Partial Differential Equations and Related Analytic-Geometric Inequalities Politecnico di Milano, February 17-18, 2017

ABSTRACTS OF THE TALKS

Gianni Arioli
Existence and stability of periodic solutions for some Hamiltonian PDEs

We consider the nonlinear wave equation $u_{tt} - u_{xx} = \pm u^3$ and the beam equation $u_{tt} + u_{xxxx} = \pm u^3$ on an interval. Numerical observations indicate that time-periodic solutions for these equations are organized into structures that resemble branches and seem to undergo bifurcations. We prove the existence of time-periodic solutions for various periods (a set of positive measure in the case of the beam equation) along the main nontrivial branch. Then we consider a simplified version of a bridge model, consisting in a nonlinear beam equation coupled with a nonlinear wave equation, and we describe some results on the linear stability/instability of the system. Our proofs are computer-assisted. This is joint work with Hans Koch.

Elvise Berchio
Improved Poincaré inequalities on the hyperbolic space

The validity of the Poincaré inequality in the hyperbolic space, and of a generalized higher order Poincaré inequality, where the best constants are not achieved, suggests to look for possible improvements of these inequalities. In the talk we will discuss the existence of Hardy-type and Rellich type remainder terms, namely positive corrections to add on the r.h.s. of the inequalities. The optimality issue of the constants involved will be also addressed. The talk is based on the joint papers [BGG] and [BG].

[BGG] E. Berchio, D. Ganguly, G. Grillo J. Funct. Anal. 2016
[BG] E. Berchio, D. Ganguly, Commun. Pure Appl. Anal. 2016.

Marco Bramanti
A proof of Hörmander's hypoellipticity theorem for sublaplacians on Carnot groups

A linear differential operator $\mathcal{L}$ with smooth coefficients is called *hypoelliptic* in a domain $\Omega \subseteq \mathbb{R}^N$ if, for every open $\Omega' \subseteq \Omega$, whenever a distribution u is such that $\mathcal{L}u \in C^\infty(\Omega')$ then $u \in C^\infty(\Omega')$. If $\mathcal{L}$ has the form

$$\mathcal{L} = \sum_{j=1}^{q} X_j^2 + X_0$$

where $X_0, X_1, \ldots, X_q$ ($q + 1 < N$) are real smooth vector fields (i.e., first order differential operators), then $\mathcal{L}$ is a degenerate elliptic-parabolic operator. Nevertheless, a famous theorem by Hörmander [Acta Math. 1967], gives a sufficient (and "almost necessary") condition for $\mathcal{L}$ to be hypoelliptic: one requires that the vector fields X_i and their commutators $[X_i, X_j]$, $[[X_i, X_j], X_k], \ldots$ span $\mathbb{R}^N$ at every point of the domain ("Hörmander's condition"). The original proof of this theorem is deep and difficult; an alternative proof, due to Kohn (1973), exploits the theory of pseudodifferential operators. In the paper by Bramanti-Brandolini, Nonlinear Analysis TMA (2015) that here will be outlined, we give a proof of this theorem in the case of a sublaplacian on a Carnot group, i.e. with $\mathcal{L} = \sum_{j=1}^{q} X_j^2$, where $X_1, ..., X_q$ are left invariant and 1-homogeneous with respect to a structure of Carnot group in $\mathbb{R}^N$, and satisfy Hörmander's condition. This is a model situation which is widely studied, for several reasons that will be recalled. The double simplification consisting in assuming the existence of an underlying group structure and the lacking of the drift term X_0 allows to give a more simple, transparent and selfcontained proof. In particular, the subelliptic estimates which we prove involve Sobolev spaces induced by the vector fields, instead of Euclidean fractional Sobolev spaces, as in the classical approach.

Andrea Cianchi

Second-order regularity for p-Laplacian type elliptic problems

We deal with boundary value problems for a class of quasilinear elliptic equations in divergence form, including the p-Laplace equation. Under minimal regularity assumptions on the boundary, we prove the membership to a Sobolev space of a nonlinear expression of the gradient of the solution, for any square-integrable right-hand side. This is a joint work with V. Maz'ya.

Fabio Cipriani

Logarithmic Sobolev inequalities seen from Quantum Statistical Mechanics

It will be shown how Gross Logarithmic Sobolev Inequalities have a natural interpretation in the framework of equilibrium Quantum Statistical Mechanics and how they can be reduced to spectral semi-boundedness.

Andrea Colesanti

Valuations on spaces of functions

We will present the notion of valuation defined on a space of functions, along with some examples. We will motivate the introduction of these functionals comparing them with valuations defined on classes of sets, having in mind, as main example, valuations on convex bodies. We will review some characterization results that have been recently proved for valuations on function spaces, and in the last part we will describe our ongoing research concerning valuations on the space of convex functions.

Francesco Dalla Pietra

Anisotropic Hardy inequalities

The aim of the talk is to present some results, obtained in a joint paper with G. di Blasio and N. Gavitone, about Hardy-type inequalities involving a general norm in $\mathbb{R}^n$. More precisely, given a smooth norm F of $\mathbb{R}^n$, we investigate the validity of Hardy inequalities of the type

$$\int_\Omega F(\nabla u)^2\,dx \geq C_F(\Omega)\int_\Omega \frac{u^2}{d_F^2}\,dx, \qquad \forall u \in H_0^1(\Omega), \tag{1}$$

where Ω is a domain of $\mathbb{R}^n$, and d_F is the anisotropic distance to the boundary with respect to the dual norm. The case of the optimality of the constant is also addressed. Moreover, improved versions of (1), obtained by adding remainder terms which depend on suitable norms of u, will be considered.

