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SPECIAL CHAPTERS OF PROJECTIVE GEOMETRY

1 Introduction

These notes are the outcome of a Ph.D. course I gave in the Spring 1999 at the Dipartimento di Matematica, Università degli Studi di Milano. The main aim was to introduce the Ph.D. students to some modern aspects and methods of projective geometry. The lectures have been conceived to touch three important themes: the connection between projective geometry and deformations of quasi-homogeneous singularities, cohomological dimension and connectivity results, and applications of formal geometry to projective geometry.

As far as the first theme is concerned, I started with the classical problem of classifying the extensions in $\mathbb{P}^{n+1}$ of a given subvariety Y in $\mathbb{P}^n$, by proving a remarkable result due to Zak-L'vovsky (theorem 2.1). The method of the proof given here relies on another fundamental result due to Mori-Sumihiro-Wahl (theorem 2.2) which is also interesting in itself. The first three sections provide the proofs of these two results and give some applications, comments and examples. The condition involved in the Zak-L'vovsky theorem (the surjectivity of the Zak map) is better understood in the case of curves in terms of the so-called Gaussian maps (see section 5). The study of the Gaussian maps has been initiated in 1987 by J. Wahl with the main motivation of understanding the geometry of the curves lying on $K3$-surfaces. In section 6 we prove a result of Schlessinger which relates the deformation theory of the vertex of the affine cone C_Y over a smooth projectively normal subvariety Y in $\mathbb{P}^n$ with the projective geometry of Y. In particular, it becomes transparent that the surjectivity of the Zak map is naturally interpreted in terms of deformations of the vertex of the cone C_Y.

The second theme is presented in section 8. First we prove a special case of a result of Hartshorne-Lichenbaum which says that the cohomological dimension of a quasi-projective variety U of dimension n is $\leq n - 1$ if and only if U is not a projective variety. This result is then applied to prove a generalization to weighted projective spaces of the Fulton-Hansen connectivity theorem. Then some applications of this connectivity result are given.

Unfortunately, there was no time to deal with formal geometry and its applications to projective geometry. However, in sections 7 and 9 we present two results whose proofs involve in an essential way considering the first infinitesimal neighbourhood in $\mathbb{P}^n$ of a closed subvariety $X \subset \mathbb{P}^n$. The first one, due to Van de Ven, characterizes the linear subspaces as the only irreducible smooth subvarieties of $\mathbb{P}^n$ for which the normal sequence splits. The second result, due to Ellingsrud-Gruson-Peskine-Stromme, gives necessary and sufficient conditions for a curve Y lying on a complete intersection surface X in $\mathbb{P}^n$ to be the scheme-theoretic intersection of X with a hypersurface of $\mathbb{P}^n$ (see theorem 9.1). The methods of proving this latter result also yield a geometric proof of a result of Barth which asserts that $\mathrm{Pic}(X) = \mathbb{Z}$ for every smooth subvariety X of $\mathbb{P}^n$ of dimension $\geq \frac{n+2}{2}$. We hope that the method of using the first infinitesimal neighbourhood will convince the reader less familiar with formal methods that formal geometry deserves to be studied and applied to projective geometry.

Finally, I am grateful to Antonio Lanteri for inviting me at the

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2 Extensions of projective varieties

We shall fix throughout an algebraically closed ground field k (of arbitrary characteristic, unless otherwise specified).

Let Y be a smooth connected closed subvariety of dimension ≥ 1 of the n-dimensional projective space $\mathbb{P}^n$ over k.

DEFINITION 2.1 *An irreducible closed subvariety X of the $(n+1)$-dimensional projective space $\mathbb{P}^{n+1}$ is said to be extension of Y (in $\mathbb{P}^{n+1}$) if the following two conditions are satisfied:*

1. $\dim(X) = \dim(Y) + 1$.
2. *There exists a linear embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ such that $Y = X \cap H$, where $H := i(\mathbb{P}^n)$ and the intersection is taken in the scheme-theoretical sense.*

EXAMPLE 2.1 Fix $Y \subset \mathbb{P}^n$ as above and a linear embedding $i : P^n \hookrightarrow \mathbb{P}^{n+1}$, and set $H := i(\mathbb{P}^n)$. Pick an arbitrary point $x \in \mathbb{P}^{n+1} \setminus H$, and let us denote by X the projective cone in $\mathbb{P}^{n+1}$ over Y of vertex x. Clearly, X is an extension of Y in $\mathbb{P}^{n+1}$. These kind of extensions will be called *trivial extensions.*

One of the fundamental problems of the classical projective geometry is to classify all extensions in $\mathbb{P}^{n+1}$ of a given subvariety $Y \subset \mathbb{P}^n$. We shall prove a remarkable result due to Zak-L'vovsky in connection with this problem. In order to state it we need to consider two exact sequences. First let us fix some notation. For every algebraic variety Z we shall denote by Ω^1_Z the sheaf of differential forms of degree one on Z (over k). Then we shall define the tangent sheaf T_Z of Z as the dual $(\Omega^1_Z)^* = \underline{\mathrm{Hom}}_Z(\Omega^1_Z, \mathcal{O}_Z)$ of Ω^1_Z. If Z is smooth then T_Z is locally free, i.e. is a vector bundle on Z. Moreover, if Z is a closed subscheme of a scheme W of ideal sheaf $\mathcal{I}$, then the $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ is called the conormal sheaf of Z in W. The normal sheaf $N_{Z|W}$ of Z in W is by definition the dual $(\mathcal{I}/\mathcal{I}^2)^* = \underline{\mathrm{Hom}}_Z(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$ of $\mathcal{I}/\mathcal{I}^2$. If Z is smooth and W is smooth along Z (i.e. at each point of Z) then $N_{Z|W}$ is locally free, i.e. is a vector bundle on Z.

Coming back to our situation, let

$$0 \longrightarrow T_Y \longrightarrow T_{\mathbb{P}^n}|Y \xrightarrow{a} N_{Y|\mathbb{P}^n} \longrightarrow 0$$

be the normal sequence of Y in $\mathbb{P}^n$. Consider also the Euler sequence of $\mathbb{P}^n$ restricted to Y:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow (n+1)\mathcal{O}_Y(1) \xrightarrow{b} T_{\mathbb{P}^n}|Y \longrightarrow 0,$$

where $\mathcal{O}_Y(1)$ is the sheaf of hyperplane sections of Y (with respect to the embedding $Y \hookrightarrow \mathbb{P}^n$), and $(n+1)\mathcal{O}_Y(1)$ denotes the direct sum of $n+1$ copies of $\mathcal{O}_Y(1)$.

In particular, we get the surjective maps

$$a(-1) : T_{\mathbb{P}^n}(-1)|Y \to N_{Y|\mathbb{P}^n}(-1)$$

and

$$b(-1) : (n+1)\mathcal{O}_Y \to T_{\mathbb{P}^n}(-1)|Y,$$

and therefore the surjective map of vector bundles

$$c := a(-1) \circ b(-1) : (n+1)\mathcal{O}_Y \to N_{Y|\mathbb{P}^n}(-1).$$

Passing to global sections we get the following map of k-vector spaces

$$z := H^0(c) : H^0(Y, (n+1)\mathcal{O}_Y) \to H^0(Y, N_{Y|\mathbb{P}^n}(-1)), \tag{2.1}$$

which we call *the Zak map* of Y in $\mathbb{P}^n$.

