L^p and Schauder estimates for nonvariational operators structured on Hörmander vector fields with drift

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Abstract

Let $X_0, X_1, ..., X_q$ be a system of real smooth vector fields defined in some bounded domain Ω of $\mathbb{R}^n$ $(q+1<n)$ and satisfying the Hörmander's condition. We are interested in studying the class of nonvariational operators structured on these vector fields, namely:

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0$$

where the matrix $\{a_{ij}(x)\}$ is uniformly elliptic , $a_0(x)$ is bounded away from 0, and all these coefficients are bounded. Note that the "drift" vector field X_0 has weight two. We can define the Hölder and Sobolev spaces $C^{2,\alpha}(\Omega), S^{2,p}(\Omega)$, as well as the space $VMO(\Omega)$, with respect to the vector fields X_i and the subelliptic metric they induce. We are then interested in proving suitable local a priori estimates on the second order derivatives (with respect to the vector fields). Namely: assuming the coefficients $a_{ij}, a_0 \in C^{\alpha}(\Omega)$, we prove Schauder-type estimates:

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq c\left\{\|\mathcal{L}u\|_{C^{\alpha}(\Omega)} + \|u\|_{\mathcal{L}^{\infty}(\Omega)}\right\}$$

for any $\Omega' \Subset \Omega$, while assuming $a_{ij}, a_0 \in VMO(\Omega)$ we prove analogous $\mathcal{L}^p$-estimates:

$$\|u\|_{S^{2,p}(\Omega')} \leq c\left\{\|\mathcal{L}u\|_{\mathcal{L}^{p}(\Omega)} + \|u\|_{\mathcal{L}^{p}(\Omega)}\right\}$$

for any $1<p<\infty$. Similar estimates for higher order derivatives (of even order) are also established.

When the drift term X_0 is lacking, the above estimates have been proved by Bramanti and Brandolini. One of the main difficulties to overcome in this new situation is due to the fact that in presence of a drift, the distance d_X induced by the vector fields X_i does not satisfy the segment property. This prevents us from proving that a metric ball is a space of homogeneous type, in the sense of Coifman-Weiss, so that the corresponding theory of singular integrals cannot be immediately applied, and this requires some new ideas.