la trasformata Mojette : venti anni di Tomografia (molto) Discreta

Jeanpierre GUEDON et al

Lab IRCCyN – Polytech Univ Nantes FRANCE

Tomo & Applications – Politectenico di Milano ITALIA

The Mojette transform : 20 years of (very) Discrete Tomography

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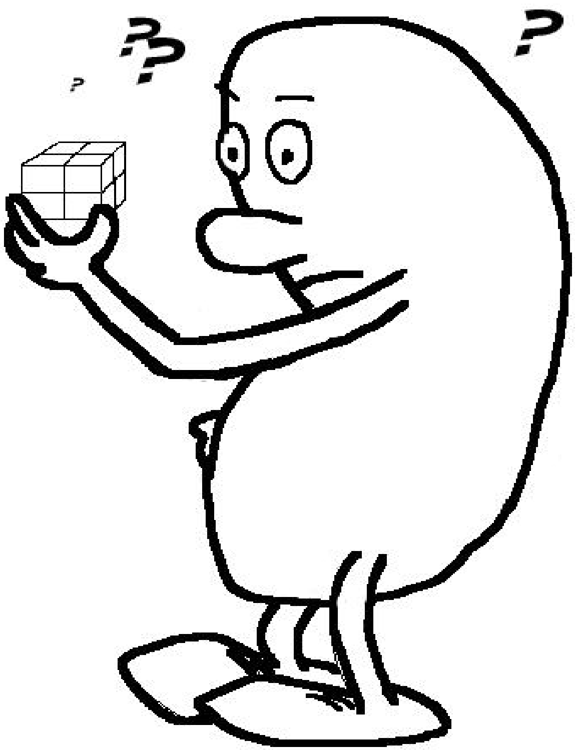
la trasformata Mojette : 20 anni

Grazie

Nicolas Normand
Olivier Philippé
Florent Autrusseau
Benoit Parrein
Pierre Evenou
Pierre Verbert
Myriam Servières
Andrew Kingston
Imants Svalbe
Chuanlin Liu
Aurore Arlicot
Henri Der Sarkissian



Yves Bizais
Dominique Barba
Nicole Burger
Mark Barratt
Dietmar Eggeman
Sandrine Lecoq
Eloise Denis
Peggy Subirats
Tarraf Torfeh
Hadi Fayad
Jiazy Li
Shakes Chandra
Zeeshan Ahmed


introduction
Mojette = bean

Mojette = fagiolo = bean

augenbohne

pois chique

black-eyed-pea

costeño, frijol de costa

voamba (Madagascar)

cowpea

dolique from China

cornille

mongette

dolique à œil noir

dolique mongette

niébé (Sénégal)

bannette (Provence)

coco œil noir

cow bean

pois à vaches


Saint Helier
Nantes
Mojette land
France
Image NASA
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Google
47°19'37.98" N
0°32'41.79" O
Altitude 512.93 km

introduction

$$Mf(k,l) = proj_f(b,p,q)$$

$$proj_f(b,p,q) = \sum_k \sum_l \left(f(k,l)\,\Delta\left(b + qk - pl\right)\right)$$

With k,l,b,p,q integers

Initial work : Myron Katz 1978

outline

1. The Mojette transform with exact data

2. Tomographic reconstruction

3. Links with Fourier and FRT

1. The Mojette transform with exact data

1.1 The Mojette transform definition

1.2 Reconstruction theorem

1.3 Null space and fantoms

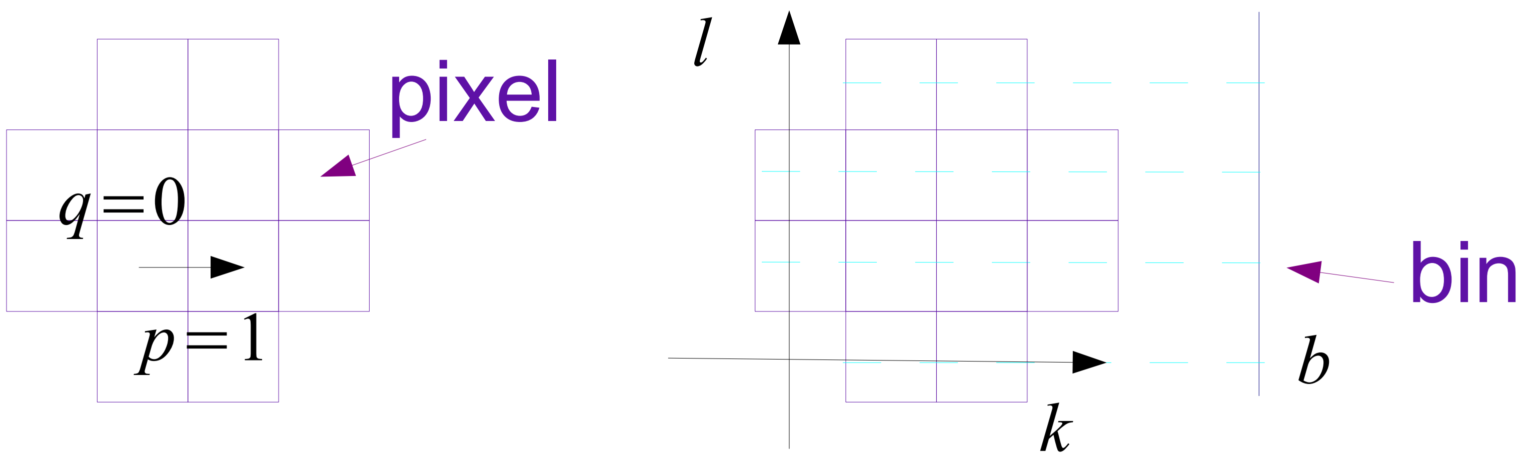
1.4 Applications

1. The Mojette transform with exact data

1.1 The Mojette transform definition

Acts onto a regular pixel grid (k,l)
e.g. rectangular hexagonal etc.

$$proj_f(b,p,q)=\sum_k \sum_l f(k,l)\Delta(b+qk-pl)$$



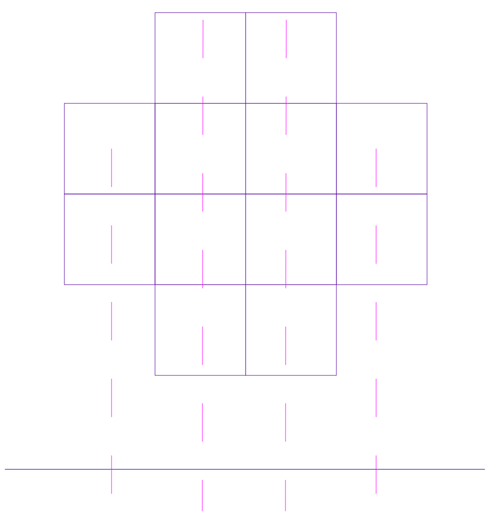
$$proj_f(b,p=1,q=0)=\sum_k \sum_l f(k,l)\Delta(b-l)$$

1. The Mojette transform with exact data

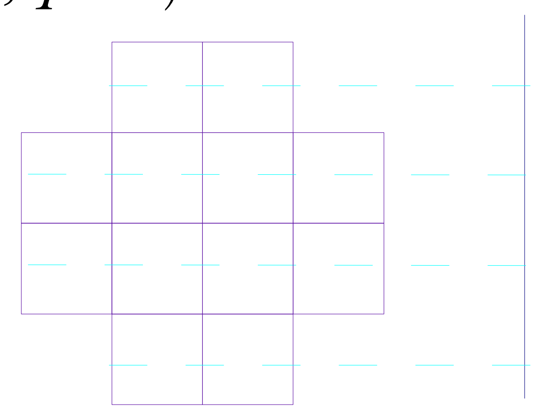
1.1 The Mojette transform definition

Projection direction (p,q) $\left(p \in Z, q \in N\right)$
q>0 except for (p=1,q=0)

$\left(p=0, q=1\right)$



$\left(p=1, q=0\right)$

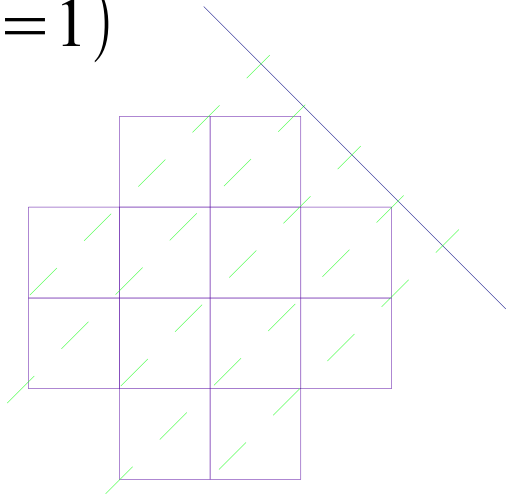


1. The Mojette transform with exact data

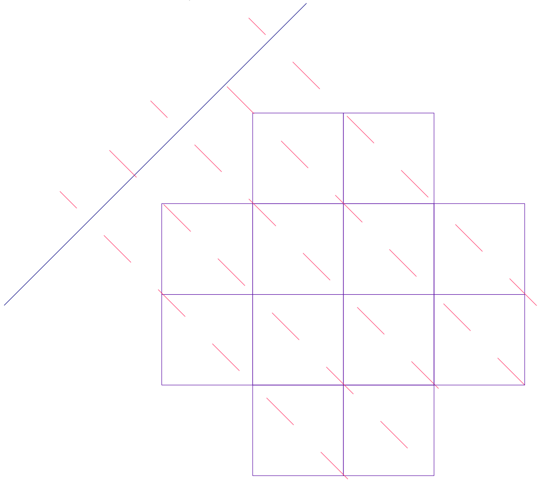
1.1 The Mojette transform definition

Projection direction (p,q)

$(p=1, q=1)$



$(p=-1, q=1)$



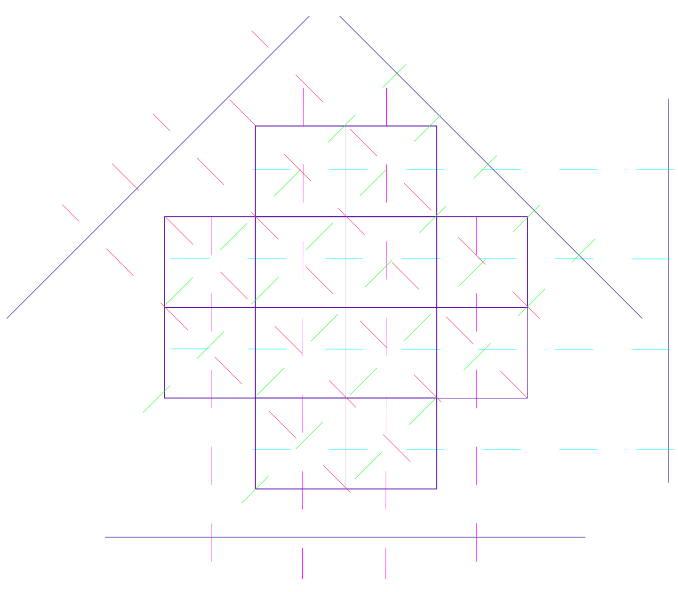
$$proj_f(b,1,1)=\sum_k \sum_l \left(f(k,l)\,\Delta(b+k-l)\right)$$

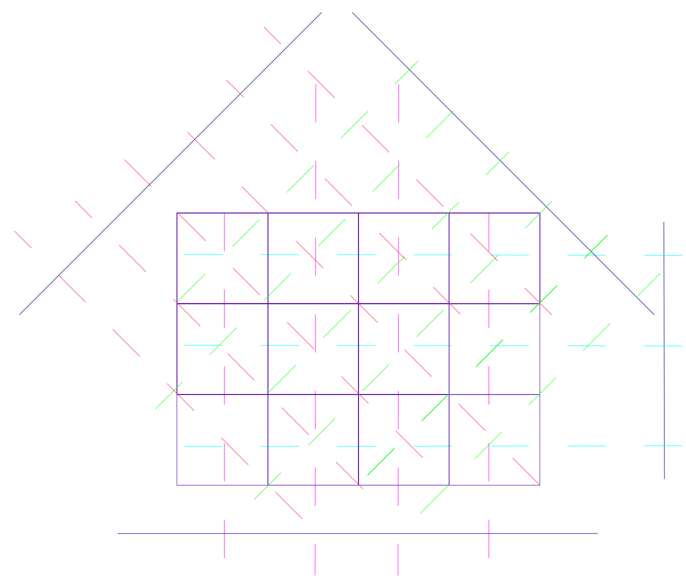
$$proj_f(b,-1,1)=\sum_k \sum_l \left(f(k,l)\,\Delta(b-k-l)\right)$$

1. The Mojette transform with exact data

1.1 The Mojette transform definition

The shape can be rectangular, convex , or not





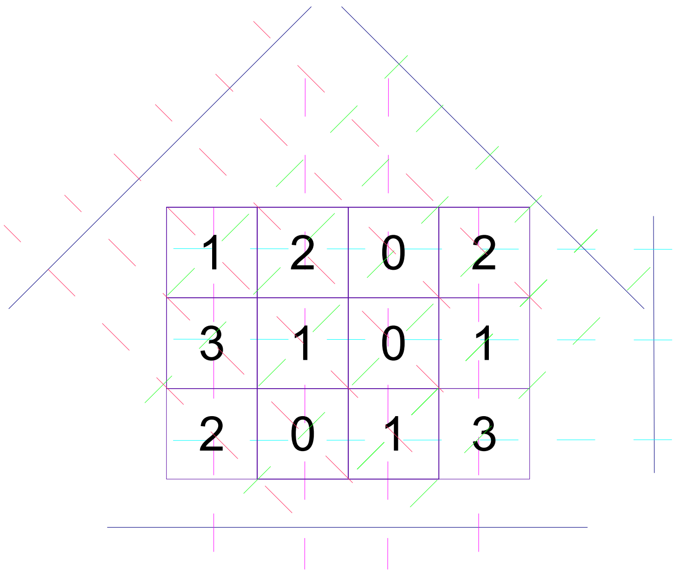
2 figures with 12 pixels
and a same set of projections

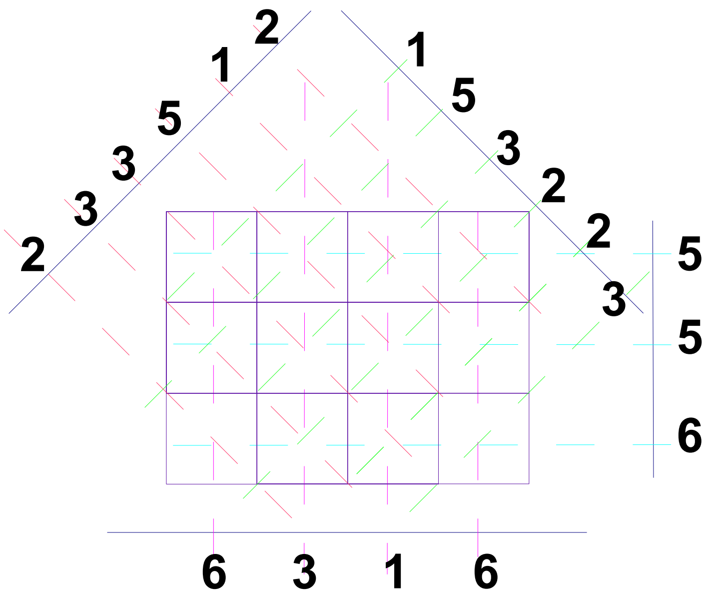
$$S = \{(1\,0),(0\,1),(1\,1),(-1\,1)\}$$

1. The Mojette transform with exact data

1.1 The Mojette transform definition

$$proj_f(b,p,q)=\sum_k \sum_l f(k,l)\Delta(b+qk-pl)$$



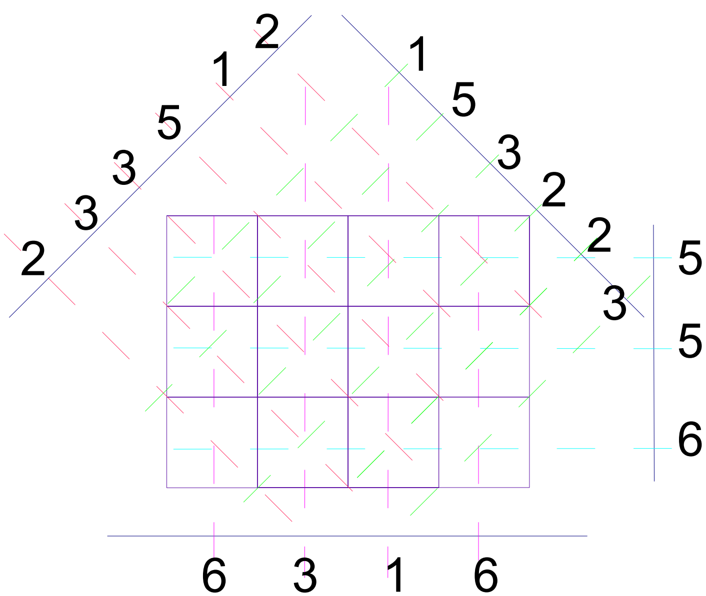


Computing the projections of set $S=\{(1\,0),(0\,1),(1\,1),(-1\,1)\}$

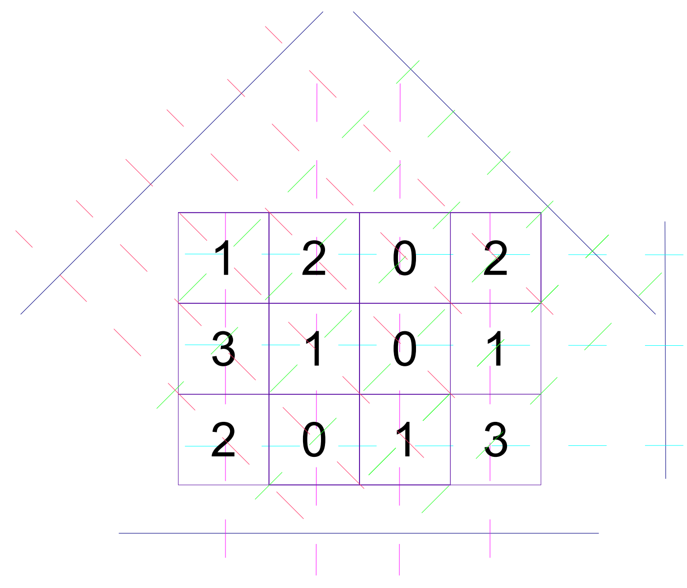
1. The Mojette transform with exact data

1.1 The Mojette transform definition

$$proj_f(b,p,q)=\sum_k \sum_l f(k,l)\,\Delta(b+qk-pl)$$



the projections

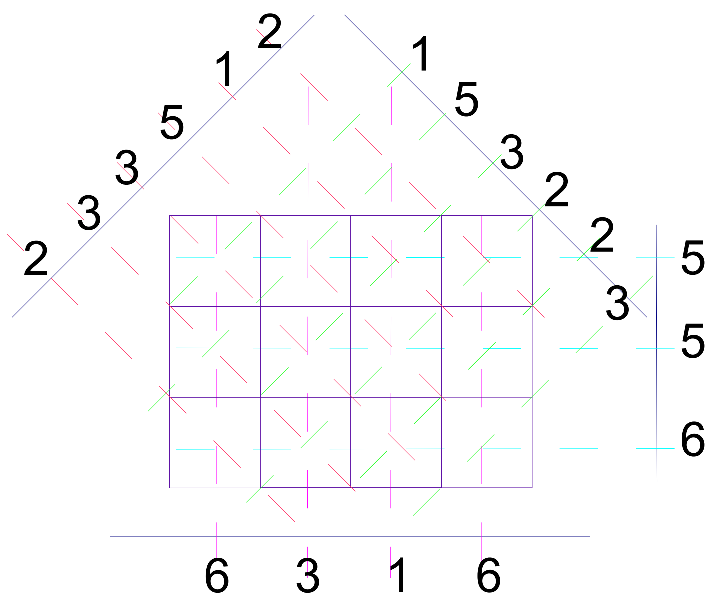


the image

1. The Mojette transform with exact data

1.1 The Mojette transform definition

$$proj_f(b,p,q)=\sum_k \sum_l f(k,l)\,\Delta(b+qk-pl)$$



the projections

Typical Questions :

Can I reconstruct this shape from such a set of projections ?

If yes, which is the shortest algorithm (if no error)?

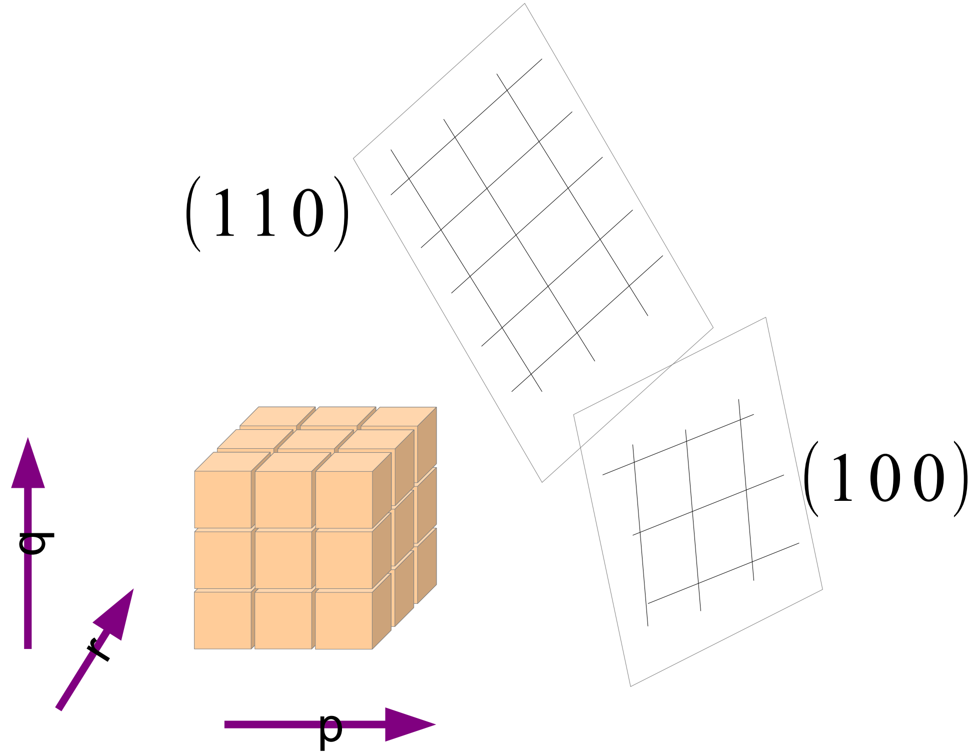
If no, how many solutions match the projections set ?

1. The Mojette transform with exact data

1.1 The Mojette transform definition in 3D and nD

Same definition in 3D and nD

$$proj_f(b_1, b_2, p, q, r) = \sum_k \sum_l \sum_m f(k, l, m) \Delta\left(\mathbf{B} - det\begin{pmatrix} k & l & m \\ p & q & r \end{pmatrix}\right)$$



1. The Mojette transform with exact data

1.1 The Mojette transform definition in 3D and nD

$$proj_f(b, p, q) = \sum_k \sum_l f(k, l)\,\Delta\left(b + \mathbf{P_{21}}\begin{pmatrix} k \\ l \end{pmatrix}\right)$$

$$\mathbf{P_{21}} = \begin{pmatrix} 1 - \frac{q}{p} \end{pmatrix}$$

Same definition in 3D and nD

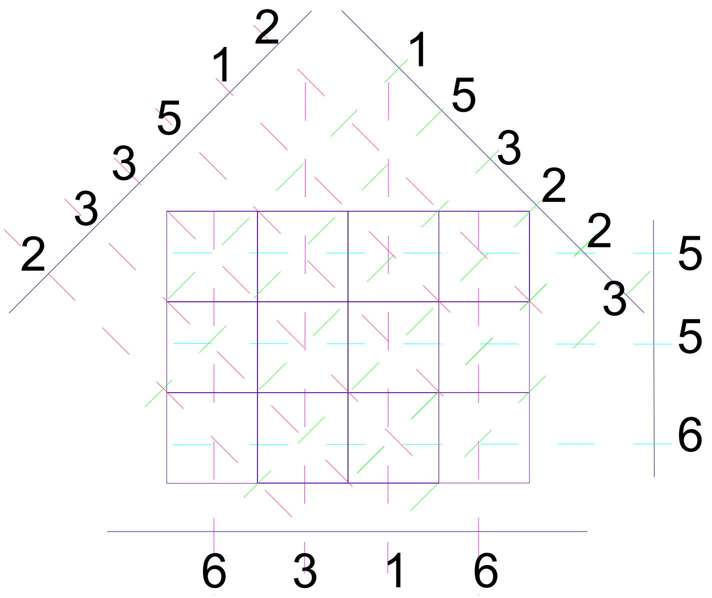
$$proj_f(b_1, b_2, p, q, r) = \sum_k \sum_l \sum_m f(k, l, m)\,\Delta\left(\mathbf{B} + \mathbf{P_{32}}\begin{pmatrix} p \\ q \\ r \end{pmatrix}\right)$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 1\;0 - p/r \\ 0\;1 - q/r \end{pmatrix}$$

1. The Mojette transform with exact data

1.2 Reconstruction theorem

$$proj_f(b,p,q) = \sum_k \sum_l f(k,l)\,\Delta(b+qk-pl)$$



the projections

Questions :

Can I reconstruct this shape from such a set of projections ?

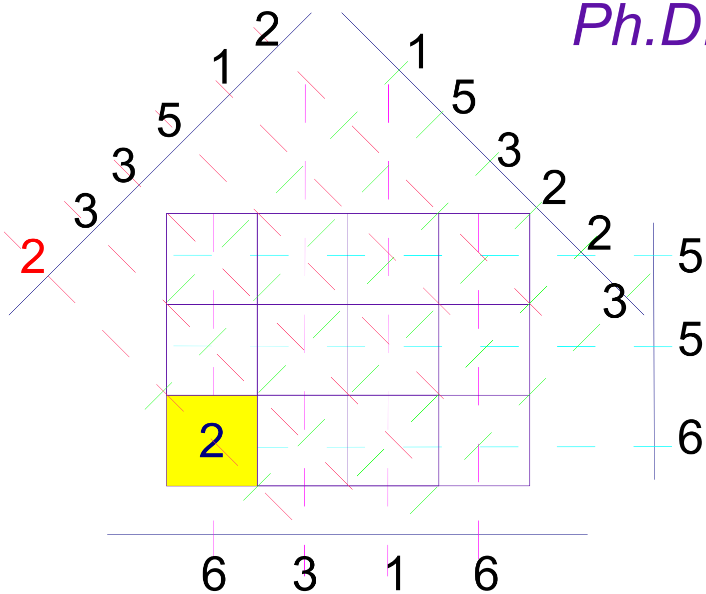
If yes, which is the shortest algorithm ?

1. The Mojette transform with exact data

1.2 Reconstruction theorem

$$proj_f(b,p,q)=\sum_k \sum_l f(k,l)\,\Delta(b+qk-pl)$$

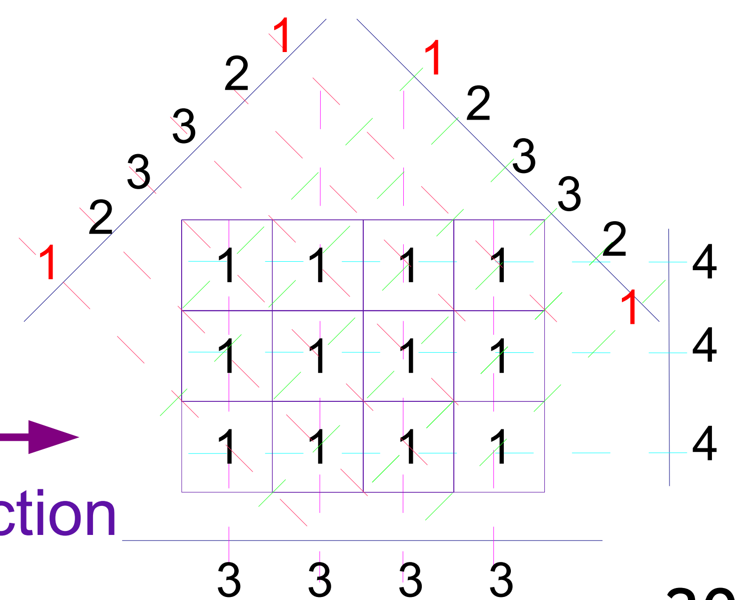
Ph.D. Nicolas Normand 1997



Some pixels can be « backprojected » from their bin value since 1-1 relationship

the reconstruction process

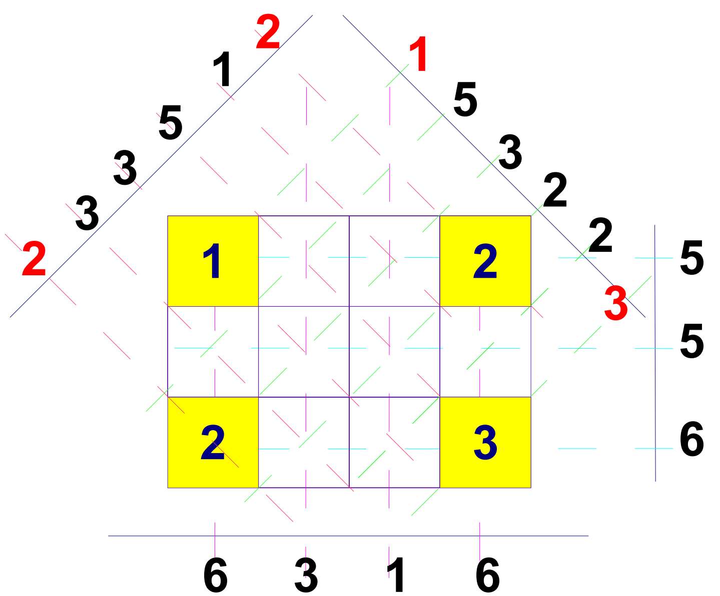
the characteristic shape function



1. The Mojette transform with exact data

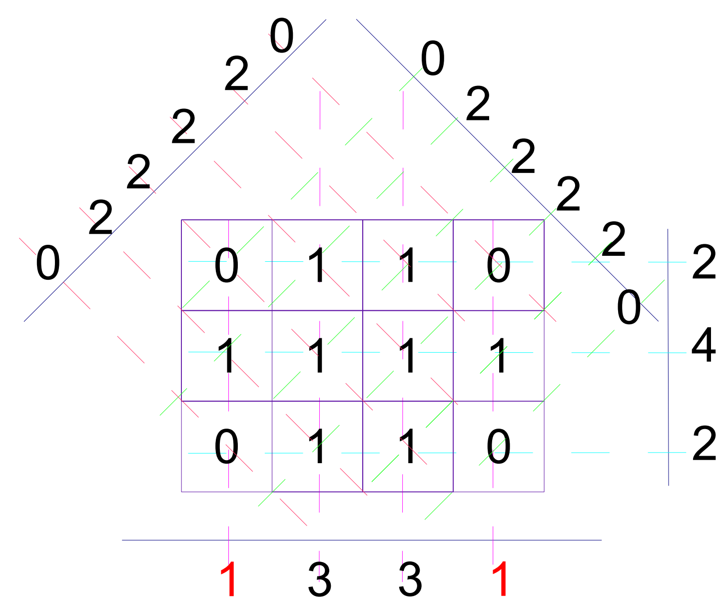
1.2 Reconstruction theorem

Ph.D. Nicolas Normand 1997



the reconstruction process

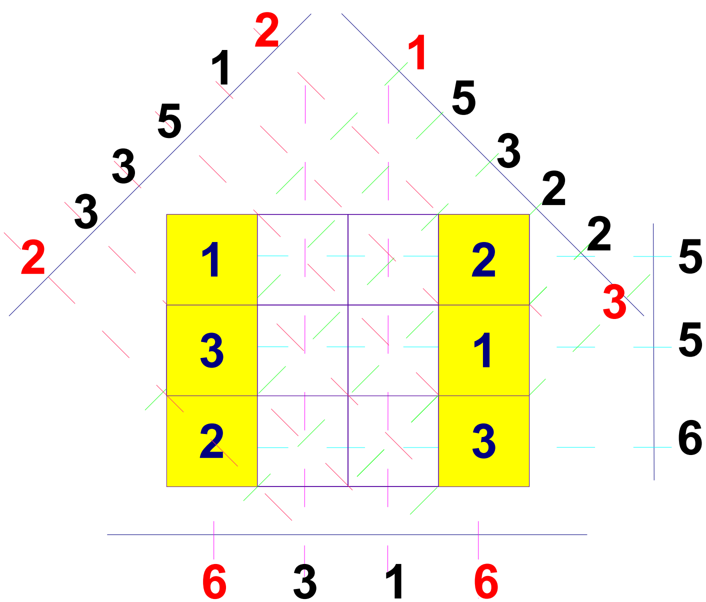
When this is done, new pixels become in 1-1 relationships



1. The Mojette transform with exact data

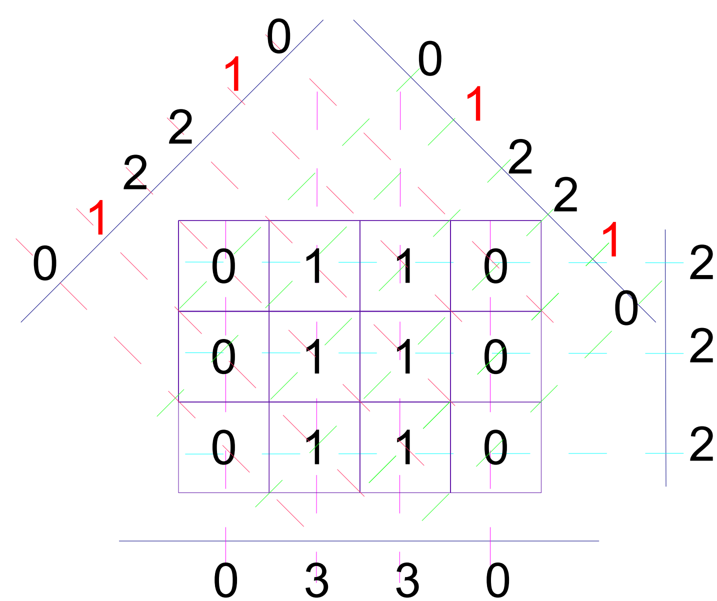
1.2 Reconstruction theorem

Ph.D. Nicolas Normand 1997



the reconstruction process

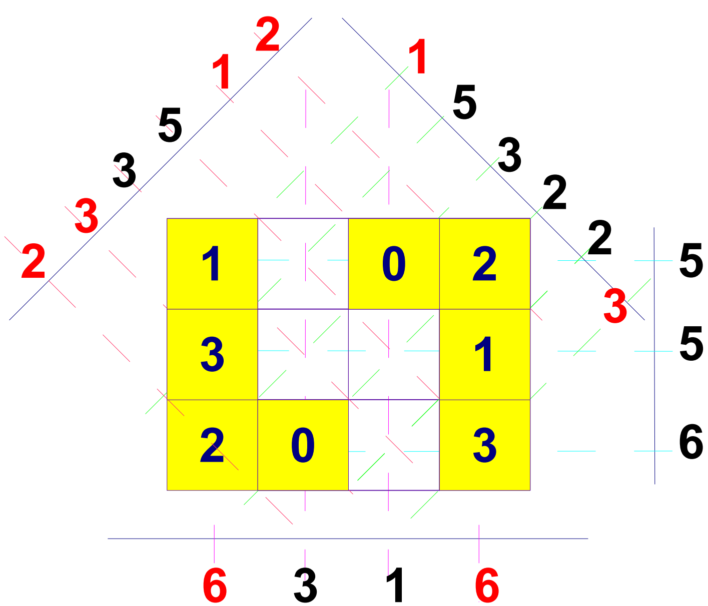
new pixels become in 1-1 relationships



1. The Mojette transform with exact data

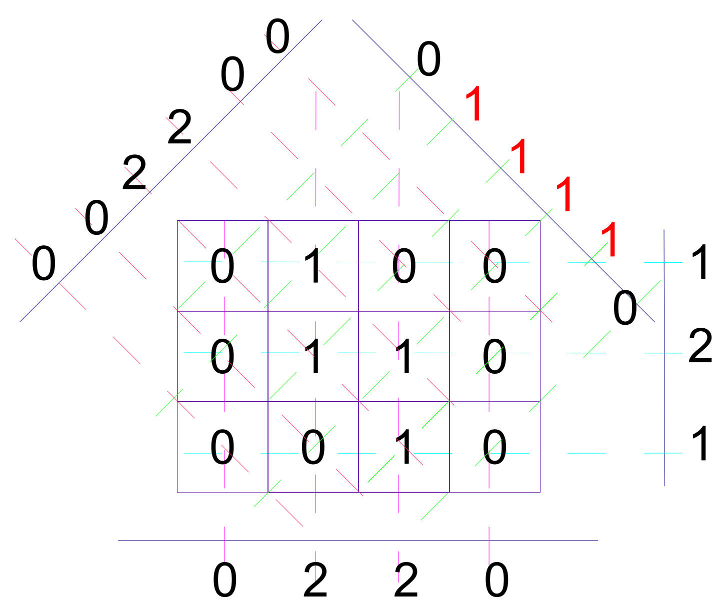
1.2 Reconstruction theorem

Ph.D. Nicolas Normand 1997



the reconstruction process

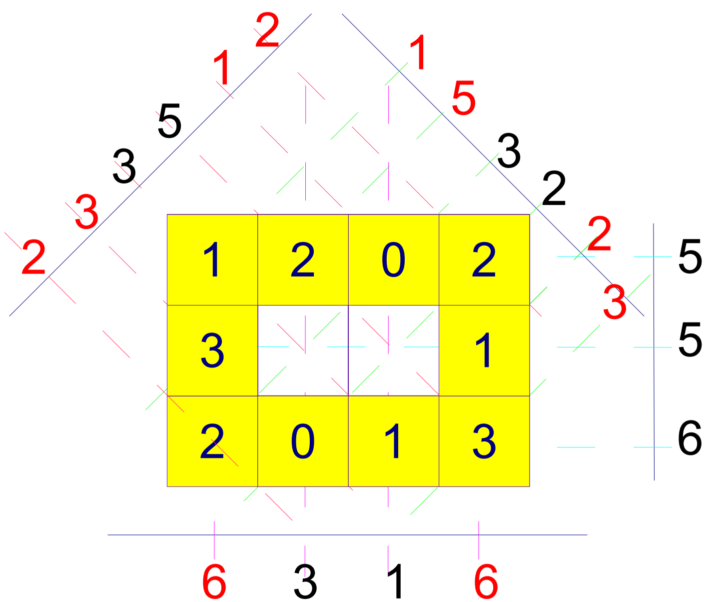
and again ...



1. The Mojette transform with exact data

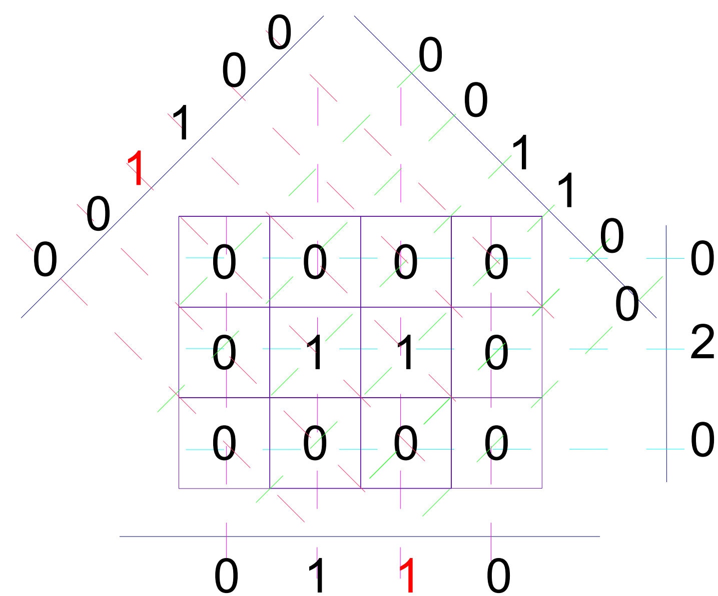
1.2 Reconstruction theorem

Ph.D. Nicolas Normand 1997



the reconstruction process

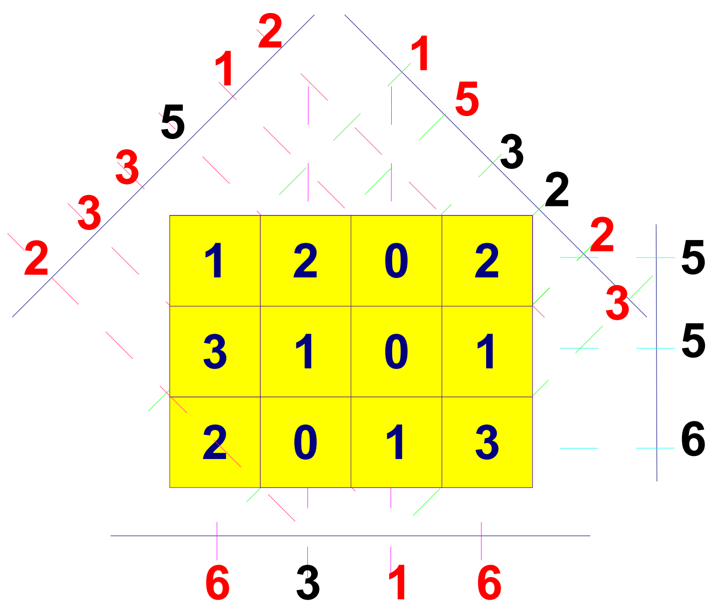
and again ...



1. The Mojette transform with exact data

1.2 Reconstruction theorem

Ph.D. Nicolas Normand 1997



We complete the reconstruction

One projection was not used

12 bins were used
(= number of pixels)

We could have used other bins

the reconstruction process has linear
complexity in # pixels & # projections

1. The Mojette transform with exact data

1.2 Reconstruction theorem

For a rectangular shape, Myron KATZ (78) proved:

For a rectangular shape of size (PxQ) , and a set of projections

$$S = \{ (p_i, q_i), i = 1 \ldots N \}$$

Reconstruc-ability of shape PxQ from set S
<=>

$$\left(\sum_{i=1}^{N} |p_i| \geqslant P\right) \text{or} \left(\sum_{i=1}^{N} |q_i| \geqslant Q\right)$$

1. The Mojette transform with exact data

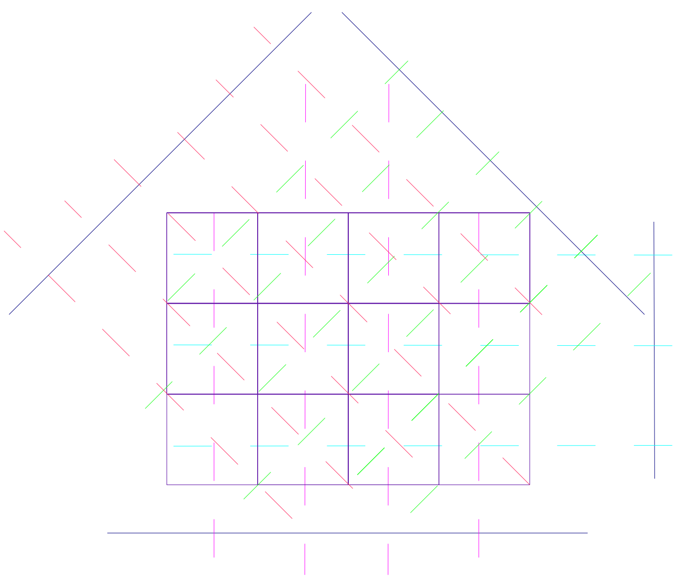
1.2 Reconstruction theorem

rectangular shape P=4 Q=3

Reconstruc-ability of shape PxQ
from $S=\{(1\,0),(0\,1),(1\,1),(-1\,1)\}$

<=>

$$\left(3=\sum_{i=1}^{N}|p_i|\geqslant P=4\right)\text{or}\left(3=\sum_{i=1}^{N}|q_i|\geqslant Q=3\right)$$



Explain why : $S_1=\{(0\,1),(1\,1),(-1\,1)\}$
is a sufficient set to reconstruct

1. The Mojette transform with exact data

1.2 Reconstruction theorem

For a convex shape ?