Alberto Ferrero

Existence and stability for entire solutions of nonlinear polyharmonic equations

In the talk, existence and stability for entire solutions of polyharmonic equations with exponential non-linearity, will be discussed. Special attention will be devoted to radial solutions of such kind of equations, for which a quite complete picture has been obtained at least in the biharmonic case while the general case still exhibits several open questions. The study of the stability problem exploits in a fundamental way some higher order Hardy-Rellich type inequalities. The results contained in the talk were obtained in collaboration with Alberto Farina. In the final part of the talk, some time will be also devoted to the recent results obtained by other authors after the publication of our paper.

Enzo Ferone

On the Stability of the Bossel-Daners Inequality

The Bossel-Daners inequality is a Faber-Krahn type inequality for the first Laplacian eigenvalue with Robin boundary conditions. We prove a stability result for such inequality.

Elisa Francini

Stable reconstruction of coefficients in elliptic equations from knowledge of their solutions

In this talk I will review some results concerning stable reconstruction of a coefficient in an elliptic partial differential equation from knowledge of one of its solutions. I will also discuss some application of these results to hybrid inverse problems.

Nunzia Gavitone

An overdetermined boundary value problem for Monge-Ampère operator

The study of the optimal constant in an Hessian-type Sobolev inequality leads to a fully nonlinear boundary value problem, overdetermined with non standard boundary conditions. We show that all the solutions have ellipsoidal symmetry. In the proof we use the maximum principle applied to a suitable auxiliary function in conjunction with an entropy estimate from affine curvature flow. This is a joint work with B. Brandolini, C. Nitsch and C. Trombetti.

Matteo Muratori

Fractional Laplacians in weighted L^p spaces and applications to nonlinear diffusions

This talk is divided in two parts. In the first one I will discuss a self-adjointness property of the Euclidean s-fractional Laplacian ($0 < s < 1$) in L^2 spaces with power-type weights. Depending on the behaviour of the (negative) power at infinity, self-adjointness of the corresponding weighted operator may or may not hold: in fact there exists a precise threshold, coinciding with the spatial dimension, below which it holds and above which it fails. Similar results can then be extended to the case of general L^p spaces, referring in such framework to the validity of an "integration-by-parts" formula. The proofs rely on nonlocal cut-off arguments and Riesz-potential techniques; a gap is left open in dimension one, since there both methods seem to fail. In the second part I will briefly talk about the application of the above results to the study of well-posedness and asymptotic properties of weighted fractional diffusion equations of porous medium type. In particular, they turn out to be crucial to prove uniqueness of solutions with Radon measures as initial data, which are deeply linked to asymptotics. As for the latter, the (negative) threshold power $2s$ associated with the weight discriminates between a Barenblatt-type (self-similar) and a separable-solution behaviour as time goes to infinity.

Nicola Soave

Variational problems with long-range interaction

We consider a class of variational problems for densities that repel each other at distance. Typical examples are given by the Dirichlet functional and the Rayleigh quotient

$$D(\mathbf{u}) = \sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2 \quad \text{and} \quad R(\mathbf{u}) = \sum_{i=1}^{k} \frac{\int_\Omega |\nabla u_i|^2}{\int_\Omega u_i^2},$$

minimized in the class of $H^1(\Omega, \mathbb{R}^k)$ functions attaining some boundary conditions on $\partial\Omega$, and subject to the constraint

$$\operatorname{dist}(\{u_i > 0\}, \{u_j > 0\}) \geq 1 \qquad \forall i \neq j.$$

For these problems, we investigate the optimal regularity of the solutions, prove a free-boundary condition, and derive some preliminary results characterizing the free boundary $\partial\{\sum_{i=1}^{k} u_i > 0\}$. This is a joint work with H. Tavares, S. Terracini and A. Zilio.

Margherita Solci
Minimizing movement for oscillating energies

We consider the problem of the characterization of variational motions along a sequence of functionals depending on a parameter ε (the space scale), and give conditions to obtain that the limit of the minimizing movements coincides with the minimizing movement (gradient flow) of the Γ-limit of the sequence. This holds for a class of functionals containing convex functionals. In general, minimizing movements depend on the interaction of the time and space scales, and the conditions for the commutativity are violated for oscillating functionals or if there is no coercivity. We discuss some examples and related issues.

Alessandro Zilio
Predators-prey model with competition, the emergence of packs and territoriality

I will present a series of works in collaboration with Henri Berestycki, dealing with systems of predators interacting with a single prey. The system is linked to the Lotka-Volterra model of interaction with diffusion, but unlike more classic works, we are interested in studying the case where competition between predators is very strong: in this context, the original domain is partitioned in different sub-territories occupied by different predators. The question that we ask is under which conditions the predators segregate in packs and whether there is a benefit to the hostility between the packs. More specifically, we study the stationary states of the system, the stability of the solutions and the bifurcation diagram, and the asymptotic properties of the system when the intensity of the competition becomes infinite: these results have interesting connections with spectral properties of the Laplacian.