Now, we can state the following fundamental result:

THEOREM 2.1 (ZAK-L'VOVSKY [55], [35]) *In the above situation, assume furthermore that* Y *is of codimension* $r \geq 2$ *and non-degenerate in* $\mathbb{P}^n$*, and that the Zak map* (2.1) *is surjective. Then every extension of* Y *in* $\mathbb{P}^{n+1}$ *is trivial.*

Theorem 2.1 is valid in arbitrary characteristic, even under more general hypotheses (see e.g. [4]). The proof we shall give below is one of the proofs of [4] and is valid only in characteristic zero. This proof is based on the following fundamental result (which will be proved in the next section):

THEOREM 2.2 (MORI-SUMIHIRO-WAHL [52]) *Let* (X, L) *be a normal polarized variety (i.e. a normal projective variety* X *endowed with an ample line bundle* L*) of dimension* ≥ 2*. Assume that the characteristic of* k *is zero and that* $H^0(X, T_X \otimes L^{-1}) \neq 0$*. Then there exists an effective divisor* E *in the complete linear system* $|L|$ *such that* X *is isomorphic to the projective cone over the polarized scheme* $(E, L_E := L|E)$*. In other words,* $X \cong \operatorname{Proj}(A[T])$*, where* $A := \oplus_{i=0}^{\infty} H^0(E, L_E^i)$*, with* T *an indeterminate over* A*, and the gradation of* $A[T]$ *is given by* $\deg(aT^m) = \deg(a) + m$ *whenever* $a \in A$ *is a homogeneous element. Moreover,* $L \cong \mathcal{O}_{\operatorname{Proj}(A[T])}(1)$*.*

We shall also make use of the following two elementary results (which will also be proved later):

PROPOSITION 2.1 (SCHLESSINGER [44]) *Let* X *be a normal variety over* k *of dimension* ≥ 2*,* Y *a closed subvariety of* X *of codimension* ≥ 2*, and* F *an* $\mathcal{O}_X$*-module which is the dual of a coherent* $\mathcal{O}_X$*-module* G*. Then the restriction map* $H^0(X, F) \to H^0(X \setminus Y, F)$ *is an isomorphism of* k*-vector spaces.*

PROPOSITION 2.2 (BERTINI-SERRE) *Let* E *be a vector bundle of rank* r *on an algebraic variety* X *over* k*. Assume that* V *is a finite dimensional* k*-vector subspace of* $H^0(X, E)$ *which generates* E *(this means that for every* $x \in X$ *the* $\mathcal{O}_{X,x}$*-module* E_x *is generated by* V*). Then there is a non-empty Zariski open subset* V_0 *of* V *such that* $\operatorname{codim}_X(Z(s)) \geq r$ *for every* $s \in V_0$*, where* $Z(s)$ *denotes the zero locus of* s*, and* $\operatorname{codim}_X(Z(s)) > \dim(X)$ *means* $Z(s) = \emptyset$*.*

PROOF OF THEOREM 2.1. The proof which follows is taken from [4] and works only in characteristic zero (because it makes use of theorem 2.2 which is in general false in positive characteristic). However, theorem 2.2 is valid in arbitrary characteristic (see [4], for another proof which is characteristic free).

CLAIM 2.1 (MUMFORD [38]) *In the hypotheses of theorem 2.1, for every* $i \geq 2$ *one has* $H^0(Y, N_{Y|\mathbb{P}^n}(-i)) = 0$.

Indeed, since $N_{Y|\mathbb{P}^n}(-i-1) \subseteq N_{Y|\mathbb{P}^n}(-i)$ for all i (via the multiplication by a global equation of a hyperplane in $\mathbb{P}^n$), it will be sufficient to prove the statement for $i = 2$. Assume that there exists a non-zero section $s \in H^0(Y, N_{Y|\mathbb{P}^n}(-2))$. Since Y is non-degenerate in $\mathbb{P}^n$ the k-linear map of vector spaces

$$H^0(P^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(Y, N_{Y|\mathbb{P}^n}(-1)), \text{ given by } h \to hs,$$

is injective. Moreover, the surjectivity of the Zak map (2.1) implies that the second space is of dimension $\leq n+1$. Since the first space is of dimension $n+1$ it follows that this map is an isomorphism. In particular, every global section of $N_{Y|\mathbb{P}^n}(-1)$ is of the form hs, with $h \in H^0(P^n, \mathcal{O}_{\mathbb{P}^n}(1))$, whence the zero locus of every global section of $N_{Y|\mathbb{P}^n}(-1)$ contains the support of a non-zero divisor of Y.

On the other hand, the surjective map $c : (n+1)\mathcal{O}_Y \to N_{Y|\mathbb{P}^n}(-1)$ considered above shows that the vector bundle $N_{Y|\mathbb{P}^n}(-1)$ of rank $r = \operatorname{codim}_{\mathbb{P}^n}(Y) \geq 2$ is generated by its global sections. Then, by proposition 2.2, the zero locus of a general section of $H^0(Y, N_{Y|\mathbb{P}^n}(-1))$ should be of codimension $\geq r \geq 2$, a contradiction. The claim is proved.

Let X be an arbitrary extension of Y in $\mathbb{P}^{n+1}$. The hypotheses imply that Y is a Cartier divisor on X. Since Y is smooth, it follows that X is smooth at each point of Y. In other words, Y is contained in the smooth locus $V := \operatorname{Reg}(X)$ of X. Moreover, since Y is a hyperplane section of X, $\operatorname{Sing}(X) = X \setminus V$ is a finite (possibly empty) set of points.

On the other hand, the equality $Y = X \cap H$ (scheme-theoretically) tells us that Y is the proper intersection of X with H. Then a general property of proper intersections implies that

$$N_{X|\mathbb{P}^{n+1}} \otimes \mathcal{O}_Y \cong N_{Y|\mathbb{P}^n}. \tag{2.2}$$

Let $f : X' \to X$ be the normalization of X (in its field of rational functions). Clearly, $f|f^{-1}(V) : f^{-1}(V) \to V$ is an isomorphism. In other words, V can be identified with a Zariski open subset of X', denoted again by V; in particular, Y is contained both in X and in X' as an ample Cartier divisor ($\mathcal{O}_{X'}(Y) = f^*(\mathcal{O}_X(Y))$ is ample because Y is ample on X and ampleness is preserved under inverse images of finite morphisms). Note also that $\mathcal{O}_{X'}(Y)$ is generated by its global sections because it is the inverse image of the very ample line bundle $\mathcal{O}_X(Y) = \mathcal{O}_X(1)$.

Set $N_{X'} := f^*(N_{X|\mathbb{P}^{n+1}})^{**}$ (bidual). Clearly, $N_{X'}|V \cong N_{X|\mathbb{P}^{n+1}}|V$. Set $N_{X'}(i) := N_{X'} \otimes \mathcal{O}_{X'}(iY)$ for all $i \in \mathbb{Z}$. Now, using (2.2) and all these observations, for every $i \geq 1$ we get the exact sequence

$$0 \longrightarrow N_{X'}(-i-1) \xrightarrow{h'} N_{X'}(-i) \longrightarrow N_{Y|\mathbb{P}^n}(-i) \longrightarrow 0,$$

which yields the exact sequence

$$0 \to H^0(X', N_{X'}(-i-1)) \to H^0(X', N_{X'}(-i)) \to H^0(Y, N_{Y|\mathbb{P}^n}(-i)). \tag{2.3}$$

Here $h' \in H^0(X', \mathcal{O}_{X'}(Y))$ is a global equation of Y in X'. By the above claim the last space is zero for every $i \geq 2$, therefore the first map (between the H^0's) is an isomorphism for all $i \geq 2$.