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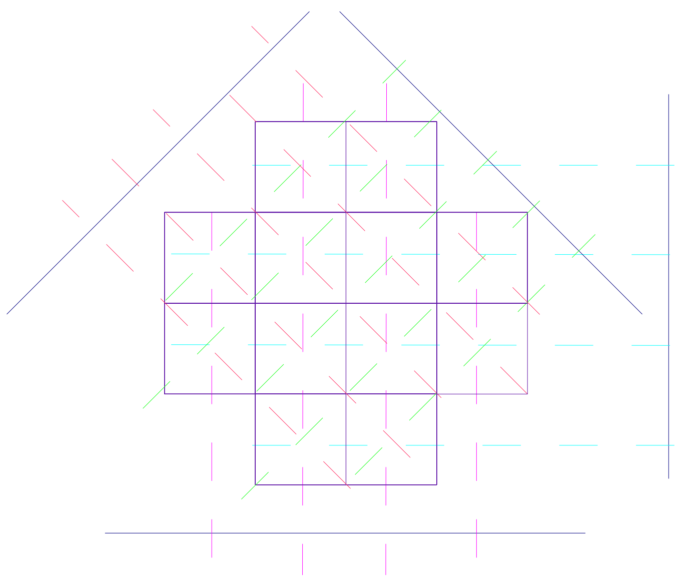
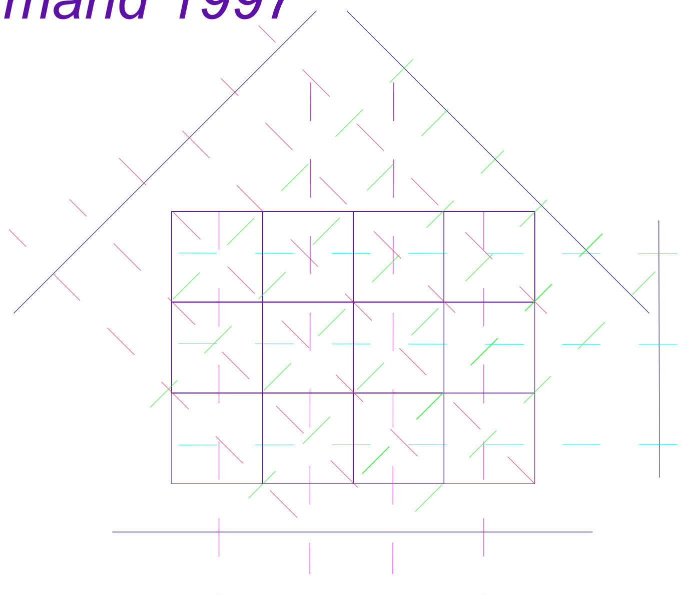




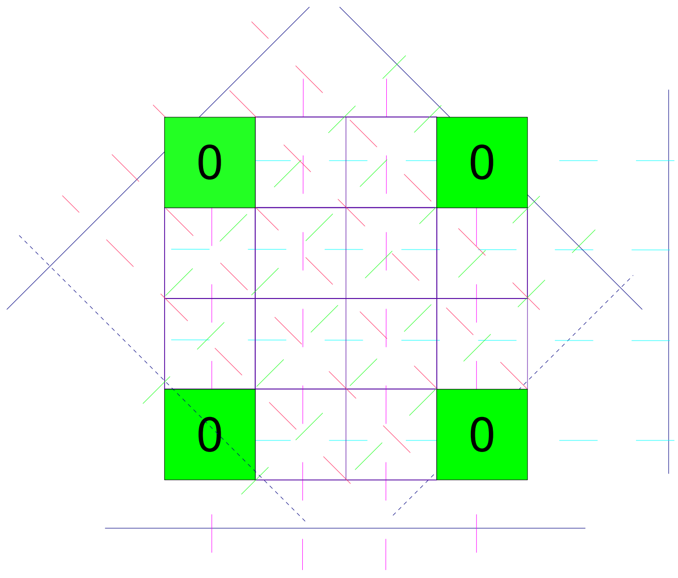
2 figures with 12 pixels
and a same set of projections

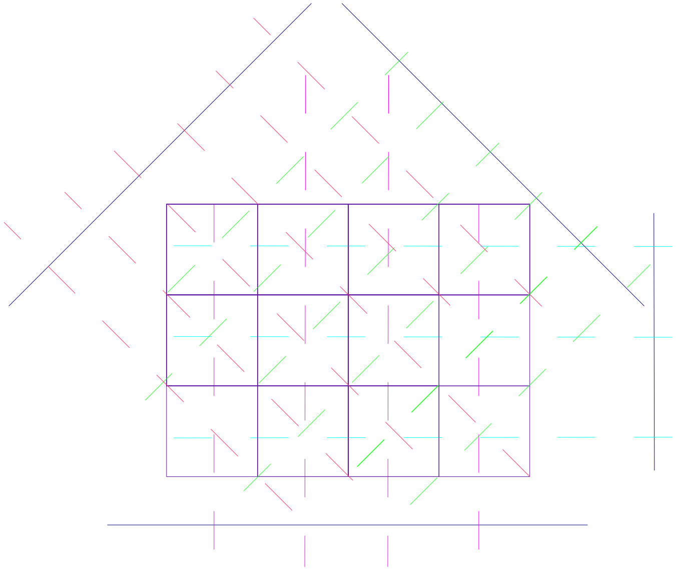
$$S = \{(1\,0),(0\,1),(1\,1),(-1\,1)\}$$

1. The Mojette transform with exact data

1.2 Reconstruction theorem

For a convex shape ? Non optimal solution : insert it into a rectangular shape with 0 values





$$S = \{(1\,0),(0\,1),(1\,1),(-1\,1)\}$$

Does not change the projection values (change the number of bins)

1. The Mojette transform with exact data

1.2 Reconstruction theorem

For a convex shape ?

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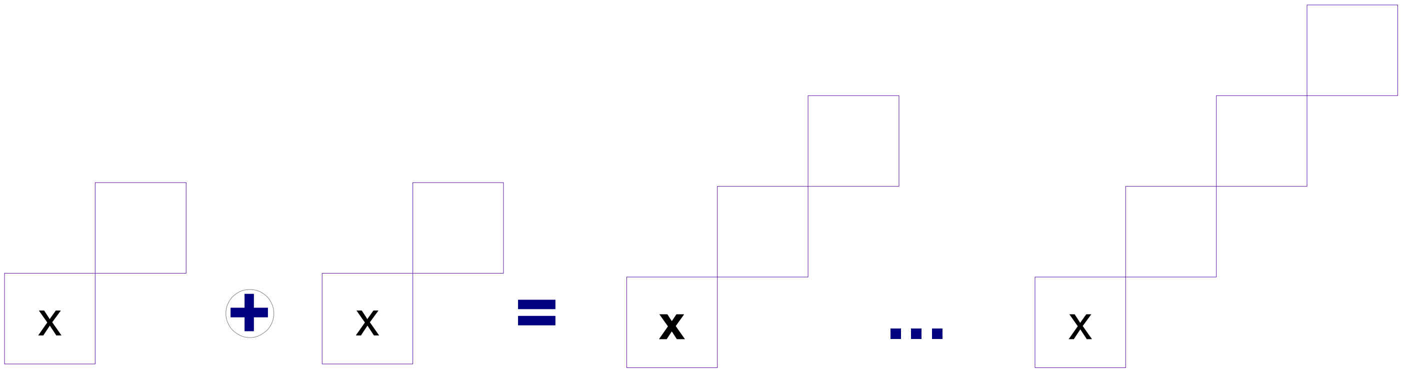
Mathematical morphology with 2 Pixels Structuring Element (2PSE)

A projection direction <=> A 2PSE : {(0 0) ,(p,q)}

2PSE : eigenvector of dilation



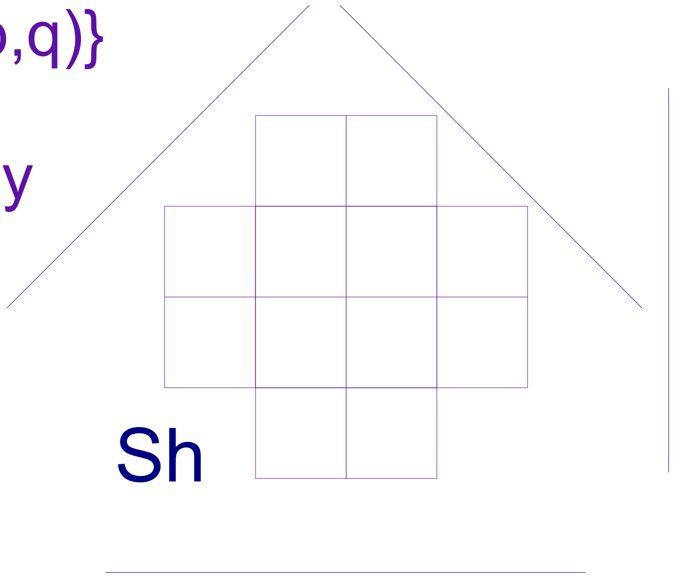
1. The Mojette transform with exact data

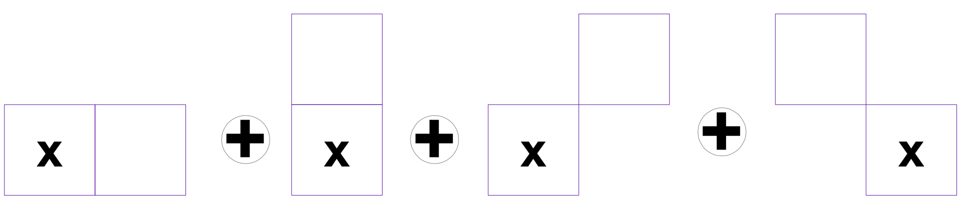
1.2 Reconstruction theorem

A projection direction <=>a 2PSE {(0 0) ,(p,q)}

A set of projections <=> shape obtained by
the series of
2PSE dilations

(Normand-Guédon 97)



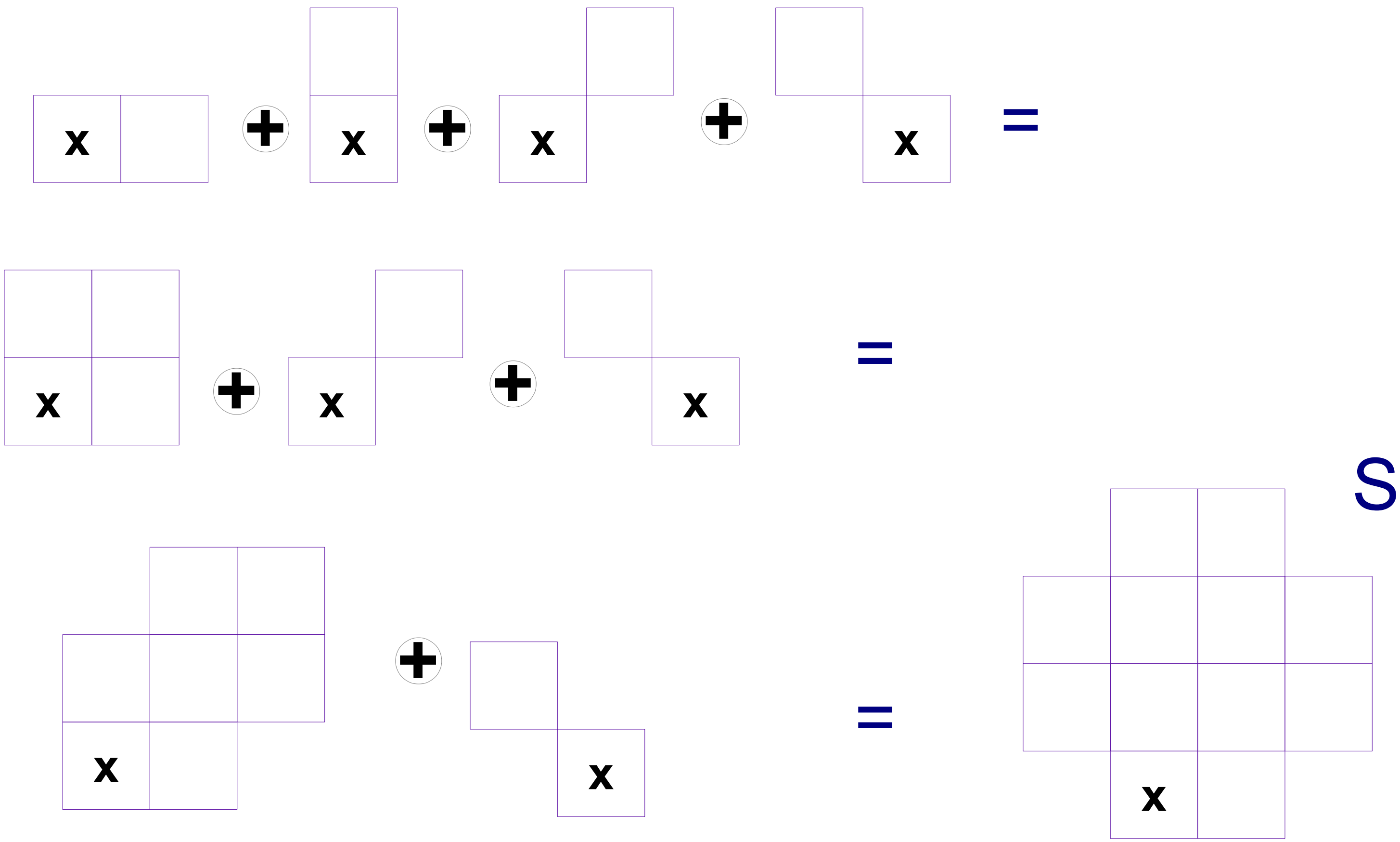


$$S = \{(1\,0),(0\,1),(1\,1),(-1\,1)\}$$

1. The Mojette transform with exact data

1.2 Reconstruction theorem

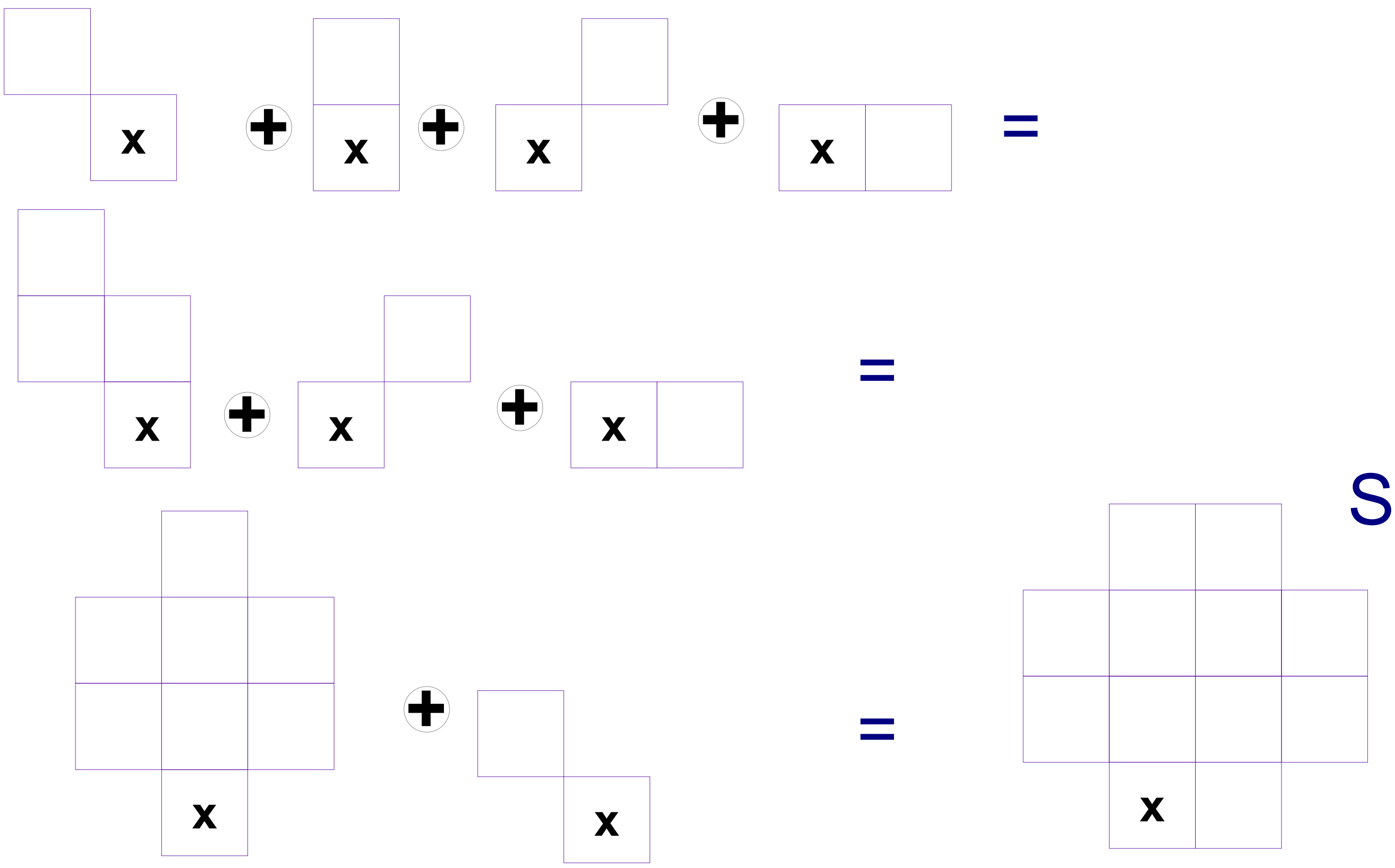
A set of projections S corresponds to the shape obtained by the series of dilations of 2PSE



1. The Mojette transform with exact data

1.2 Reconstruction theorem

the series of dilations of 2PSE is commutative, associative has a neutral element (p=q=0)

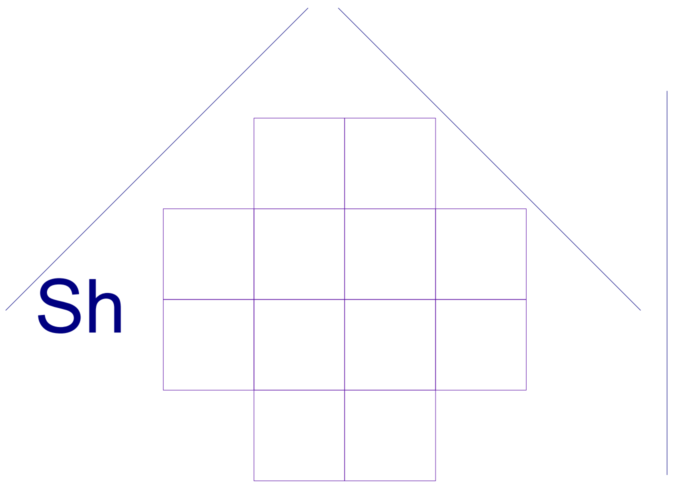


1. The Mojette transform with exact data

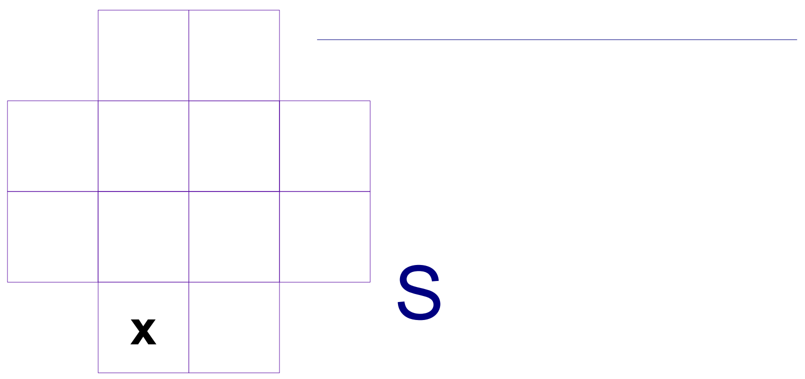
1.2 Reconstruction theorem

For a convex shape Sh , and a set of projections S={(pi, qi), i=1...N}

Reconstruc-ability of shape Sh from set S
<=>
The shape S generated by projections set S
Can not be inserted into Sh



In the example, S just identically fit Sh
=> the reconstruction is not possible



1. The Mojette transform with exact data

1.2 Reconstruction theorem

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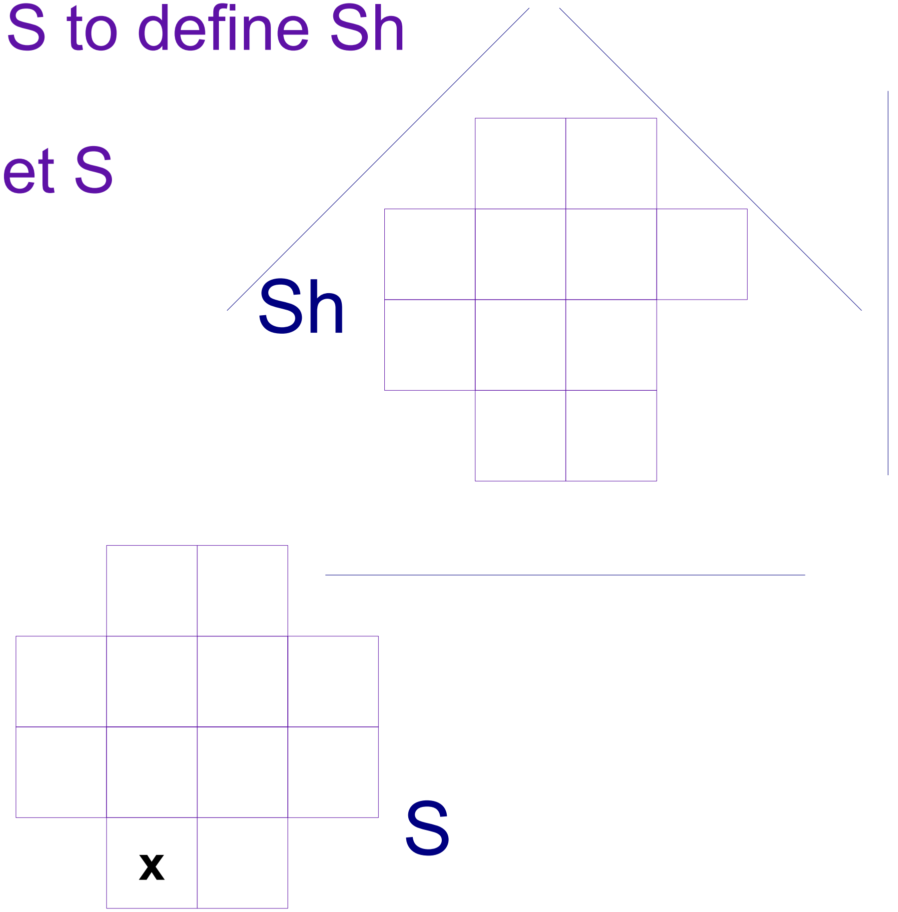
Corollary
When building a shape S by 2PSE dilation series of S
And suppressing ANY border pixel of S to define Sh
Allows for
Reconstruc-ability of shape Sh from set S

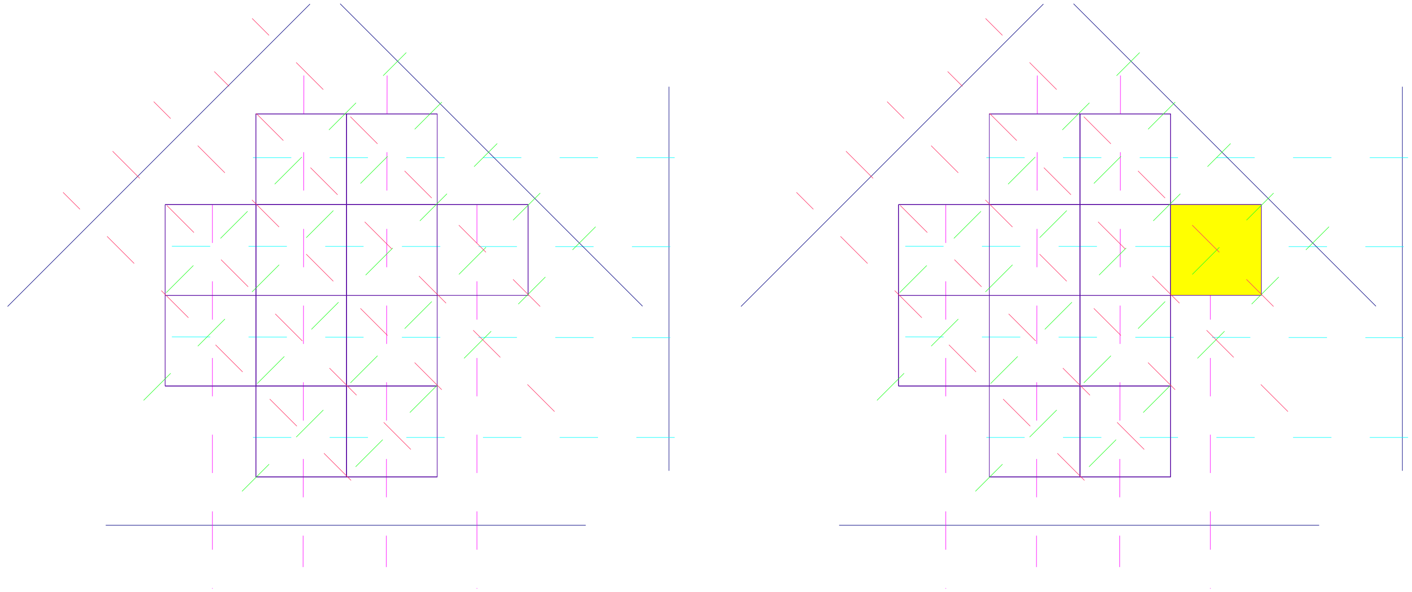


In the example, S can not fit Sh
=> the reconstruction is possible

1. The Mojette transform with exact data

1.2 Reconstruction theorem

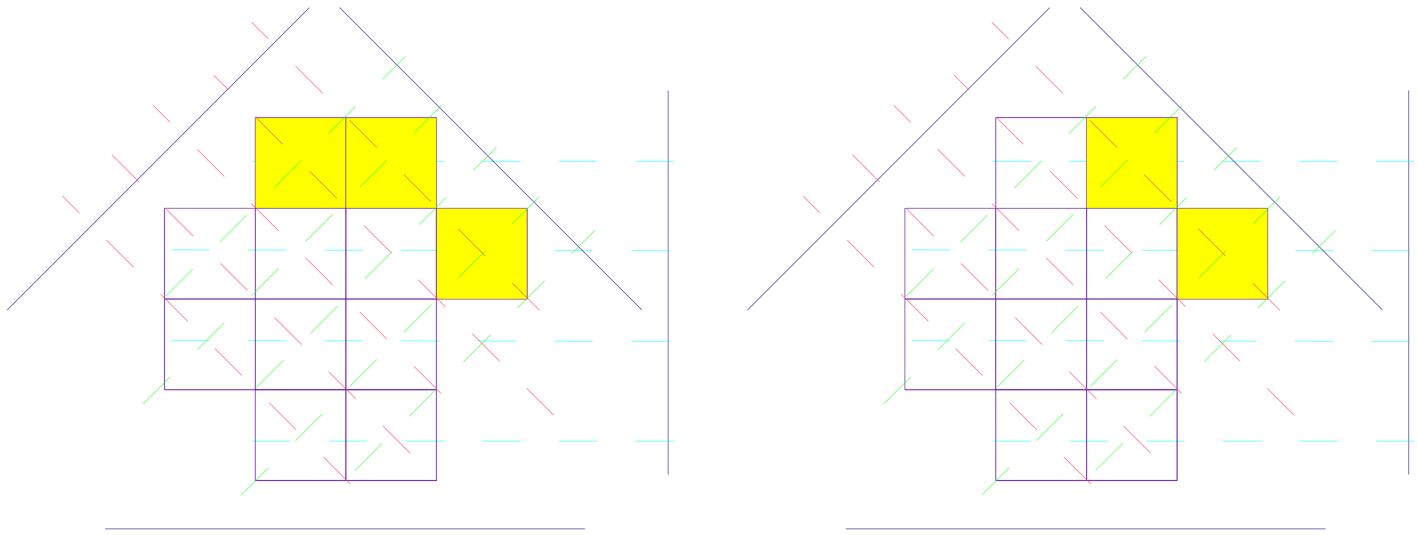
Checking ...



1. The Mojette transform with exact data

1.2 Reconstruction theorem

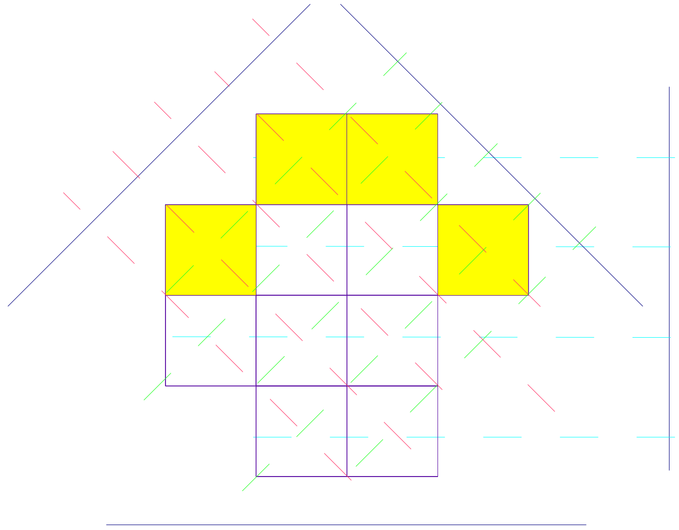
Checking ...

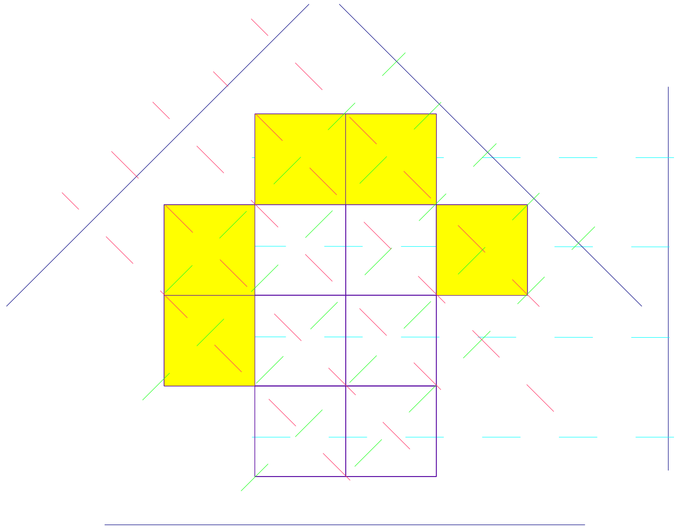


1. The Mojette transform with exact data

1.2 Reconstruction theorem

Checking ...

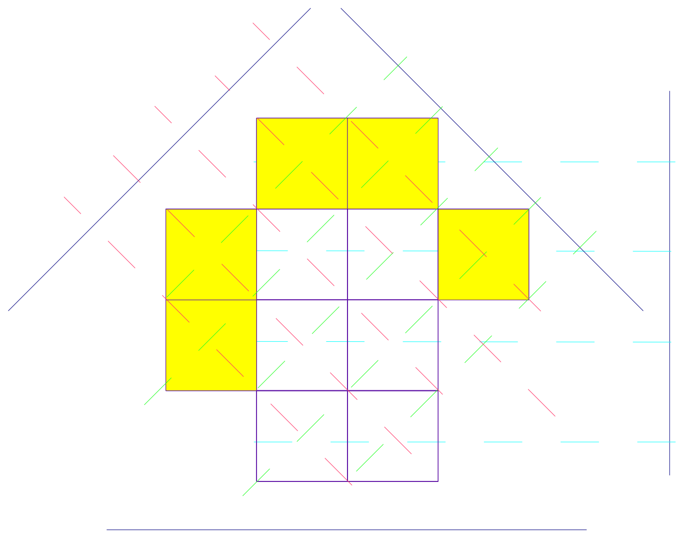


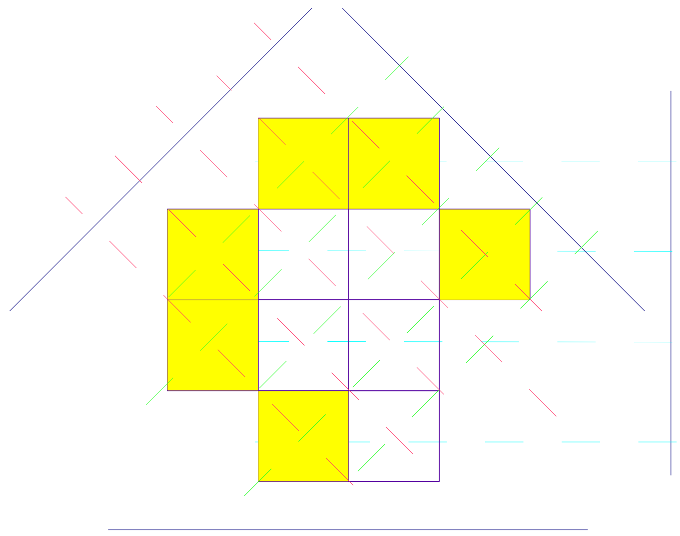


1. The Mojette transform with exact data

1.2 Reconstruction theorem

Checking ...

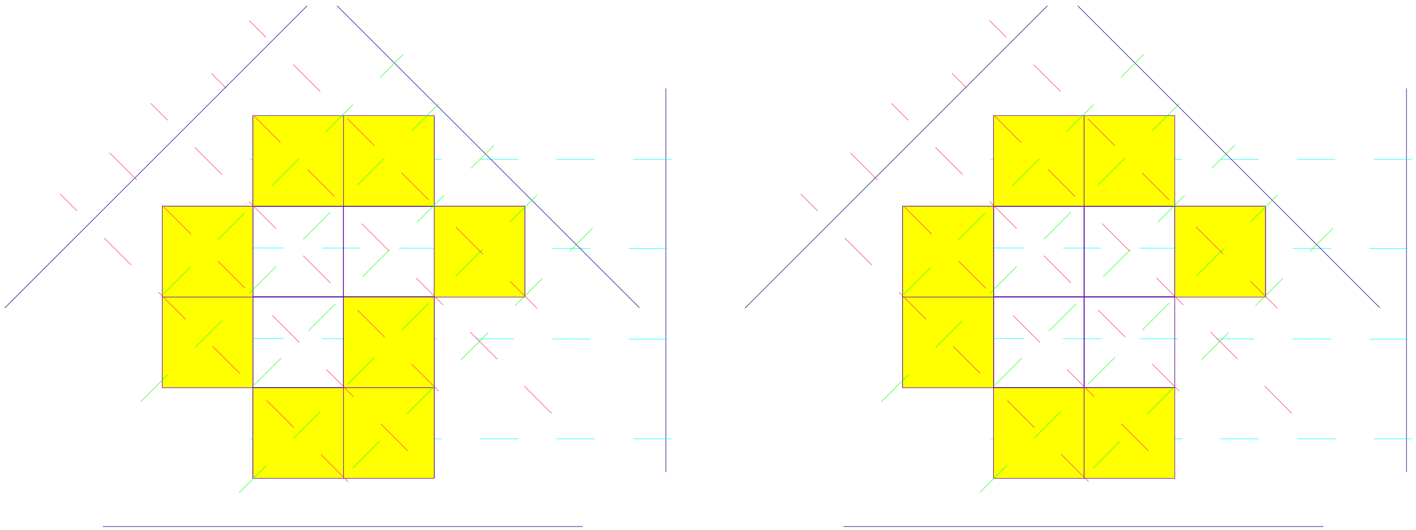




1. The Mojette transform with exact data

1.2 Reconstruction theorem

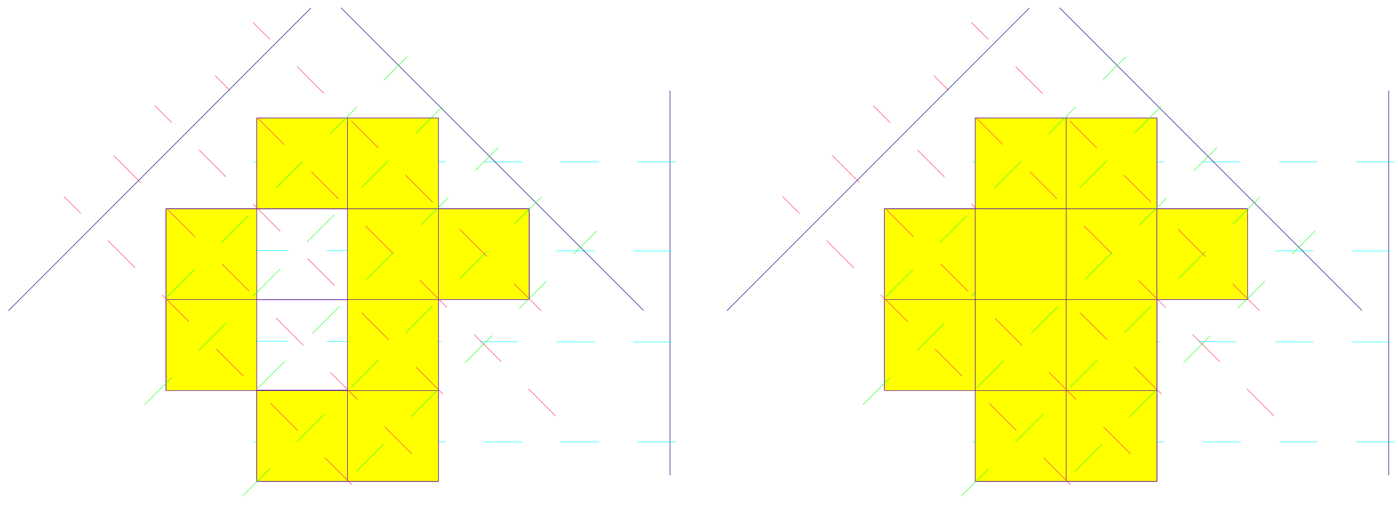
Checking ...



1. The Mojette transform with exact data

1.2 Reconstruction theorem

Checking ...



1. The Mojette transform with exact data

1.2 Reconstruction theorem

Ph.D. Nicolas Normand 1997

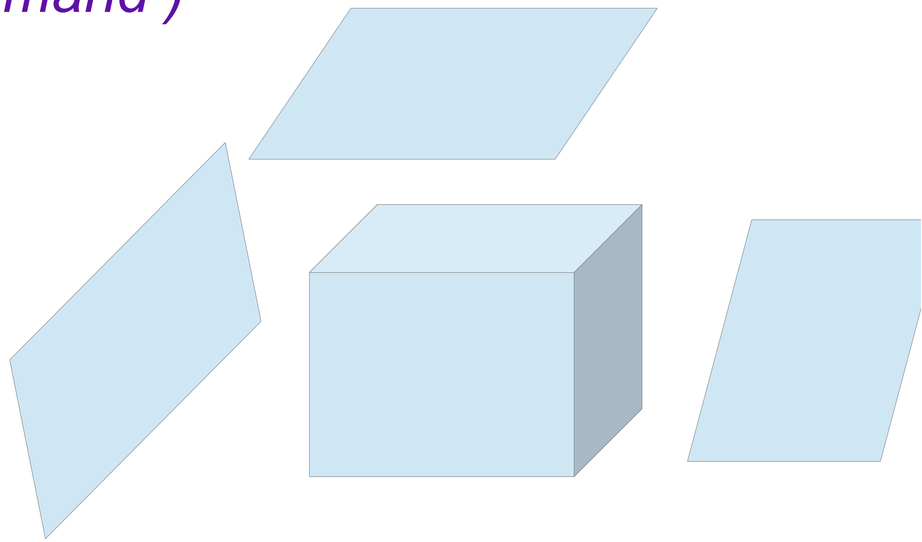
For a convex shape Sh , and a set of projections S={(pi, qi), i=1...N}

Reconstruc-ability of shape Sh from set S with Normand's algorithm
<=>
The shape S , generated by the series of dilations using the 2PSE corresponding to projections set S, can not be inserted into Sh

1. The Mojette transform with exact data

1.2 Reconstruction theorem in 3D and nD

Ph.D. Pierre Verbert 2004 (with Nicolas Normand)



For a 3D convex shape Sh ,
and a set of projections
S={(pi, qi, ri), i=1...N}

Reconstruc-ability of shape Sh from set S with Normand's algorithm
<=>
The shape S , generated by the series of dilations using the 2PSE corresponding to projections set S, can not be inserted into Sh

1. The Mojette transform with exact data

1.3 Null space and phantoms

What is a phantom (or ghost or switching component) ?

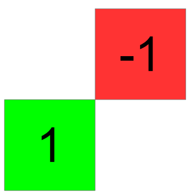
How to build a phantom ?

What is the use of a phantom ?

