd row).



weighted wavelet   constant padding l/r 20   constant padding l/r 40

Conclusions/Questions

- add noise and do the FBP reconstruction;
- reduce number of projections; compare to FBP;
- create a phantom for which FBP introduces artefacts; maybe half a circle (should be non-symmetric)