On the other hand, since $\mathcal{O}_{X'}(1)$ is ample, by a general simple statement, $H^0(X', N_{X'}(-i)) = 0$ for every $i >> 0$. Therefore by induction on i we get that $H^0(X', N_{X'}(-i)) = 0$ for all $i \geq 2$. Then the exact sequence (2.3) (with $i = 1$), the claim and the surjectivity of the Zak map yield

$$\dim_k(H^0(X', N_{X'}(-1))) \leq \dim_k(H^0(Y, N_{Y|\mathbb{P}^n}(-1))) \leq n+1. \tag{2.4}$$

On the other hand, since X' is normal, $V \subseteq X'$, $\operatorname{codim}_{X'}(X' \setminus V) \geq 2$ and $N_{X'}$ is reflexive, by proposition 2.1 we get

$$\dim_k H^0(X', N_{X'}(-1)) = \dim_k H^0(V, N_{X|\mathbb{P}^{n+1}}(-1)). \tag{2.5}$$

Then (2.4) and (2.5) yield

$$\dim_k H^0(V, N_{X|\mathbb{P}^{n+1}}(-1)) \leq n+1. \tag{2.6}$$

Now look at the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_V \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{O}_V \arrow[d] & & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & (n+2)\mathcal{O}_V(1) \arrow[r] \arrow[d] & N_{X|\mathbb{P}^{n+1}}|V \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & T_V \arrow[r] \arrow[d] & T_{\mathbb{P}^{n+1}}|V \arrow[r] \arrow[d] & N_{X|\mathbb{P}^{n+1}}|V \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

in which the last row is the normal sequence of X in $\mathbb{P}^{n+1}$ restricted to $V = \operatorname{Reg}(X)$, the second column is the Euler sequence of $\mathbb{P}^{n+1}$ restricted to V, and $F := \operatorname{Ker}((n+1)\mathcal{O}_V \to N_{X|\mathbb{P}^{n+1}}|V)$. By proposition 2.1 we have

$$\dim_k H^0(V, (n+2)\mathcal{O}_V) = \dim_k H^0(X', (n+2)\mathcal{O}_{X'}) = n+2.$$

The top row yields the exact sequence

$$0 \to H^0(V, F(-1)) \to H^0(V, (n+2)\mathcal{O}_V) \to H^0(V, N_{X|\mathbb{P}^{n+1}}(-1)),$$

where for every coherent $\mathcal{O}_{X'}$-module G we put $G(-1) := G \otimes \mathcal{O}_{X'}(-Y)$. Therefore the last equalities together with the inequality (2.6) yield $H^0(V, F(-1)) \neq 0$. Then from the first column of the above diagram, taking into account that

$$H^0(V, \mathcal{O}_V(-1)) = H^0(X', \mathcal{O}_{X'}(-1)) = 0,$$

it follows that $H^0(V, T_V(-1)) \neq 0$. Recalling that $T_{X'} = (\Omega^1_{X'})^*$, X' is normal and $\operatorname{codim}_{X'}(X' \setminus V) \geq 2$, this last fact translates - via proposition 2.1 - into

$$H^0(X', T_{X'}(-1)) \neq 0. \tag{2.7}$$

Now, (2.7) allows one to apply theorem 2.2 to the normal polarized variety $(X', \mathcal{O}_{X'}(Y))$ (in which $\mathcal{O}_{X'}(Y)$ is not only ample, but also generated by its global sections) to deduce that X' is isomorphic to the projective cone over $(E, \mathcal{O}_{X'}(Y) \otimes \mathcal{O}_E)$ for some $E \in |\mathcal{O}_{X'}(Y)|$. This implies that X' is in fact isomorphic to the cone over $(Y, \mathcal{O}_Y(Y))$. Finally, a simple standard argument shows that this implies that X must be the cone over Y. □

PROOF OF PROPOSITION 2.1. We shall make use of the following well-known general facts (see [25]):

a) Let X be a scheme, Y a closed subscheme of X, and set $U := X \setminus Y$. Let F be a coherent $\mathcal{O}_X$-module. Then one can define the cohomology spaces $H^i_Y(X,F)$, $\forall i \geq 0$, with support in Y such that there is a canonical exact sequence (called the exact sequence of local cohomology)

$$0 \to H^0_Y(X,F) \to H^0(X,F) \to H^0(U,F) \to H^1_Y(X,F) \to \cdots$$

$$\cdots \to H^q_Y(X,F) \to H^q(X,F) \to H^q(U,F) \to H^{q+1}_Y(X,F) \to \cdots,$$

where the maps $H^q(X,F) \to H^q(U,F)$ are the restriction maps.

b) Assume now that $X = \mathrm{Spec}(A)$ is affine and $Y = V(I)$ is given by the ideal I of the commutative Noetherian ring A. Let M be a finitely generated A-module, and let $F := \tilde{M}$ be the coherent sheaf on X associated to M. Let $f_1, \dots, f_p \in A$ be p arbitrary elements of I. $f_1, \dots, f_p$ is said to be an M-sequence if f_1 is not a zero divisor in M (i.e. $f_1 m = 0$, with $m \in M$, implies $m = 0$), and f_{i+1} is not a zero divisor in $M/(f_1M + \ldots + f_iM)$ for all $i = 1, \dots, p-1$. The maximal non-negative integer p such that there is an M-sequence $f_1, \dots, f_p \in I$ is called the I-depth of the A-module M (denoted by I-depth(M)). One can prove that the following equality holds:

$$I - \mathrm{depth}(M) = \inf_{p \in V(I)} \{\mathrm{depth}(M_p)\},$$

where $V(I) := \{p \in \mathrm{Spec}(A) \mid I \subseteq p\}$,
and $\mathrm{depth}(M_p) := pA_p - \mathrm{depth}(M_p)$.

c) Let $r \geq 0$ be a non-negative integer. Then the following two statements are equivalent:

$$I - \operatorname{depth}(M) \geq r, \text{ i.e. } pA_p - \operatorname{depth}(M_p) \geq r \text{ for all } p \in V(I), \quad (2.8)$$

$$H^i_Y(X,F) = 0 \quad \forall\, i < r, \text{ where } X = \operatorname{Spec}(X), Y = V(I), \text{ and } F = \tilde{M}, \quad (2.9)$$

In particular, if I-depth$(M) \geq 2$ then the restriction map $H^0(X,F) \to H^0(U,F)$ is an isomorphism, where $X = \operatorname{Spec}(A)$, $Y = V(I)$ and $F = \tilde{M}$.

d) If A is a normal ring (i.e. a Noetherian domain which is integrally closed in its fraction field) and if I is an ideal of A of height ≥ 2 (which by definition means that every minimal prime ideal of A containing I has height ≥ 2), then I-depth$(A) \geq 2$. This follows from a well known criterion of normality due to Serre (see e.g. [48]).

Now we can prove proposition 2.1. The conclusion of our proposition is local, so we may assume $X = \operatorname{Spec}(A)$ affine, $Y = V(I)$, with I an ideal of A, and $F = \tilde{M}$, with M an A-module of the form $M = \operatorname{Hom}_A(N,A)$, with N an A-module. Since A is normal of dimension ≥ 2 and $\operatorname{codim}_X(Y) \geq 2$, a well known criterion of normality due to Serre (see [48, IV-44, Théorème 11]) implies that pA_p-depth$(A) \geq 2$ for every $p \in Y = V(I)$. By what we have said above it follows that I-depth$(A) \geq 2$, i.e. there is an A-sequence $f_1, f_2 \in I$. Using the properties recalled above, the conclusion of the proposition is a consequence of the following:

CLAIM 2.2 *f_1, f_2 is an M-sequence, i.e.* I-depth$(M) \geq 2$.