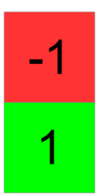
1. The Mojette transform with exact data

1.3 Null space and phantoms

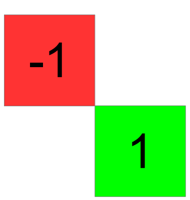
Definition of a phantom : object added into the image and not seen onto the projection



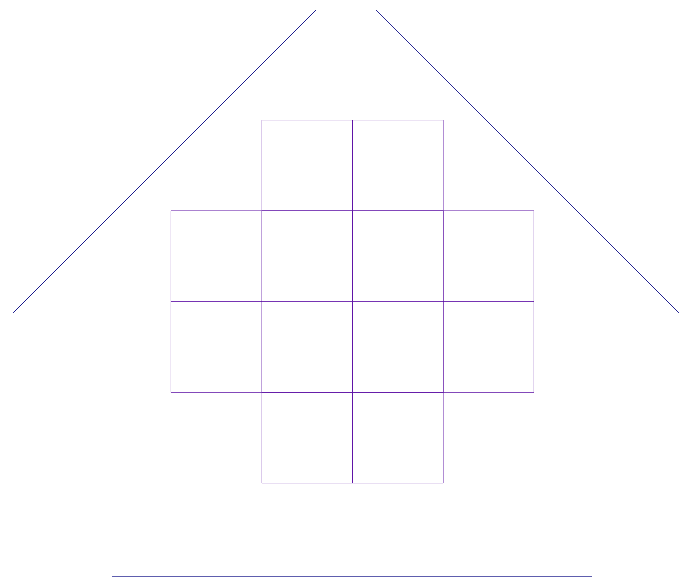
direction (1 1)



direction (0 1)



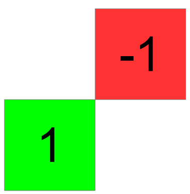
direction (-1 1)



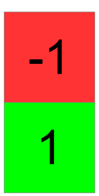
1. The Mojette transform with exact data

1.3 Null space and phantoms

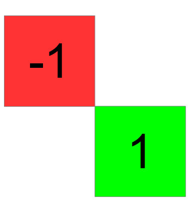
Definition of a phantom : object added into the image and not seen onto the projection



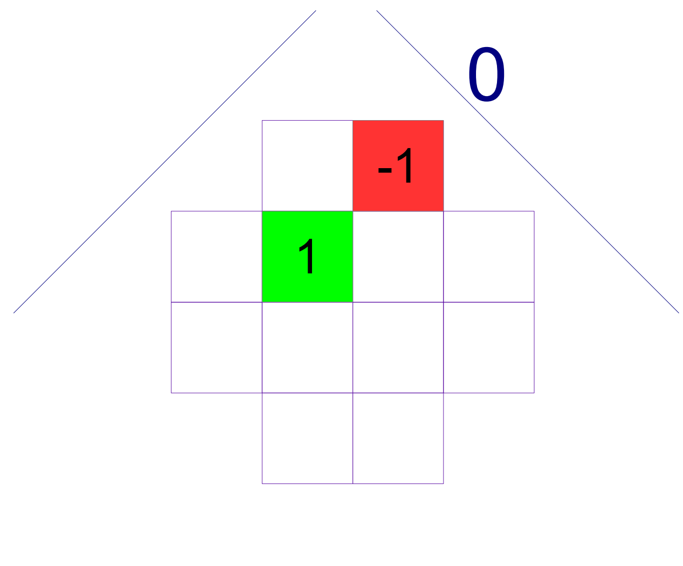
direction (1 1)



direction (0 1)



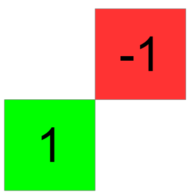
direction (-1 1)



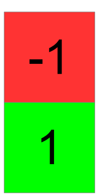
1. The Mojette transform with exact data

1.3 Null space and phantoms

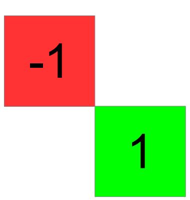
Definition of a phantom : object added into the image and not seen onto the projection



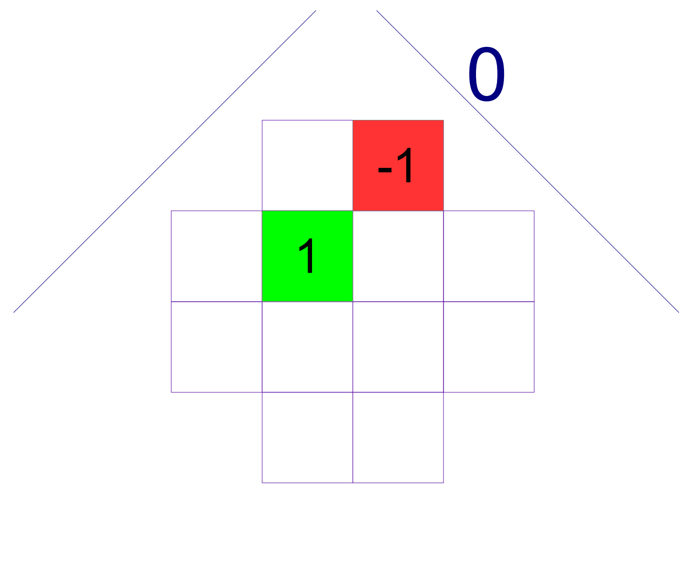
direction (1 1)



direction (0 1)



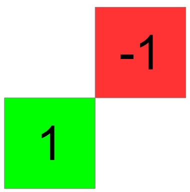
direction (-1 1)



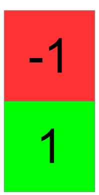
1. The Mojette transform with exact data

1.3 Null space and phantoms

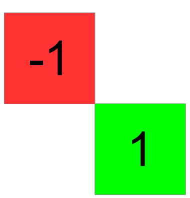
Definition of a phantom : object added into the image and not seen onto the projection



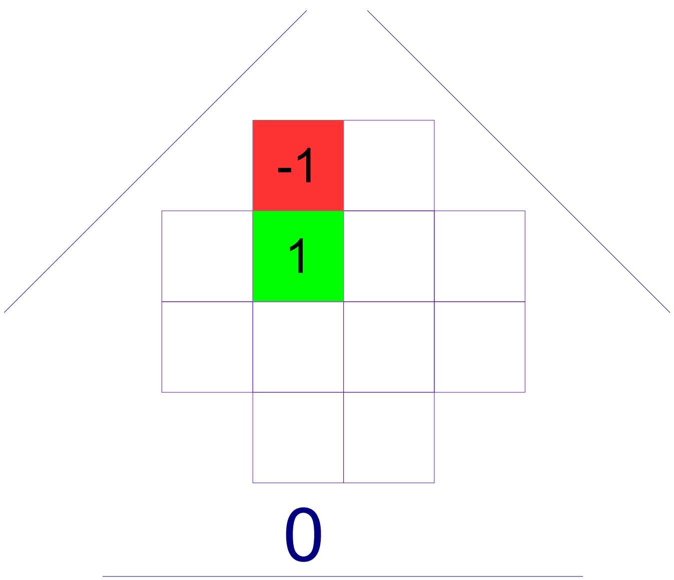
direction (1 1)



direction (0 1)



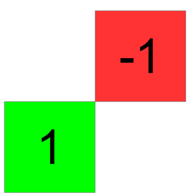
direction (-1 1)



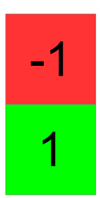
1. The Mojette transform with exact data

1.3 Null space and phantoms

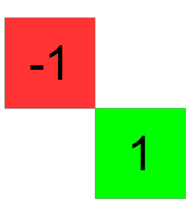
Definition of a phantom : object added into the image and not seen onto the projection



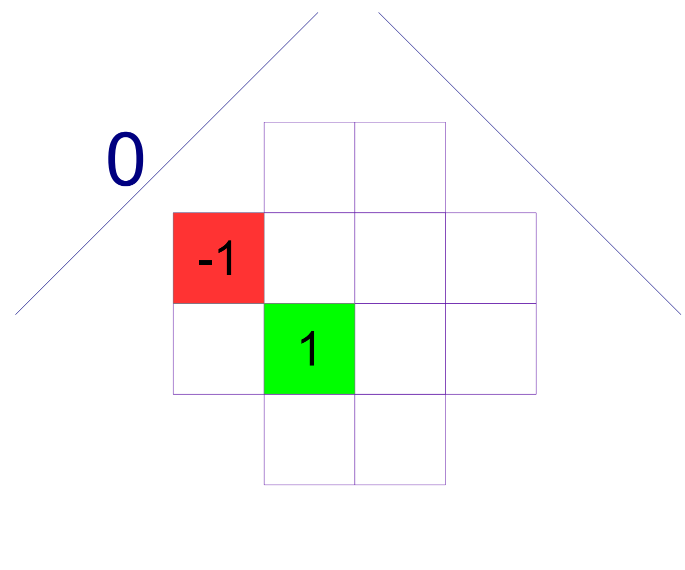
direction (1 1)



direction (0 1)



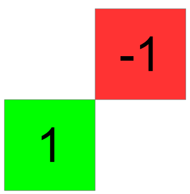
direction (-1 1)



1. The Mojette transform with exact data

1.3 Null space and phantoms

Definition of a phantom : object added into the image and not seen onto the projection



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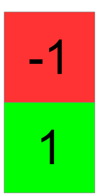
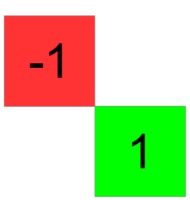
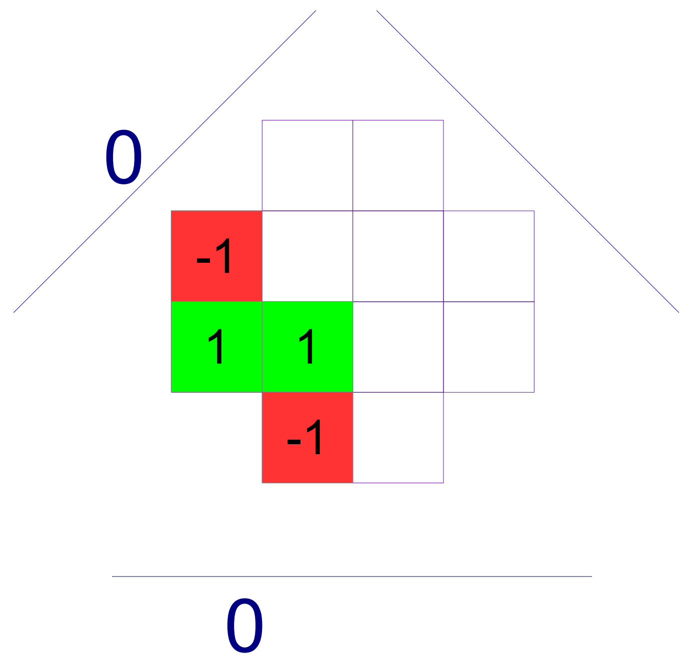


direction (0 1)



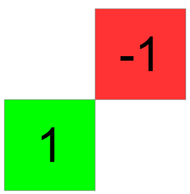
direction (-1 1)



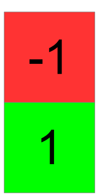
1. The Mojette transform with exact data

1.3 Null space and phantoms

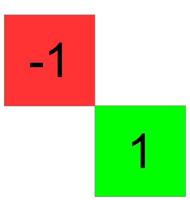
Definition of a phantom : object added into the image and not seen onto the projection



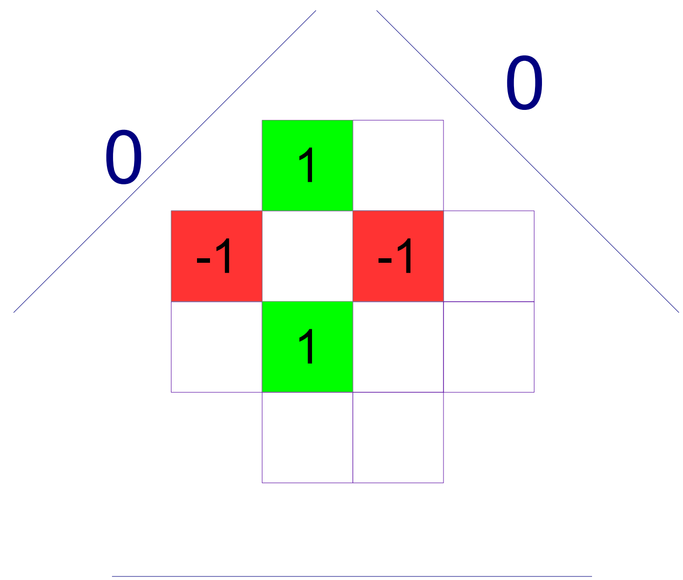
direction (1 1)



direction (0 1)



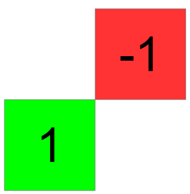
direction (-1 1)



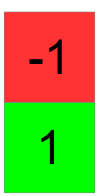
1. The Mojette transform with exact data

1.3 Null space and phantoms

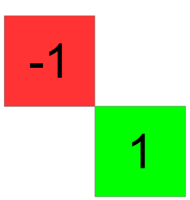
Definition of a phantom : object added into the image and not seen onto the projection



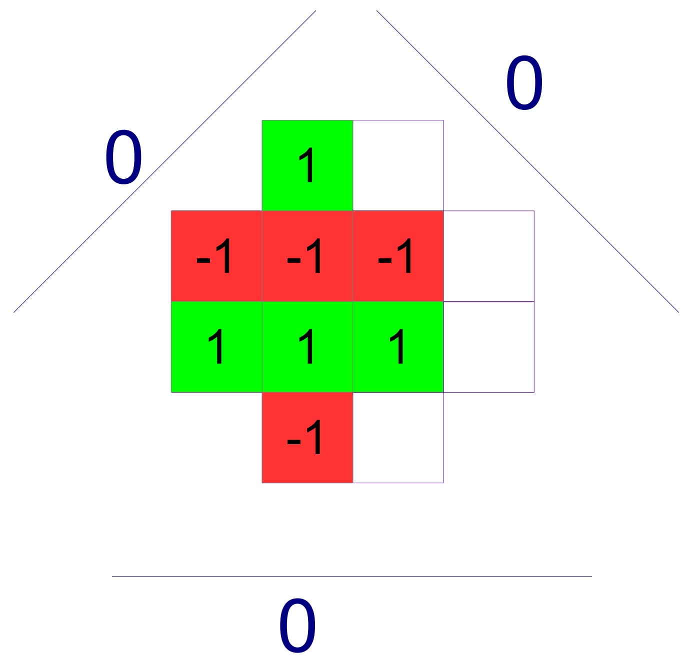
direction (1 1)



direction (0 1)



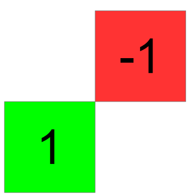
direction (-1 1)



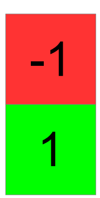
1. The Mojette transform with exact data

1.3 Null space and phantoms

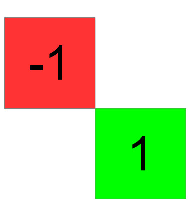
Definition of a phantom : object added into the image and not seen onto the projection



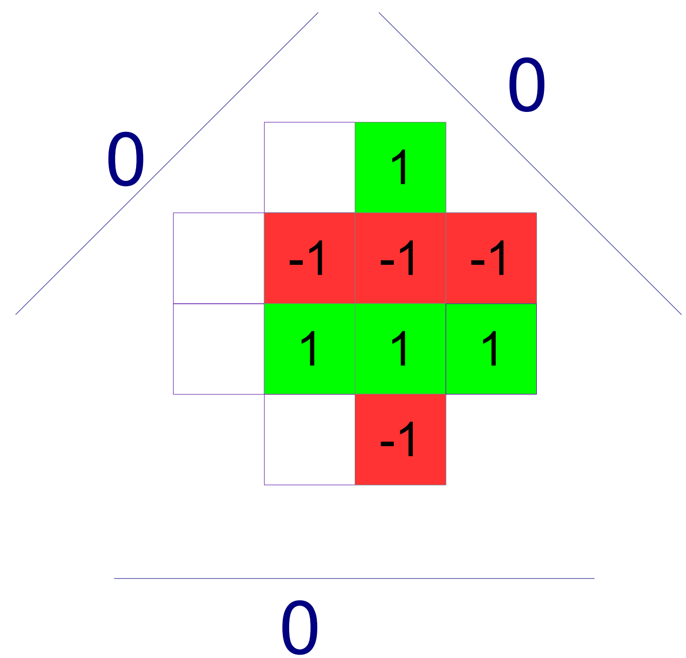
direction (1 1)



direction (0 1)



direction (-1 1)



1. The Mojette transform with exact data

1.3 Null space and fantoms

building a phantom : by simple convolution

0	1	0
-1	0	-1
0	1	0

S2={(1 1)(-1 1)}

0	0	1
-1	0	0

S1={(2 1)}

0	1	0
-1	0	-1
0	1	0

*

0	0	1
-1	0	0

=

			-1	
	1	1	0	1
-1	0	-1	-1	
	1			

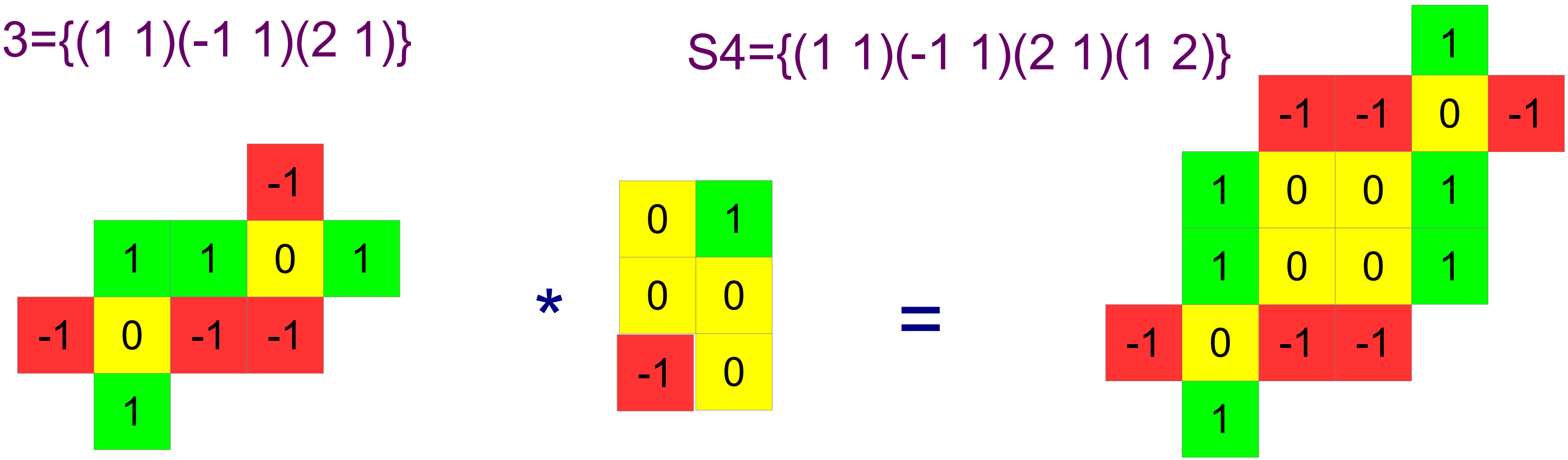
S3={(1 1)(-1 1)(2 1)}

1. The Mojette transform with exact data

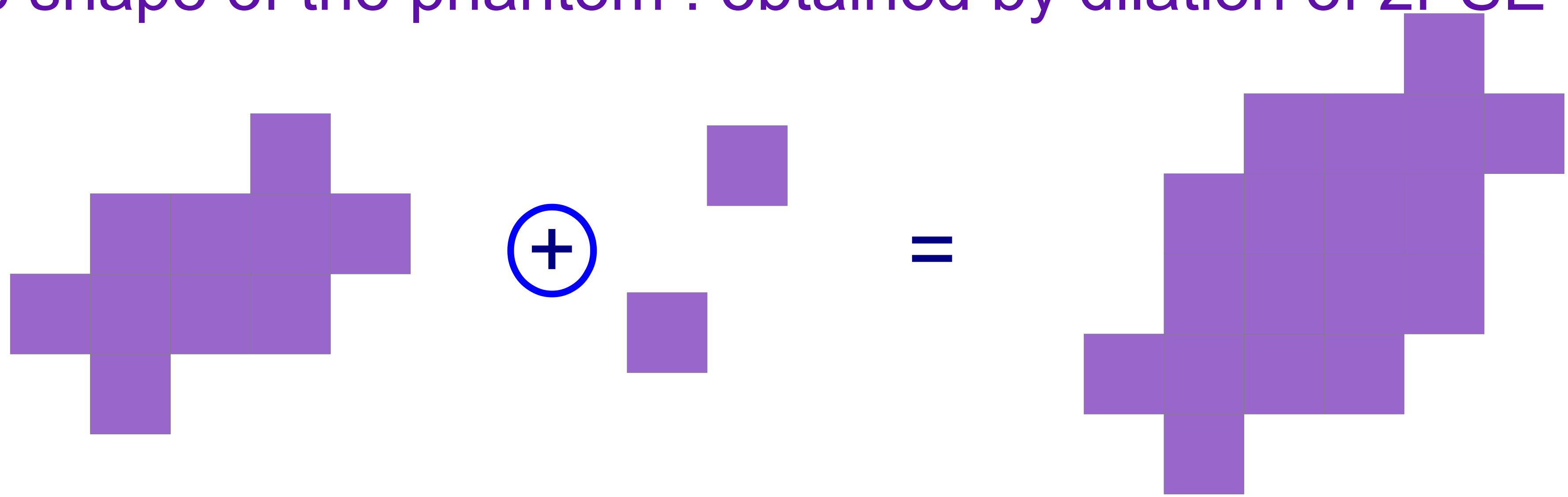
1.3 Null space and fantoms

S3={(1 1)(-1 1)(2 1)}

S4={(1 1)(-1 1)(2 1)(1 2)}



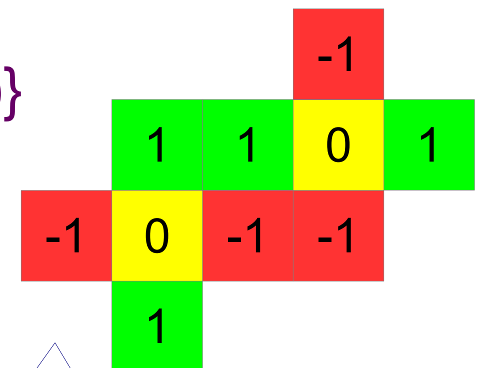
The shape of the phantom : obtained by dilation of 2PSE

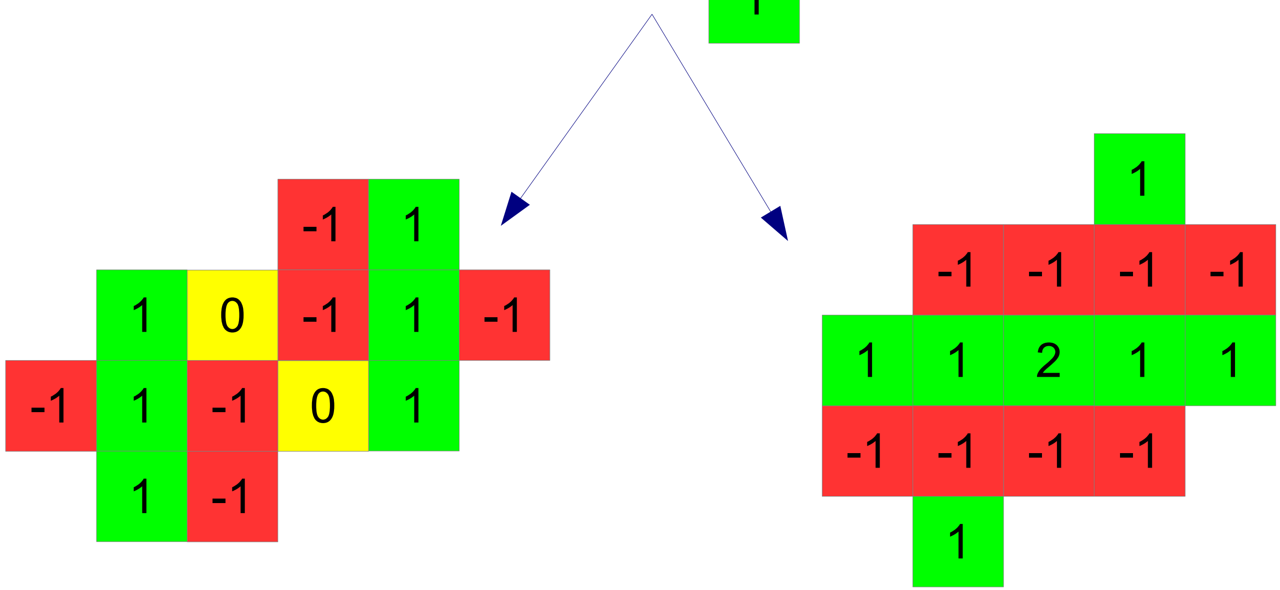


1. The Mojette transform with exact data

1.3 Null space and fantoms

From S3={(1 1)(-1 1)(2 1)}





S'4={(1 1)(-1 1)(2 1)(1 0)}   S''4={(1 1)(-1 1)(2 1)(0 1)}

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1. The Mojette transform with exact data

1.3 Null space and fantoms

S6={(1 1)(-1 1)(2 1)(1 2)(-1 2)(1 0)}

				-1	1		
		1	0	-1	1	-1	
	-1	1	0	-1	2	-1	
	-1	1	-1	-1	2	-1	1
1	-1	2	-1	-1	1	-1	
	-1	2	-1	0	1	-1	
	-1	1	-1	0	1		
		1	-1				

la trasformata Mojette : 20 anni

Larger switching components

				1	-1				
		-1	0	0	0	0	1		
	1	0	0	1	-1	0	0	-1	
	0	0	0	0	0	0	0	0	
-1	0	-1	0	0	0	0	1	0	1
1	0	1	0	0	0	0	-1	0	-1
	0	0	0	0	0	0	0	0	
	-1	0	0	-1	1	0	0	1	
		1	0	0	0	0	-1		
				-1	1				

S8={
(1,0)(0,1)
(1 1)(-1 1)
(2 1)(1,2)
(-1,2)(-2 1)}

la trasformata Mojette : 20 anni

Larger switching components

							1	-1				
				-1	0	0	0	0	0	1		
		1	0	1	0	0	0	-1	0	0	-1	
	-1	0	0	-1	1	0	0	1	0	0	0	
	0	0	-1	0	-1	0	0	0	0	1	0	1
1	0	1	1	0	1	0	-1	0	-1	-1	0	-1
-1	0	-1	0	0	0	0	1	0	1	0	0	
	0	0	0	-1	0	0	-1	1	0	0	1	
	1	0	0	1	0	0	0	-1	0	-1		
		-1	0	0	0	0	0	1				
				1	-1							

S9={
(1,0)(0,1)
(1 1)(-1 1)
(2 1)(1,2)
(-1 2)(-2 1)
(3 1)}

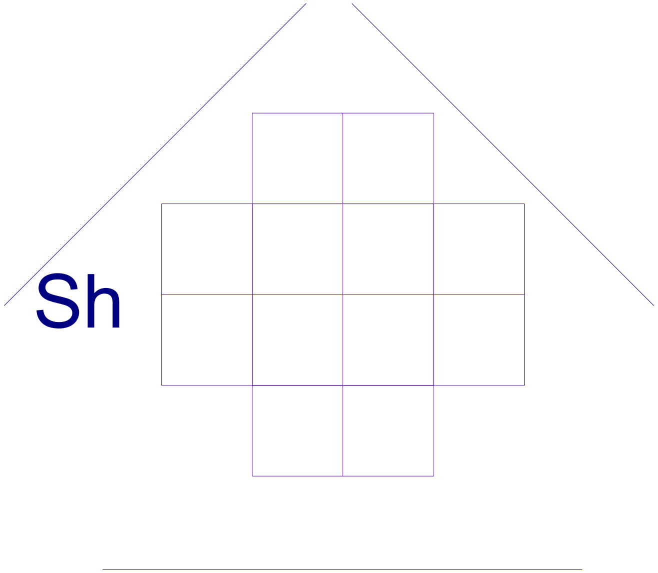
1. The Mojette transform with exact data

1.3 Null space and fantoms

Example

Take our convex shape Sh

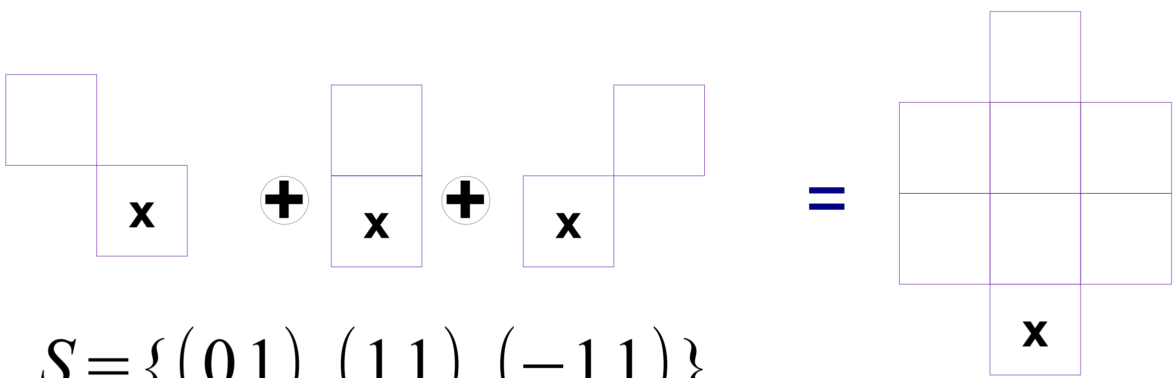
$$S = \{(0\,1),(1\,1),(-1\,1)\}$$



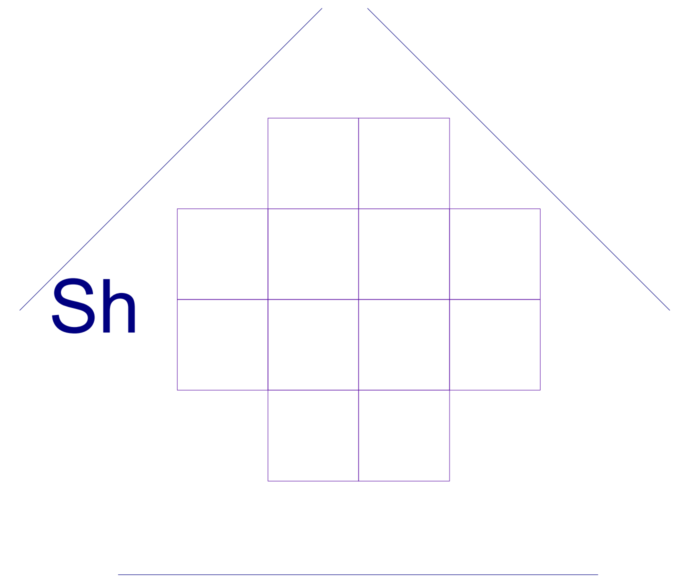
1. The Mojette transform with exact data

1.3 Null space and fantoms

Perform dilation series



$$S = \{(0\ 1), (1\ 1), (-1\ 1)\}$$

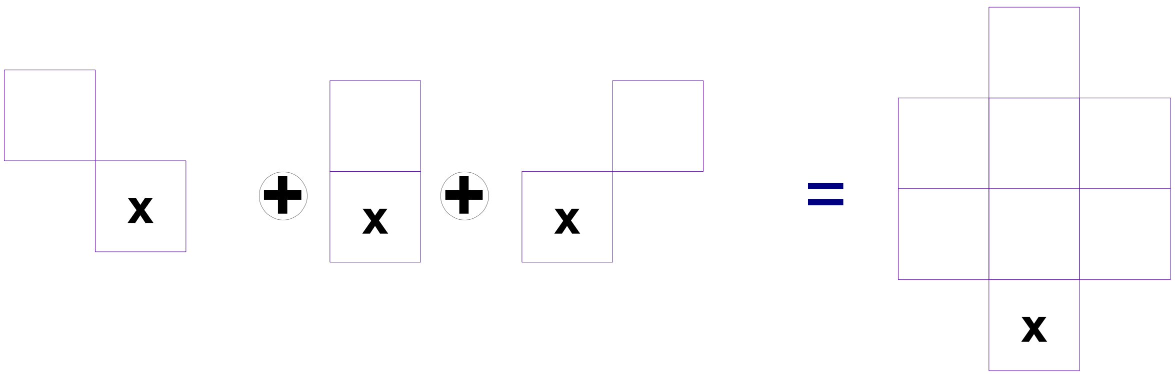


The resulting shape S
Can be inserted into Sh

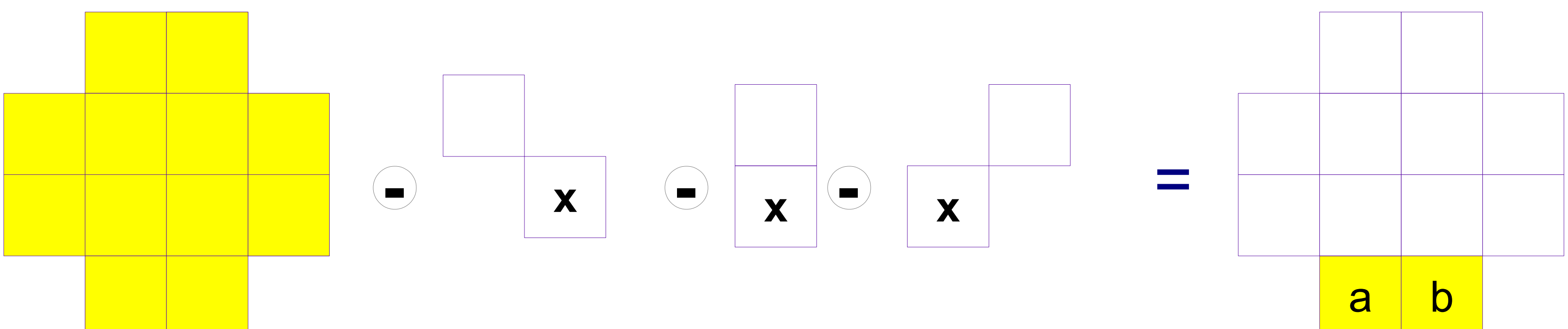
=> No reconstruct-ability
(no unicity of the reconstruction)

1. The Mojette transform with exact data

1.3 Null space and fantoms



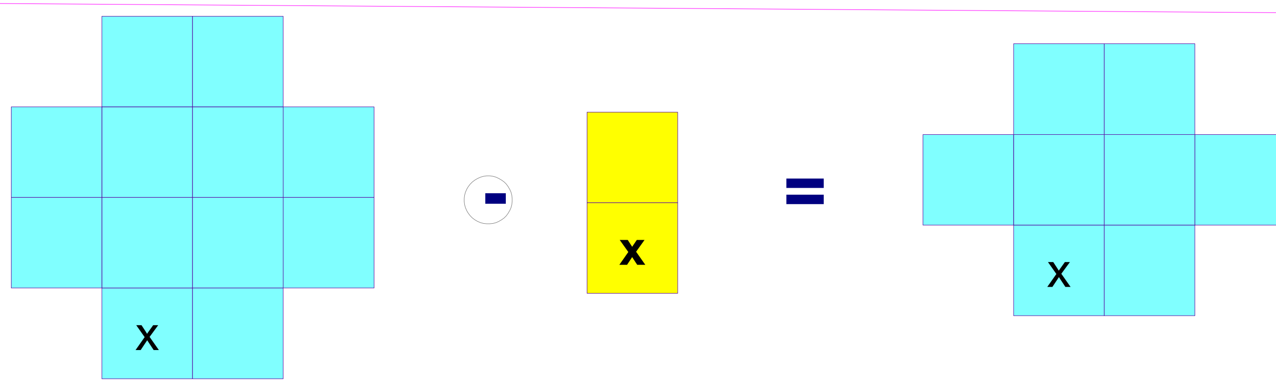
Equivalent process : making erosion series of Sh do not lead to empty set



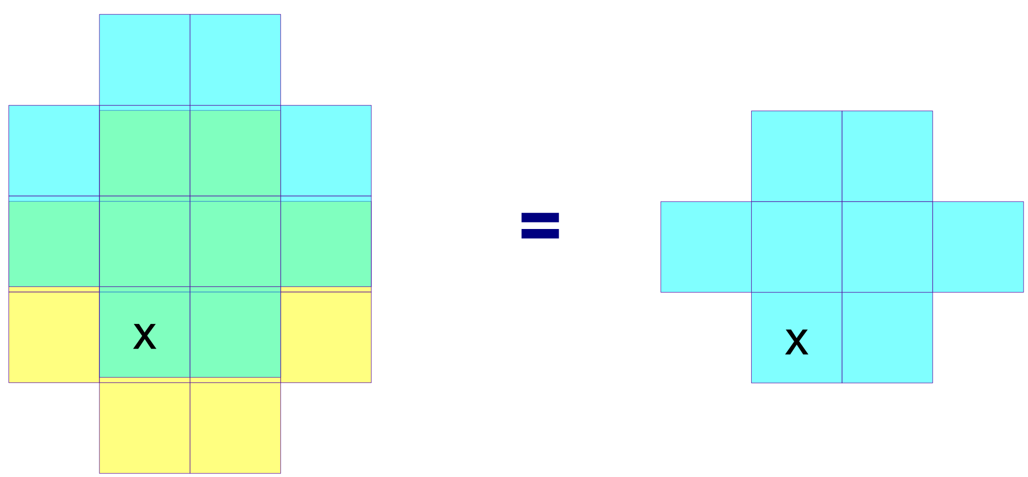
$$S = \{(0\,1), (1\,1), (-1\,1)\}$$

1. The Mojette transform with exact data

1.3 Null space and fantoms

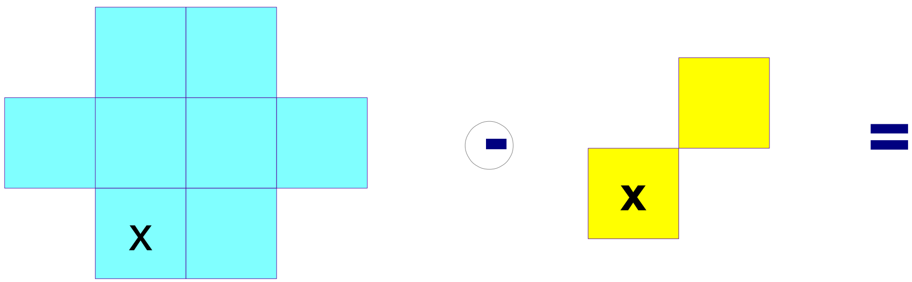


Erosion of Sh by 2PSE (0 1)

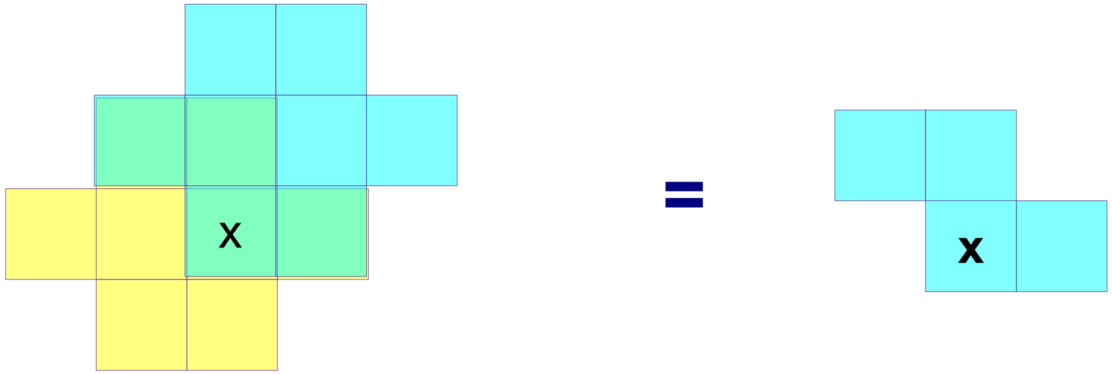


1. The Mojette transform with exact data

1.3 Null space and fantoms

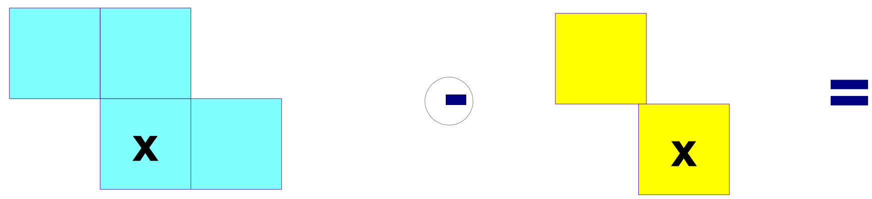


Erosion of resulting shape by 2PSE (1 1)

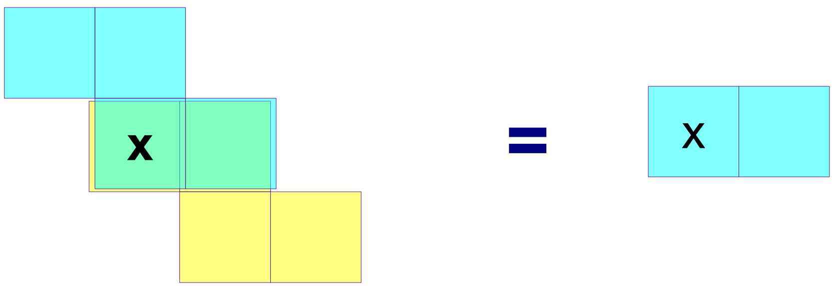


1. The Mojette transform with exact data

1.3 Null space and fantoms



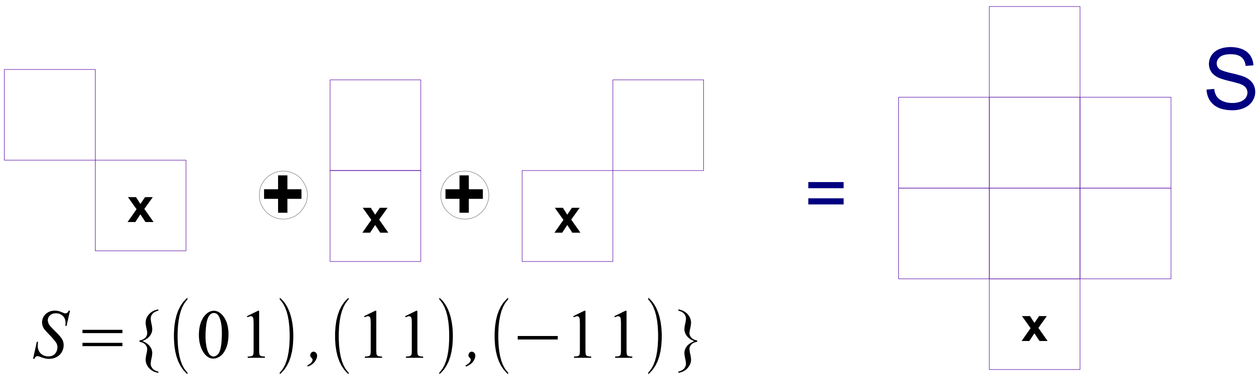
Erosion of resulting shape by 2PSE (-1 1)



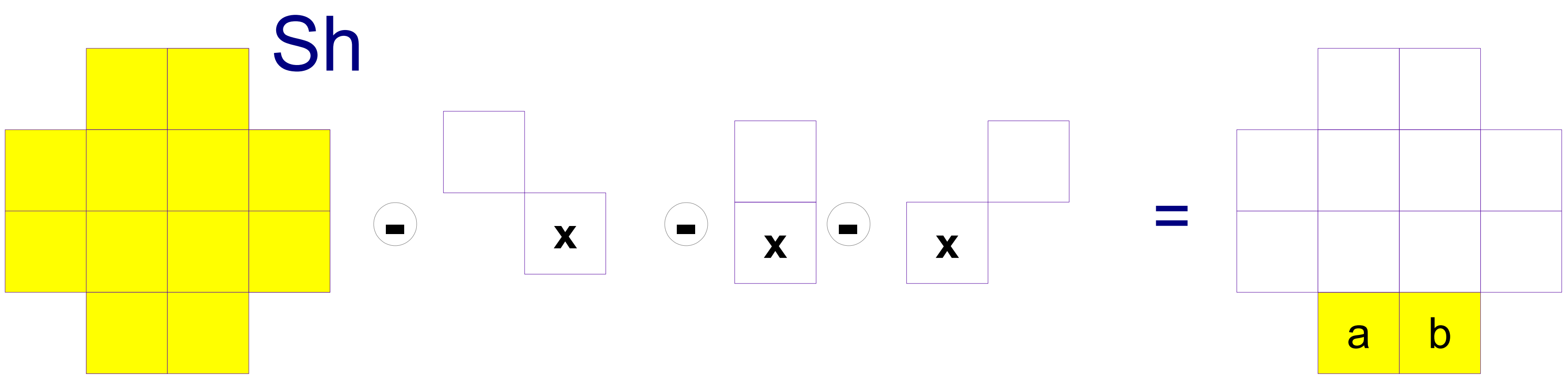
1. The Mojette transform with exact data

1.3 Null space and fantoms

Incomplete reconstruction theorem
(Philippé-Guédon 1997)



$$S = \{(0\,1),(1\,1),(-1\,1)\}$$

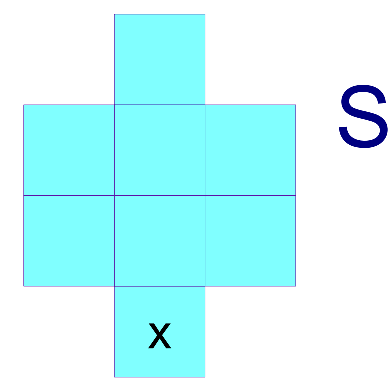


1. The Mojette transform with exact data

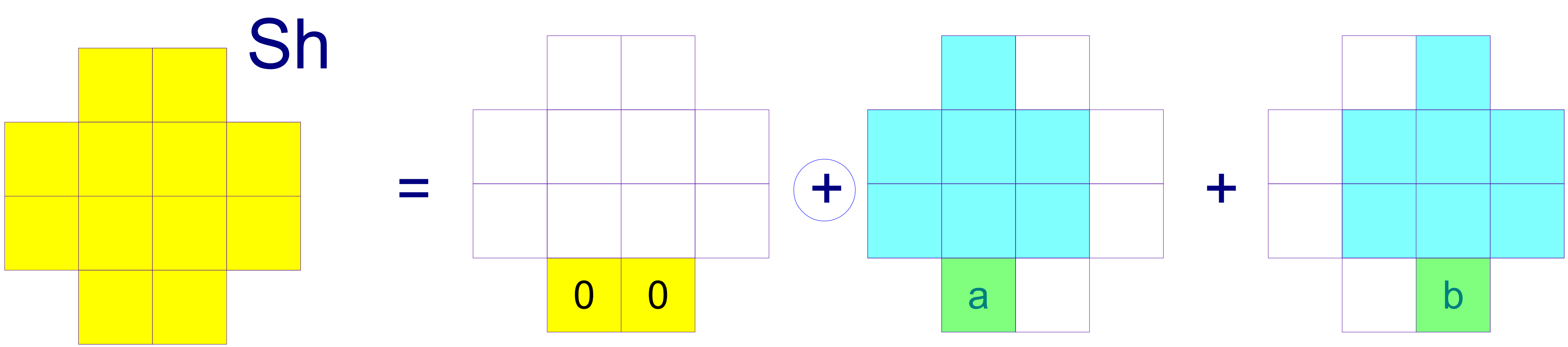
1.3 Null space and fantoms

Incomplete reconstruction theorem
(Philippé-Guédon 1997)

Linear algebra – projector :



$E = \mathrm{Im}(f) \oplus \mathrm{Ker}(f)$

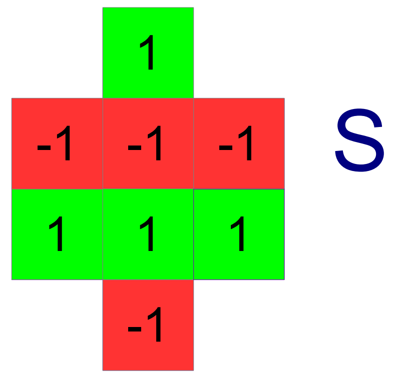


1. The Mojette transform with exact data

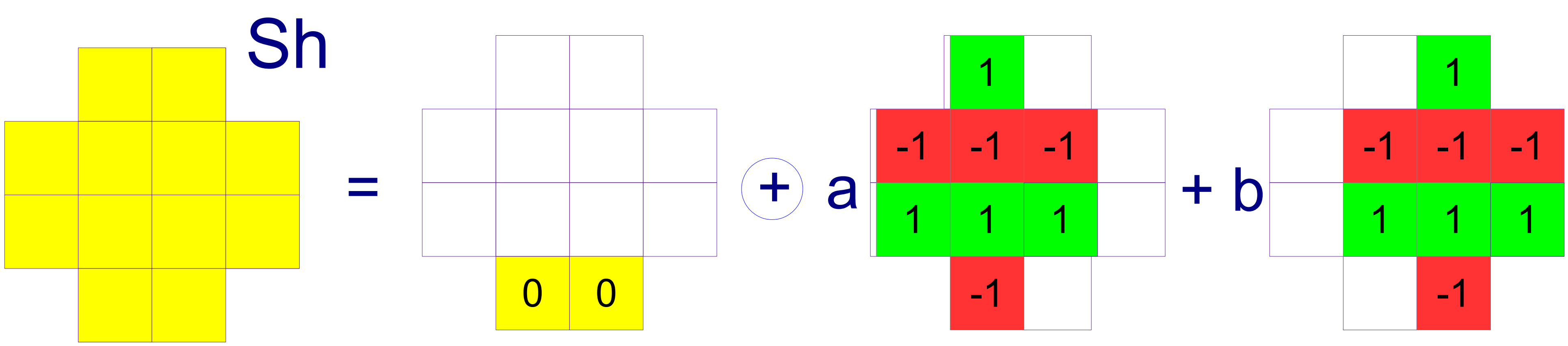
1.3 Null space and fantoms

Incomplete reconstruction theorem
(Philippé-Guédon 1997)

Linear algebra – projector :



$E = Im(f) \oplus Ker(f)$



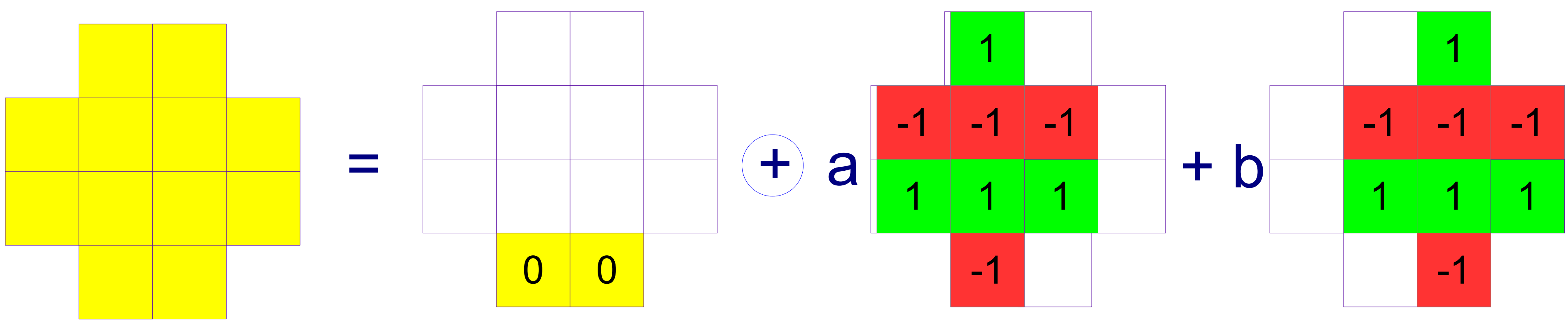
1. The Mojette transform with exact data

1.3 Null space and fantoms

Incomplete reconstruction theorem

The Mojette kernel is totally described by locations of non-eroded pixels onto which the phantom can be placed (= rank of Ker). Image is recovered by inverse Mojette algorithm after setting 0 onto non eroded pixels

E = Im(f) ⊕ Ker(f)



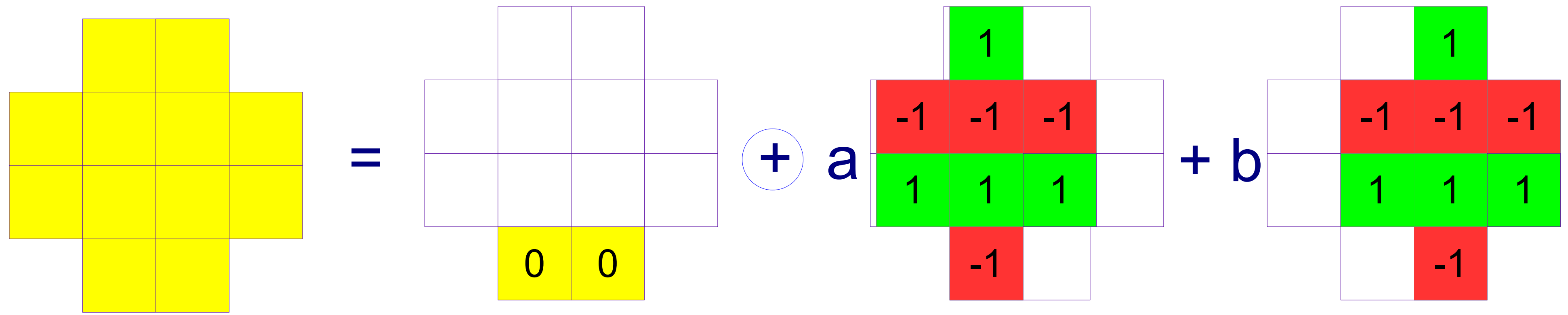
1. The Mojette transform with exact data

1.3 Null space and fantoms

Incomplete reconstruction theorem

Notice that E is just a vectorial subspace composed of pixels

$$E = Im(f) \oplus Ker(f)$$



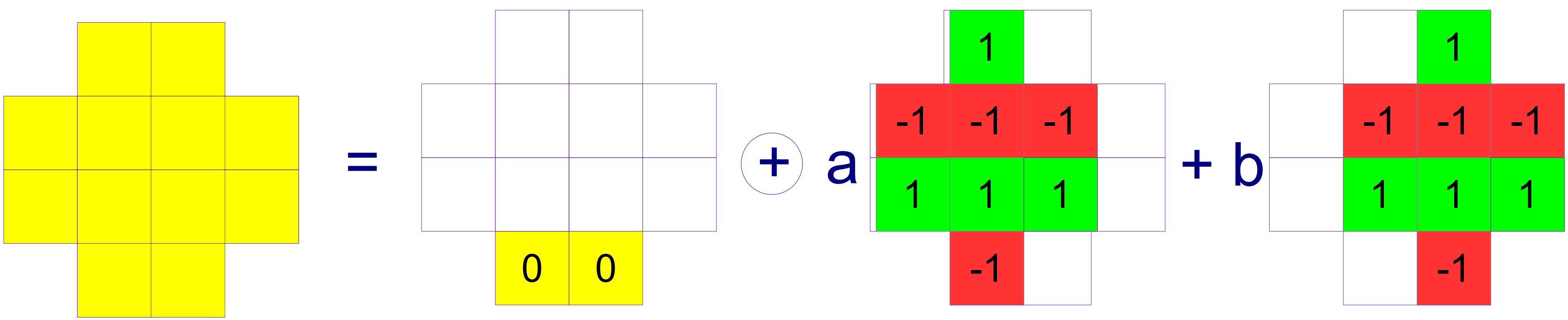
1. The Mojette transform with exact data

1.3 Null space and fantoms

Incomplete reconstruction theorem

when E has quantized pixels (ie binary / ternary) then the number of solutions is finite and all solutions are described here

E = Im(f) ⊕ Ker(f)



1. The Mojette transform with exact data

1.4 Applications

Error correcting code

Distributed storage

Network protocol

Watermarking

1. The Mojette transform with exact data

1.4 Applications

Error correcting code, optimal codes ?

Hamming H(7,4) H(15,11) H(31,26) ... :
detect and correct 1 binary error

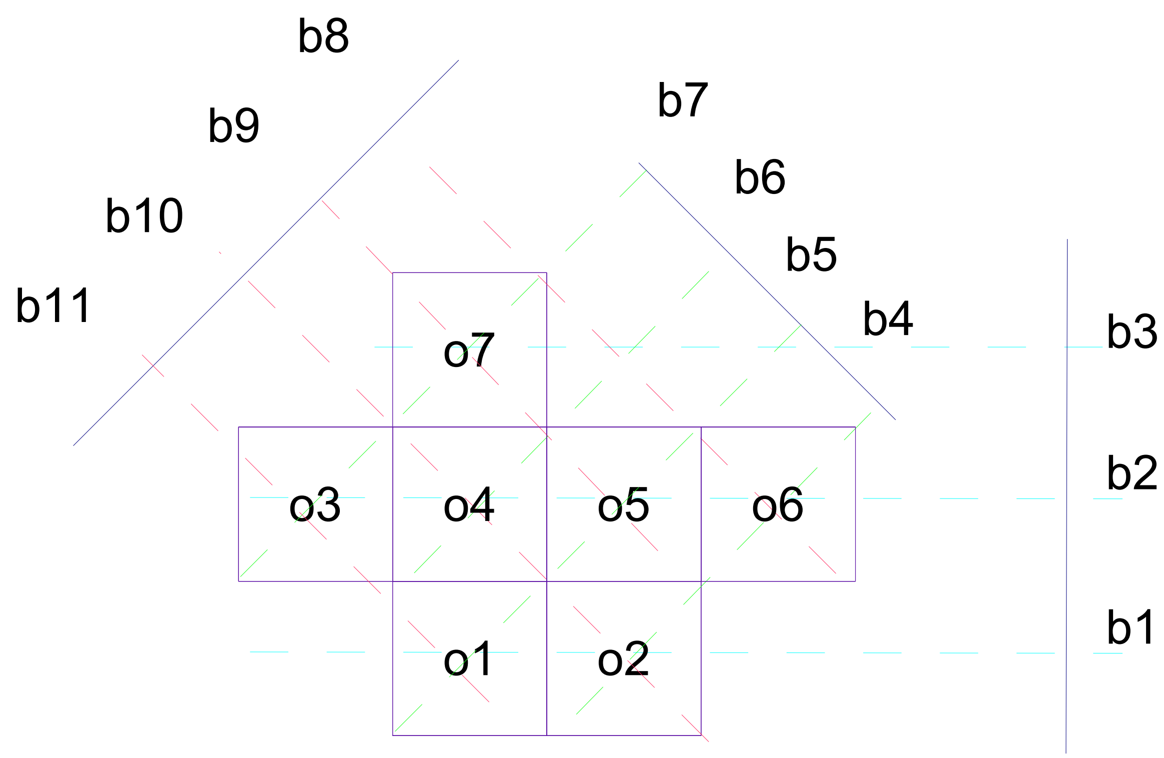
Other linear codes in between ?

1. The Mojette transform with exact data

1.4 Applications binary values

between Hamming H(7,4) and H(15,11) :

Moj(11,7)

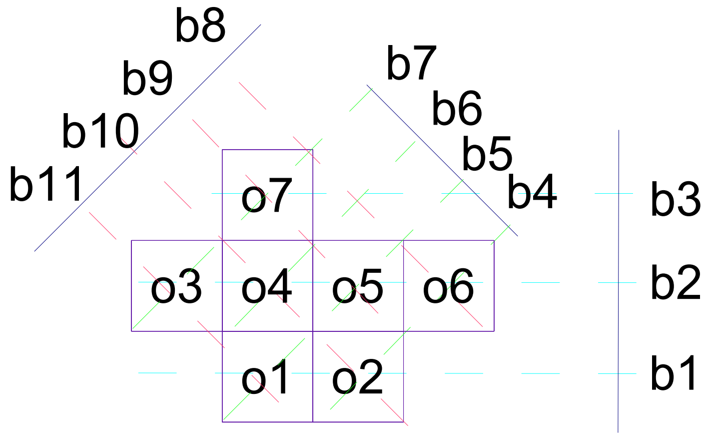


1. The Mojette transform with exact data

1.4 Applications

Moj(11,7)

7 initial bits oi
11 send bj



$$\left(\Sigma_0 = \sum_{i=1}^{7} o_i\right)$$

$$\left(\begin{array}{l}\Sigma_1 = b_1 + b_2 + b_3 \\ \Sigma_2 = b_4 + b_5 + b_6 + b_7 \\ \Sigma_3 = b_8 + b_9 + b_{10} + b_{11}\end{array}\right)$$

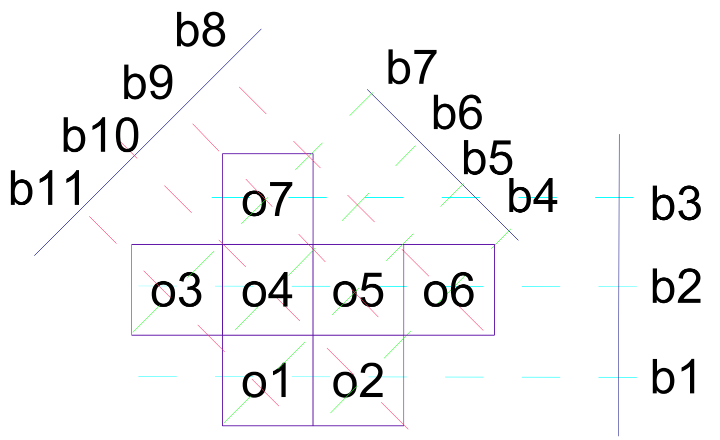
$$\left(\Sigma_0 = \Sigma_1 = \Sigma_2 = \Sigma_3\right)$$

1. The Mojette transform with exact data

1.4 Applications

between Hamming H(7,4) and H(15,11) :

Moj(11,7)



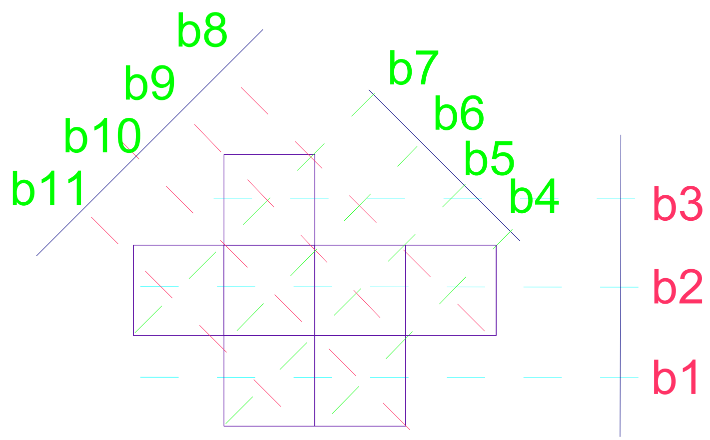
At reception, computes

$$\left(\begin{array}{l}\Sigma_1 = b_1 + b_2 + b_3 \\ \Sigma_2 = b_4 + b_5 + b_6 + b_7 \\ \Sigma_3 = b_8 + b_9 + b_{10} + b_{11}\end{array}\right)$$

$$\left(CASE\,0 : \left(\Sigma_1 = \Sigma_2 = \Sigma_3\right) : NO\ ERROR\ DETECTED\right)$$

1. The Mojette transform with exact data

1.4 Applications



At reception, computes

$$\Sigma_1 = b_1 + b_2 + b_3$$
$$\left(\Sigma_2 = b_4 + b_5 + b_6 + b_7 + b_8\right)$$
$$\Sigma_3 = b_9 + b_{10} + b_{11} + b_{12}$$

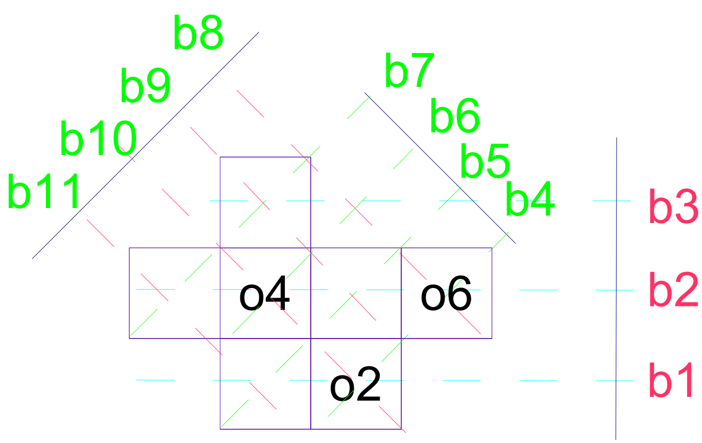
$$\left(CASE\ 1 : \left(\Sigma_1 \neq \left(\Sigma_2 = \Sigma_3\right)\right)\right)$$

The error is located on projection (1 0)
=> first, only use the 2 other projections

1. The Mojette transform with exact data

1.4 Applications

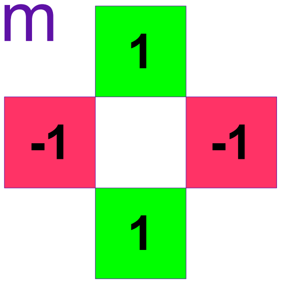
$$(CASE\,1:(\Sigma_1 \neq (\Sigma_2 = \Sigma_3)))$$



The error is located on projection (1 0)

$$\begin{pmatrix} \Sigma_2 = b_4 + b_5 + b_6 + b_7 \\ \Sigma_3 = b_8 + b_9 + b_{10} + b_{11} \end{pmatrix}$$

After partial reconstruction, the phantom is left :



$$o_1 + o_3 = b_{11}$$

$$\begin{pmatrix} o_1 + o_5 = b_5 \\ o_5 + o_7 = b_9 \end{pmatrix}$$

$$o_3 + o_7 = b_7$$

Now, we also know that 2 of 3 binary value of (1 0) are good and one is not

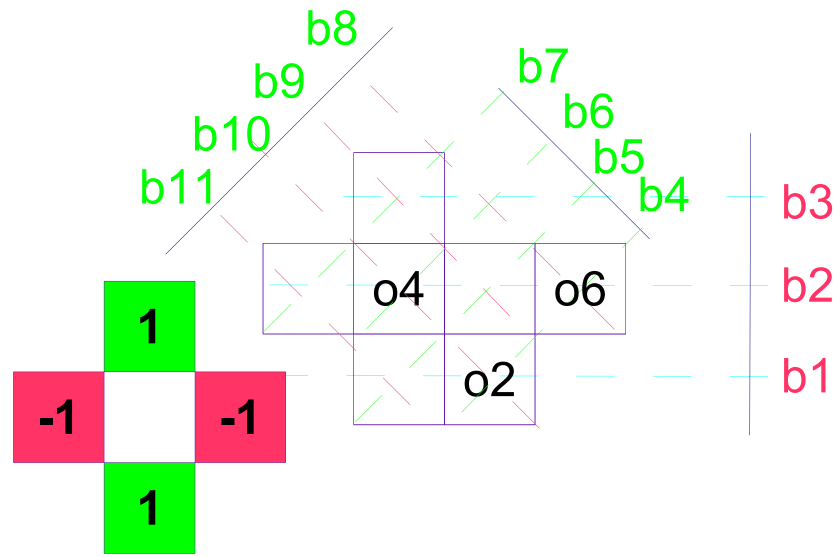
1. The Mojette transform with exact data

1.4 Applications

$$\left(CASE\,1:\left(\Sigma_1\neq\left(\Sigma_2=\Sigma_3\right)\right)\right)$$

Always true :

$$o_1+o_3=b_{11}$$
$$\left(\begin{matrix}o_1+o_5=b_5\\ o_5+o_7=b_9\end{matrix}\right)$$
$$o_3+o_7=b_7$$



With only 1 system over 3 true:

$$\left(\begin{matrix}\bar{b}_1=o_1+o_2\\ b_2=o_3+o_4+o_5+o_6\\ b_3=o_7\end{matrix}\right) \text{ or } \left(\begin{matrix}b_1=o_1+o_2\\ \bar{b}_2=o_3+o_4+o_5+o_6\\ b_3=o_7\end{matrix}\right) \text{ or } \left(\begin{matrix}b_1=o_1+o_2\\ b_2=o_3+o_4+o_5+o_6\\ \bar{b}_3=o_7\end{matrix}\right)$$

And with respect to : $\left(\Sigma_1\neq\left(\Sigma_2=\Sigma_3\right)\right)$ => Locate the error

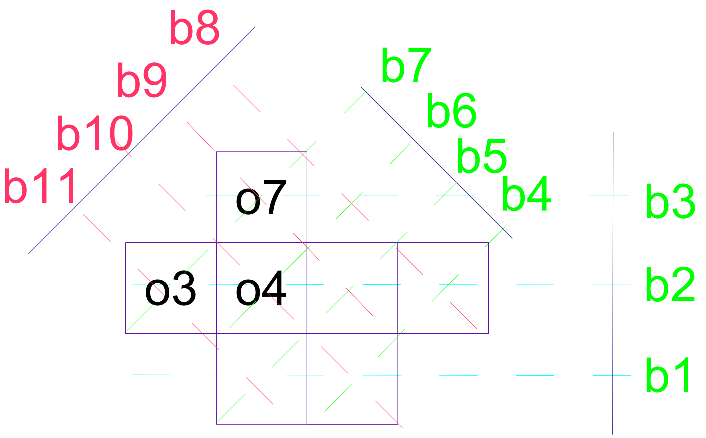
1. The Mojette transform with exact data

1.4 Applications

$$\left(CASE:(\Sigma_1=\Sigma_2)\neq\Sigma_3\right)$$

Computes :

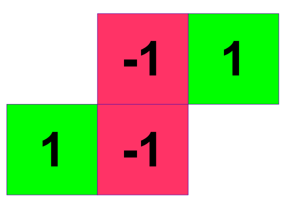
$$\begin{pmatrix} o_2+o_6=b_4 \\ o_1+o_5=b_5 \\ o_1+o_2=b_1 \\ o_3+o_4+o_5+o_6=b_2 \end{pmatrix}$$



Only 1 system over 4 is correct

Since only 1 location for the phantom

$$\begin{pmatrix} \bar{b}_8=o_6 \\ b_9=o_5+o_7 \\ b_{10}=o_2+o_4 \\ b_{11}=o_1+o_3 \end{pmatrix} \quad \begin{pmatrix} b_8=o_6 \\ \bar{b}_9=o_5+o_7 \\ b_{10}=o_2+o_4 \\ b_{11}=o_1+o_3 \end{pmatrix} \quad \begin{pmatrix} b_8=o_6 \\ b_9=o_5+o_7 \\ \bar{b}_{10}=o_2+o_4 \\ b_{11}=o_1+o_3 \end{pmatrix} \quad \begin{pmatrix} b_8=o_6 \\ b_9=o_5+o_7 \\ b_{10}=o_2+o_4 \\ \bar{b}_{11}=o_1+o_3 \end{pmatrix}$$



And with respect to : $\left((\Sigma_1=\Sigma_2)\neq\Sigma_3\right)$ => Locate the error

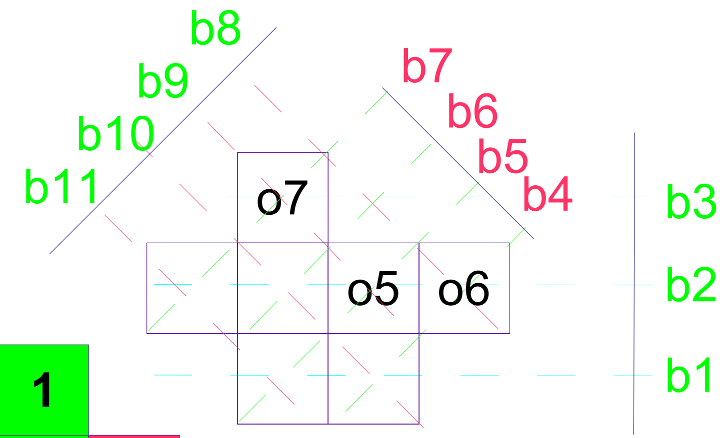
1. The Mojette transform with exact data

1.4 Applications

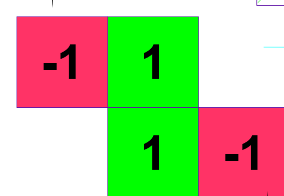
$$\left(CASE:\left(\Sigma_1=\Sigma_3\right)\neq\Sigma_2\right)$$

Computes :

$$\begin{pmatrix} o_2+o_4=b_{10} \\ o_1+o_3=b_{11} \\ o_1+o_2=b_1 \\ o_3+o_4+o_5+o_6=b_2 \end{pmatrix}$$



Only 1 system over 4 is correct
Since only 1 location for the phantom



$$\begin{pmatrix} \bar{b}_4=o_2+o_6 \\ b_5=o_1+o_5 \\ b_6=o_4 \\ b_7=o_4+o_7 \end{pmatrix} \begin{pmatrix} b_4=o_2+o_6 \\ \bar{b}_5=o_1+o_5 \\ b_6=o_4 \\ b_7=o_4+o_7 \end{pmatrix} \begin{pmatrix} b_4=o_2+o_6 \\ b_5=o_1+o_5 \\ \bar{b}_6=o_4 \\ b_7=o_4+o_7 \end{pmatrix} \begin{pmatrix} b_4=o_2+o_6 \\ b_5=o_1+o_5 \\ b_6=o_4 \\ \bar{b}_7=o_4+o_7 \end{pmatrix}$$

And with respect to : $\left(\left(\Sigma_1=\Sigma_3\right)\neq\Sigma_2\right)$ => Locate the error

1. The Mojette transform with exact data

1.4 Applications Distributed storage

What exactly happens when you store
A file of 1MB onto Google, Facebook or other ?

In order to recover your file with a sufficient probability (> 0.9999) the file has to be duplicated 3 times : 3MB stored

Instead, with the same probability (> 0.9999) store 1.5 MB !

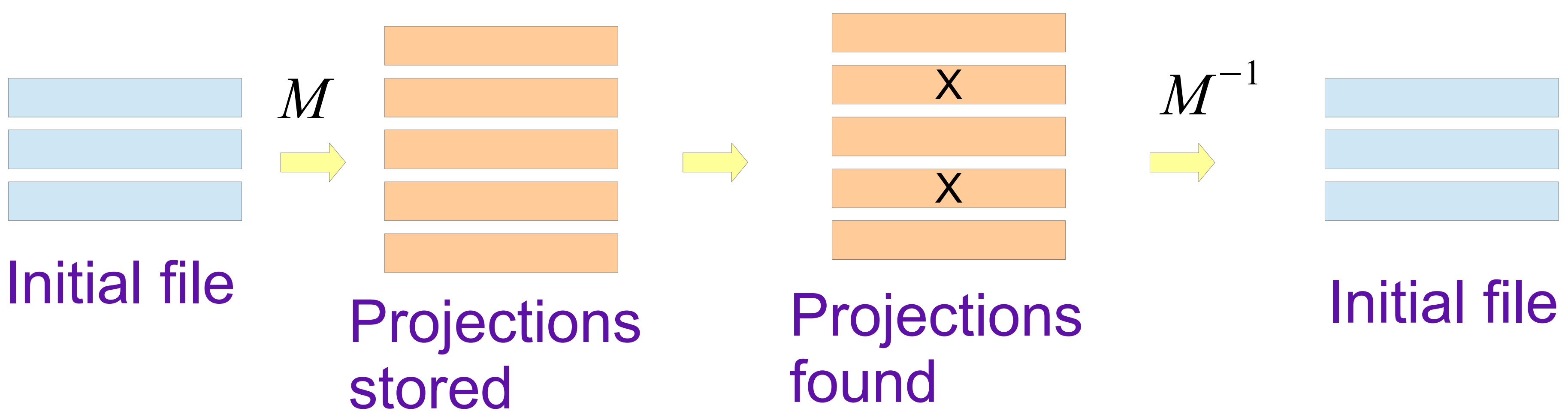
Store redundant Mojette projections

1. The Mojette transform with exact data

1.4 Applications   Distributed storage

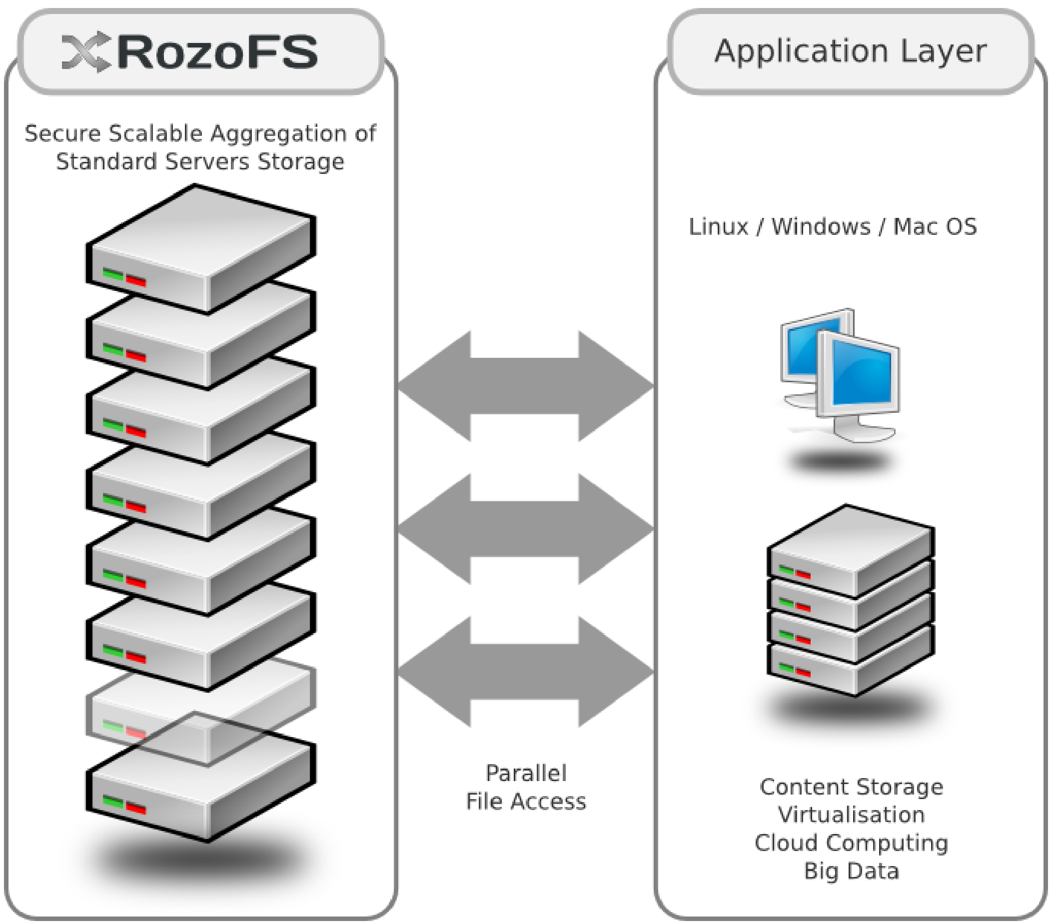
Put the file into rectangular shape (Px3)
Compute projections set : $S=\{(0\,1),(1\,1),(-1\,1)(2\,1)(-2\,1)\}$

Any subset of 3 over 5 reconstruct the shape
Store each projection onto a different disk



1. The Mojette transform with exact data

1.4 Applications   Distributed storage



RozoFS inc

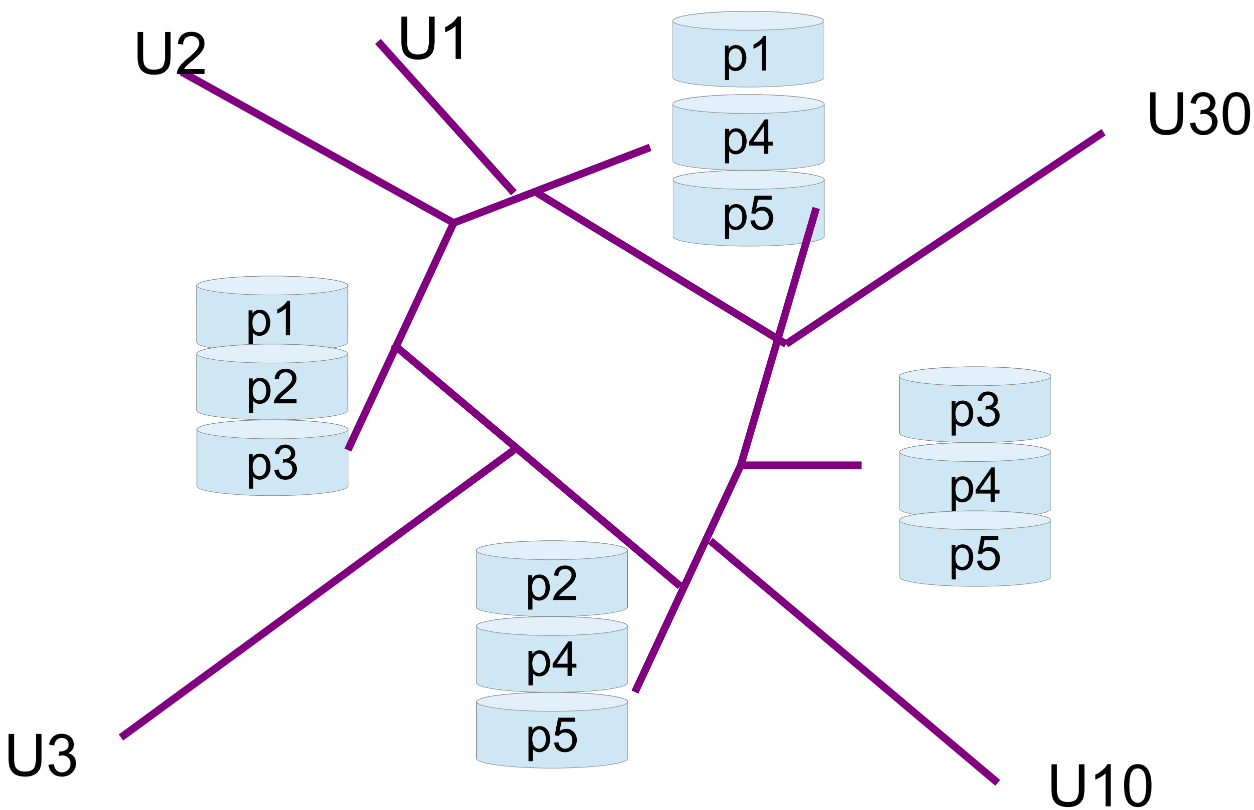
Uses Mojette transform
to store files
1 server = 1 projection

Interest 1: 1 server failed
=>only 1 projection missing
Interest 2: read projections
in a parallel way

1. The Mojette transform with exact data

1.4 Applications   Distributed storage

RozoFS



RozoFS inc

5 projections
4 needed
3 stored on a node

=> distributed security

1. The Mojette transform with exact data

1.4 Applications wavelet image transmission

2001 Ph.D. Benoit Parrein

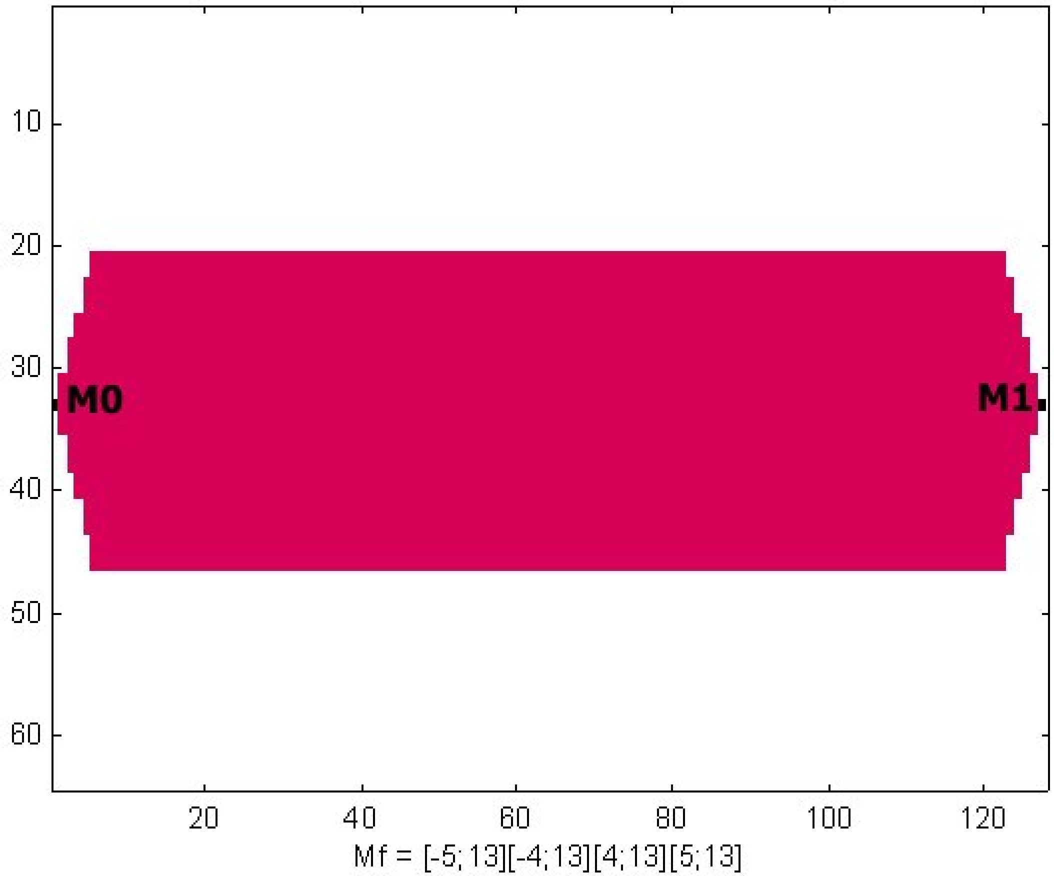
« geometric buffer »

1. The Mojette transform with exact data

1.4 Applications wavelet image transmission

2001 Ph.D. Benoit Parrein

« geometric buffer »

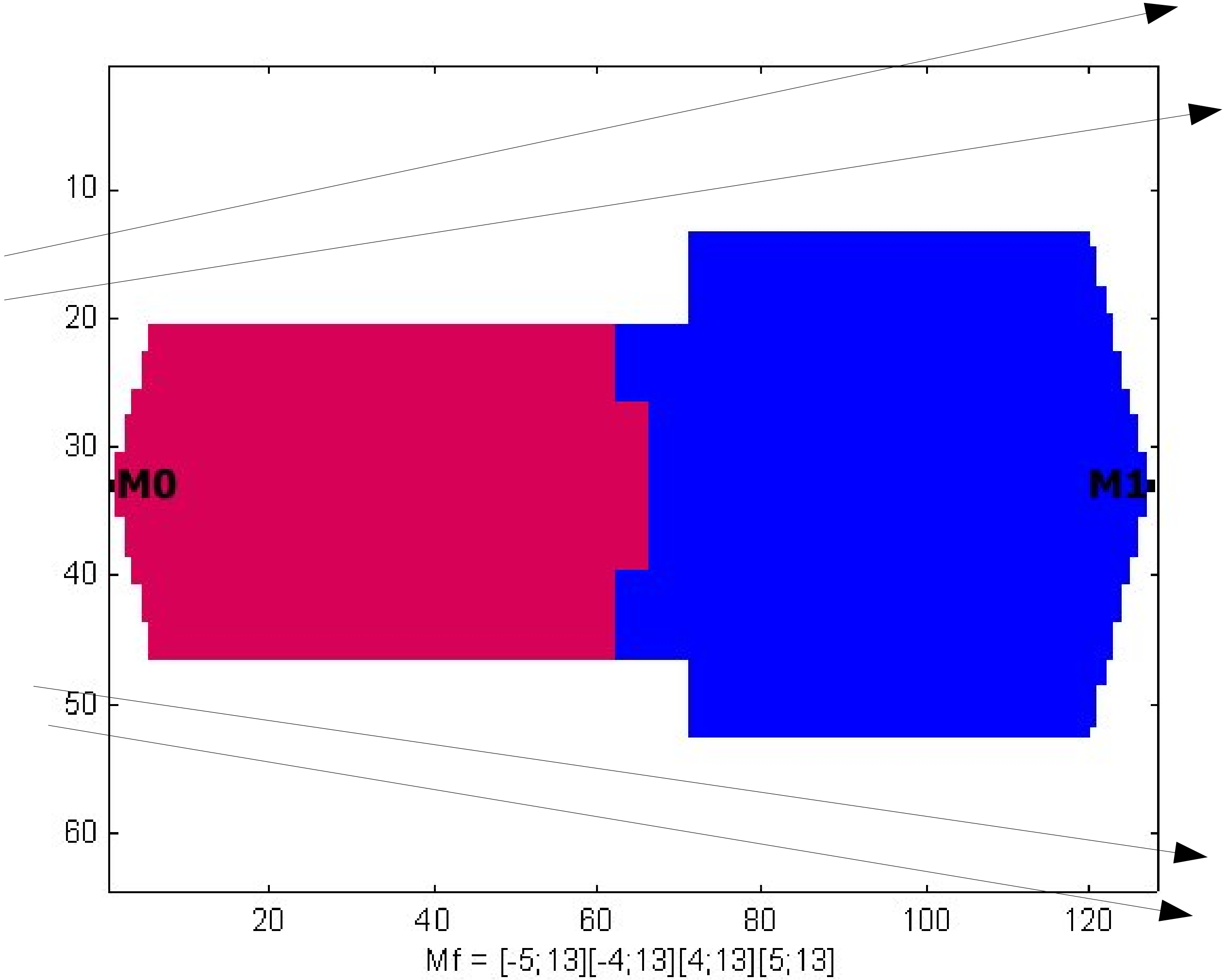


1. The Mojette transform with exact data

1.4 Applications wavelet image transmission

2001 Ph.D. Benoit Parrein

« geometric buffer »



« Grey packet »

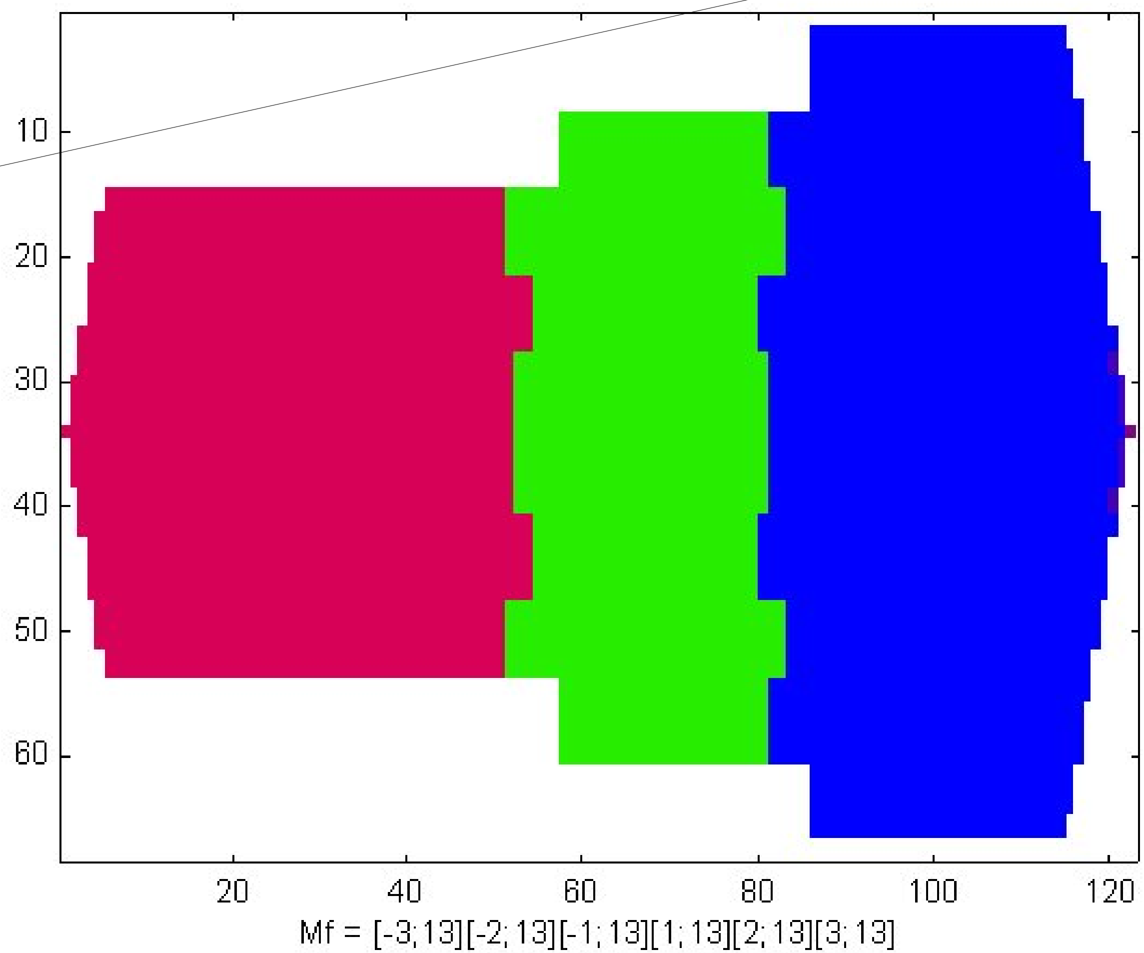
Contains a projection both for LF & HF

1. The Mojette transform with exact data

1.4 Applications wavelet image transmission

2001 Ph.D. Benoit Parrein

« geometric buffer »



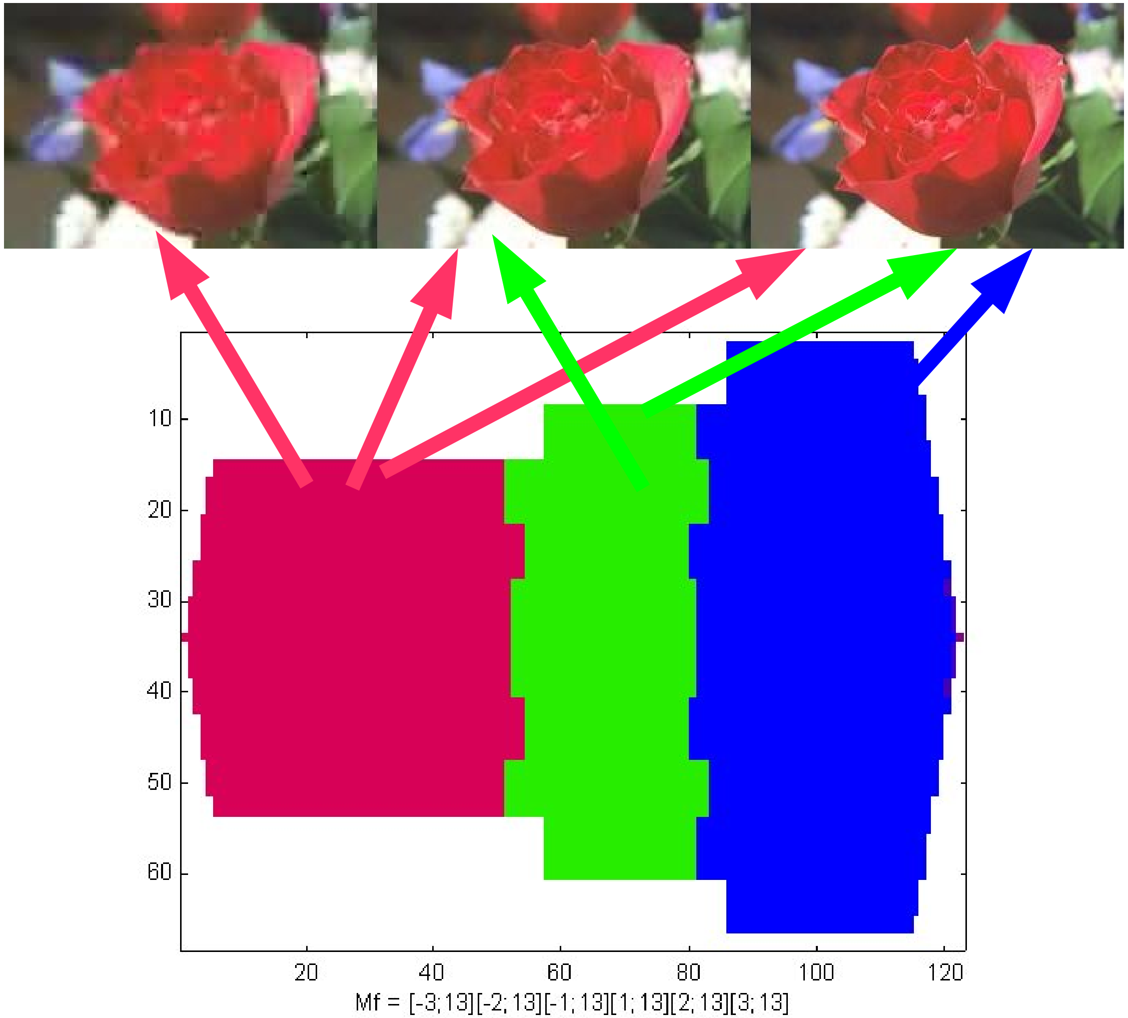
« Grey packet »