To prove claim 2.2 observe that since f_1 is not a zero divisor in A we have the following exact sequence

$$0 \longrightarrow A \xrightarrow{f_1} A \longrightarrow B := A/f_1A \longrightarrow 0,$$

in which the map f_1 is the multiplication by f_1. Since the functor Hom is left exact we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(N,A) \xrightarrow{f_1} \operatorname{Hom}_A(N,A) \longrightarrow \operatorname{Hom}_A(N,B).$$

Since $M = \mathrm{Hom}_A(N, A)$, this shows first that f_1 is not a zero-divisor in M, and second, that

$$M/f_1M \subseteq \mathrm{Hom}_A(N, B) \cong \mathrm{Hom}_B(\overline{N}, B),$$

where $\overline{N} := N/f_1N$ (it is immediate to see that $\overline{N}$ becomes a B-module and that there is the above identification of B-modules).

Now we can apply the same argument to $B = A/f_1A$ and to the non-zero divisor $f' := f_2 \bmod f_1A$ in B to prove that f_2 is not a zero divisor in $\mathrm{Hom}_B(\overline{N}, B)$, whence, *a fortiori*, not a zero divisor in the B-submodule M/f_1M. The claim (and thereby proposition 2.1) is proved. □

PROOF OF PROPOSITION 2.2. (See [20] for a more general formulation.) Since V generates E, the canonical evaluation map $\varphi : X \times V \to E$ defined by $\varphi(x, s) := s(x)$, is a surjective smooth morphism such that every fibre of φ is a k-vector subspace of V of dimension $v - r$, where $v := \dim_k(V)$. Let C be the zero section of the canonical projection $\pi : E \to X$, so that $C \cong X$, and in particular, $\dim(C) = \dim(X) =: d$. It follows that $\varphi^{-1}(C) = \{(x, s) \mid s(x) = 0\}$ is a closed irreducible subset of $X \times V$ of dimension $d + v - r$.

Let $p : X \times V \to V$ be the second projection of $X \times V$, and let us denote by Y the closure of $p(\varphi^{-1}(C))$ in V. Then there are two cases to be considered:

1. $Y \neq V$. Then setting $V_0 := V \setminus Y \neq \emptyset$, we get that for every $s \in V_0$ we have

$$p^{-1}(s) \cap \varphi^{-1}(C) \cong \{x \in X \mid s(x) = 0\} = \emptyset,$$

i.e. $Z(s) = \emptyset$, whence $\mathrm{codim}_X(Z(s)) \geq r$.

2. $Y = V$. Then we get the dominant morphism $q := p|\varphi^{-1}(C) : \varphi^{-1}(C) \to V$ of irreducible varieties. By the theorem of the dimension of fibres (see e.g. [50, page 60, theorem 7]) there is a non-empty open subset V_0 of V which is contained in $q(\varphi^{-1}(C)) = p(\varphi^{-1}(C))$ such that

$$\begin{aligned} \dim(q^{-1}(s)) &= \dim(p^{-1}(s) \cap \varphi^{-1}(C)) \\ &= \dim(\varphi^{-1}(C)) - \dim(V) \\ &= (d + v - r) - v = d - r, \quad \forall s \in V_0. \end{aligned}$$

In other words, $\operatorname{codim}_X(Z(s)) = r$ for all $s \in V_0$. Proposition 2.2 is proved. □

We end this section by a generalization of theorem 2.1 of Zak-L'vovsky. First we shall need a definition.

DEFINITION 2.2 *Let Y be a smooth connected closed subvariety of $\mathbb{P}^n$ of dimension $d \geq 1$. Let $m \geq 1$ be an integer. A closed irreducible subvariety X of $\mathbb{P}^{n+m}$ of dimension $d+m$ is called extension of Y in $\mathbb{P}^{n+m}$ if there is a linear embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+m}$ such that $Y = X \cap H$ (in the scheme-theoretic sense), where $H := i(\mathbb{P}^n)$. If there exists an extension X of Y in $\mathbb{P}^{n+m}$ which not a cone, we sometimes also say that Y extends non-trivially m steps.*

THEOREM 2.3 *Let Y be a smooth connected non-degenerate subvariety of codimension ≥ 2 of $\mathbb{P}^n$. Assume* $\operatorname{char}(k) = 0$, $\dim(Y) \geq 1$, *and* $H^0(Y, N_{Y|\mathbb{P}^n}(-2)) = 0$. *Set* $\dim(H^0(Y, N_{Y|\mathbb{P}^n}(-1))) =: n + r + 1$ *(by proposition* 4.1 *below, $r \geq 0$). If $m \geq r+1$ then every extension of Y in $\mathbb{P}^{n+m}$ is a cone over a subvariety of $\mathbb{P}^{n+m-1}$.*

NOTE. By the claim in the proof of theorem 2.1, the surjectivity of the Zak map (2.1) implies that $H^0(Y, N_{Y|\mathbb{P}^n}(-2)) = 0$. Therefore theorem 2.3 is a generalization of theorem 2.1 in characteristic zero.

PROOF. We shall proceed in a similar way as in the last part of the proof of theorem 2.1. Since $\dim(X) = \dim(Y) + m$ and $Y = X \cap H$ (scheme-theoretically), $N_{Y|X} \cong m\mathcal{O}_Y(1)$ and $N_{X|\mathbb{P}^{n+m}}|Y \cong N_{Y|H}$. (Note that since X is smooth at each point of Y, $N_{X|\mathbb{P}^{n+m}}$ is a vector bundle along Y.) Let $f : X' \to X$ be the normalization of X. Then, exactly as in the proof of theorem 2.1, $Y \subset X'$ and $N_{Y|X'} \cong m\mathcal{O}_Y(1)$. Moreover, $N_{X'} := f^*(N_{X|\mathbb{P}^{n+m}})^{**}$ is a vector bundle along Y such that $N_{X'}|Y \cong N_{Y|H}$. Since f is a finite morphism, $\mathcal{O}_{X'}(1) := f^*(\mathcal{O}_X(1))$ is an ample line bundle on X' generated by its global sections. For every coherent sheaf F on X' and for every integer i, set $F(i) := F \otimes \mathcal{O}_{X'}(i)$. Fix an integer $i > 0$ and consider the exact sequences ($p \geq 0$) coming from these isomorphisms

$$0 \to \mathcal{I}^{p+1} \otimes N_{X'}(-i) \to \mathcal{I}^p \otimes N_{X'}(-i) \to \mathcal{I}^p/\mathcal{I}^{p+1} \otimes N_{X'}(-i) \to 0, \tag{2.10}$$

where $\mathcal{I}$ is the ideal sheaf of Y in X. Moreover, we have

$$\begin{aligned}\mathcal{I}^p/\mathcal{I}^{p+1} \otimes N_{X'}(-i) &\cong S^p(m\mathcal{O}_Y(-1)) \otimes N_{Y|H}(-i) \\ &\cong \binom{m-1+p}{p} N_{Y|H}(-i-p). \qquad (2.11)\end{aligned}$$

The hypothesis that $H^0(Y, N_{Y|\mathbb{P}^n}(-2)) = 0$ implies that, for every $i \geq 2$, $H^0(Y, N_{Y|\mathbb{P}^n}(-i)) = 0$. Then by (2.11) we get $H^0(Y, \mathcal{I}^p/\mathcal{I}^{p+1} \otimes N_{X'}(-1)) = 0$ for all $p \geq 1$. Therefore (2.10) implies that the canonical maps

$$H^0(X', \mathcal{I}^{p+1} \otimes N_{X'}(-1)) \to H^0(X', \mathcal{I}^p \otimes N_{X'}(-1))$$

are isomorphisms for every $p \geq 1$. This yields

$$H^0(X', \mathcal{I} \otimes N_{X'}(-1)) \cong H^0(X', \mathcal{I}^p \otimes N_{X'}(-1)), \quad \forall p \geq 1. \qquad (2.12)$$