Contains a projection both for LF , MF & HF

1. The Mojette transform with exact data

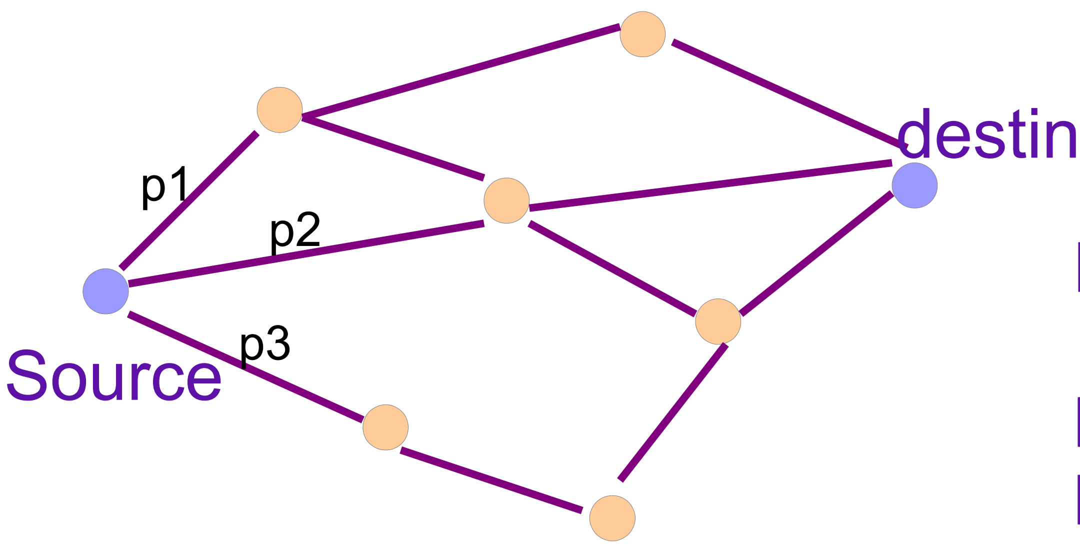
1.4 Applications wavelet image transmission

2001 Ph.D. Benoit Parrein



1. The Mojette transform with exact data

1.4 Applications adhoc Network protocol



From OLSR

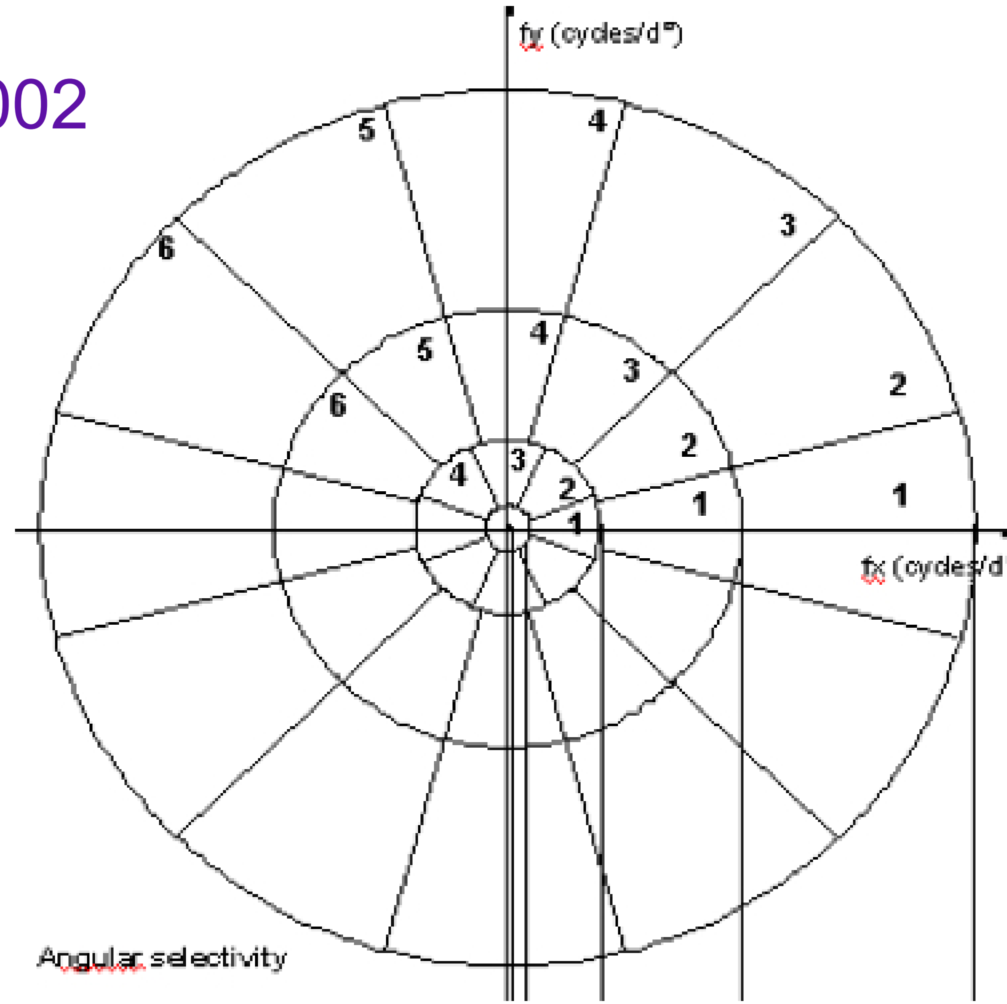
MultiPath – OLSR
B. Parrein, J. Yi, S. David

1. The Mojette transform with exact data

1.4 Applications Watermarking

Ph.D. Florent Autrusseau 2002

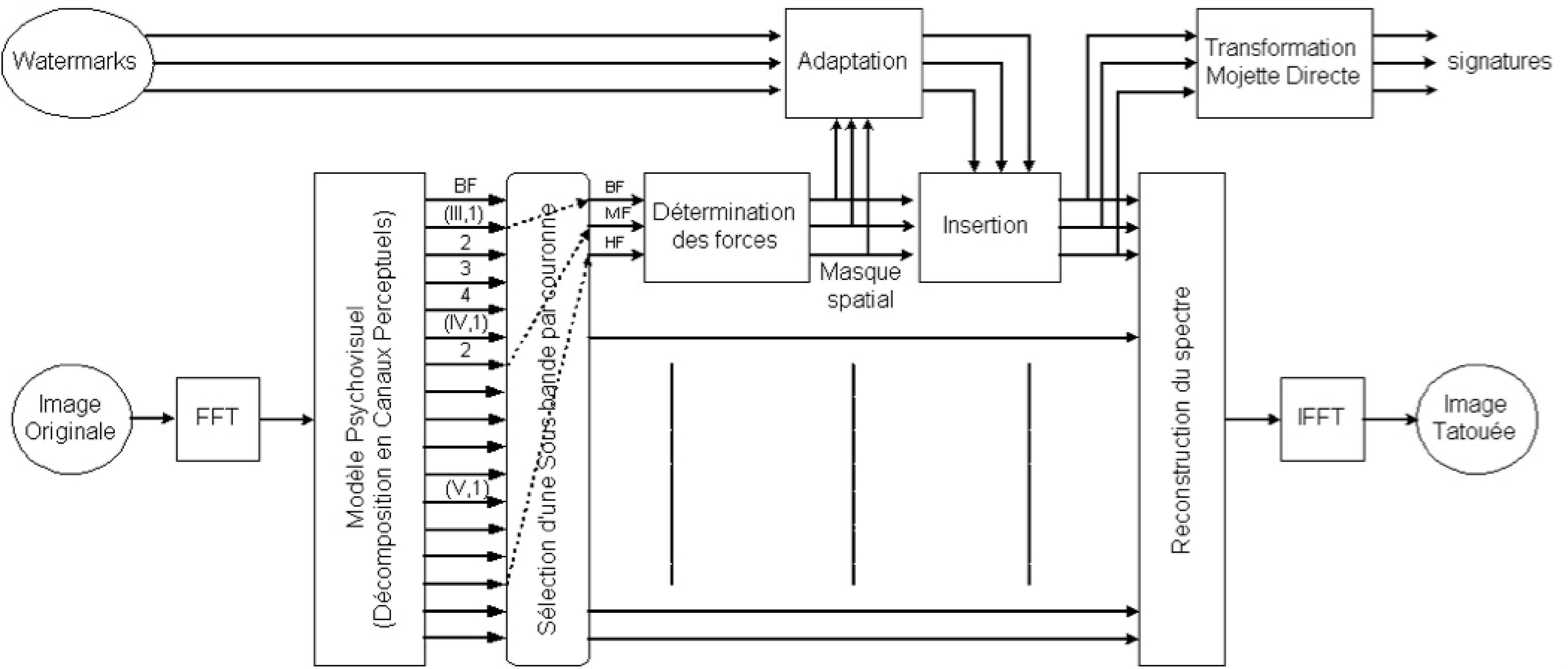
Psycho Visual
Image
decomposition
In Fourier Space



1. The Mojette transform with exact data

1.4 Applications Watermarking

Ph.D. Florent Autrusseau 2002



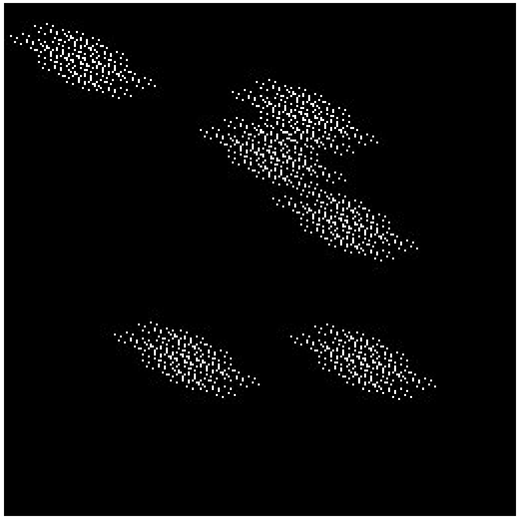
1. The Mojette transform with exact data

1.4 Applications Watermarking

original

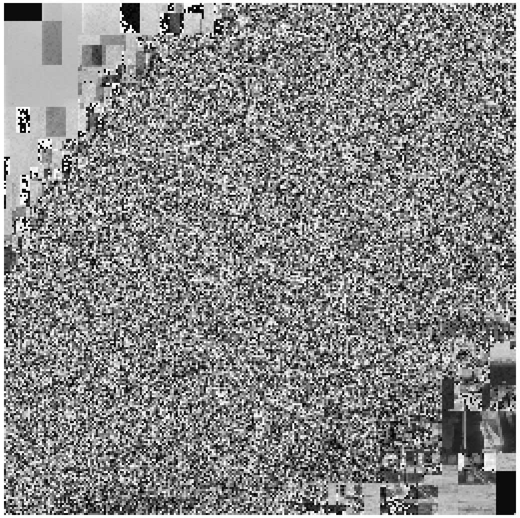
+

watermark



=

Watermarked image



Ph.D. Florent Autrusseau 2002

outline

1. The Mojette transform with exact data
2. Tomographic reconstruction
3. Links with Fourier and FRT

2. Tomographic Mojette reconstruction

2.1 Binary-Ternary Mojette

2.2 From Radon to Mojette space

2.3 Discrete rotations

2. Tomographic Mojette reconstruction

2.1 Binary-Ternary Mojette

2.2 From Radon to Mojette space

2.3 Discrete rotations

2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

Katz criteria holds for real values

Idea :
Negative redundancy & Use of phantoms

Works with Chuanlin LIU – Imants Svalbe

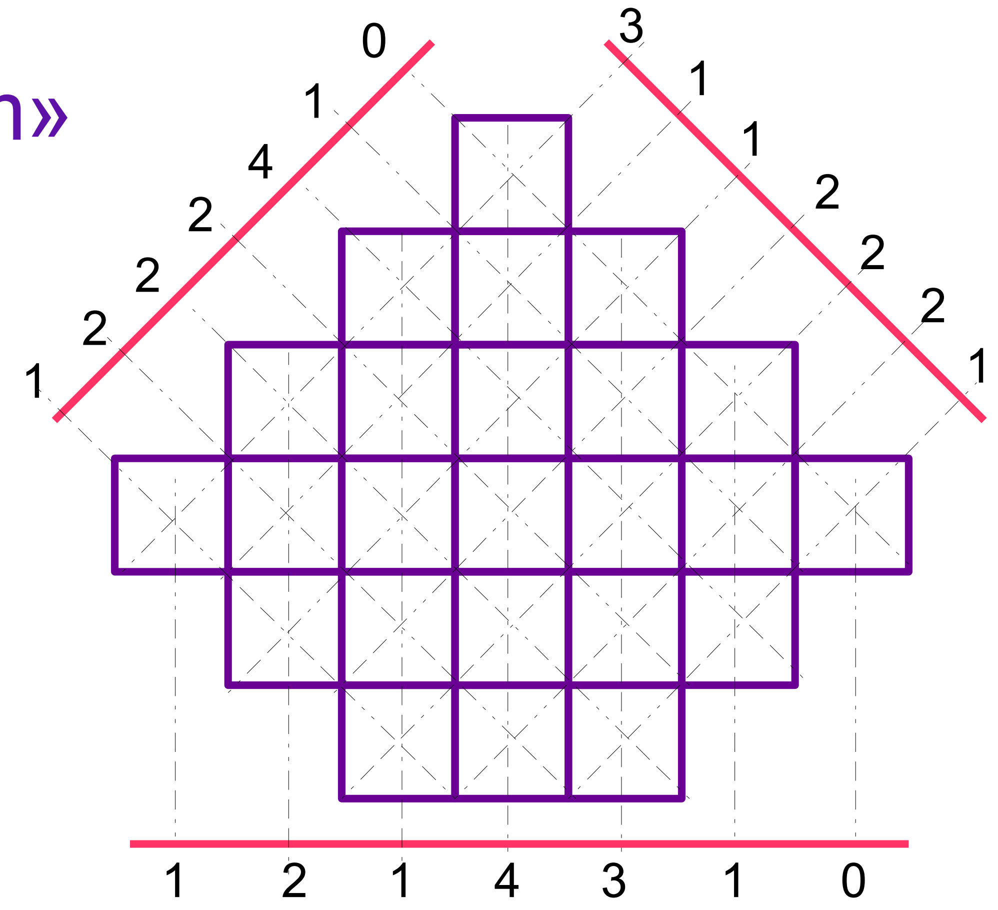
2. Tomographic Mojette reconstruction

2.1 Binary Mojette

« The Mojette line algorithm»

2004 Ph. D. of Pierre Verbert

Binary image
Integer addition



2. Tomographic Mojette reconstruction

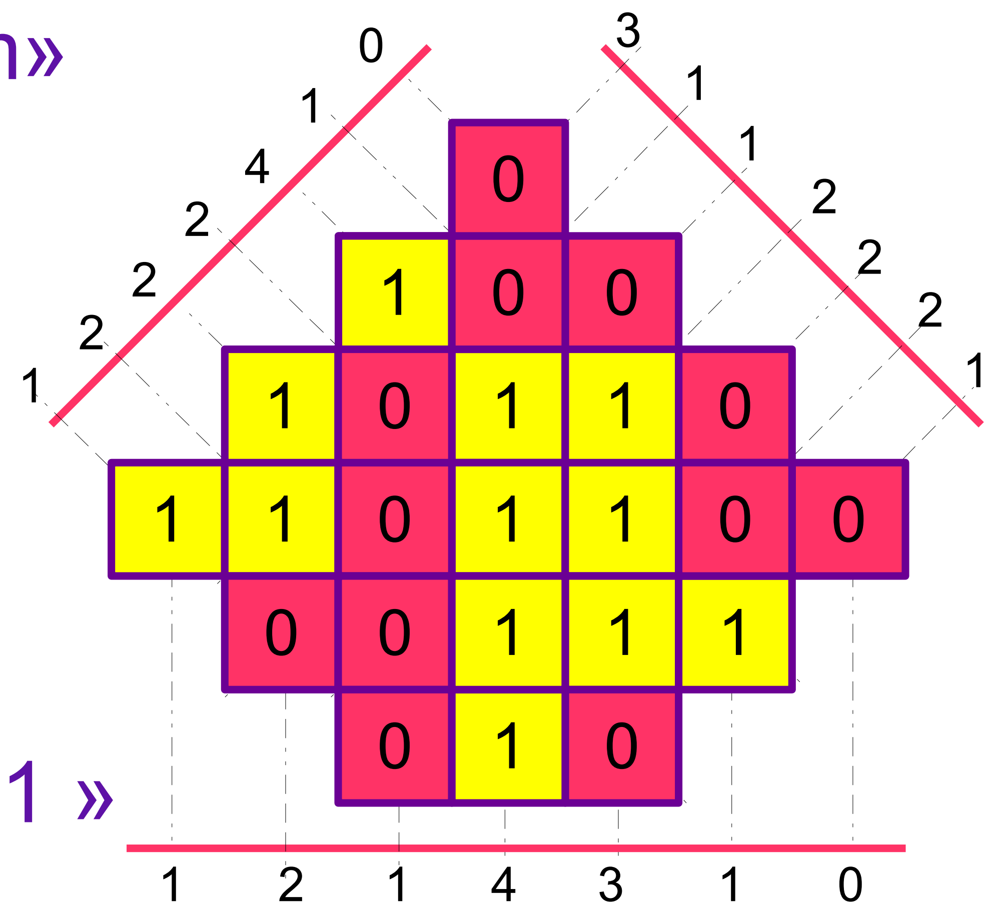
2.1 Binary Mojette

« The Mojette line algorithm»

2004 Ph. D. Pierre Verbert

A bin sums of n pixels

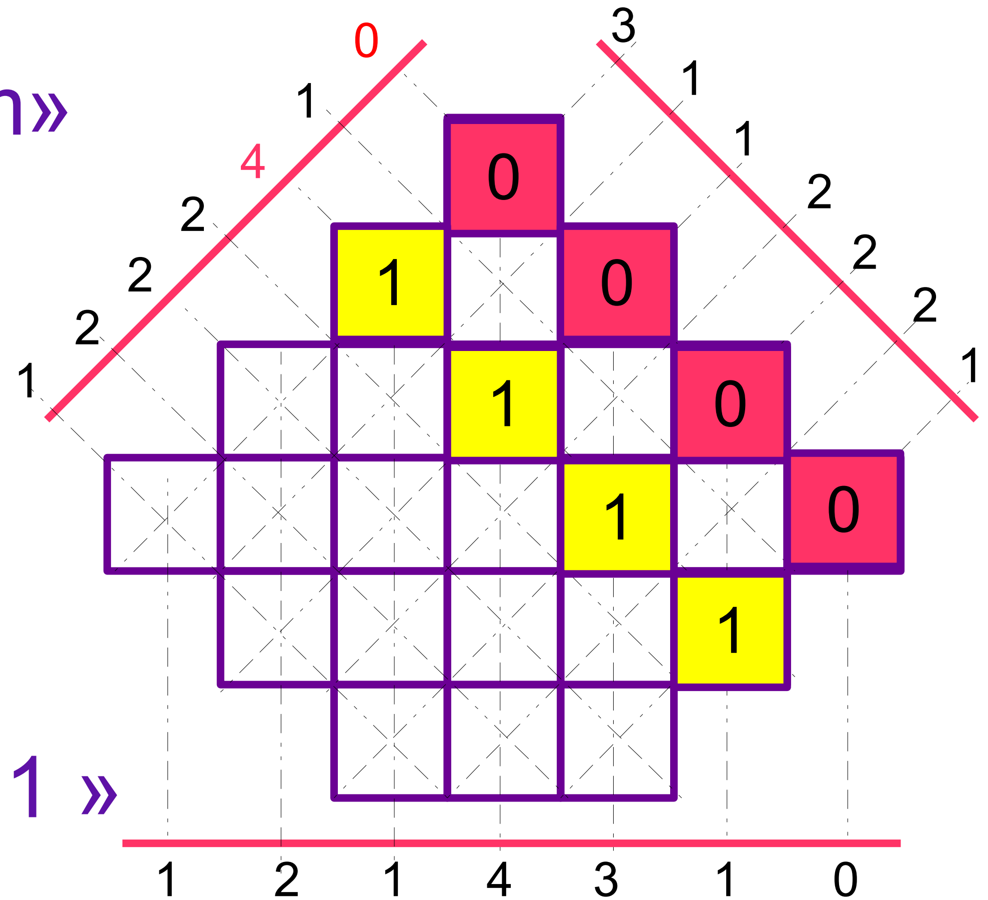
If bin = 0 the line is zero
If bin=n the line is full of « 1 »



2. Tomographic Mojette reconstruction

2.1 Binary Mojette

« The Mojette line algorithm»



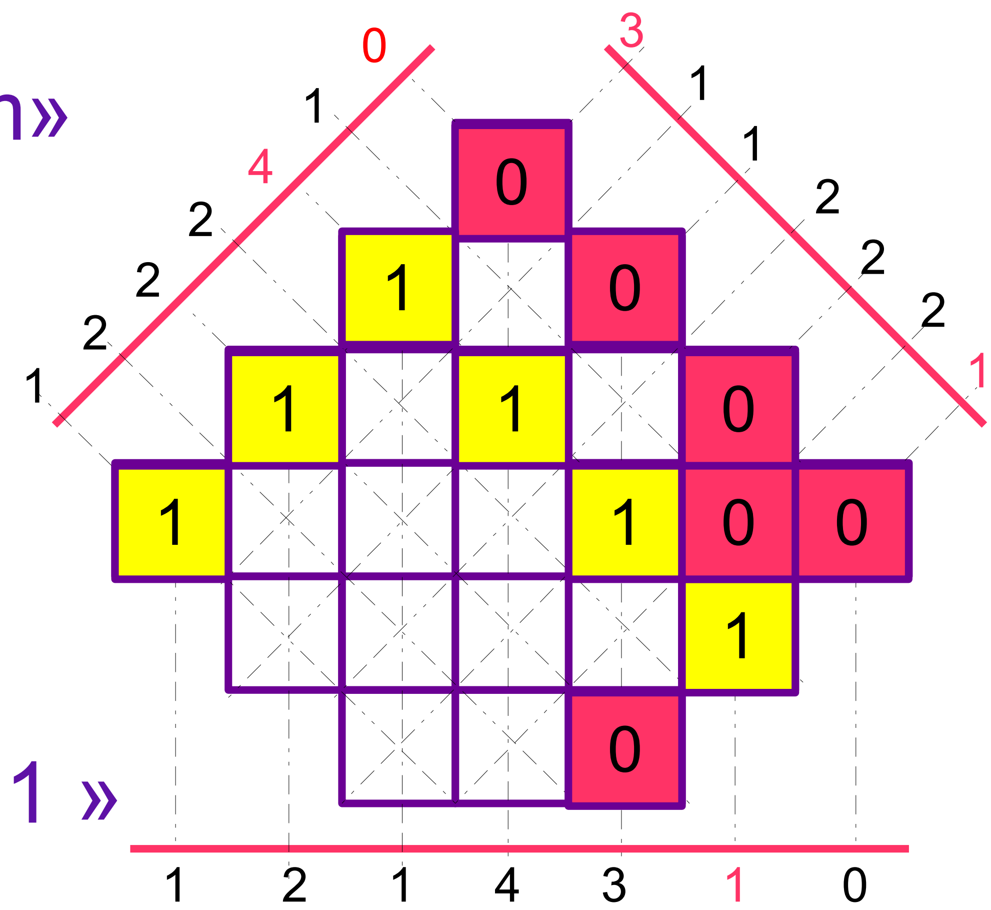
Bin sum of n pixels

If bin = 0 the line is zero
If bin=n the line is full of « 1 »

2. Tomographic Mojette reconstruction

2.1 Binary Mojette

« The Mojette line algorithm»



Bin sum of n pixels

If bin = 0 the line is zero

If bin=n the line is full of « 1 »

2. Tomographic Mojette reconstruction

2.1 Binary Mojette

« The Mojette line algorithm»

Bin sum of n pixels

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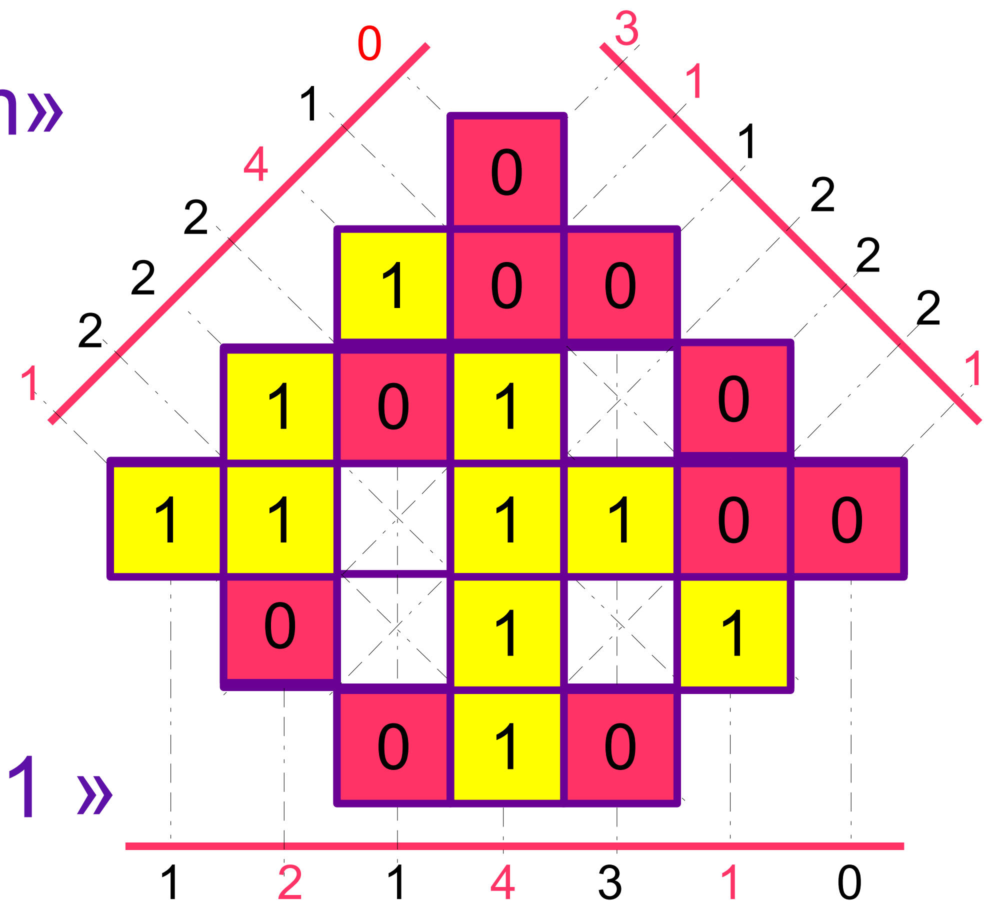
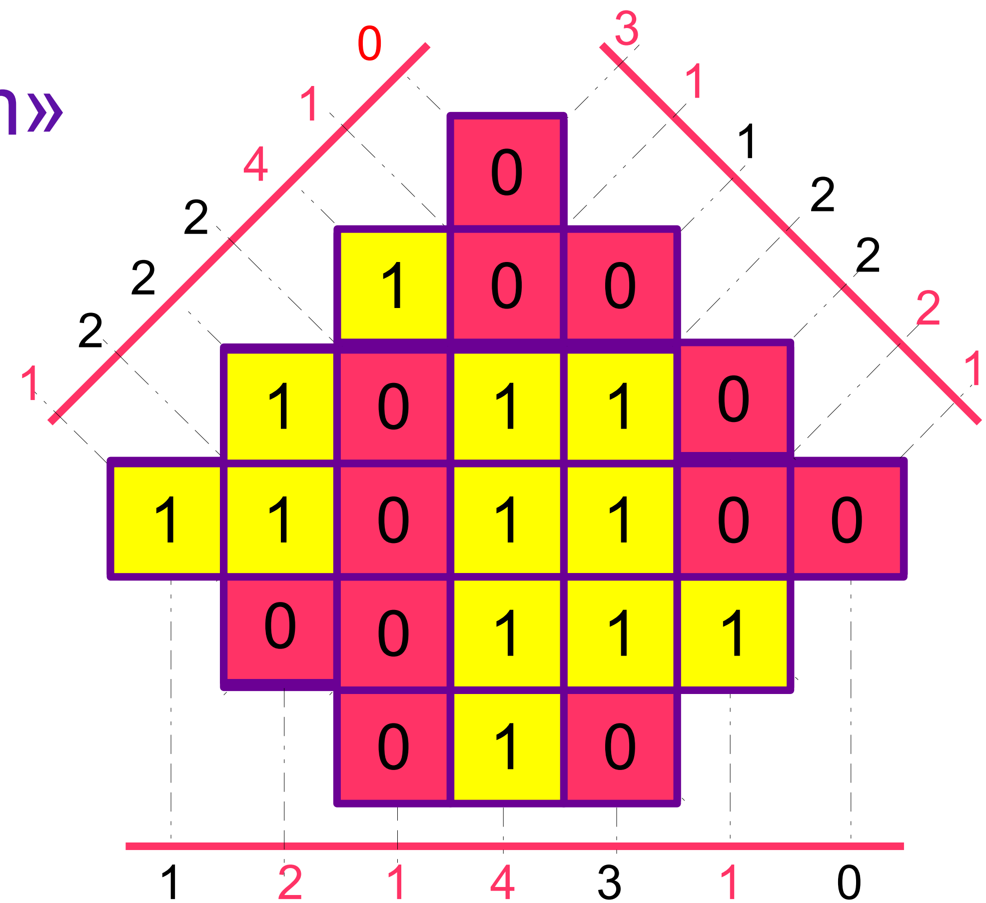


2. Tomographic Mojette reconstruction

2.1 Binary Mojette

« The Mojette line algorithm»

12 bins reconstruct
24 pixels ...



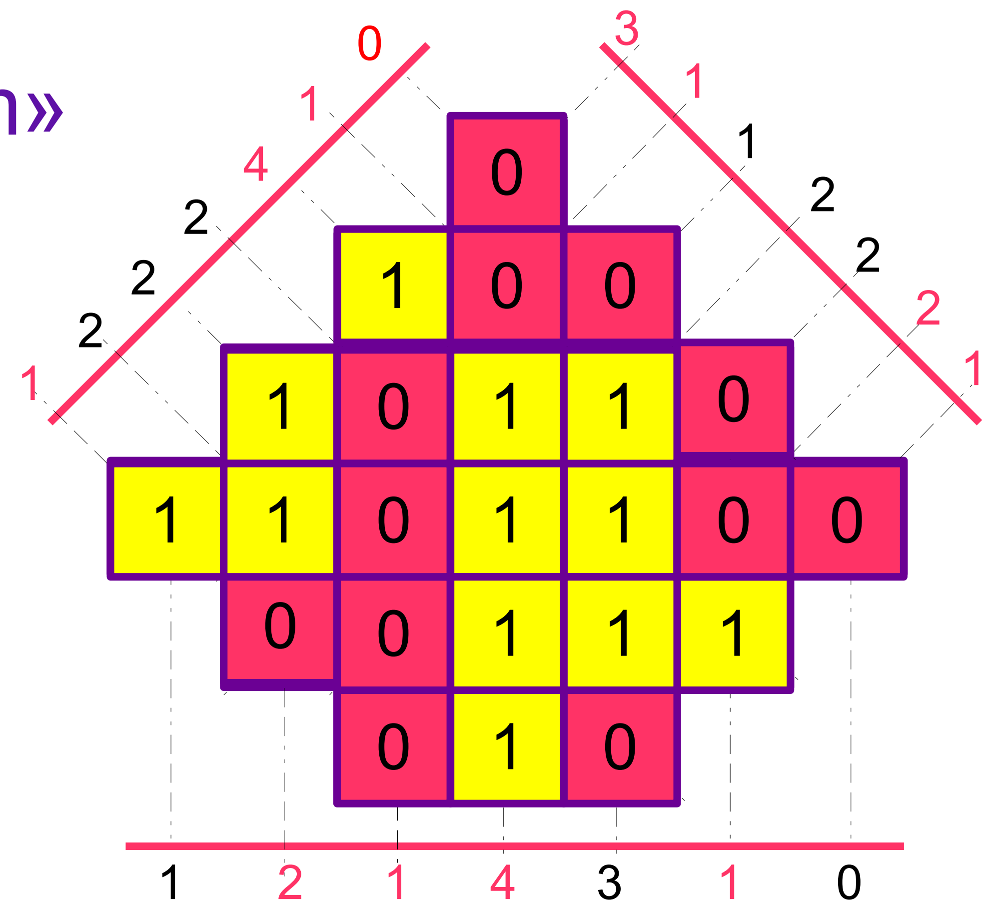
2. Tomographic Mojette reconstruction

2.1 Binary Mojette

« The Mojette line algorithm»

conclusion

When the image is composed of compact objects the Mojette line can be of interest (for compression purposes)



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

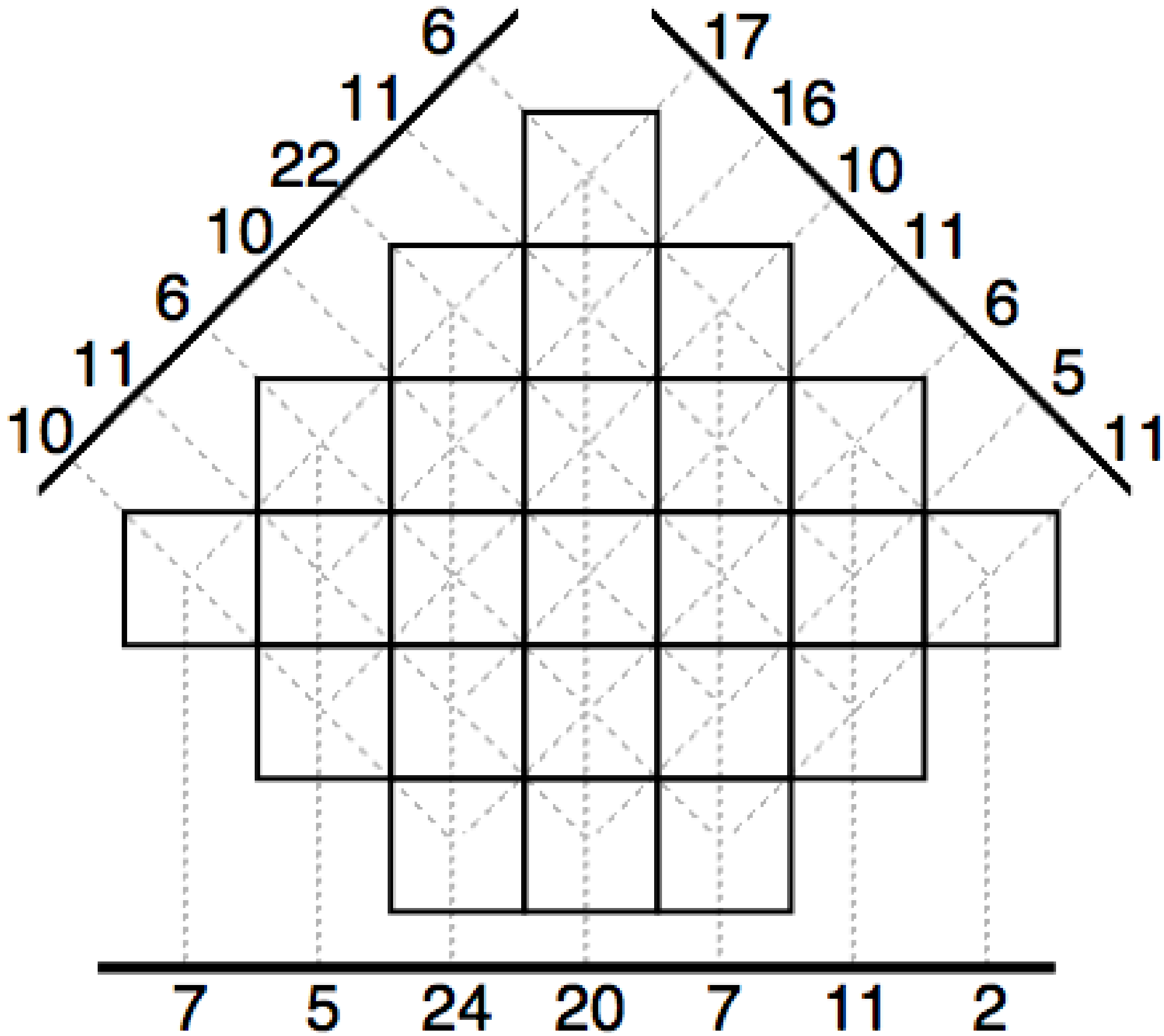
2010 : The Mojette game

3 rules :

1-Each pixel is a number(0 to 9).

2-There are only 3 different numbers per grid.

3- A bin sums the pixels along the discrete line



2. Tomographic Mojette reconstruction

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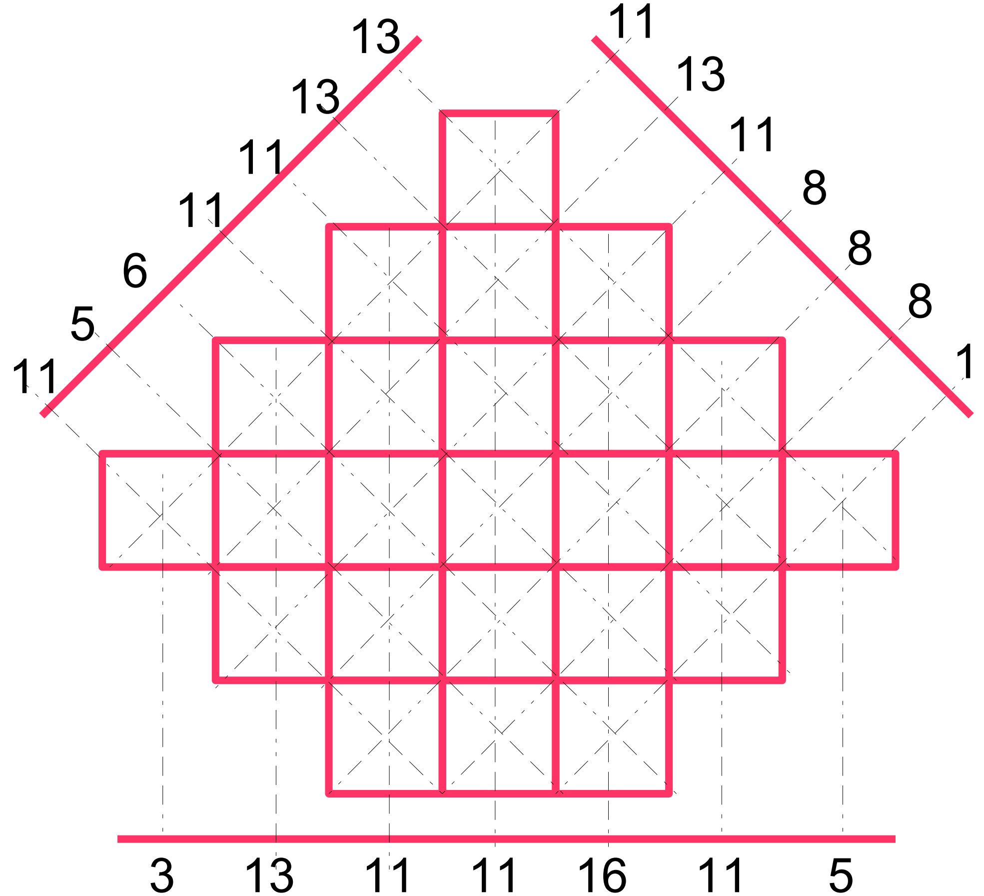
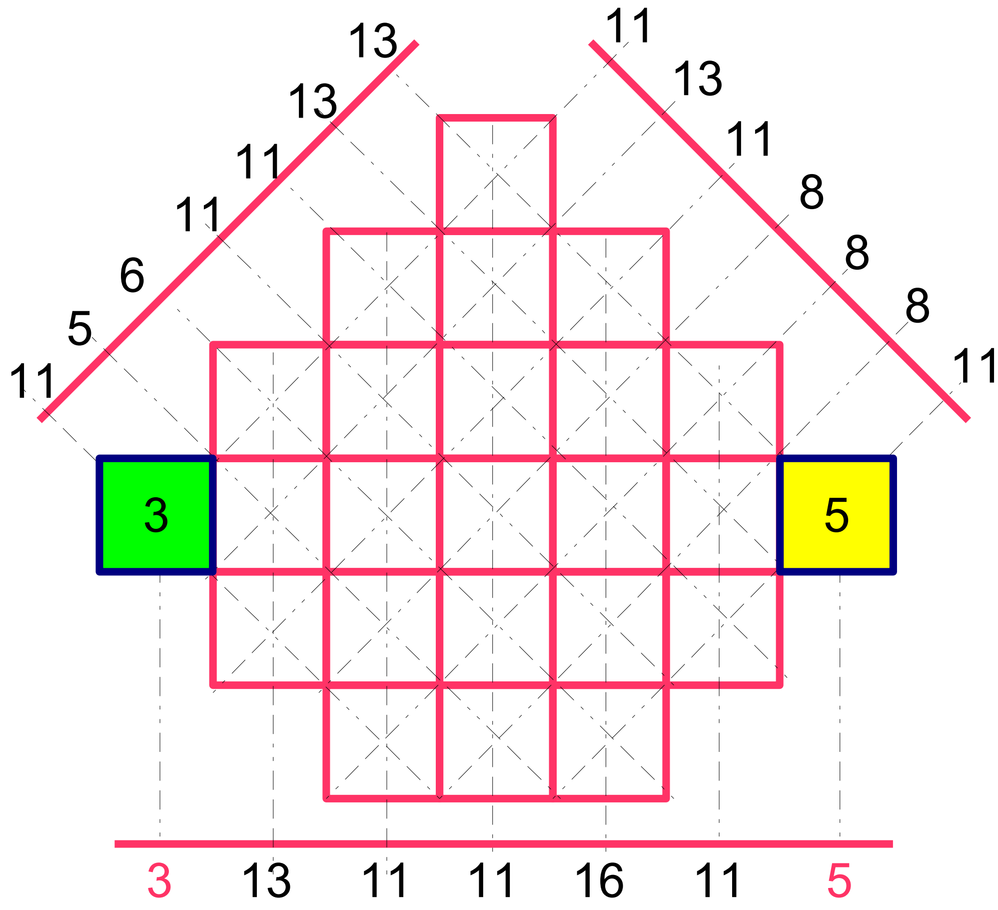


2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

2010 : The Mojette game

Start with 1-1 relationship



2. Tomographic Mojette reconstruction

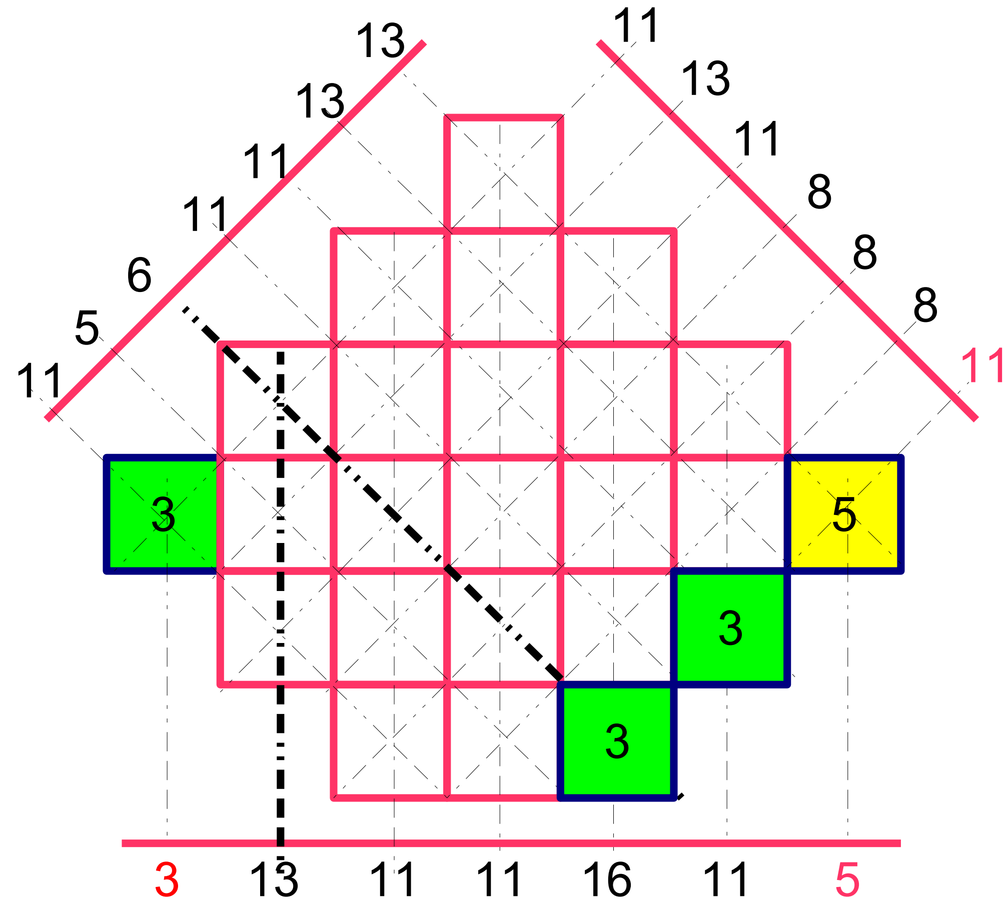
2.1 Binary or Ternary reconstruction

2010 : The Mojette game

Find third value

Third value = 0 or 1

If 1 : 5=3+1+1
And 6=3+1+1+1
But 8=?+?+? no



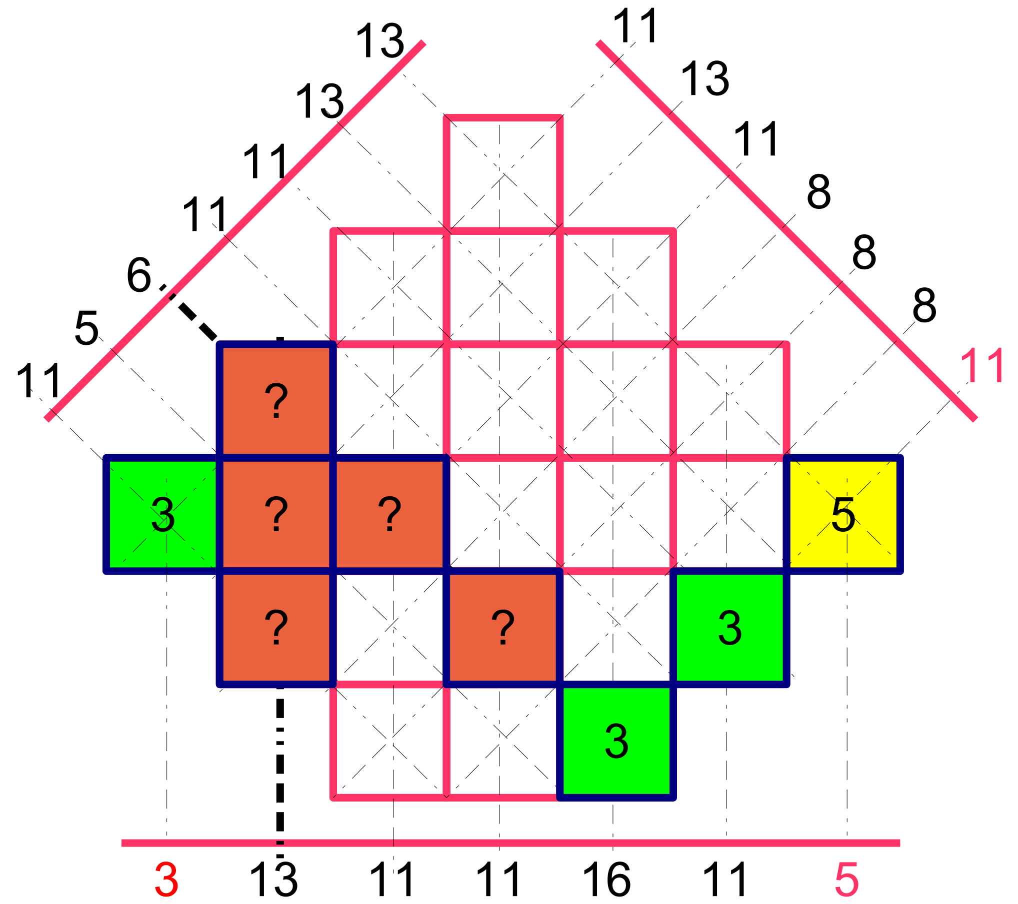
2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

2010 : The Mojette game

Find intersecting lines

6 = 3+3+0+0
13 = 5+5+3



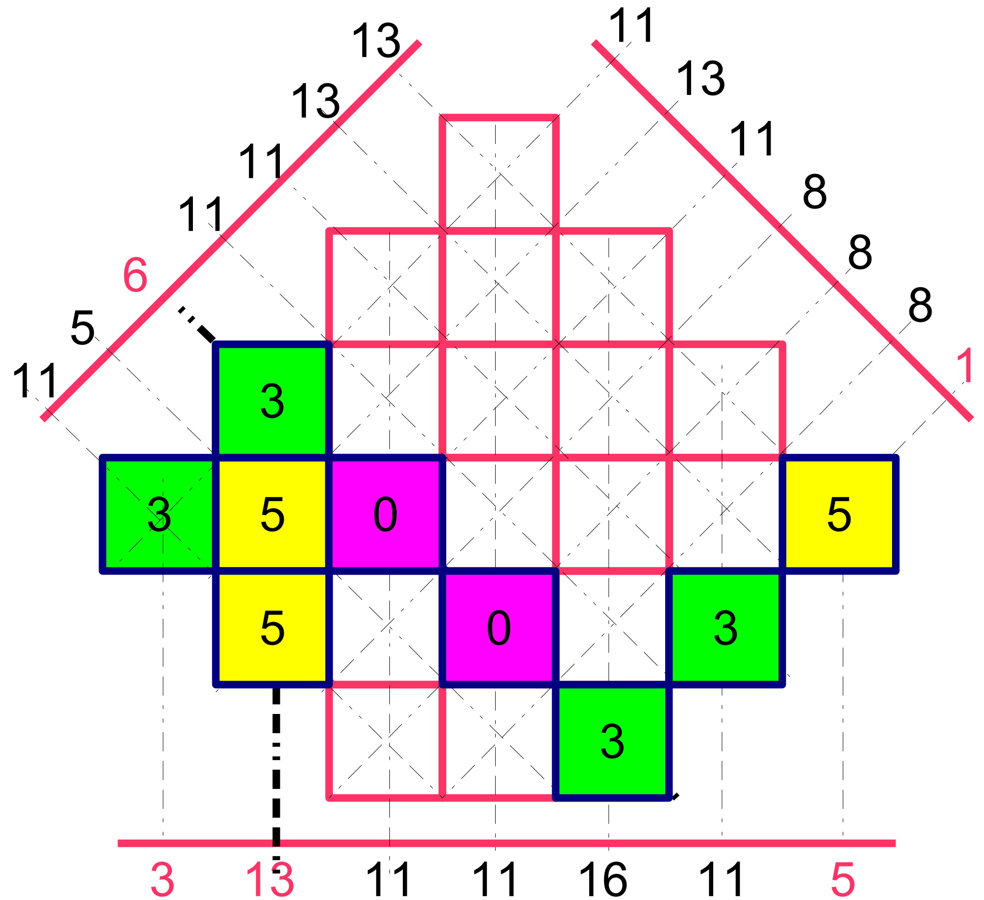
2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

2010 : The Mojette game

Find intersecting lines

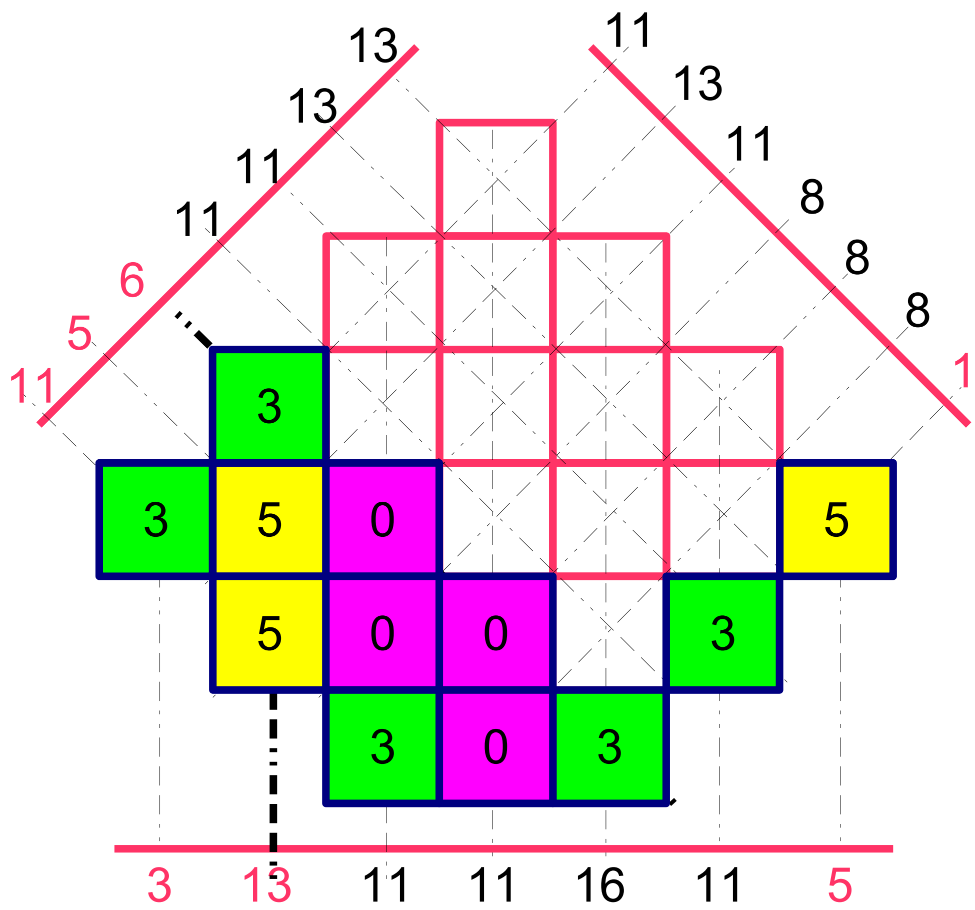
6 = 3+3+0+0
13 = 5+5+3



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

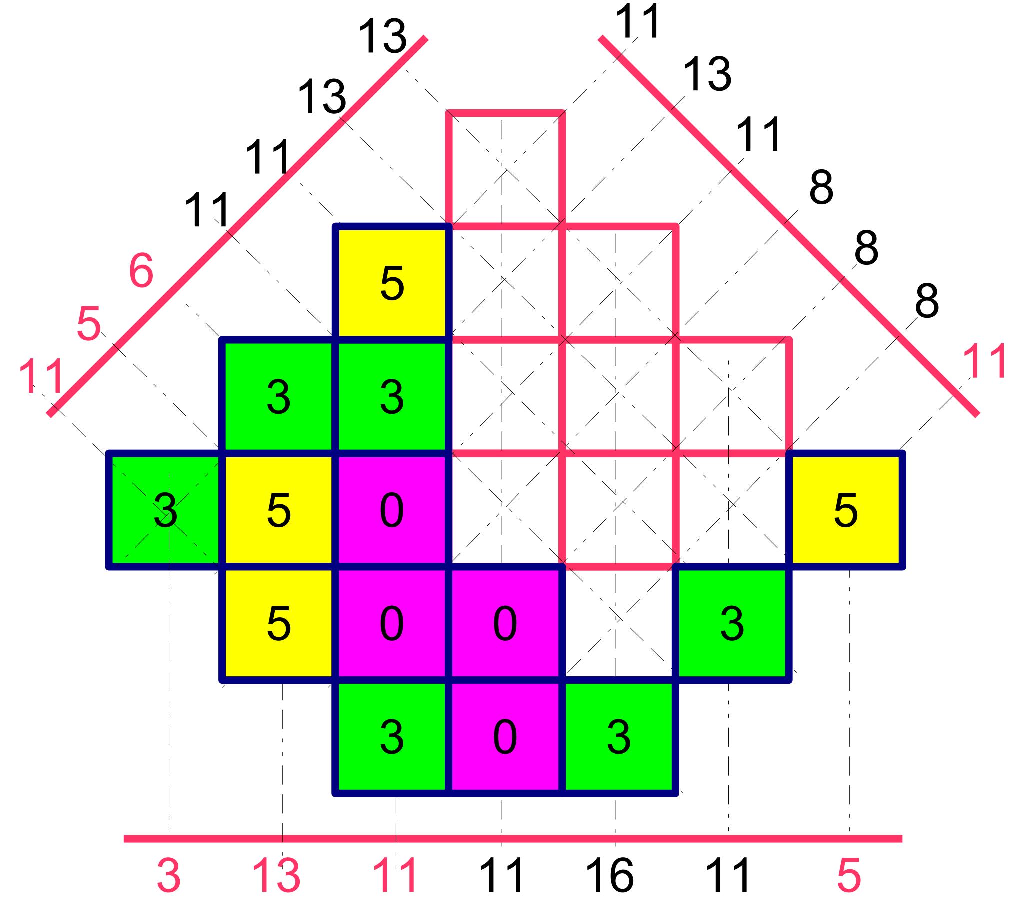
2010 : The Mojette game



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

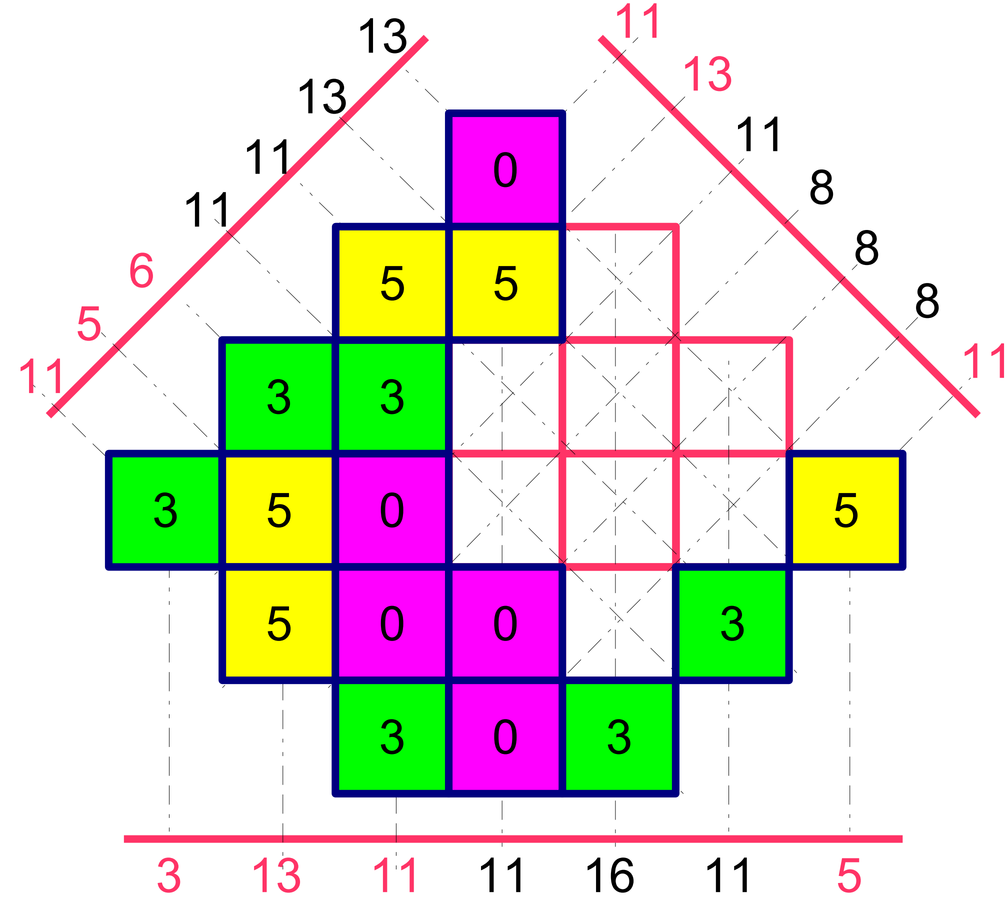
2010 : The Mojette game



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

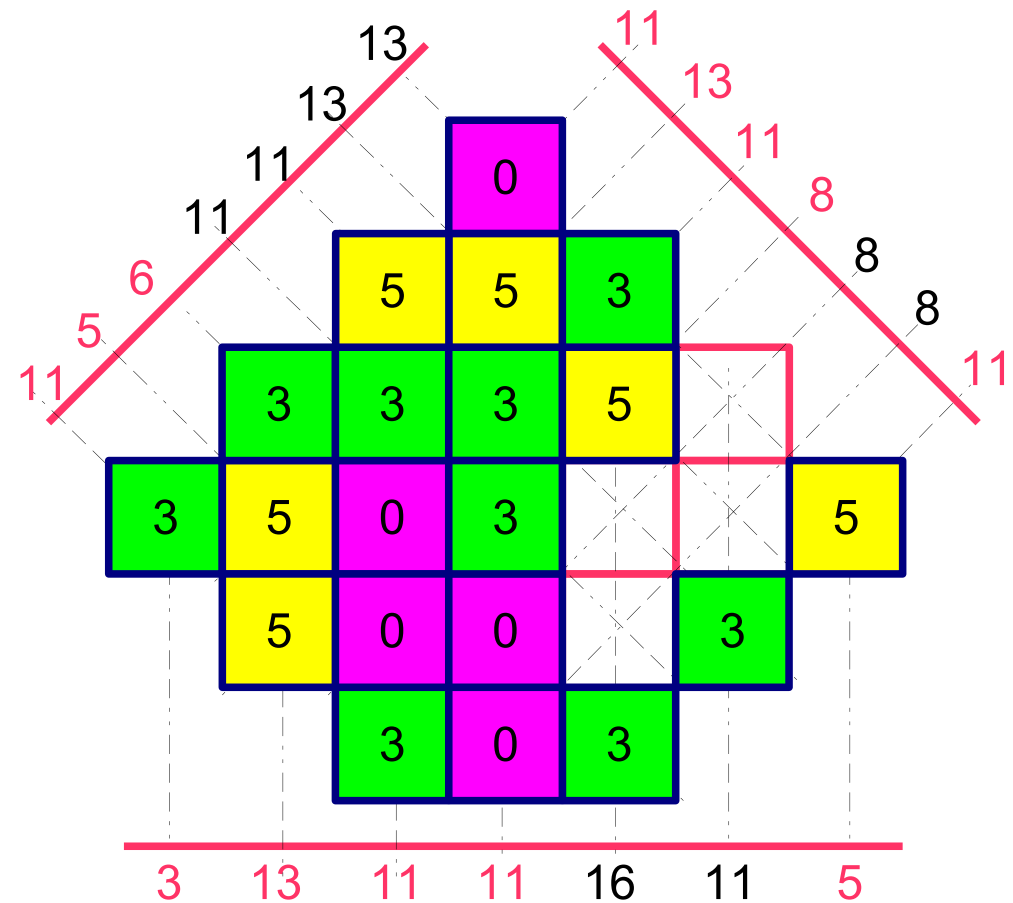
2010 : The Mojette game



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

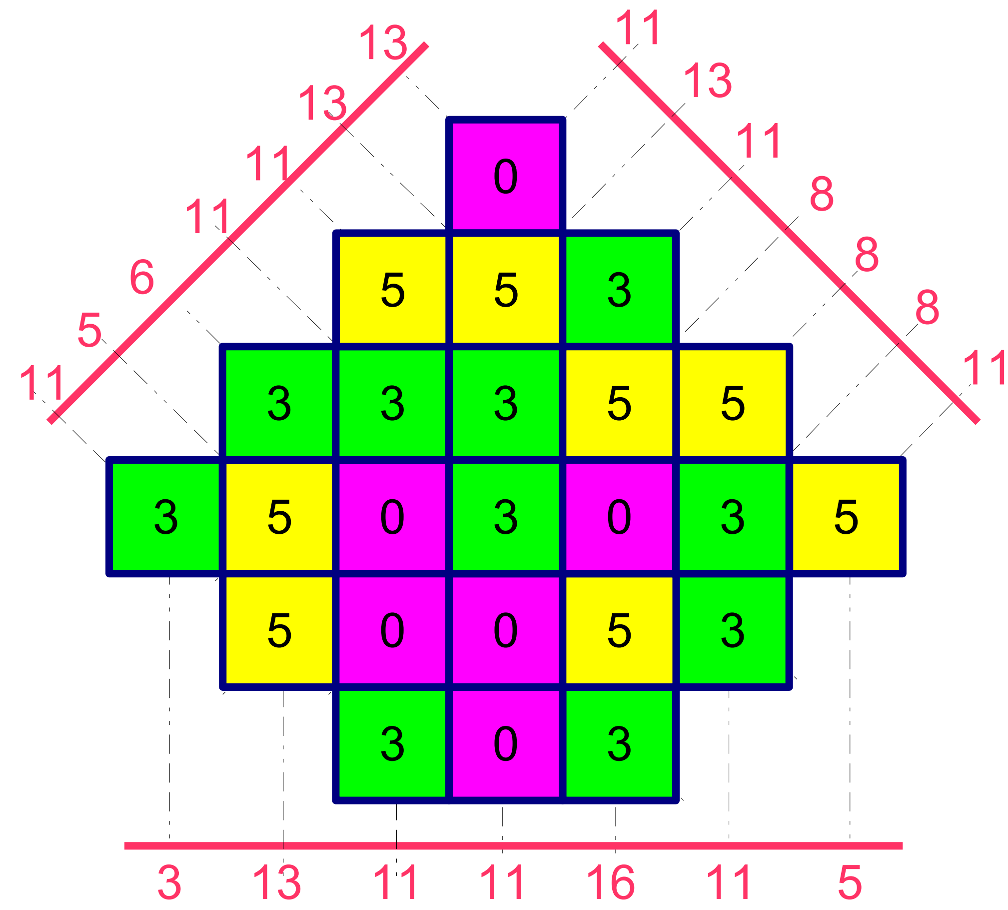
2010 : The Mojette game



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

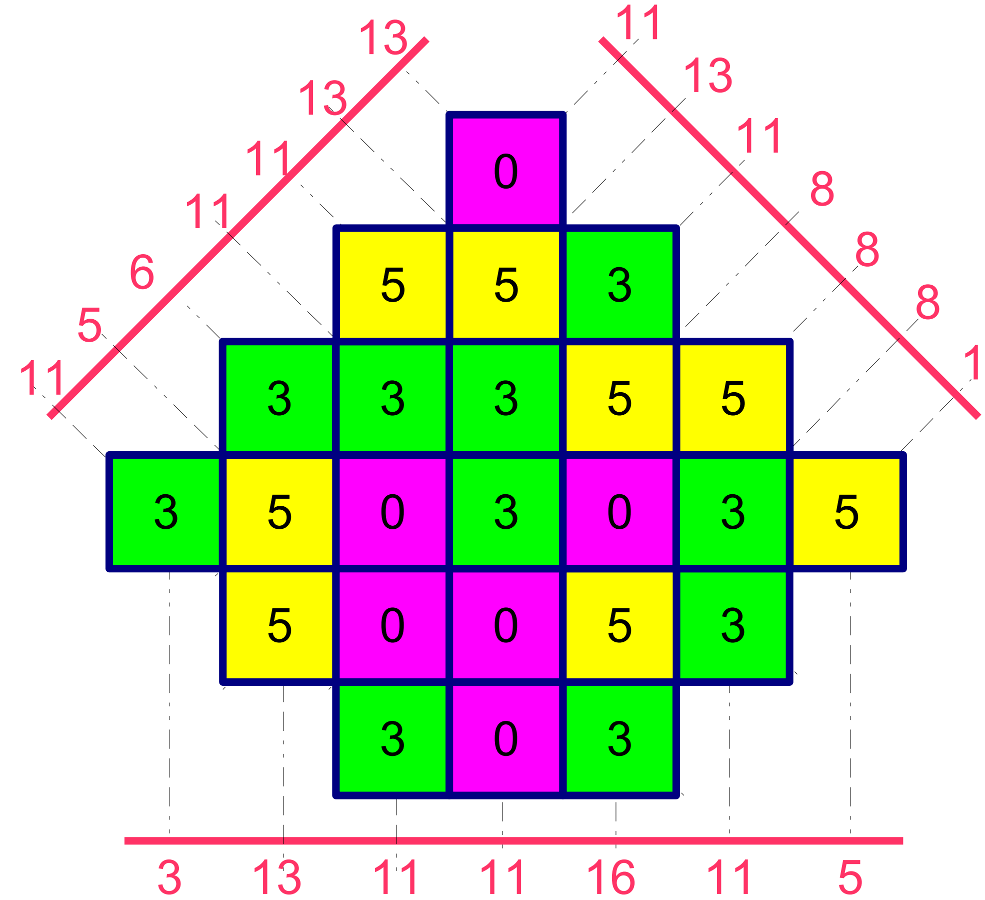
2010 : The Mojette game



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

2010 : The Mojette game



2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

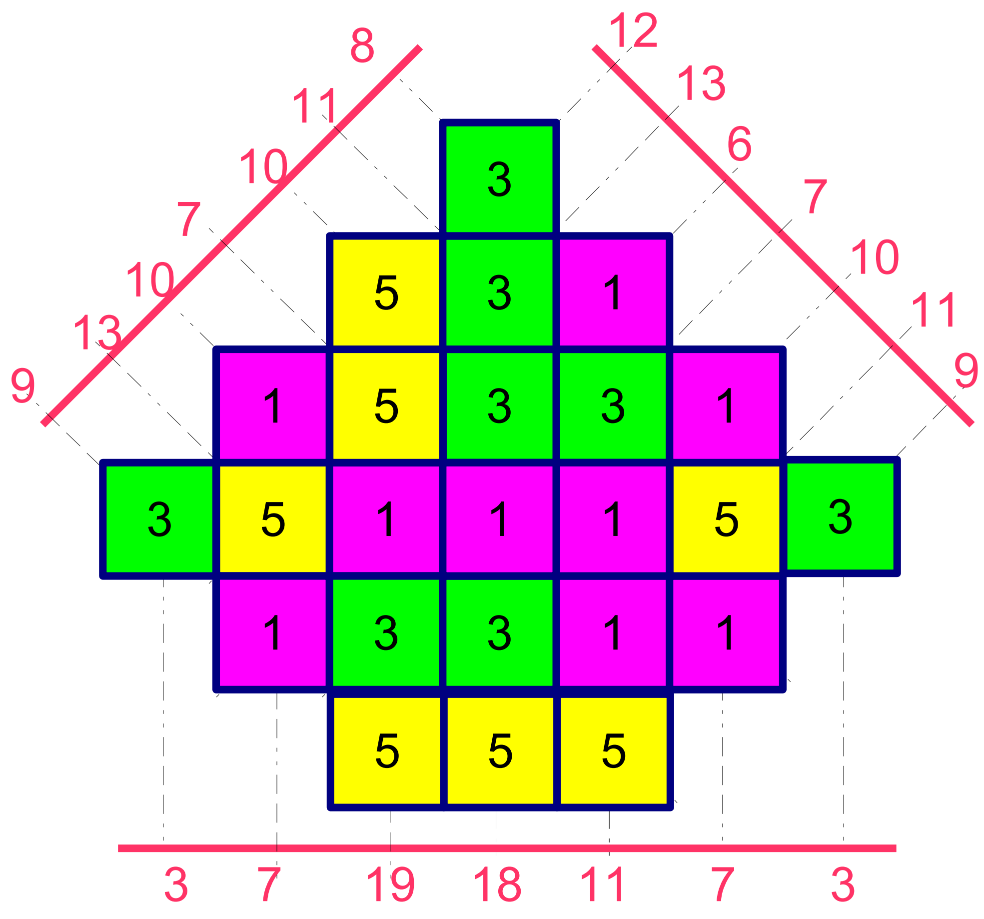
2010 : The Mojette game

Grid with possible ghost

Because a<b<c values
Such that

$$b - a = c - b$$

Here b – a = 2



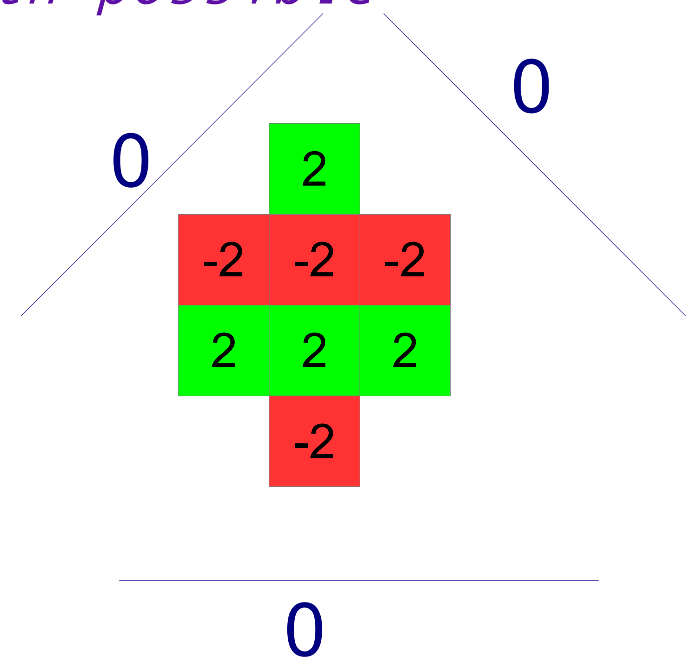
2. Tomographic Mojette reconstruction

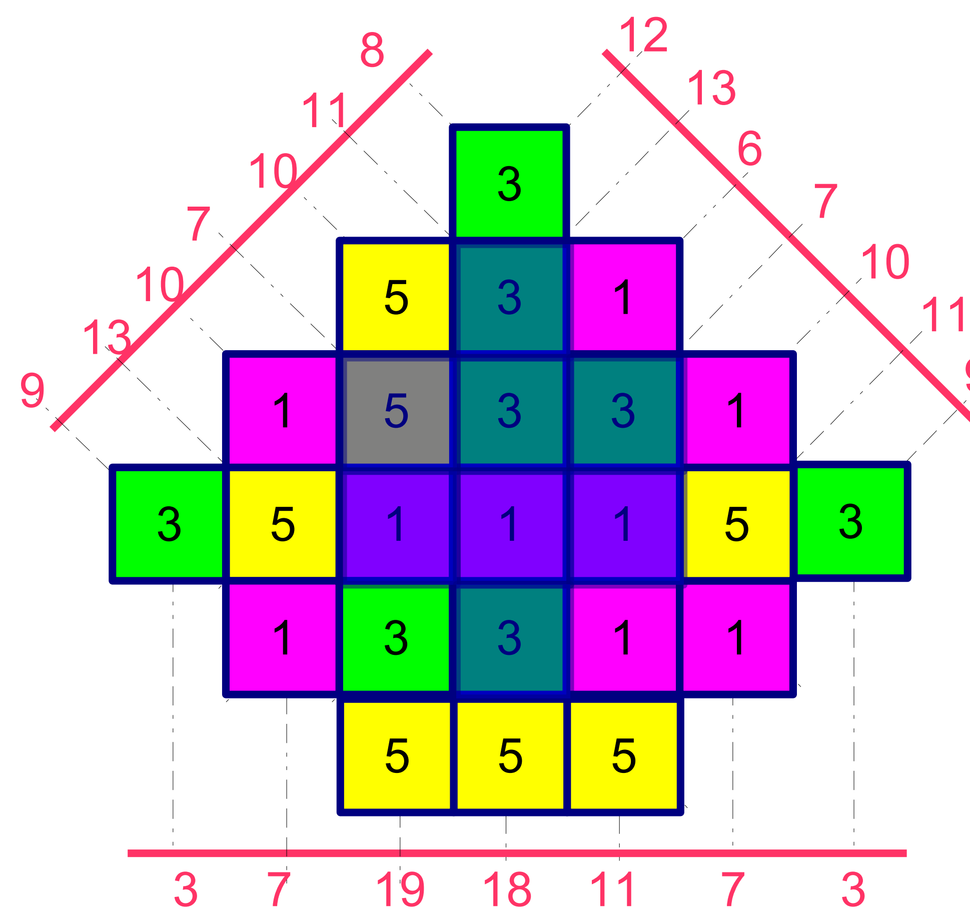
2.1 Binary or Ternary reconstruction

2010 : The Mojette game

3 rules :

Grid with possible ghost

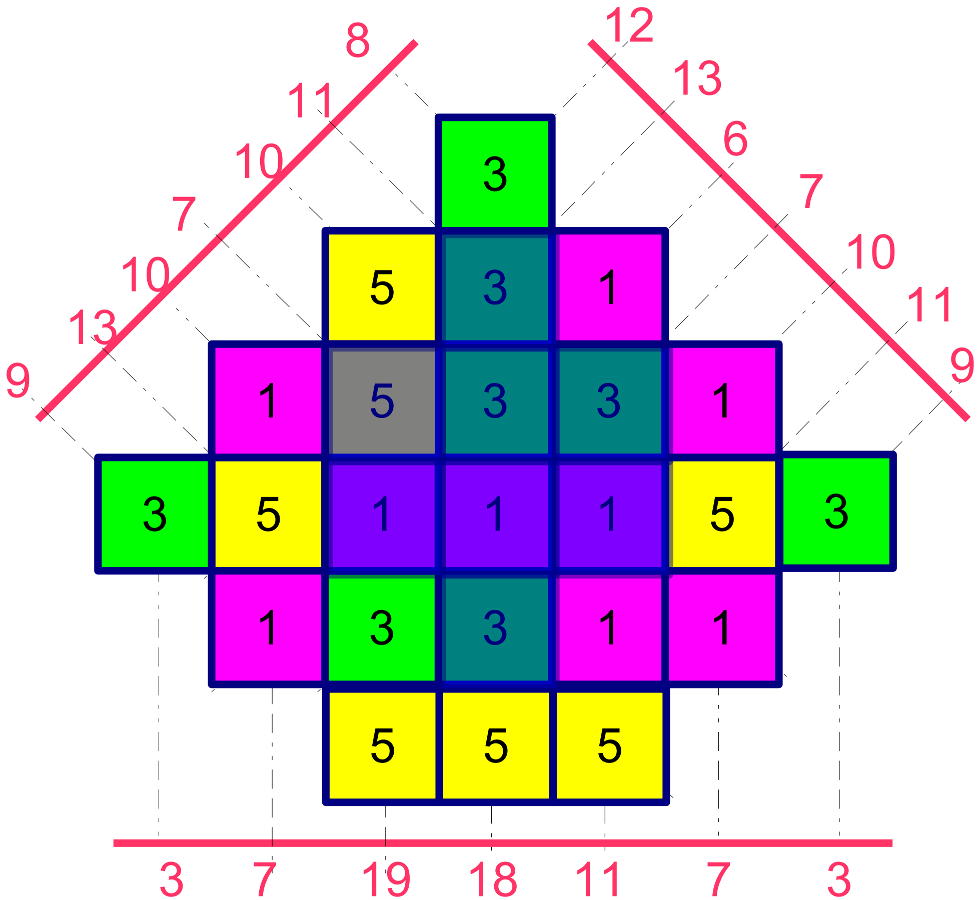


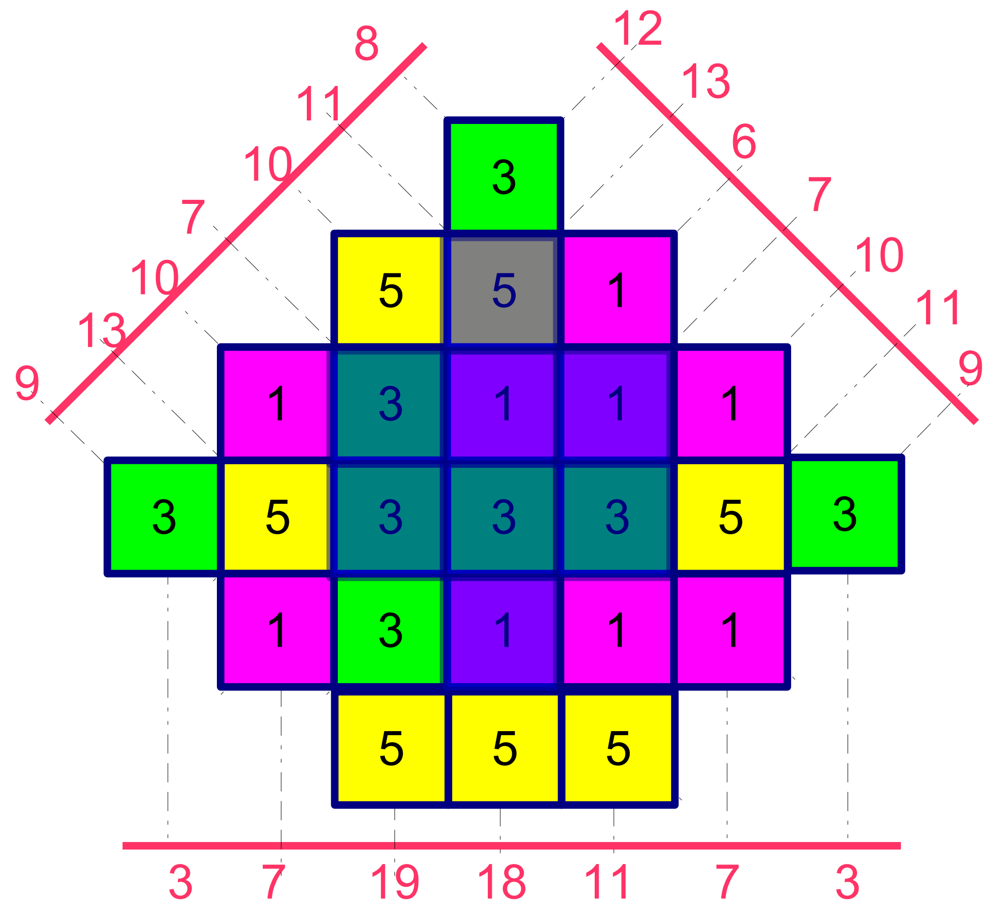


2. Tomographic Mojette reconstruction

2.1 Binary or Ternary reconstruction

2010 : The Mojette game





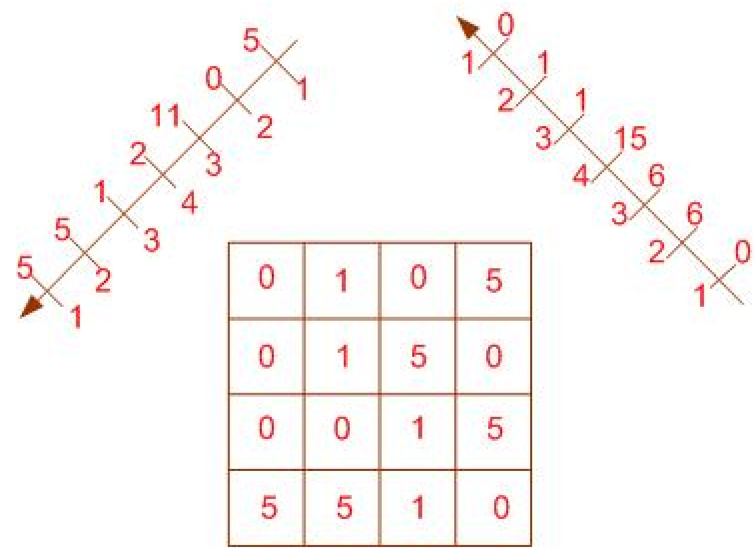
2. Tomographic Mojette reconstruction

2.1 Ternary reconstruction

2013 : Ph.D. Chuanlin LIU

Suppose the image composed of 3 very different materials

(e.g . bone air water)
Here 0 1 and 5



2. Tomographic Mojette reconstruction

2.1 Ternary reconstruction

2013 : Ph.D. Chuanlin LIU

Summing 2 pixels

Build tables for all possible sums with 0 1 5

	0	1	2	5	6	10
0	2	1	0	1	0	0
1	0	1	2	0	1	0
5	0	0	0	1	1	2

Summing 4 pixels

sums

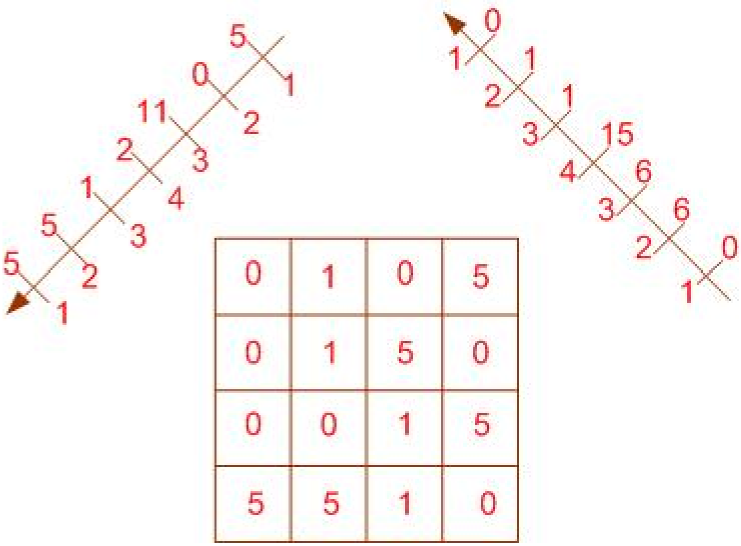
Number of

	0	1	2	3	4	5	6	7	8	10	...
0	4	3	2	1	0	3	2	1	0	2	...
1	0	1	2	3	4	0	1	2	3	0	...
5	0	0	0	0	0	1	1	1	1	2	...

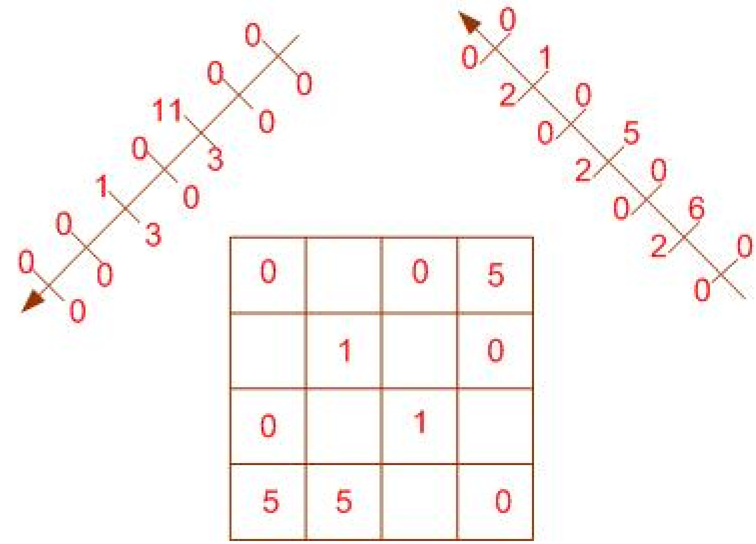
2. Tomographic Mojette reconstruction

2.1 Ternary reconstruction

2013 : Ph.D. Chuanlin LIU



Backproject 0 1 5
according to tables



This is a new « line backprojection »

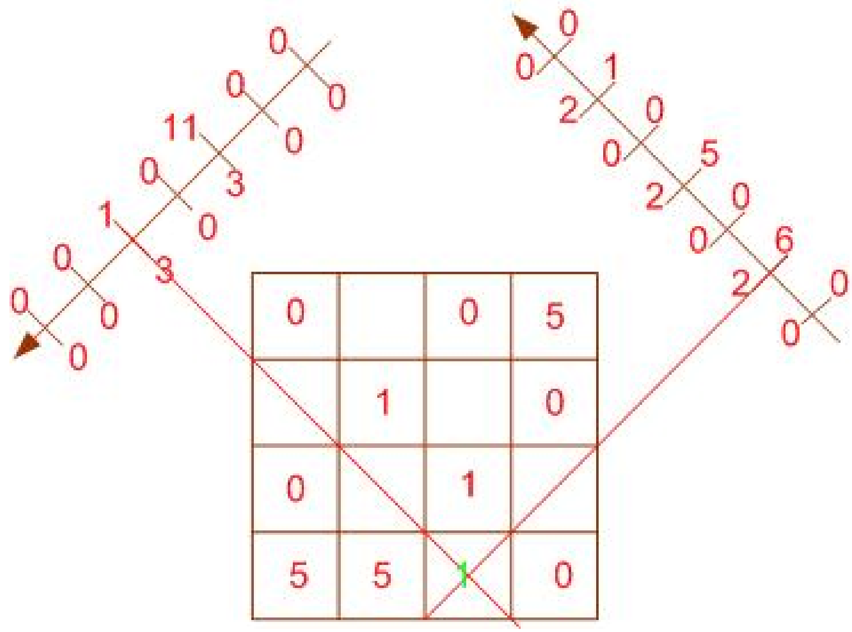
2. Tomographic Mojette reconstruction

2.1 Ternary reconstruction

2013 : Ph.D. Chuanlin LIU

When no more corespondence bins,

Find pixel from intersected lines (2 or several bins)



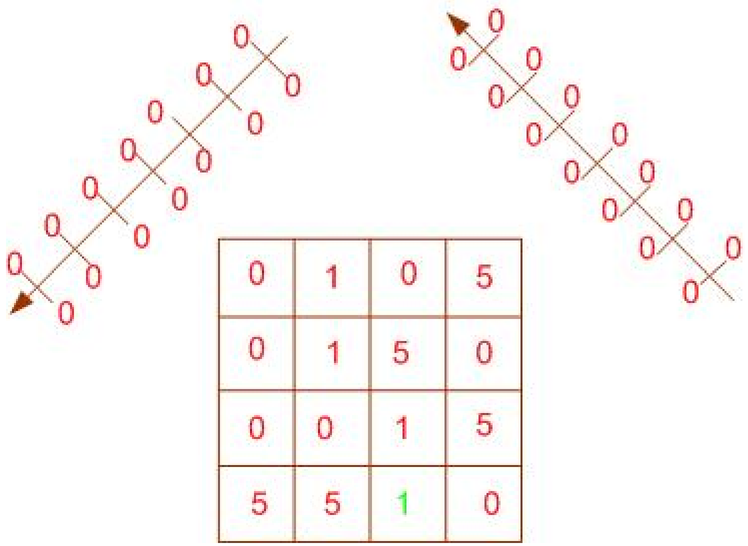
Here « 1 » is the pixel value from bins values 6=5+1 and 1=1+0+0

2. Tomographic Mojette reconstruction

2.1 Ternary reconstruction

2013 : Ph.D. Chuanlin LIU

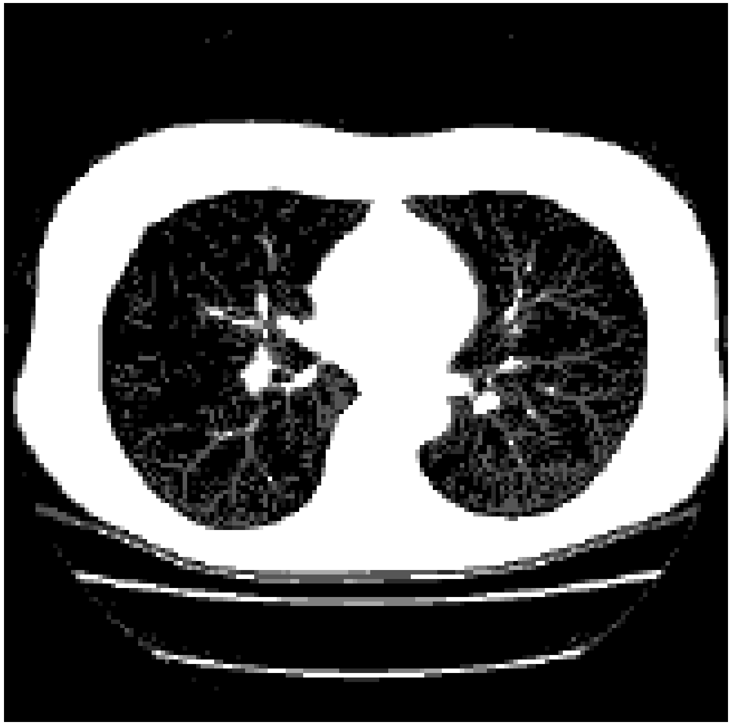
Here, we recover the entire image



2. Tomographic Mojette reconstruction

2.1 Ternary reconstruction

2013 : Ph.D. Chuanlin LIU



Ternary 256x256 image
the set

$$(S=\{\begin{matrix}(1,0),(0,1),(1,1),(-2,1),(2,1),(1,2),(-1,2),\\(1,3),(-1,3),(3,1),(-3,1),(2,3),(-2,3),(3,2),(-3,2),\\(1,4),(-1,4),(3,4),(-3,4)\\(1,5),(-1,5),(5,1),(-5,1),(2,5),(-2,5)\end{matrix}\})$$

reconstructs the image with

$$(\sum_i |p_i|=50, \sum_i |q_i|=64) \quad (\mathrm{Red}=\frac{\#\,bins}{\#\,pixels}-1=-0.56)$$