On the other hand we have the following:

CLAIM 2.3 *Let $Y \neq \emptyset$ be a closed subvariety of the normal projective variety X' defined by the ideal sheaf $\mathcal{I}$. Assume that Y meets every hypersurface of X'. Let F is a coherent torsion free $\mathcal{O}_{X'}$-module such that the canonical maps $H^0(X', \mathcal{I}^p F) \to H^0(X', F)$ are isomorphisms for every $p \geq 1$. Then $H^0(X', F) = 0$.*

To prove claim 2.3 pick an arbitrary point $y \in Y$. Then the hypothesis implies that $s_y \in \mathcal{I}_y^p F_y$ for all $p \geq 1$. Since F_y is an $\mathcal{O}_{X',y}$-module of finite type and $\mathcal{I}_y$ is contained in the maximal ideal of the local ring $\mathcal{O}_{X',y}$, by a well known theorem of Krull (see [48]) we get

$$\bigcap_{p=1}^{\infty} \mathcal{I}_y^p F_y = 0.$$

It follows that $s_y = 0$. Since y was an arbitrary point of Y, s vanishes along Y, whence $s|V = 0$ for a certain open neighbourhood V of Y in X. By hypothesis $\operatorname{codim}_{X'}(X' \setminus V) \geq 2$. On the other hand, since F is torsion free the canonical map $F \to F^{**}$ into the bidual is injective, whence s is also a section of the reflexive sheaf F^{**}. By proposition 2.1 it follows that the restriction map $H^0(X', F^{**}) \to H^0(V, F^{**})$ is an isomorphism. Since $s|V = 0$, we infer that $s = 0$ as a section of

F^{**}, whence also as a section of F because $F \subseteq F^{**}$. This proves the claim.

Coming back to the proof of theorem 2.3, in claim 2.3 we take $F = \mathcal{I} \otimes N_{X'}(-1)$. Since $N_{X'}$ is reflexive F is torsion free. Moreover, (2.12) says that F satisfies the hypotheses of the claim. Therefore by claim 2.3 we get

$$H^0(X', \mathcal{I} \otimes N_{X'}(-1)) = 0. \tag{2.13}$$

Taking $i = 1$ and $p = 0$ in (2.10) and using (2.13) we infer that the restriction map $H^0(X', N_{X'}(-1)) \to H^0(Y, N_{Y|H}(-1))$ is injective, whence by hypotheses we get

$$\dim(H^0(X', N_{X'}(-1)) \le n + r + 1 \le n + m. \tag{2.14}$$

At this point, if we set $V := \mathrm{Reg}(X)$ we have $Y \subset V \subset X'$. Then a diagram completely similar to the diagram considered in the proof of theorem 2.1 (with $(n+2)\mathcal{O}_V(1)$ replaced by $(n+m+1)\mathcal{O}_V$), and the inequality (2.14) yield $H^1(V, T_V(-1)) \neq 0$. Since X' is normal and $\mathrm{codim}_{X'}(X' \setminus V) \ge 2$ we get

$$H^0(X', T_{X'}(-1)) \cong H^0(V, T_V(-1)) \neq 0.$$

Now the conclusion of our theorem (modulo the claim 2.3) follows applying theorem 2.2. □

REMARK 2.1 Under the hypotheses of theorem 2.3, the subvariety Y of $\mathbb{P}^n$ cannot be extended non-trivially more than r steps. In this sense the hypothesis that $H^0(Y, N_{Y|\mathbb{P}^n}(-2)) = 0$ is essential. For, let Y is a smooth complete intersection in $\mathbb{P}^n$ of multi-degree $(d_1, \ldots, d_r)$, such that $2 \le r \le n-1$ and $d_i \ge 2$, $\forall i = 1, \ldots, r$. Then $H^0(Y, N_{Y|\mathbb{P}^n}(-2)) \neq 0$, and clearly Y can be extended non-trivially in $\mathbb{P}^{n+m}$ for every $m \ge 1$. We also note that by a result of Barth (see [8]), every smooth closed subvariety Y of $\mathbb{P}^n$ of dimension ≥ 1 which can be extended smoothly in $\mathbb{P}^{n+m}$ for all $m \ge 1$ is necessarily a complete intersection. Examples of subvarieties $Y \subset \mathbb{P}^n$ satisfying $H^0(Y, N_{Y|\mathbb{P}^n}(-2)) = 0$ are given by the following:

PROPOSITION 2.3 (WAHL [53]) *Let Y be a smooth closed projectively normal subvariety of $\mathbb{P}^n$ of dimension ≥ 1. Let I be the saturated homogeneous ideal defining Y in the polynomial k-algebra $P := k[T_0, T_1, \ldots, T_n]$, and denote by $A := P/I$ the homogeneous coordinate k-algebra of Y in $\mathbb{P}^n$. Assume furthermore that there is an exact sequence*

$$uP(-3) \to sP(-2) \to I \to 0, \text{ with } s,t \geq 1, \tag{2.15}$$

where $P(i)$ is the graded P-module such that $P(i)_j = P_{i+j}$, $\forall i,j \in \mathbb{Z}$, and $aP(i)$ is the direct sum of $a \geq 1$ copies of $P(i)$. (This means that the ideal I is generated by s independent homogeneous polynomials of degree 2, and relations among them are generated by independent linear ones.) Then $H^0(Y, N_{Y|\mathbb{P}^n}(-2)) = 0$.

We shall prove this result in section 6 after we shall interpret the vanishing of $H^0(Y, N_{Y|\mathbb{P}^n}(-2))$ in terms of the deformation theory of the vertex of the affine cone over Y in $\mathbb{P}^n$. The next result produces examples satisfying the hypotheses of proposition 2.3.

THEOREM 2.4 (MUMFORD-GREEN [40], [23], SEE ALSO [33]) *Let L be a line bundle of degree $\geq 2g+3$ on a smooth projective curve Y of genus g. Let $Y \subset P := \mathbb{P}(H^0(Y,L)^*)$ be the linearly normal projective embedding of Y given by the complete linear system $|L|$. Then Y is projectively normal in P and there is an exact sequence of the form* (2.15).

REMARK 2.2 In fact, Mumford proved in [40] that there is a surjective map $sP(-2) \to I \to 0$ (if $\deg(L) \geq 2g+3$) and subsequently M. Green refined Mumford's result in [23] in the above form (see [23] for a more general result, or also [33] for a simpler proof of Green's result).

3 Proof of theorem 2.2

We shall prove theorem 2.2 under the additional hypothesis that L is generated by its global sections. Note that we used theorem 2.2 in the proofs of theorems 2.1 and 2.3 only under this additional hypothesis.

Set $R := \oplus_{i=0}^{\infty} H^0(X, L^i)$. Since L is ample and X projective, there exists a canonical isomorphism $X \cong \mathrm{Proj}(R)$ such that $L \cong \mathcal{O}_{\mathrm{Proj}(R)}(1)$. Moreover, since X is normal, R is a normal finitely generated k-algebra (see EGA III [26]). Consider the normal affine cone $\mathrm{Spec}(R)$ over the polarized variety (X, L), and let $m_R := \oplus_{i=1}^{\infty} H^0(X, L^i)$ be the irrelevant maximal ideal of R which corresponds to the vertex of $\mathrm{Spec}(R)$. Set $U := \mathrm{Spec}(R) \setminus \{m_R\}$. Then there is a canonical morphism $\pi : U \to X$. Since L is generated by its global sections, π is the projection of a locally trivial $\mathbb{G}_m$-bundle, where $\mathbb{G}_m = k \setminus \{0\}$ is the multiplicative group of the ground field k. The next lemma and its proof take care of the structure of π more closely.