2. Tomographic Mojette reconstruction

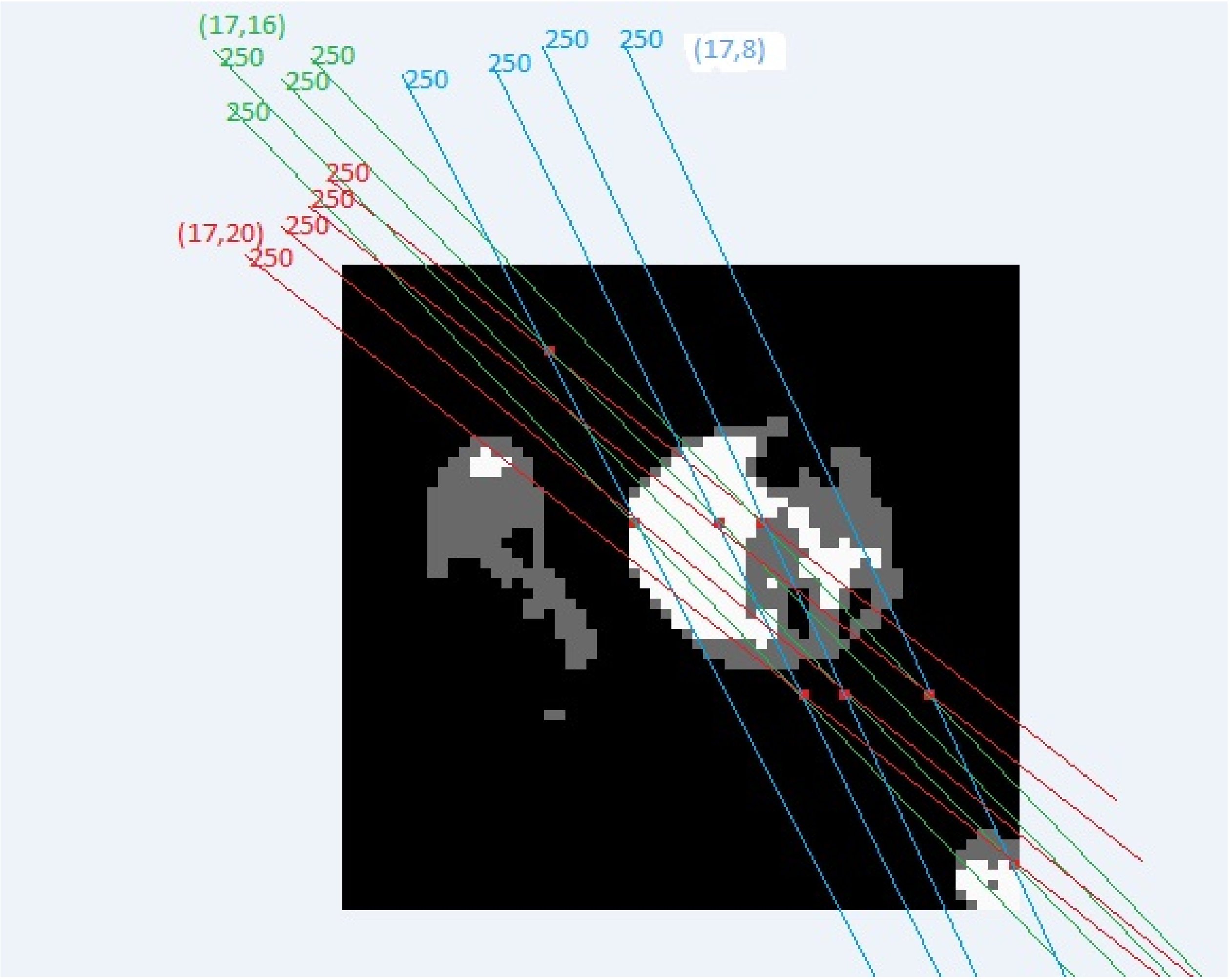
2.1 Ternary reconstruction

2013 : Ph.D. Chuanlin LIU

64x64 image with 3 projections

8 (red dots) pixels Not reconstructed

=> ghost
Putting 1 pixel value solve for the ghost



2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005
(also PhD Andrew Kingston for FRT)

$$f_i(k,l) = M_i^* M_i f(k,l)$$

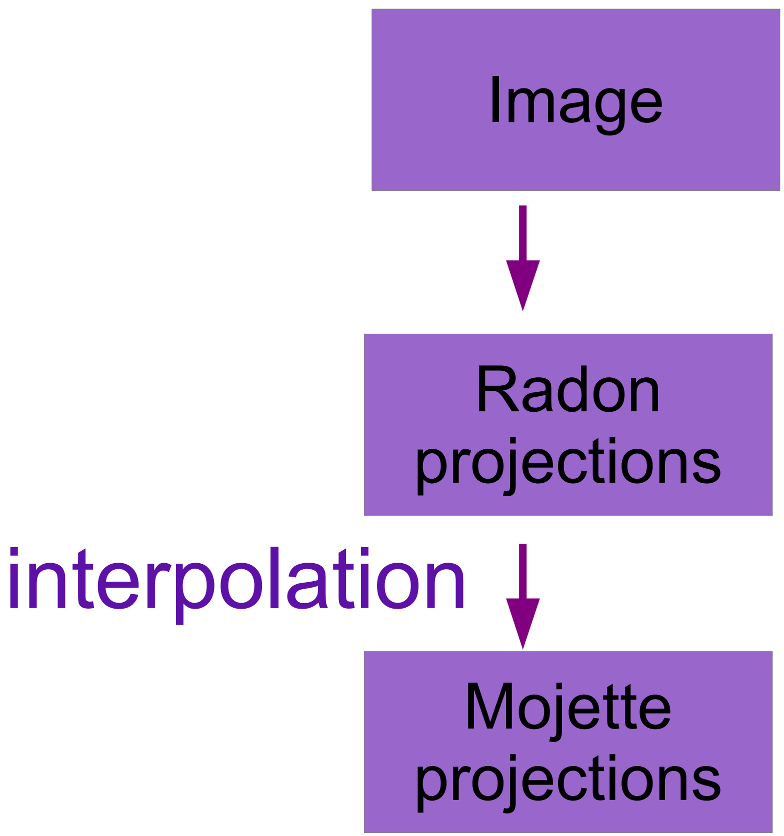
But considering only the possible I angles existing onto a NxN support (Farey-Haros N-1):

$$g(k,l) = (I-1)\, f(k,l) + \sum_{i=1}^{N} \sum_{j=1}^{N} f(i,j)$$

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005



Mojette Tomographic scheme

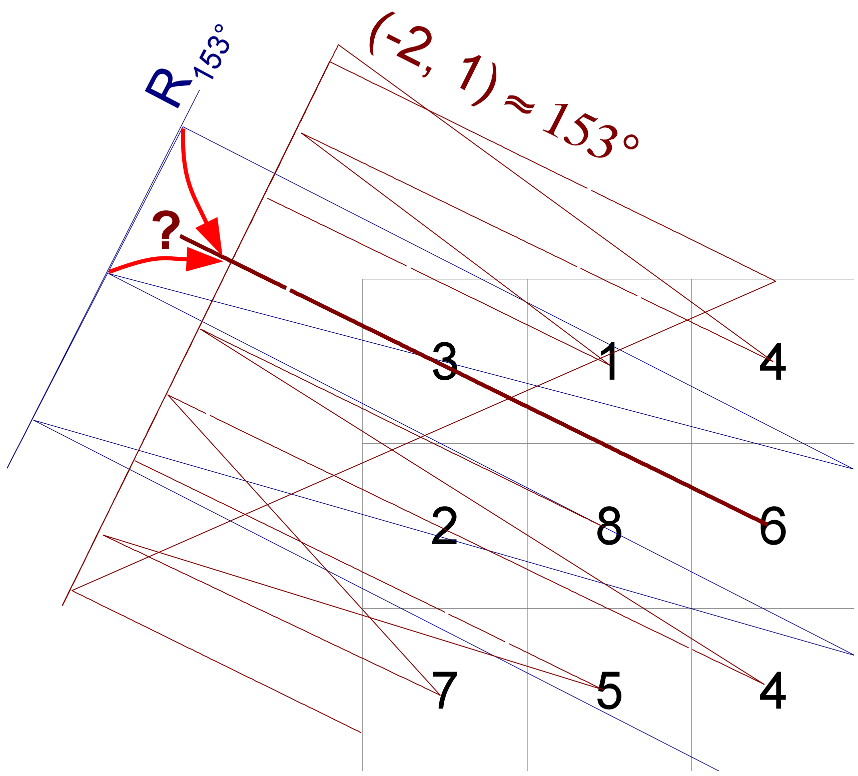
2. Tomographic Mojette reconstruction

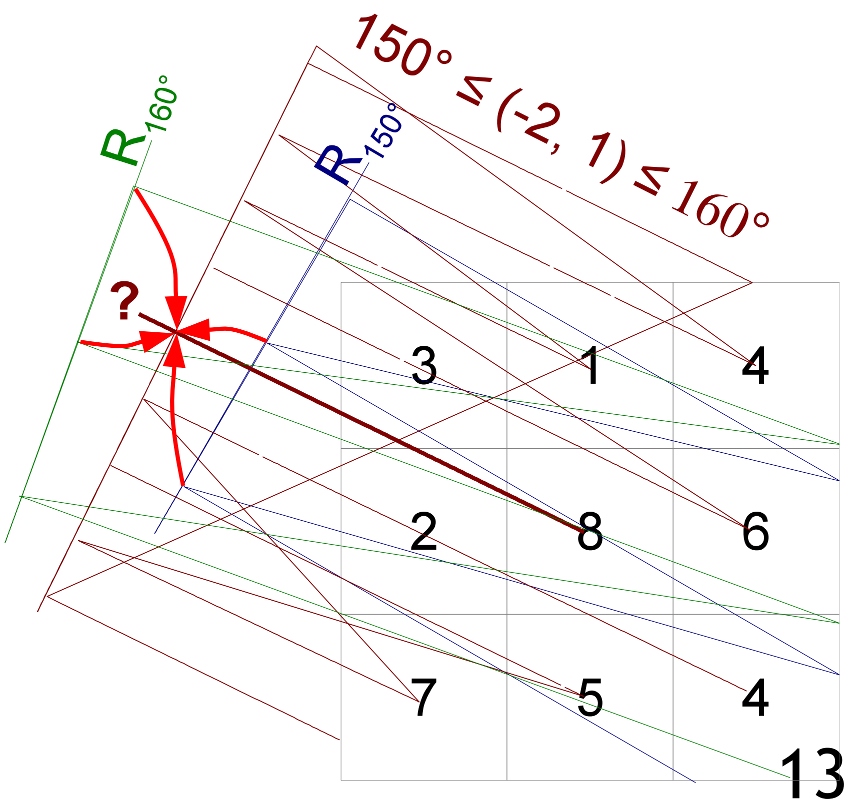
2.2 From Radon to Mojette space

PhD Myriam Servières 2005

Taking Radon projections

Approximating into (spline) Mojette space

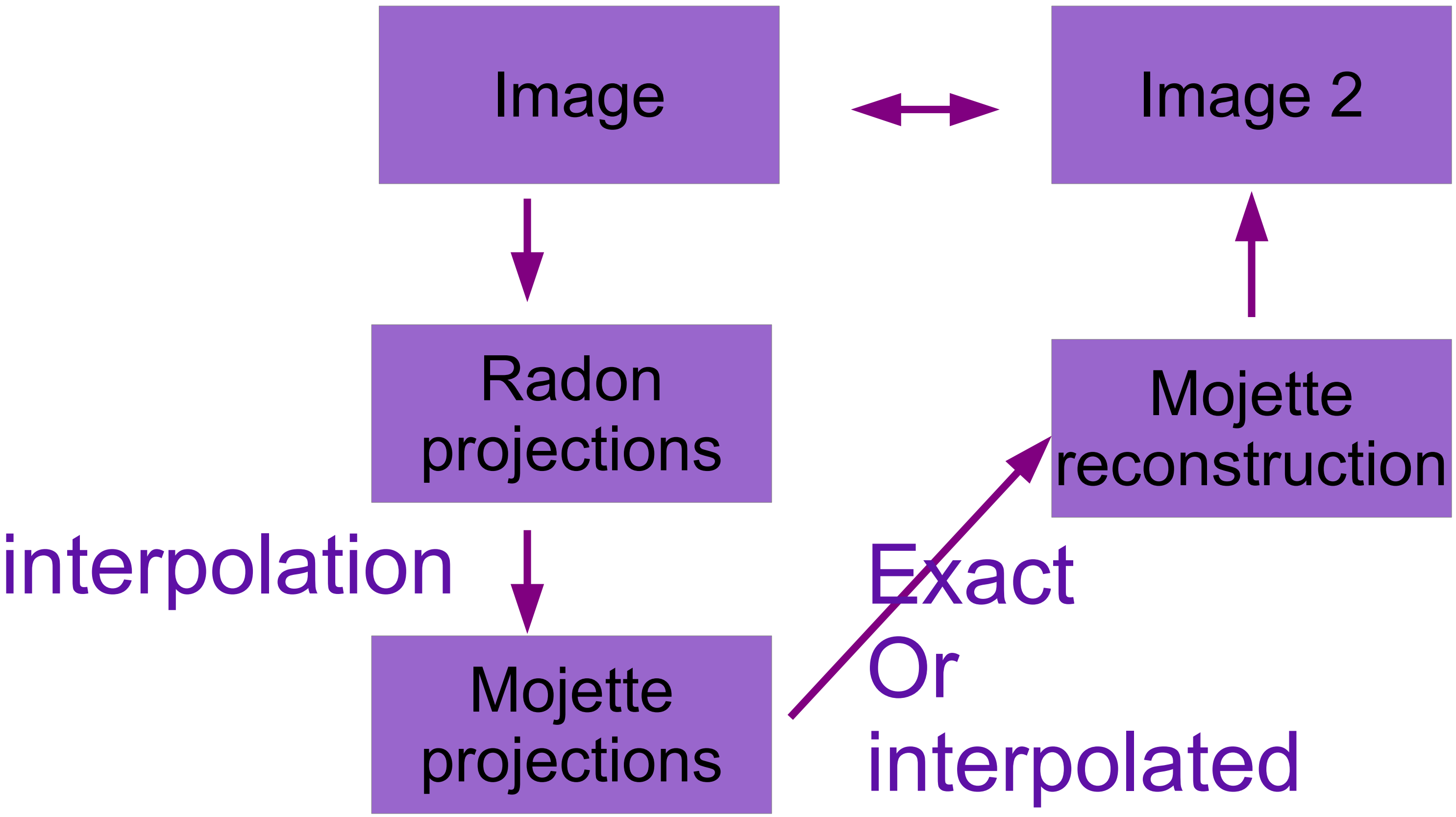




2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005

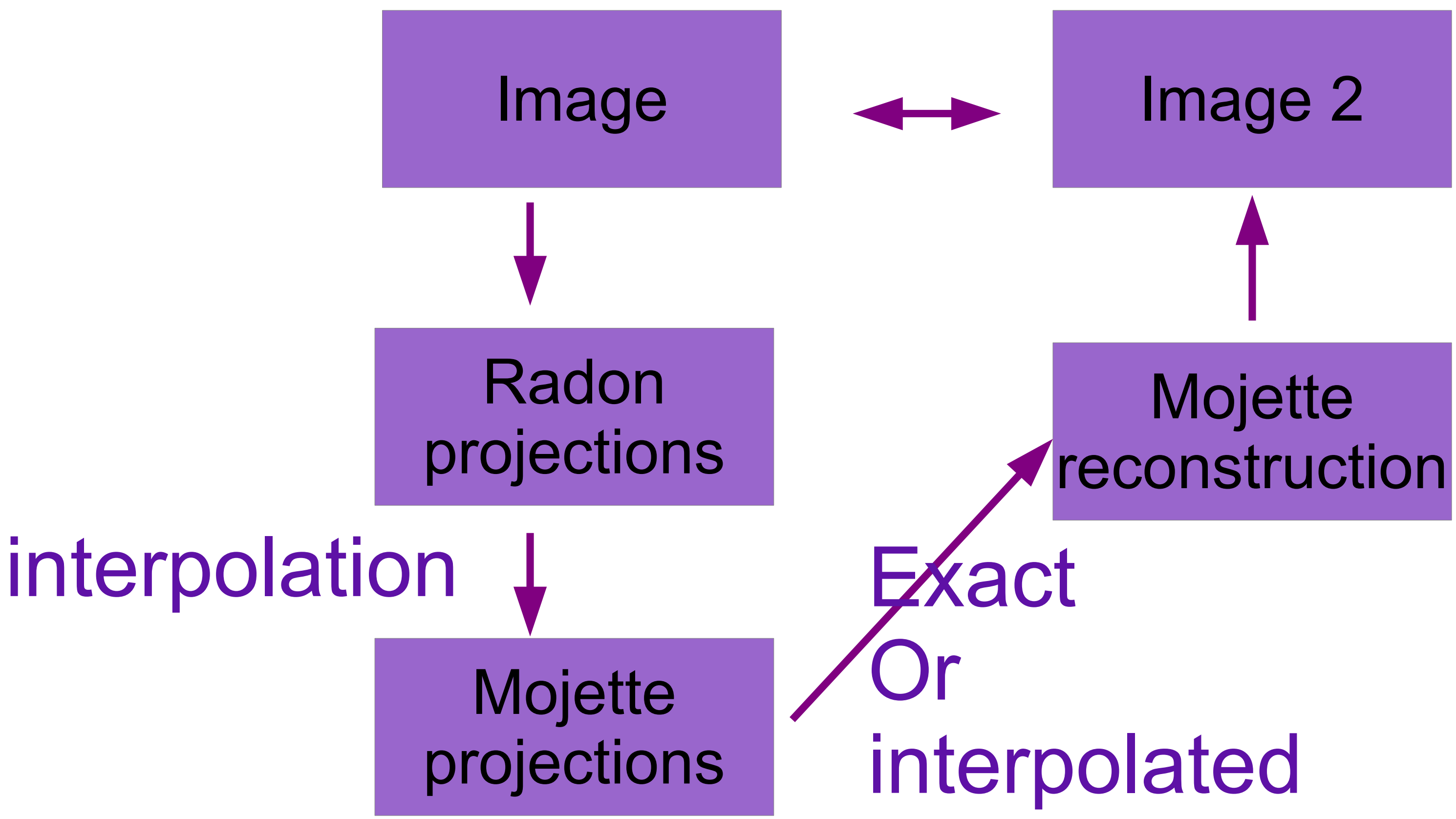


Mojette Tomographic scheme

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005



Mojette Tomographic scheme

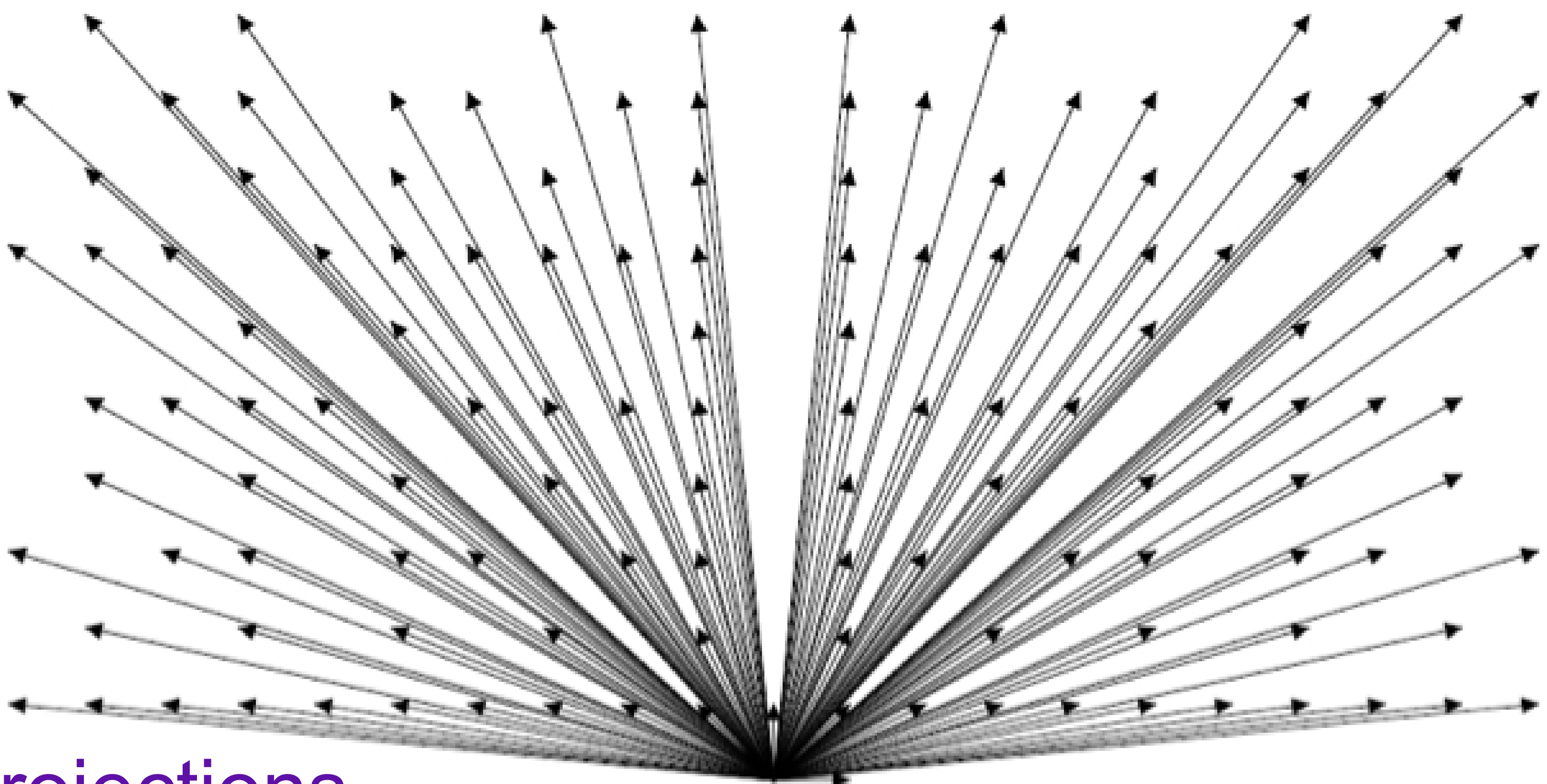
2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005

Approximating into Mojette space

Farey-Haros 10



128 projections

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005

And the constant S relying projections and image

$$\sum_{i=1}^{N}\sum_{j=1}^{N} f(i,j) = \frac{1}{I}\sum_{n}\sum_{b} proj(b, p_n, q_n) = S$$

So simple :

$$f(k,l) = \frac{1}{(I-1)}\left(g(k,l) - S\right)$$

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005

So simple :

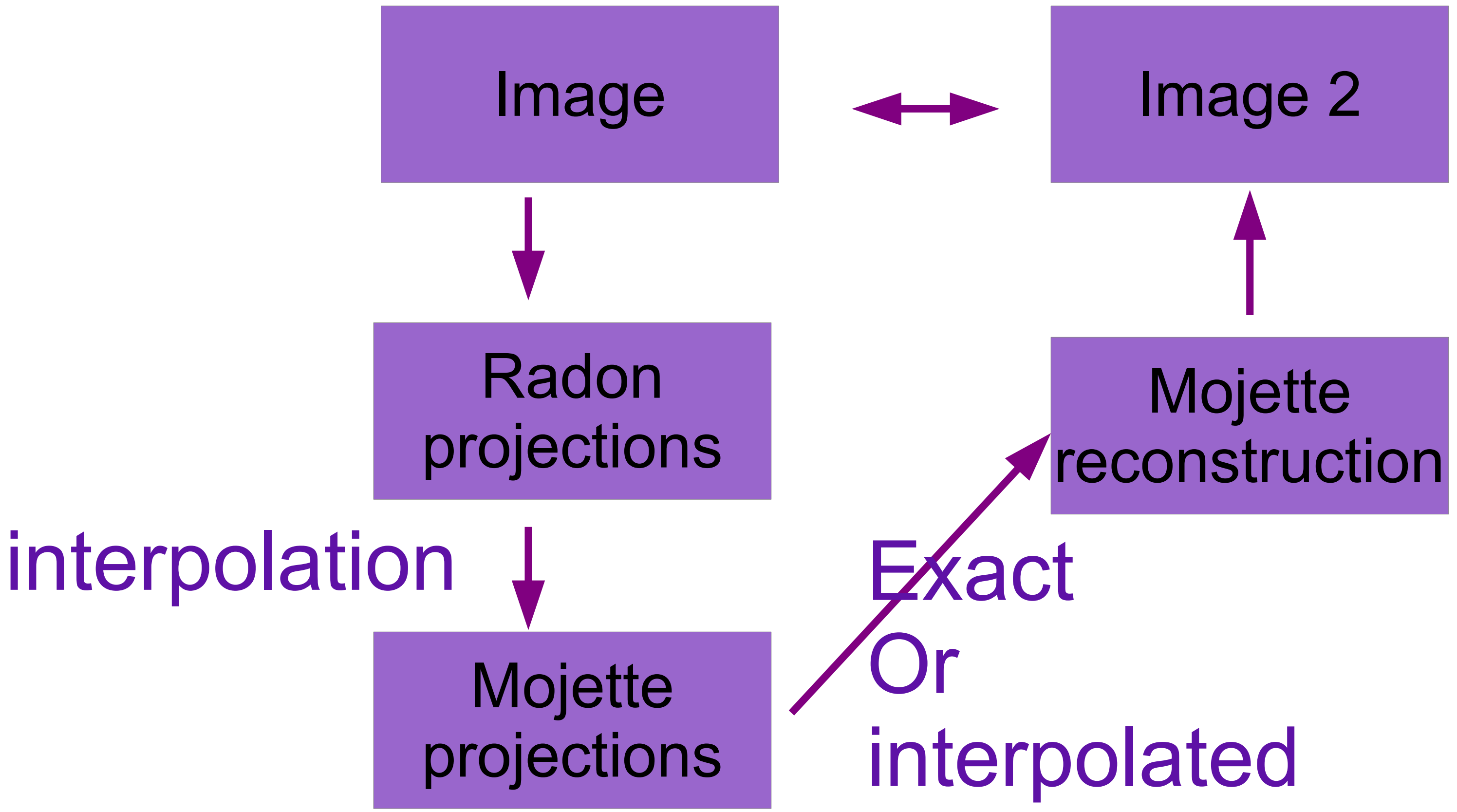
$$f(k,l) = \frac{1}{(I-1)}\left(g(k,l) - S\right)$$

However … for a 65x65 image
The Farey-Haros series is 5040 angles !

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

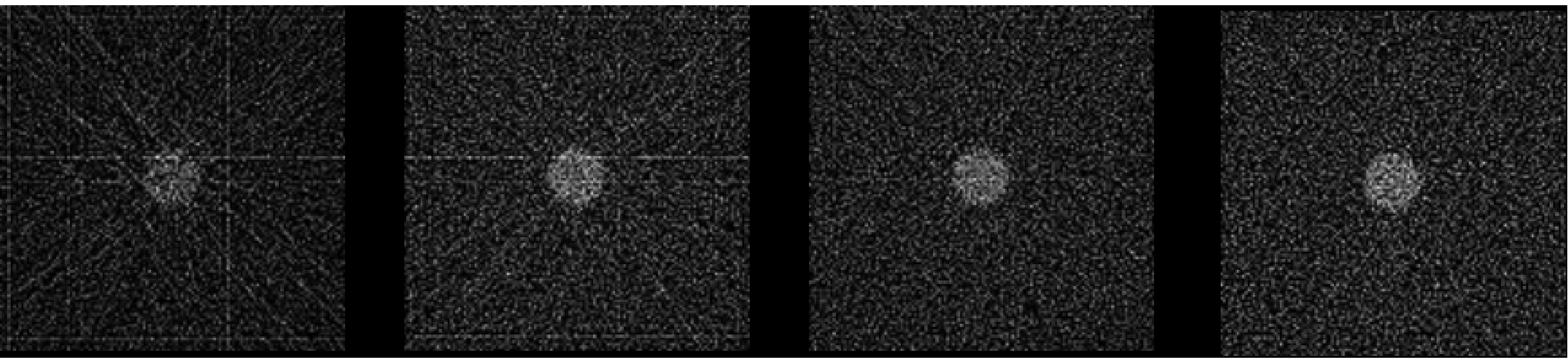
Mojette Tomographic scheme



2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005



I=64 MSE=0.40

I=128 MSE=0.31

I=256 MSE=0.31

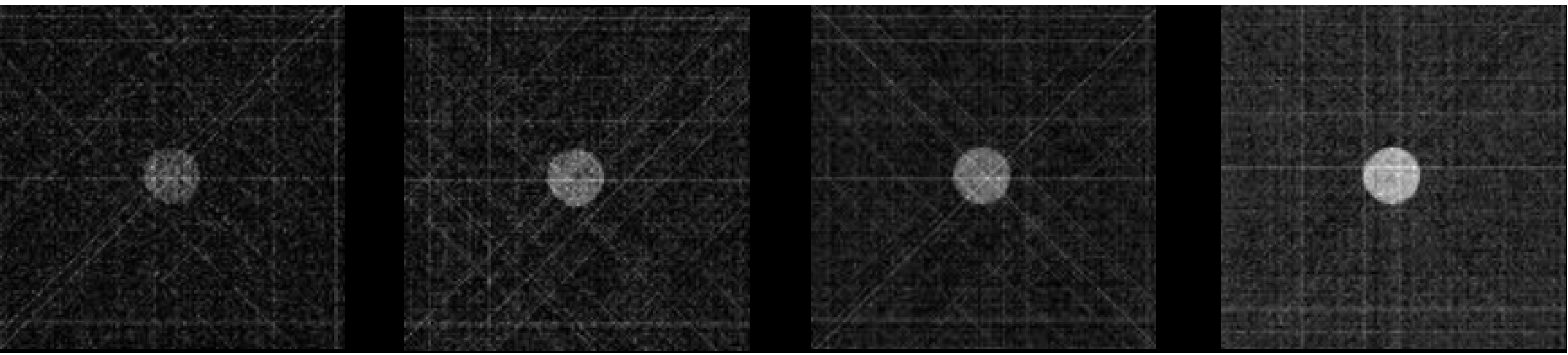
I=512 MSE=0.30

MFBP with Poisson noise

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Myriam Servières dec 2005



I=64 MSE=0.28

I=128 MSE=0.18

I=256 MSE=0.17

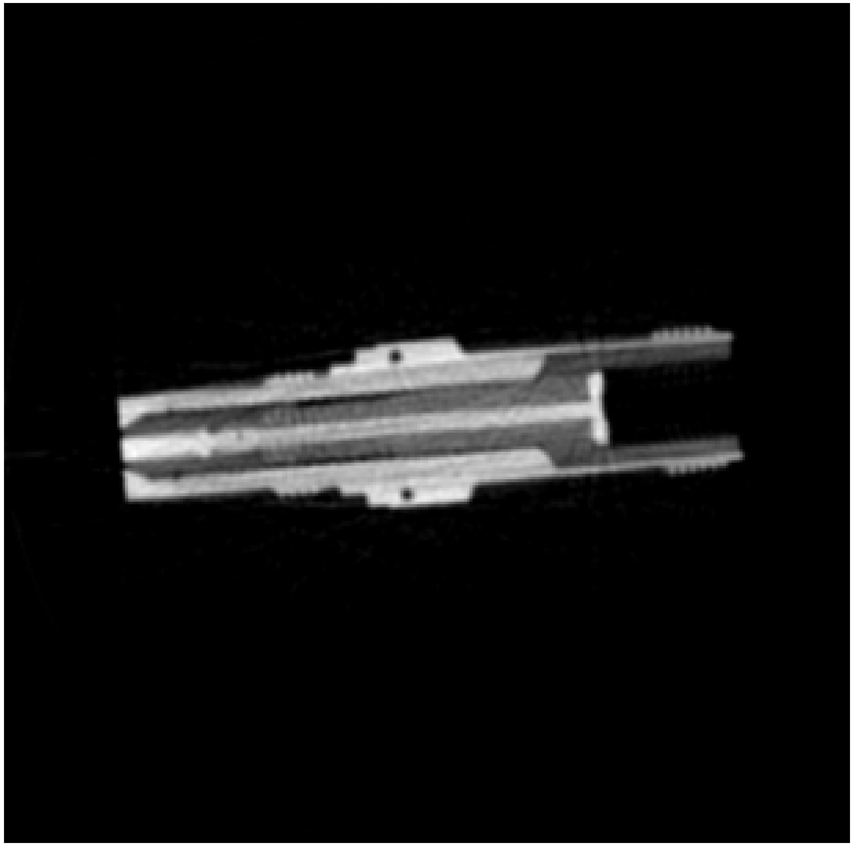
I=512 MSE=0.06

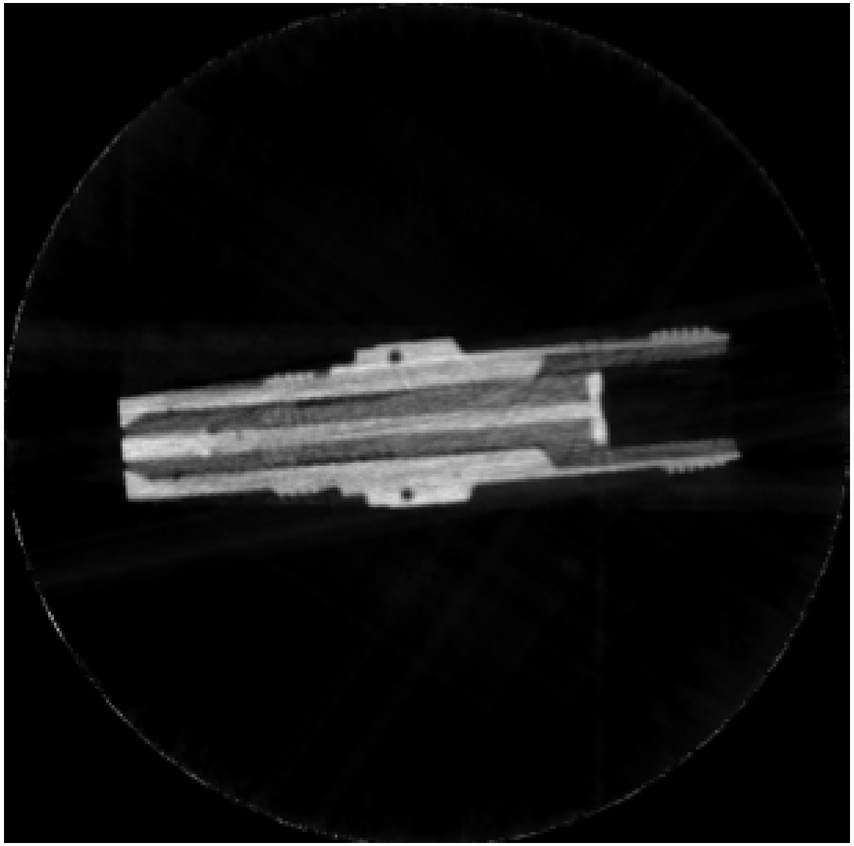
MCG with Poisson noise

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Henri der Sarkissian 2015 + Benoit Recur





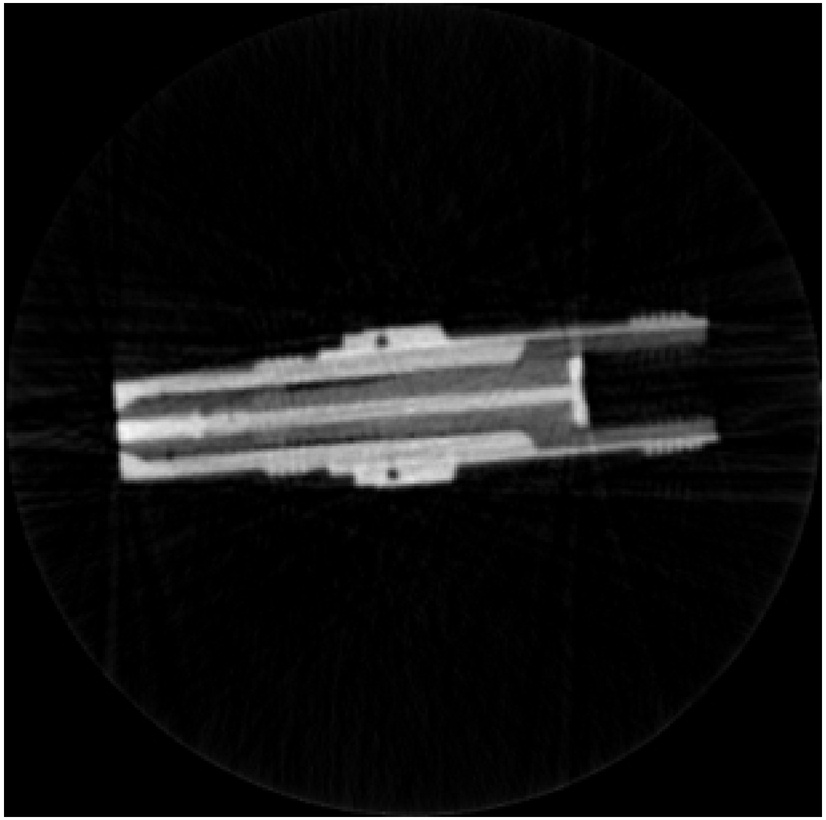
FBP SART

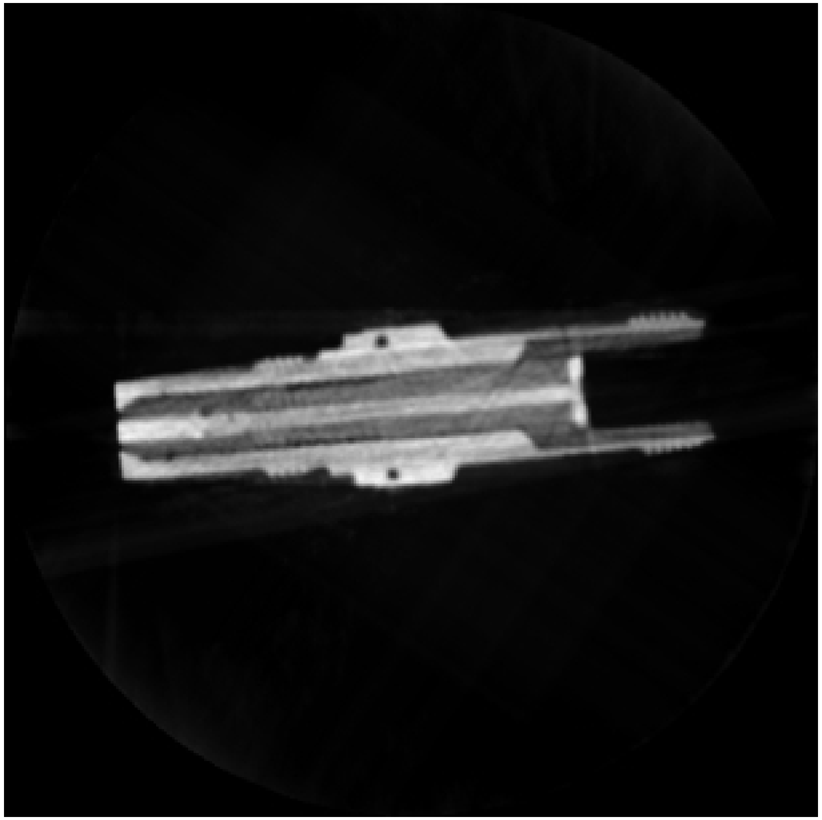
Real acquisition 403 proj – 404 bins per proj

2. Tomographic Mojette reconstruction

2.2 From Radon to Mojette space

PhD Henri der Sarkissian 2015 + Benoit Recur





Mojette FBP

Mojette SART

Real acquisition 403 proj – 404 bins per proj

outline

1. The Mojette transform with exact data
2. Tomographic reconstruction
3. Links with Fourier and FRT

3. Links between Mojette and FRT

Finite Radon Transform
Introduced by Matus & Flusser in 1993

Only for $p \times p$ grid with p prime
generates (p+1) projections of p bins
=> NO Redundancy

Used and extended by Imants Svalbe
& A. Kingston to any image size
=> torus Mojette

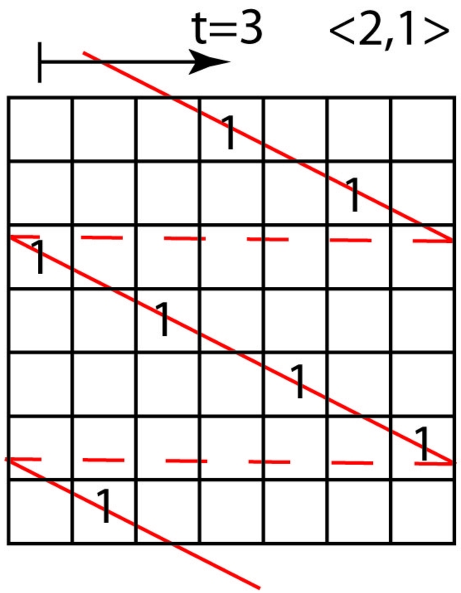
3. Links between Mojette and FRT

Finite Radon Transform

FRT:

$$R(t,m)=\sum_{i=0}^{p} f(t+mi,i) \; for\, each\; 0 \leq t < p$$

Selection of the image pixels sampled for FRT projection R(t=3, m=2) for p = 7



3. Links between Mojette and FRT

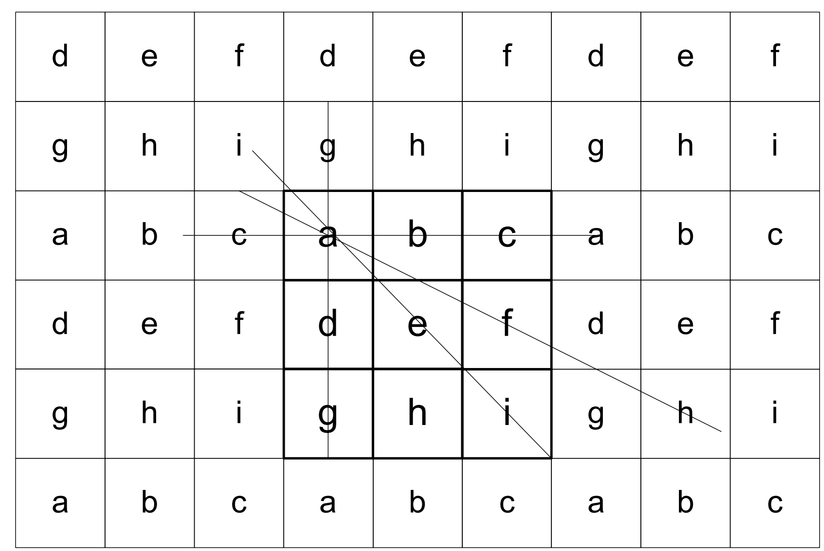
Finite Radon Transform

FRT:

$$R(t,m)=\sum_{i=0}^{p} f(t+mi,i)\ for\ each\ 0\leq t<p$$

image pxp

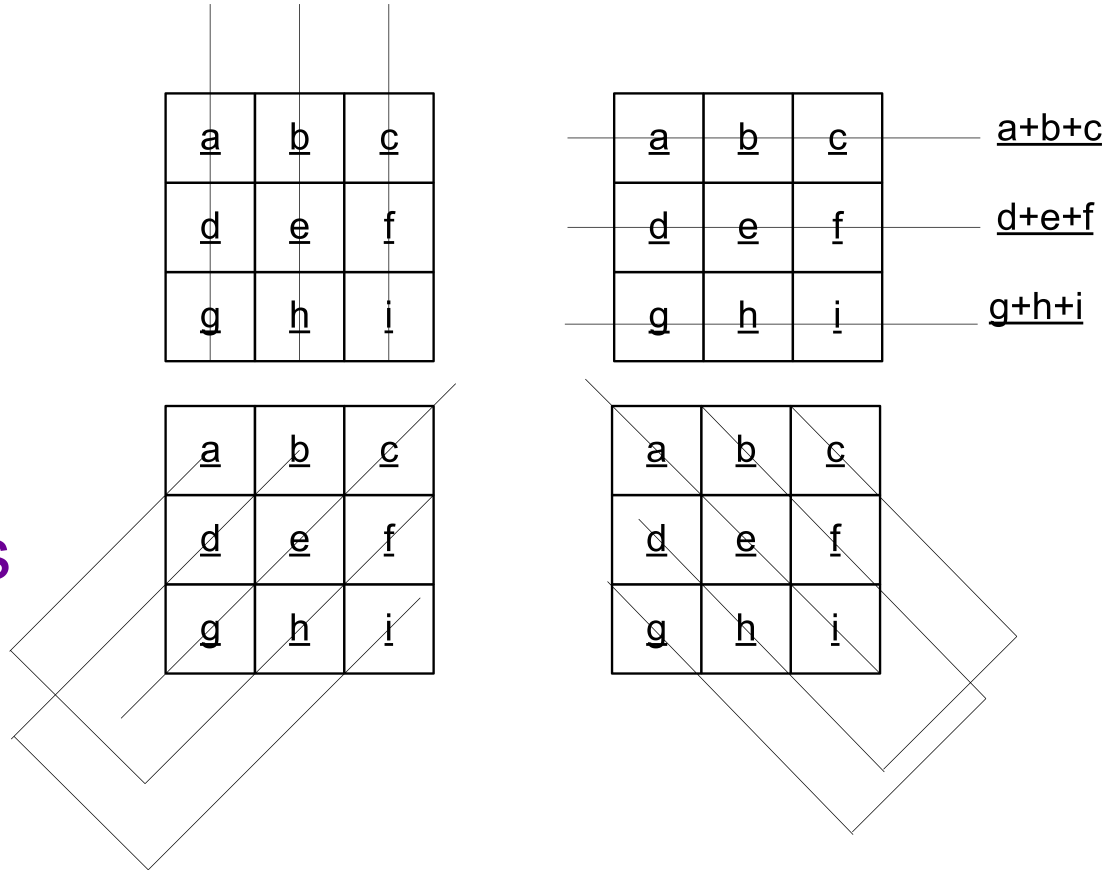
Only (p+1) projections of p bins, each summing up p pixels