LEMMA 3.1 *In the hypotheses of theorem* 2.2, *assume furthermore that L is generated by its global sections. Then there is a canonical exact sequence*

$$0 \to \mathcal{O}_U \to T_U \to \pi^*(T_X) \to 0.$$

PROOF OF LEMMA 3.1. There is a general canonical exact sequence associated to the morphism $\pi : U \to X$ (this exact sequence makes actually sense for every morphism $f : V \to W$ of algebraic varieties over k, see [28, proposition 8.11, page 176])

$$\pi^*(\Omega^1_{X|k}) \to \Omega^1_{U|k} \to \Omega^1_{U|X} \to 0,$$

which upon dualizing gives the exact sequence

$$0 \longrightarrow T_{U|X} \longrightarrow T_U \xrightarrow{\alpha} \pi^*(T_X). \tag{3.1}$$

Here $T_{U|X}$ is the relative tangent sheaf of $\pi : U \to X$. To prove lemma 3.1 it will be sufficient to check the following two facts:

$$T_{U|X} \cong \mathcal{O}_U, \text{ and} \tag{3.2}$$

$$\text{The map } \alpha \text{ is surjective.} \tag{3.3}$$

Let us first fix some notations. For every $s \in R_1 \setminus \{0\} = H^0(X, L) \setminus \{0\}$, let R_s be the ring of fractions of R with denominators in the multiplicative subset $\{1, s, s^2, ..., s^n, ...\}$. Since s is homogeneous, R_s becomes a graded k-algebra by setting $\deg(\frac{t}{s^n}) = \deg(t) - n$ whenever $t \in S$ is homogeneous. Then we may consider the subring $R_{(s)}$ of

R_s whose elements are all fractions $\frac{t}{s^n} \in R_s$ such that $t \in R_n = H^0(X, L^n)$ is a homogeneous element of degree n, i.e. $\deg(\frac{t}{s^n}) = 0$. Clearly, $R_{(s)}$ is a k-subalgebra of R_s. Moreover, since $s \in R_1$, the inclusion $R_{(s)} \subset R_s$ is identified with the inclusion of $R_{(s)}$ in the $R_{(s)}$-algebra $R_{(s)}[T, T^{-1}]$ of the Laurent polynomials in the indeterminate T. This is done by sending $T \to s$ and $T^{-1} \to \frac{1}{s}$, as is easily checked. Since L is generated by its global sections,

$$X = \operatorname{Proj}(R) = \bigcup_{s \in R_1 \setminus \{0\}} D_+(s), \text{ where } D_+(s) := \operatorname{Spec}(R_{(s)}).$$

$$U = \bigcup_{s \in R_1 \setminus \{0\}} D(s), \text{ where } D(s) := \operatorname{Spec}(R_s).$$

Then $\pi^*(D_+(s)) = D(s)$ for all $s \in R_1 \setminus \{0\}$ (and in particular, π is an affine morphism). Moreover, the restriction $\pi | D(s) : D(s) \to D_+(s)$ corresponds to the inclusion $R_{(s)} \subset R_s$, and since there is an isomorphism of $R_{(s)}$-algebras $R_s \cong R_{(s)}[T, T^{-1}]$, we see that $D(s) \cong D_+(s) \times \mathbb{G}_m$. This gives explicitly the local structure of the (locally trivial) $\mathbb{G}_m$-bundle $\pi : U \to X$.

Let A be a commutative ring, B a commutative A-algebra, and M a B-module. Denote by $\operatorname{Der}_A(B, M)$ the set of all A-derivations $D : B \to M$. One can define the sum $D + D'$ of two derivations $D, D' \in \operatorname{Der}_A(B, M)$ and the multiplication bD (with $b \in B$) in an obvious way and one easily checks that $D + D', aD \in \operatorname{Der}_A(B, M)$. In other words, $\operatorname{Der}_A(B, M)$ becomes a B-module in a natural way. If $M = B$, we shall also denote $\operatorname{Der}_A(B) := \operatorname{Der}_A(B, B)$.

The restriction of the exact sequence (3.1) to the affine open subset $D(s)$ corresponds to the obvious exact sequence

$$0 \longrightarrow \operatorname{Der}_{R_{(s)}}(R_s) \longrightarrow \operatorname{Der}_k(R_s) \xrightarrow{\alpha_s} \operatorname{Der}_k(R_{(s)}, R_s), \quad (3.4)$$

where α_s associates to any k-derivation $D \in \operatorname{Der}_k(R_s)$ the restriction $D | R_{(s)} \in \operatorname{Der}_k(R_{(s)}, R_s)$.

Now, it is clear that the map α is surjective (i.e. that condition (3.3) holds). Indeed, since this verification is local, it is sufficient to check that the map α_s (of (3.4)) is surjective for every $s \in R_1 \setminus \{0\}$. In fact, if $\delta \in \operatorname{Der}_k(R_{(s)}, R_s)$ then we can define $D \in \operatorname{Der}_k(R_s)$ by $D | R_{(s)} = \delta$ and $D(T) = u$ for an arbitrary $u \in R_s$ (via the above isomorphism of

$R_{(s)}$-algebras $R_s \cong R_{(s)}[T, T^{-1}]$). Then, of course, $D(\frac{1}{T}) = -\frac{u}{T^2}$. This proves (3.3).

It remains to prove (3.2). To this end, consider the map $D : R \to R$ defined by $D(r) := \deg(r)r$, for every homogeneous element $r \in R$. It is obvious that D is a k-derivation such that $D(R_n) \subseteq R_n$ for every $n \geq 0$. Since $\mathrm{char}(k) = 0$, it is also clear that D is surjective. Then by the universal property of $\Omega^1_{R|k}$, there is a unique homomorphism $w \in \mathrm{Hom}_R(\Omega^1_{R|k}, R)$ of R-modules which composed with the canonical derivation $d : R \to \Omega^1_{R|k}$ coincides to D. Since D is surjective, w is also surjective. Passing to sheaves on $\mathrm{Spec}(R)$ and restricting to U we get the surjective map of $\mathcal{O}_U$-modules $w : \Omega^1_{U|k} \to \mathcal{O}_U$.

If we denote by $\beta : \Omega^1_{U|k} \to \Omega^1_{U|X}$ the canonical surjection, we claim that there exists a unique map of $\mathcal{O}_U$-modules $w' : \Omega^1_{U|X} \to \mathcal{O}_U$ such that $w' \circ \beta = w$. To check this, observe that $D|R_{(s)} = 0$ for every $s \in R_1 \setminus \{0\}$, whence $D \in \mathrm{Der}_{R_{(s)}}(R_s)$, so that the existence of w' comes from the universal property of $\Omega^1_{R_s|R_{(s)}}$ (taking into account that $U = \cup_{s \in R_1 \setminus \{0\}} D(s)$). The surjectivity of w implies the surjectivity of $w' : \Omega^1_{U|X} \to \mathcal{O}_U$. But since $\pi : U \to X$ is a smooth morphism of relative dimension one, $\Omega^1_{U|X}$ is an invertible $\mathcal{O}_U$-module. Therefore w' is a surjective map between two invertible $\mathcal{O}_U$-modules, whence w' is necessarily an isomorphism.

This proves condition (3.2), and thereby lemma 3.1. □

LEMMA 3.2 *In the hypotheses of lemma* 3.1, *for every coherent* $\mathcal{O}_X$-*module* F *and for every* $p \geq 0$ *there is a natural isomorphism of graded* R-*modules*

$$H^p(U, \pi^*(F)) \cong \bigoplus_{i \in \mathbb{Z}} H^p(X, F \otimes L^i).$$

PROOF OF LEMMA 3.2. In the proof of lemma 3.1 we already observed that the morphism π is affine ($\pi^{-1}(D_+(s)) = D(s)$ for every $s \in R_1 \setminus \{0\}$). In particular, for every coherent $\mathcal{O}_X$-module F,

$$H^p(U, \pi^*(F)) \cong H^p(X, \pi_*(\pi^*(F))), \quad \forall p \geq 0.$$

To conclude the proof of lemma 3.2, it will be sufficient to show that for every coherent $\mathcal{O}_X$-module F one has a canonical identification

$$\pi_*(\pi^*(F)) \cong \bigoplus_{i\in\mathbb{Z}} F\otimes L^i. \tag{3.5}$$

To check this, since R is a finitely generated k-algebra such that $X = \mathrm{Proj}(R) = \bigcup_{s\in R_1\setminus\{0\}} D_+(s)$, the sheaf F is the sheaf $\tilde{M}$ associated to a finitely generated graded R-module M. Then by the proof of lemma 3.1, $R_s \cong R_{(s)}[T,T^{-1}]$, whence we get canonical isomorphisms of $R_{(s)}$-modules

$$M_s \cong M\otimes R_s \cong M\otimes R_{(s)}[T,T^{-1}] \cong \bigoplus_{i\in\mathbb{Z}} M_{(s)}T^i.$$

Since this happens for every $s\in R_1\setminus\{0\}$, these local isomorphisms patch together to give the isomorphism (3.5). □

The tangent sheaf $T_{\mathrm{Spec}(R)}$ of the cone $\mathrm{Spec}(R)$ corresponds to the graded R-module

$$T_R = \bigoplus_{i\in\mathbb{Z}} T_R(i),$$

where $T_R := \mathrm{Der}_k(R)$, and the piece $T_R(i)$ of weight i is given by

$$T_R(i) := \{D\in \mathrm{Der}_k(R) \mid D(R_n)\subseteq R_{n+i}\ \ \forall n\geq 0\}.$$

Now, the next step in the proof of theorem 2.2 is the following:

LEMMA 3.3 *In the above hypotheses, the k-vector space $H^0(X, T_X\otimes L^{-i})$ can be canonically identified to the k-vector space $T_R(-i)$, for all $i\geq 1$.*

PROOF OF LEMMA 3.3. Since R is normal of dimension ≥ 2 and $U = \mathrm{Spec}(R)\setminus\{m_R\}$, we can apply proposition 2.1 to deduce a canonical isomorphism of graded R-modules $T_R\cong H^0(U,T_U)$. Then the exact sequence of lemma 3.1 yields the cohomology exact sequence of R-modules

$$0\to H^0(U,\mathcal{O}_U)\to H^0(U,T_U)\to H^0(U,\pi^*(T_X))\to H^1(U,\mathcal{O}_U). \tag{3.6}$$

By lemma 3.2, for every $p\geq 0$,

$$H^p(U,\mathcal{O}_U)\cong \bigoplus_{i\in\mathbb{Z}} H^p(X,L^i),$$

and

$$H^0(U, \pi^*(T_X)) \cong \bigoplus_{i \in \mathbb{Z}} H^0(X, T_X \otimes L^i).$$

Now, by Mumford's vanishing theorem (see [39]), $H^p(X, L^i) = 0$, for every $i < 0$ and $p \leq 1$ (because $\dim(X) \geq 2$ and $\mathrm{char}(k) = 0$; note that this vanishing for $p = 0$ is trivial). Therefore the above identification and the exact sequence (3.6) yield

$$T_R(i) \cong H^0(X, T_X \otimes L^i), \quad \forall i < 0.$$

Lemma 3.3 is proved. □

In view of lemma 3.3, theorem 2.2 will be proved if we prove the following:

PROPOSITION 3.1 *Let (X, L) be a normal polarized variety of dimension ≥ 2 over k, with* $\mathrm{char}(k) = 0$. *Assume that there is a non-zero k-derivation $D : R \to R$ of weight -1, i.e. $D(R_i) \subseteq R_{i-1}$ for all $i \geq 0$, where $R = \oplus_{i=0}^{\infty} H^0(X, L^i)$. If L is generated by its global sections then the conclusion of theorem 2.2 holds true.*

To prove proposition 3.1 we need another two lemmas.

LEMMA 3.4 (ZARISKI) *Let R be a graded k-algebra such that* $\mathrm{char}(k) = 0$ *and there is a k-derivation $D : R \to R$ of weight -1 and an element $t \in R_1$ with the property that $D(t) = 1$. If $A := \{r \in R \mid D(r) = 0\}$ then A is a graded k-subalgebra of R, t is transcendental over A and $R = A[t]$. Moreover, $D = \frac{\partial}{\partial t}$ on $A[t]$.*

PROOF OF LEMMA 3.4. The fact that A is a graded k-subalgebra of R is immediate. Let $A[T]$ be the polynomial A-algebra in the indeterminate T. Grade $A[T]$ by $\deg(aT^m) = \deg(a) + m$ for every $a \in A$ homogeneous and $m \geq 0$. Consider the homomorphism of graded k-algebras $\varphi : A[T] \to R$ such that $\varphi | A = \mathrm{id}_A$ and $\varphi(T) = t$. Then it is immediate that the following diagram is commutative:

$$\begin{array}{ccc} A[T] & \xrightarrow{\varphi} & R \\ {\scriptstyle \frac{\partial}{\partial T}} \downarrow & & \downarrow {\scriptstyle D} \\ A[T] & \xrightarrow[\varphi]{} & R \end{array}$$

So, it will be enough to show that φ is bijective.

Injectivity of φ: Assume that there exists a non-zero polynomial $f(T) = \sum_{i=0}^{n} a_i T^i \in A[T]$ such that $\varphi(f(T)) = 0$. We may assume that $f(T)$ is homogeneous in $A[T]$ and of minimal degree. Thus $a_i \in A_{n-i} \subseteq R_{n-i}$. Then $\frac{\partial f(T)}{\partial T}$ is a polynomial of smaller degree and still in $\mathrm{Ker}(\varphi)$. Thus $\frac{\partial f(T)}{\partial T} = 0$, and since $\mathrm{char}(k) = 0$, $f \in A$. Since $\varphi(f) = 0$ it follows that $f = 0$, a contradiction.

Surjectivity of φ: We proceed by induction on n, the case $n = 0$ being clear. If $r \in R_n$ $(n > 0)$ then $D(r) \in R_{n-1}$, whence $D(r) = \sum_{i=0}^{n-1} a_i t^i$, with $a_i \in A_{n-1-i}$, $\forall i$ (by induction hypothesis). Thus

$$D(r - \sum_{i=0}^{n-1} \frac{a_i}{i+1} t^{i+1}) = 0,$$

i.e. $r - \sum_{i=0}^{n-1} \frac{a_i}{i+1} t^{i+1} \in A$. Lemma 3.4 is proved. $\square$

LEMMA 3.5 *Let (X, L) be a normal polarized variety over k such that L is generated by its global sections and* $\mathrm{char}(k) = 0$. *Assume that there is a non-zero k-derivation $D : R \to R$ of weight -1, where $R = \sum_{i=0}^{\infty} H^0(X, L^i)$. Then there exists an element $t \in R_1 = H^0(X, L)$ such that $D(t) = 1$.*

PROOF OF LEMMA 3.5. Let $t_0, \dots, t_n$ be a basis of the k-vector space $H^0(X, L)$. Since L is generated by its global sections we get a morphism $u : X \to \mathbb{P}^n$ such that $u^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong L$, and since L is also ample, u is a finite morphism (otherwise there would exist a curve C such that $u(C)$ is a point, whence $L|C$ cannot be ample). If for some i, $D(t_i) \neq 0$, then we can take $t = \alpha^{-1} t_i$, with $\alpha := D(t_i) \in k \setminus \{0\}$, and therefore $D(t) = 1$.