3. Links between Mojette and FRT

Finite Radon Transform FRT:

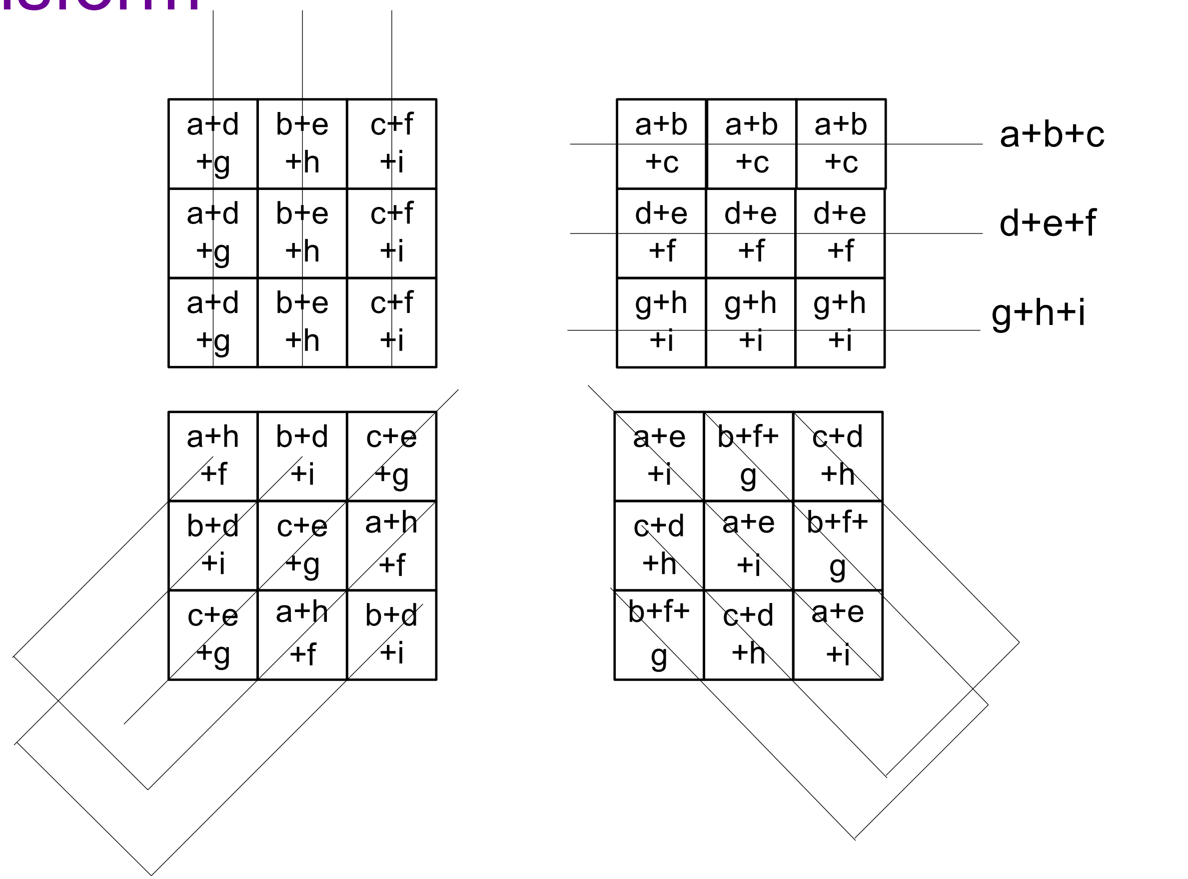
image p=3
4 projections
of 3 bins, each
summing up 3 pixels



3. Links between Mojette and FRT

Inverse Finite Radon Transform iFRT:

backprojecting each 4 projections



3. Links between Mojette and FRT

Inverse Finite Radon Transform
iFRT:

summing backprojections :

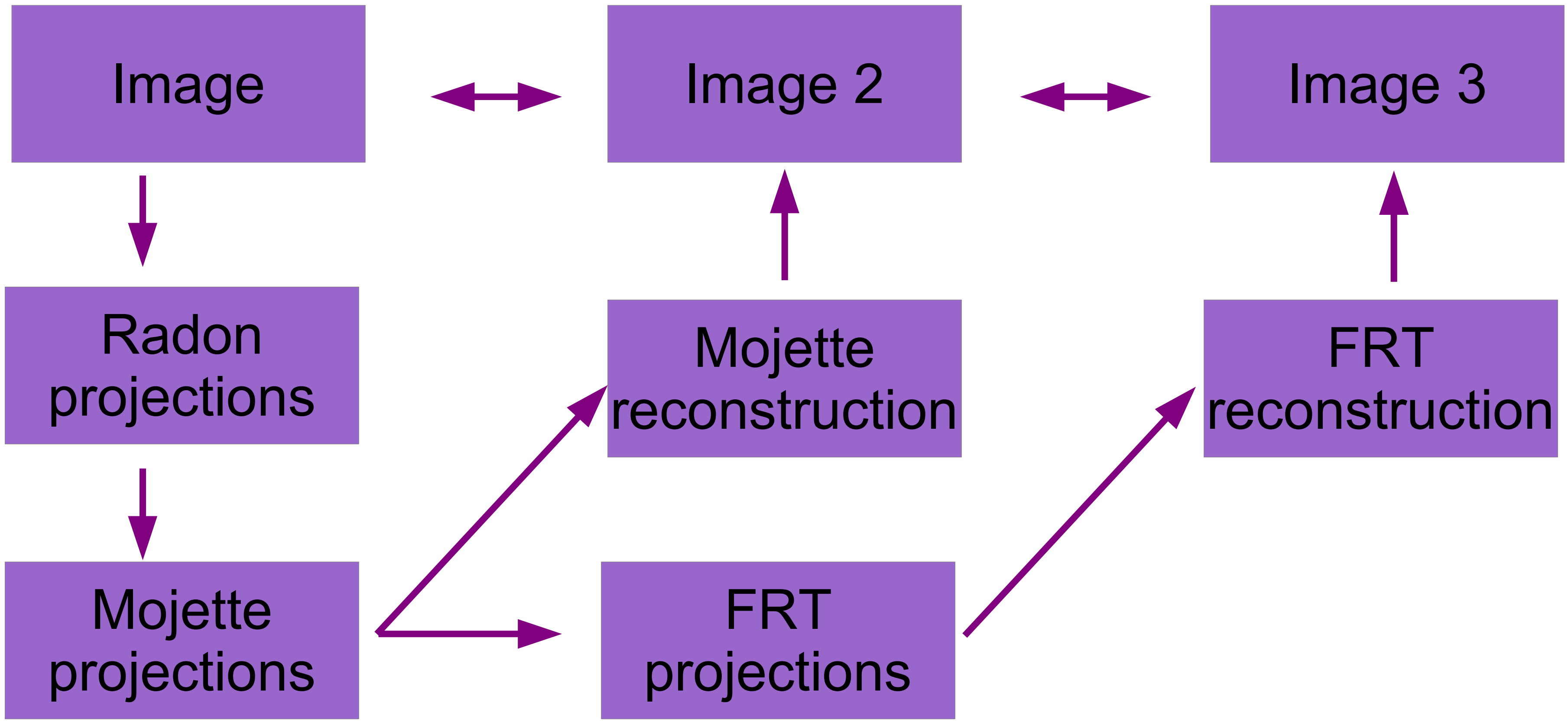
$$S = a + b + c + d + e + f + g + h + i$$

S +3a	S +3b	S +3c
S +3d	S +3e	S +3f
S +3g	S +3h	S +3i

Almost same equation than
Servières for Mojette
Andrew Kingston & Imants Svalbe

3. Links between Mojette and FRT

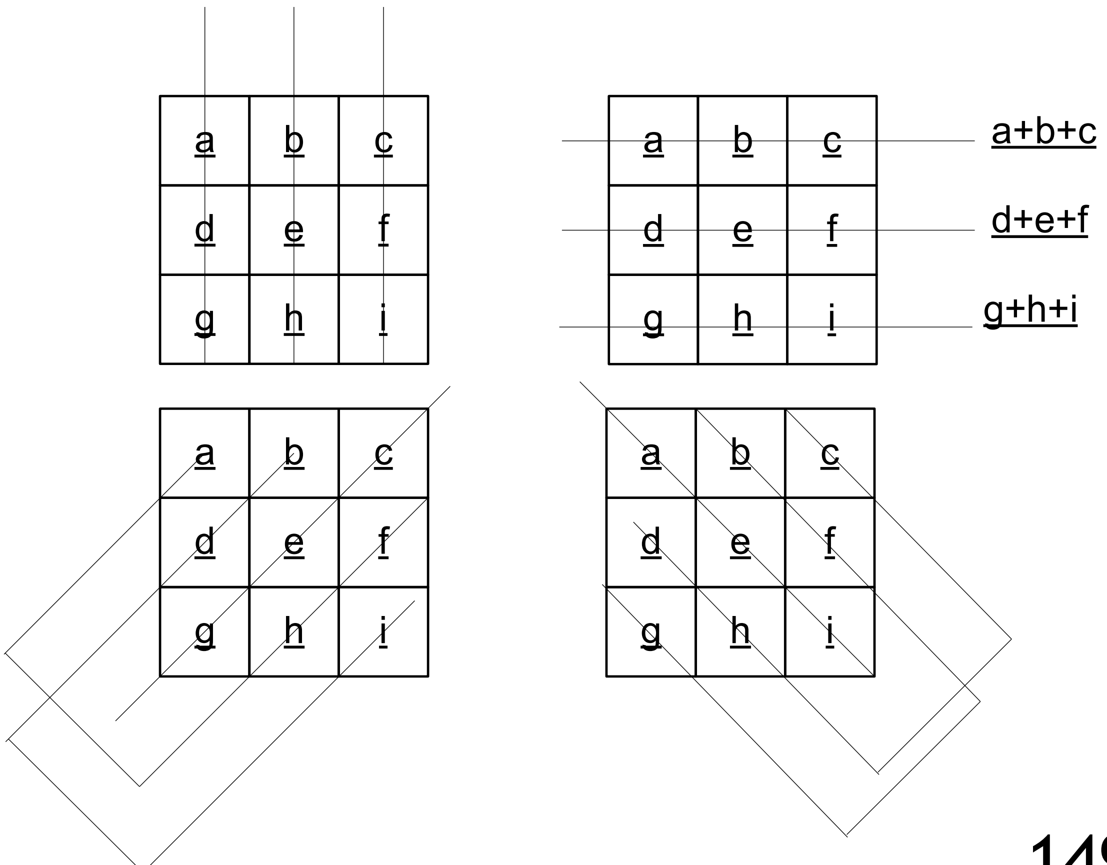
Mojette Tomographic scheme



3. Links between Mojette and FRT

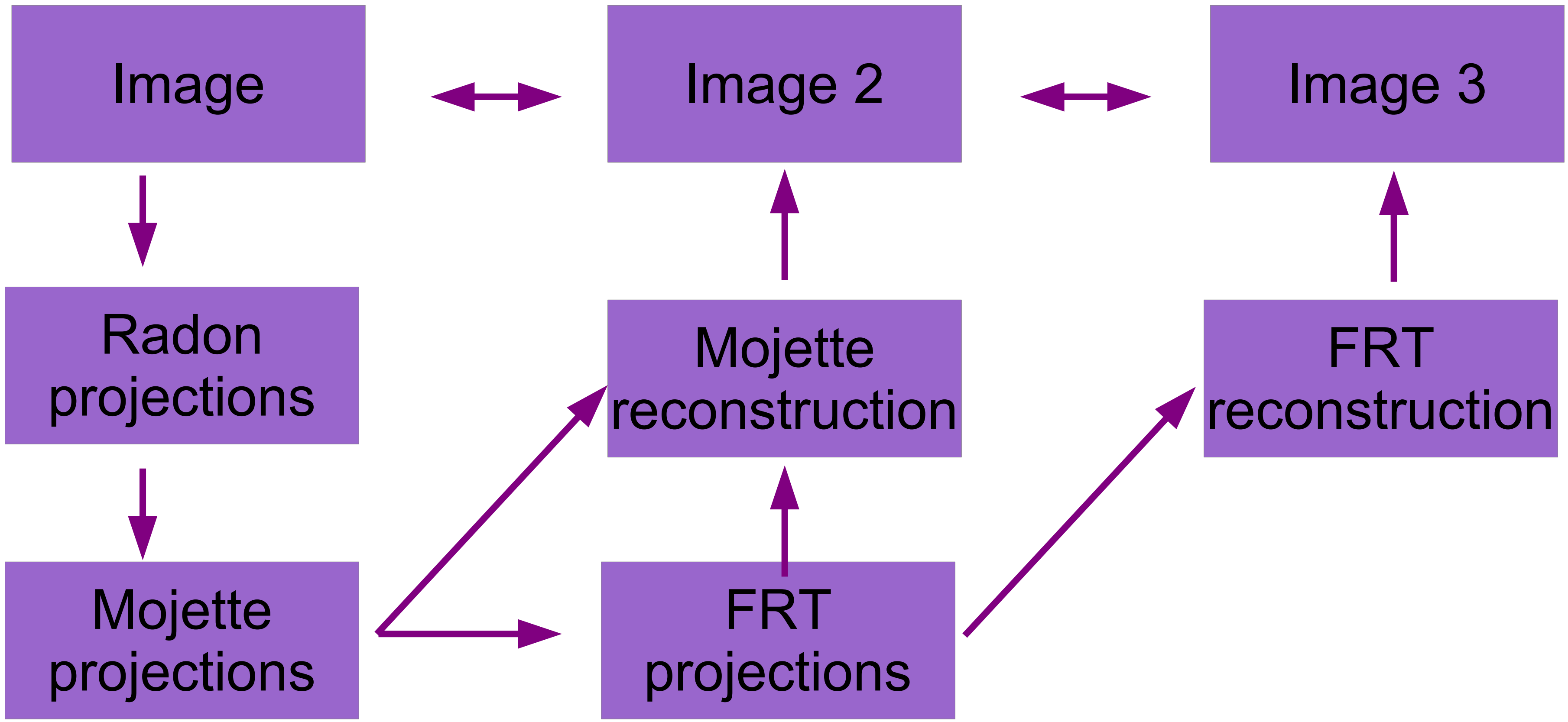
Mojette projection to Finite Radon Transform:

just summing bins
every p bin:



3. Links between Mojette and FRT

Mojette-FRT Tomographic scheme



3. Links between Mojette, Fourier and FRT transforms

3.1 Links between Mojette and Finite Radon transforms

3.2 Links with the Fourier Transform

3.1 Central Slice theorem

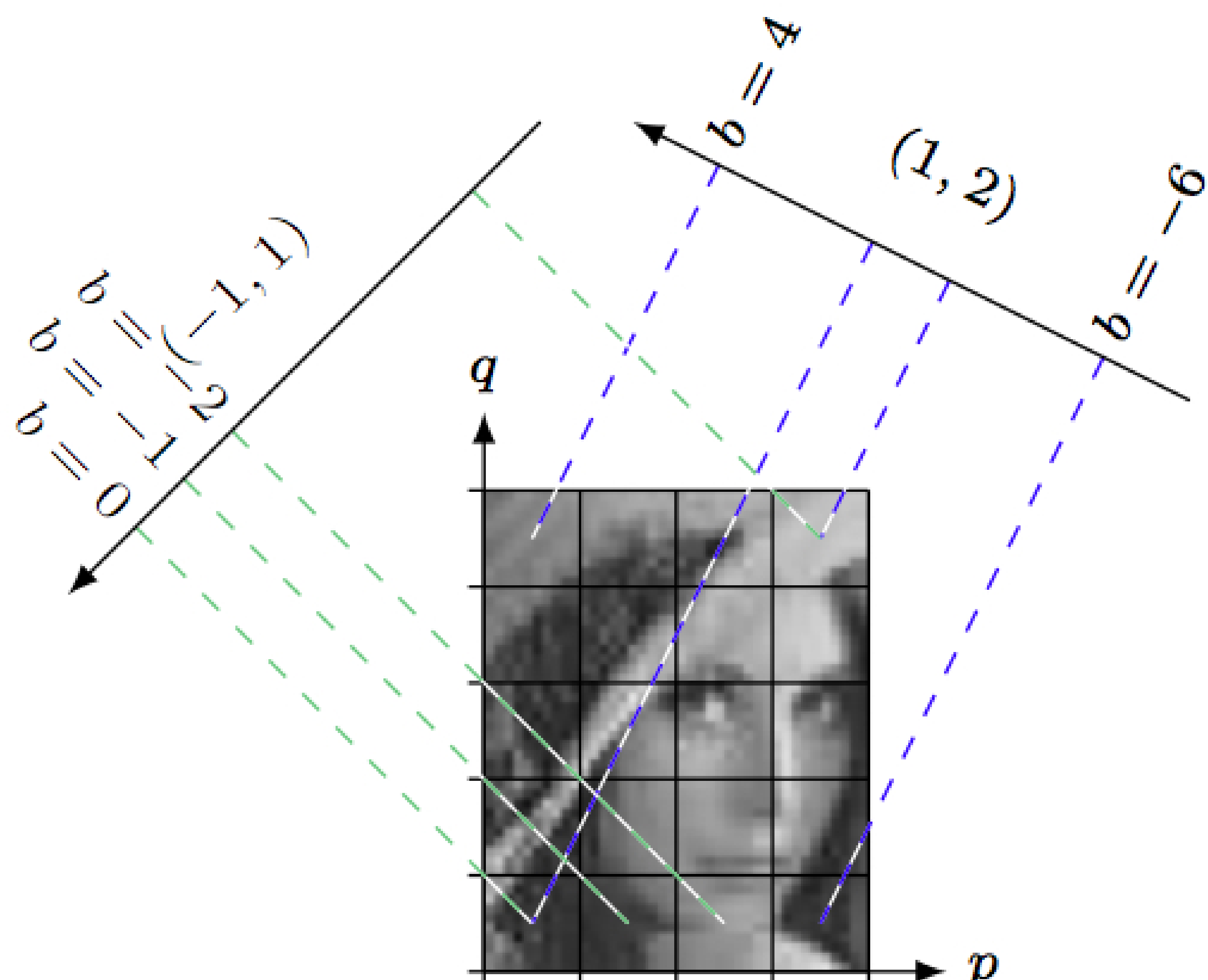
3.2 Discrete rotations

3. Links between Mojette, Fourier and FRT

3.2 Discrete rotations

Ph.D. Henri Der Sarkissian 2015

Lena image before rotation(2 1)

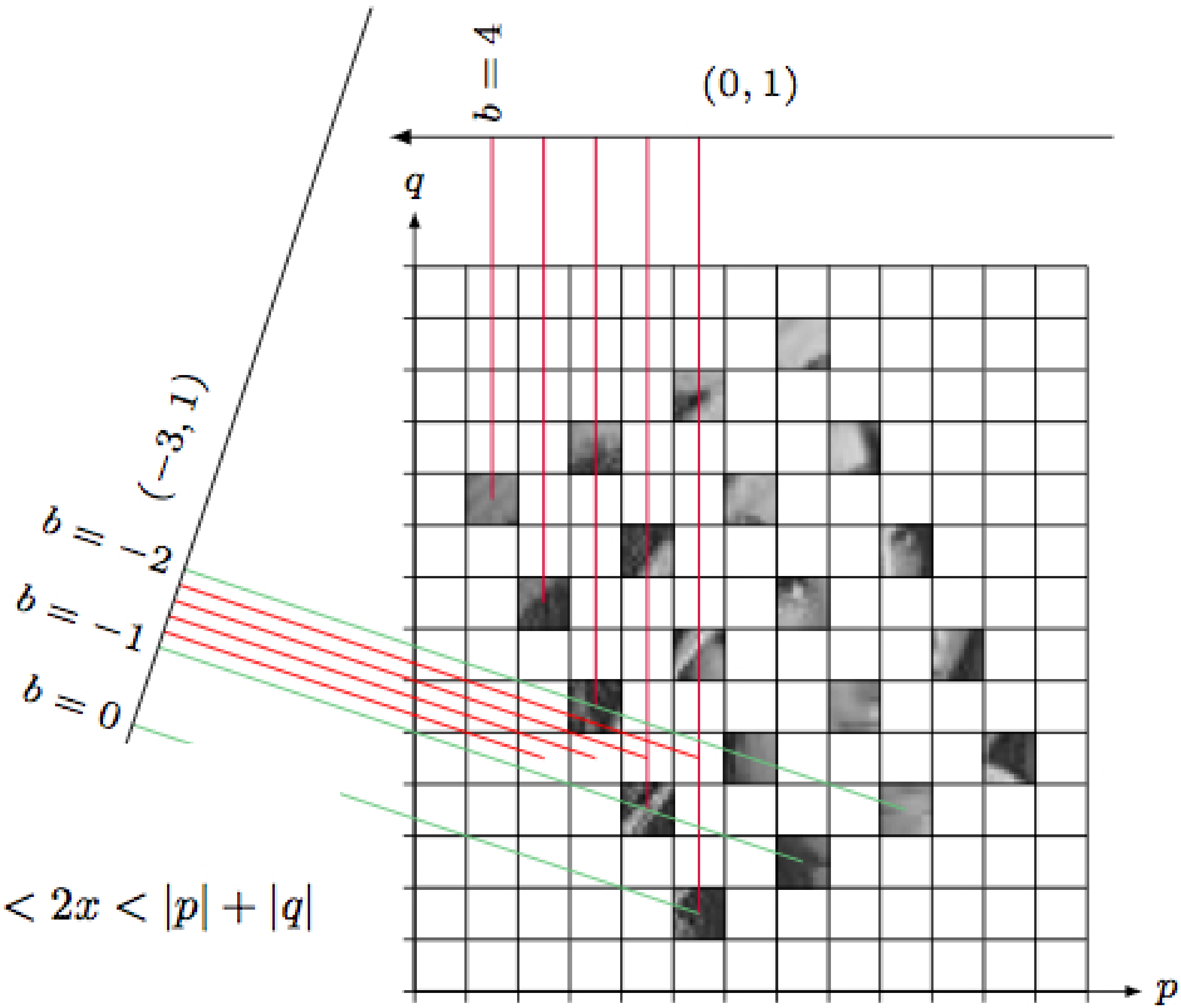


3. Links between Mojette, Fourier and FRT

3.2 Discrete rotations

Ph.D. Henri Der Sarkissian 2015

Lena image
after rotation(2 1)



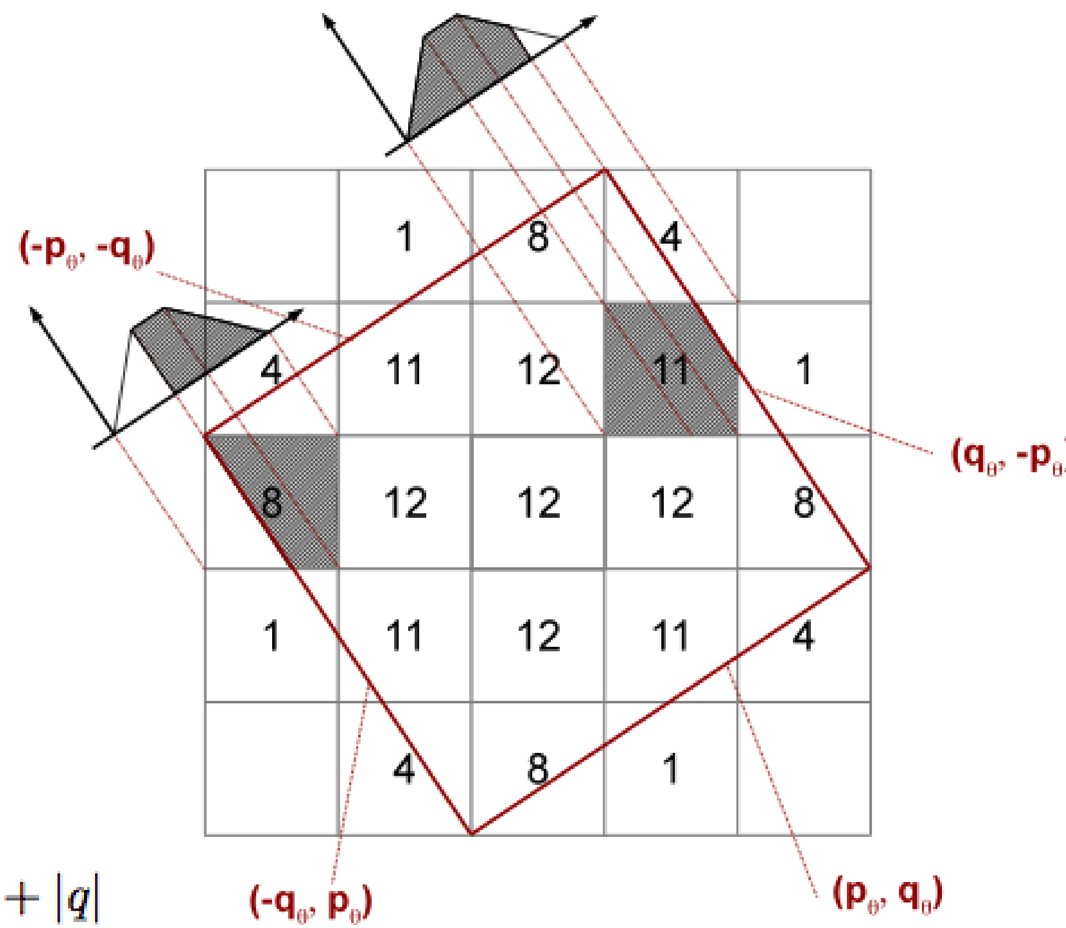
$$c) = \begin{cases} 1 \text{ if } 2x \leq ||p| - |q|| \\ \dfrac{-2x + |p| + |q|}{2\min\{|p|, |q|\}} \text{ if } ||p| - |q|| < 2x < |p| + |q| \\ 0 \text{ elsewhere} \end{cases}$$

3. Links between Mojette, Fourier and FRT

3.2 Discrete rotations

Ph.D. Henri Der Sarkissian 2015

Interpolation stage after rotation(2 1)

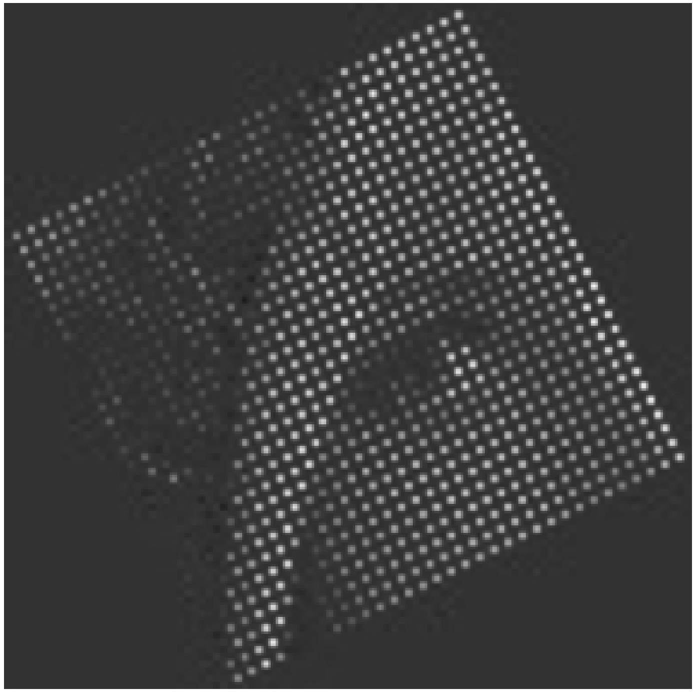


$$x) = \begin{cases} 1 \text{ if } 2x \leq ||p| - |q|| \\ \dfrac{-2x + |p| + |q|}{2\min\{|p|, |q|\}} \text{ if } ||p| - |q|| < 2x < |p| + |q| \\ 0 \text{ elsewhere} \end{cases}$$

3. Links between Mojette, Fourier and FRT

3.2 Discrete rotations

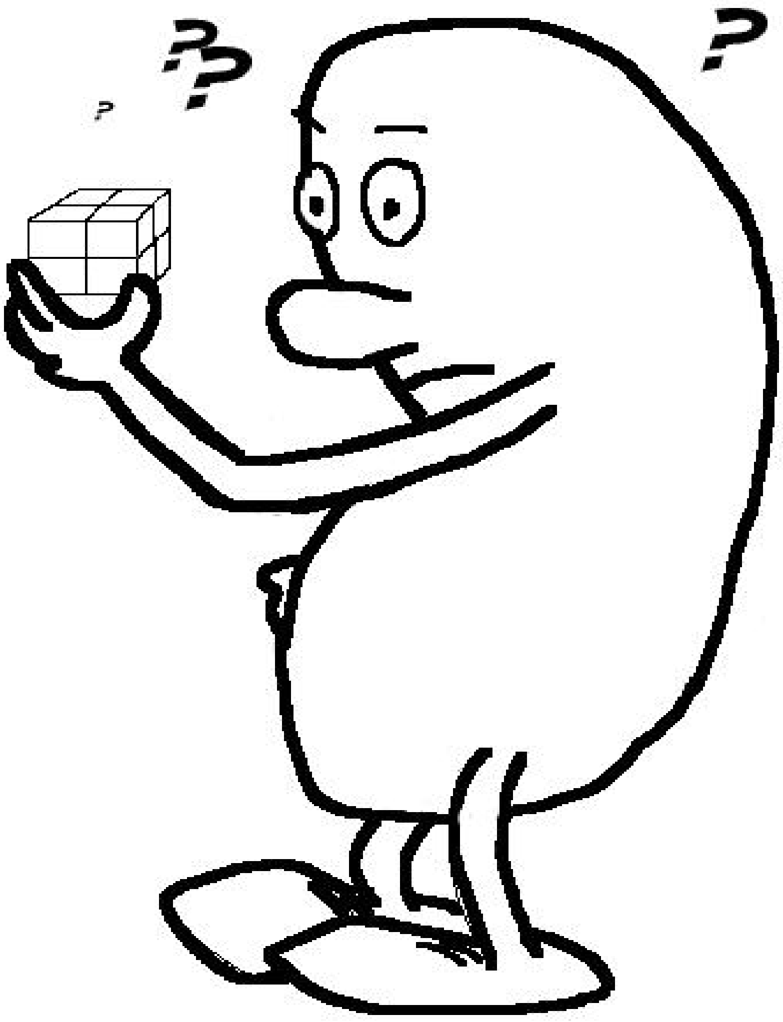
Ph.D. Henri Der Sarkissian 2015



Original 32x32 rotation(2 1) interpolation in Mojette space

la transformata Mojette : 20 anni

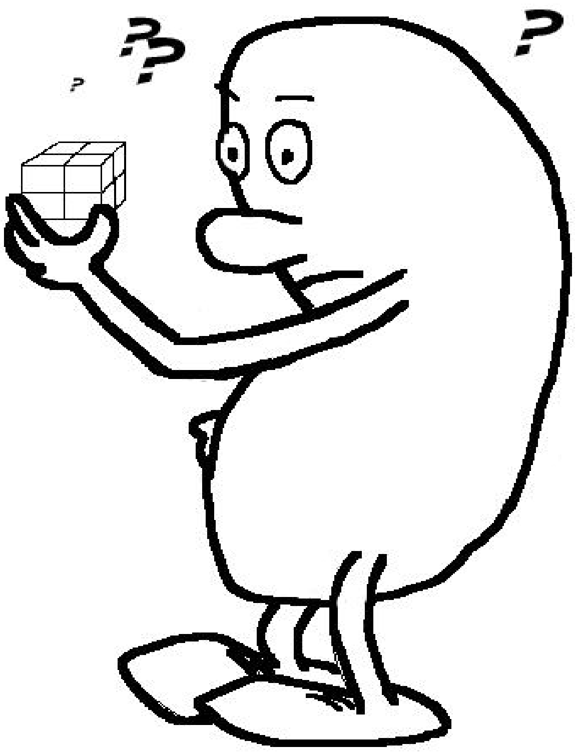
Grazie



la trasformata Mojette : 20 anni

Grazie

Nicolas Normand
Olivier Philippé
Florent Autrusseau
Benoit Parrein
Pierre Evenou
Pierre Verbert
Myriam Servières
Andrew Kingston
Imants Svalbe
Chuanlin Liu
Aurore Arlicot
Henri Der Sarkissian



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Dominique Barba
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la trasformata Mojette : 20 anni

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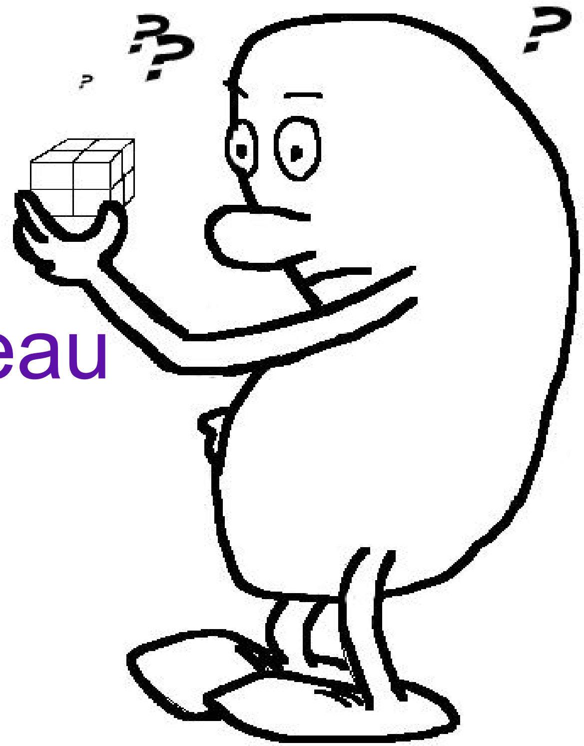
Medical Imaging

RozoFS



Jerome Fortineau

Pierre Evenou

Www.mojette.org

Www.mojette.net

Sylvain David

IRCCYN

POLYTECH
NANTES
École d'ingénieurs de l'université de Nantes

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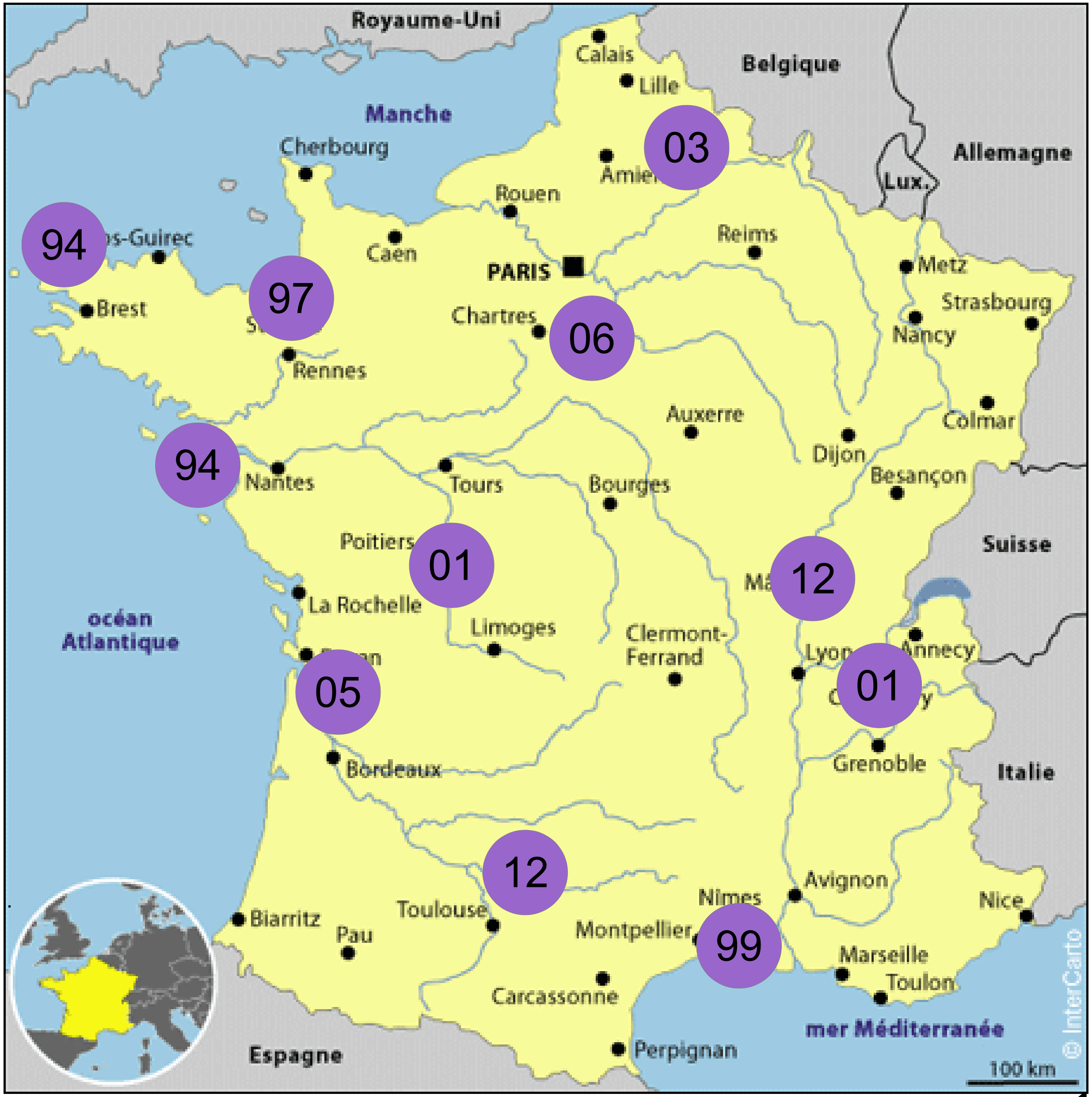

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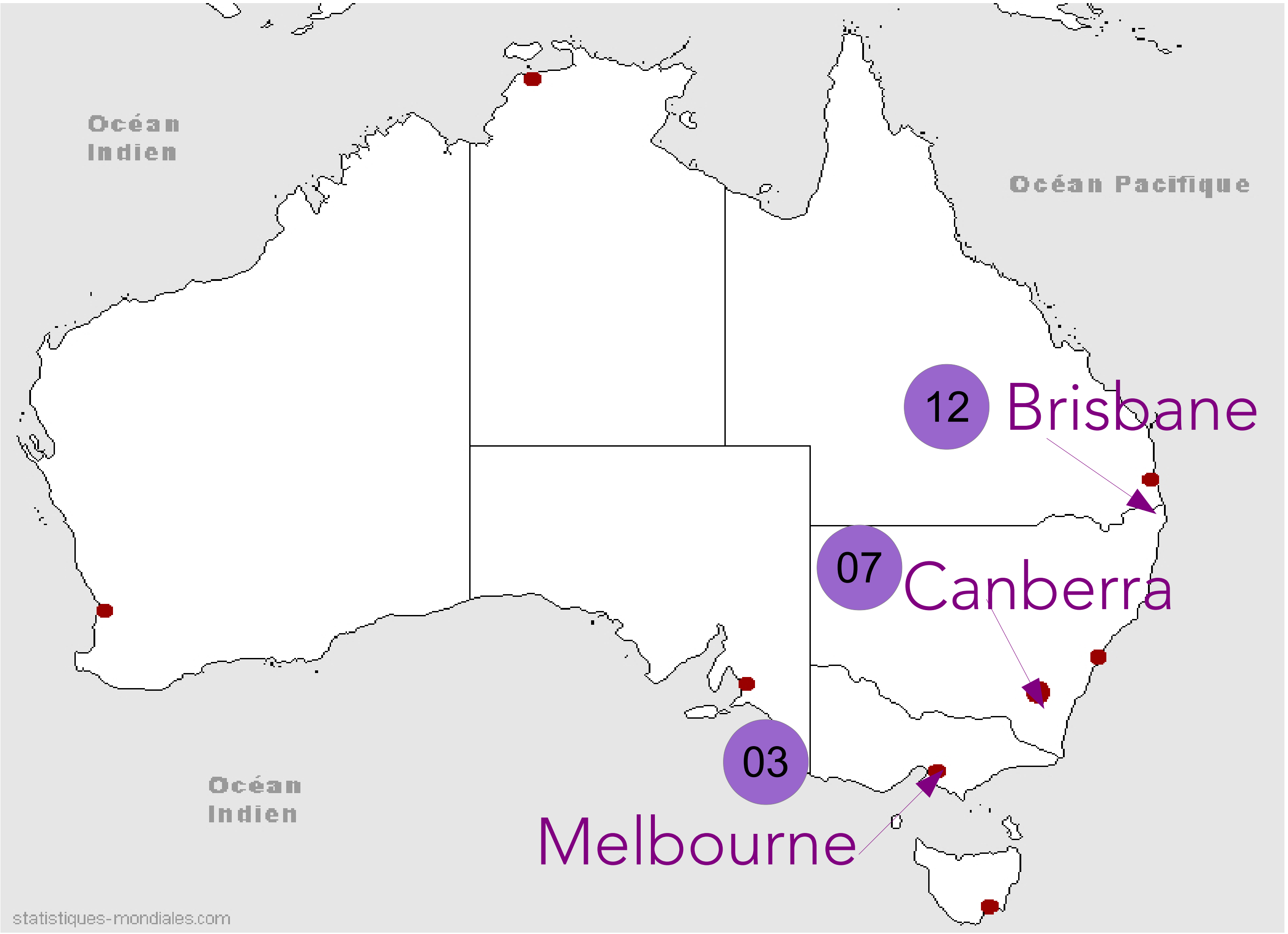

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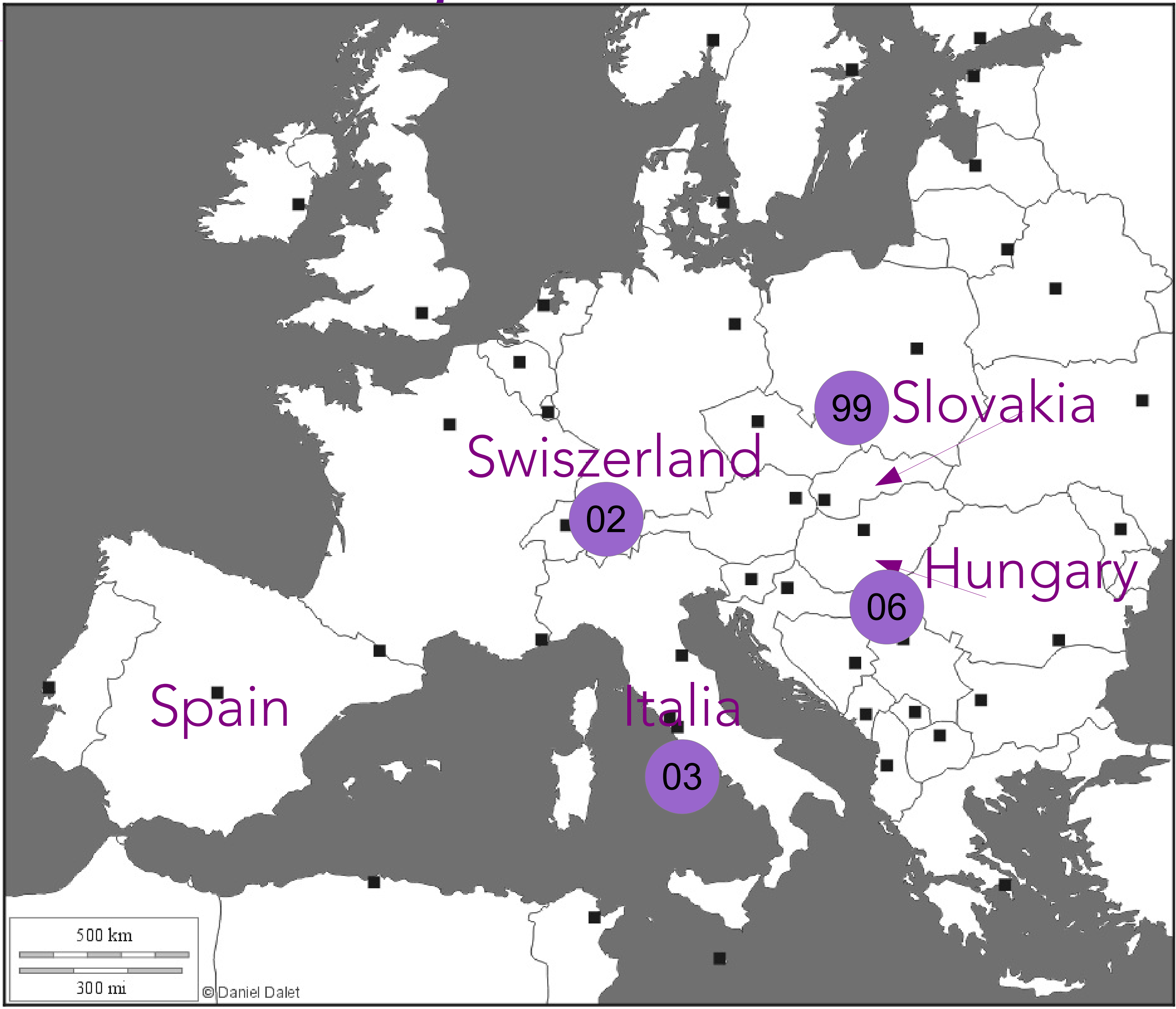

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