Assume therefore that $D(t_i) = 0$ for every $i = 0, 1, \dots, n$. We shall show that this leads to a contradiction. Consider the map of graded k-algebras

$$k[T_0, \dots, T_n] \cong \bigoplus_{i=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(i)) \to R = \bigoplus_{i=0}^{\infty} H^0(X, L^i),$$

which sends T_j to t_j, $j = 0, 1, ..., n$. Since u is finite the morphism of affine varieties $u^{-1}(D_+(T_0)) \to D_+(T_0)$ is finite, whence the map of k-algebras

$$k[T_0, ..., T_n]_{(T_0)} \to R_{(t_0)}$$

is finite. In particular, every homogeneous element $t \in R$ satisfies a non-trivial algebraic equation

$$a_0 t^m + a_1 t^{m-1} + ... + a_m = 0, \quad \text{with } a_i \in k[t_0, ..., t_n], \quad m \geq 1.$$

We may assume m minimal with this property. Since the derivation D vanishes on $k[t_0, ..., t_m]$ (by our assumption), we get by derivating

$$(ma_0 t^{m-1} + (m-1)a_1 t^{m-2} + ... + a_{m-1})D(t) = 0.$$

If $D(t) \neq 0$, since R is a domain, we get

$$ma_0 t^{m-1} + (m-1)a_1 t^{m-2} + ... + a_{m-1} = 0.$$

Recalling also that $\mathrm{char}(k) = 0$, this contradicts the minimality of m.

Therefore $D(t) = 0$ for every homogeneous $t \in R$, whence $D = 0$, a contradiction. $\square$

The lemmas 3.4 and 3.5 imply the following:

COROLLARY 3.1 *In the hypotheses of lemma* 3.5, *there exists* $t \in R_1$ *such that* $D(t) = 1$, $R = A[t]$, *and* t *is transcendental over* $A := \{r \in R \mid D(r) = 0\}$.

PROOF OF PROPOSITION 3.1. By corollary 3.1, $R = A[t]$, with $t \in H^0(X, L)$ such that $D(t) = 1$ and t is transcendental over $A = \{r \in R \mid D(r) = 0\}$. Since R is normal it follows that A is also normal. Set $E := \mathrm{div}_X(t) \in |L|$, i.e. $L = \mathcal{O}_X(E)$, and consider the canonical exact sequences ($n \geq 0$)

$$0 \longrightarrow L^{n-1} \xrightarrow{t} L^n \longrightarrow \mathcal{O}_E(nE) = L_E^n \to 0.$$

Taking cohomology we get

$$0 \to R \to R \to \bigoplus_{n=0}^{\infty} H^0(E, L_E^n) \to \bigoplus_{n=0}^{\infty} H^1(X, L^{n-1}). \qquad (3.7)$$

We claim that $H^1(X, L^{n-1}) = 0$, $\forall n \in \mathbb{Z}$. Assuming the claim, the proof of proposition 3.1 is finished, because from the exact sequence (3.7) it follows that

$$\bigoplus_{n=0}^{\infty} H^0(E, L_E^n) \cong R/tR = A[t]/tA[t] \cong A.$$

It remains therefore to prove the claim. Since A is normal, m_A-depth$(A) \geq 2$, whence m_R-depth(R)= m_R-depth$(A[t]) \geq 3$. If $U :=$ Spec$(R) \setminus \{m_R\}$, we can write the local cohomology exact sequence

$$0 = H^1(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \to H^1(U, \mathcal{O}_U) \to \tag{3.8}$$

$$\to H^2_{m_R}(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \to H^2(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = 0.$$

By lemma 3.2,

$$H^1(U, \mathcal{O}_U) \cong \bigoplus_{i=-\infty}^{\infty} H^1(X, L^i). \tag{3.9}$$

Since m_R-depth$(R) \geq 3$, $H^2_{m_R}(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = 0$ (see property c) in the proof of proposition 2.1), whence from (3.8) and (3.9) we get $H^1(X, L^i) = H^1(X, \mathcal{O}_X(iE)) = 0$ for all $i \in \mathbb{Z}$.

In this way proposition 3.1 and thereby theorem 2.2 (under the extra hypothesis that L is generated by its global sections) are completely proved. □

NOTE. The proof of theorem 2.2 without the hypothesis that L is generated by its global sections follows the same main ideas, but technically is more involved (see [52]).

4 Counterexamples and further consequences of theorem 2.2

First we show by counterexamples that theorem 2.2 is in general false in positive characteristic.

4.1 A counterexample in characteristic 2

Let $X \subset \mathbb{P}^{2n}$ $(n \geq 2)$ be the hyperquadric in $\mathbb{P}^{2n}$ of equation

$$f = T_0^2 + T_1T_{n+1} + T_2T_{n+2} + ... + T_nT_{2n}$$

over an algebraically closed field k of characteristic 2, and set $L := \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^{2n}}(1)|X$. Then X is a smooth hyperquadric in $\mathbb{P}^{2n}$, and in particular, X cannot be isomorphic to a cone over a polarized variety (E, L_E). Set

$$R = \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i)) \cong k[T_0, ..., T_{2n}]/(f)$$

(being a smooth hypersurface, X is projectively normal in $\mathbb{P}^{2n}$).

Observe that because $\mathrm{char}(k) = 2$, all the derivatives of the polynomial f vanish at the point $P = [1, 0, ..., 0]$ (and only at this point). In other words, all the projective tangent spaces at X have the point P in common, i.e. X is a strange variety. Moreover, the derivative

$$\frac{\partial}{\partial T_0} : k[T_0, ..., T_{2n}] \to k[T_0, ..., T_{2n}]$$

has the property that $\frac{\partial f}{\partial T_0} = 0$. Therefore

$$\frac{\partial(gf)}{\partial T_0} = f\frac{\partial g}{\partial T_0}, \ \ \forall g \in k[T_0, ..., T_n], \ \ i.e. \ \frac{\partial}{\partial T_0}((f)) \subseteq (f).$$

In other words, $\frac{\partial}{\partial T_0}$ yields a non-zero k-derivation $D : R \to R$ of weight -1.

On the other hand, since X is a hypersurface in $\mathbb{P}^{2n}$ $(n \geq 2)$, $H^p(X, L^{-i}) = 0$ for all $p = 0, 1$ and for all $i \in \mathbb{Z}$. Therefore the arguments in the proof of lemma 3.3 can be carried out to prove that $H^0(X, T_X(-1)) \neq 0$ (due to these vanishings, we don't have to appeal to Mumford's vanishing theorem).

This shows that (X, L) is a smooth polarized variety of dimension ≥ 3 (in particular is not a cone), such that $H^0(X, T_X \otimes L^{-1}) \neq 0$.

4.2 A counterexample in characteristic 3

Let k be an algebraically closed field of characteristic 3 and in $\mathbb{P}^3$ consider the surface X of equation

$$f = T_0^3 + T_1T_2^2 + T_2T_3^2 + T_3T_1^2.$$

We have $\frac{\partial f}{\partial T_0} = 0$, $\frac{\partial f}{\partial T_1} = T_2^2 + 2T_1T_3 = T_2^2 - T_1T_3$, $\frac{\partial f}{\partial T_2} = T_3^2 - T_1T_2$, $\frac{\partial f}{\partial T_3} = T_1^2 - T_2T_3$. The subvariety in $\mathbb{P}^3$ of equations

$$T_1^2 - T_2T_3 = T_2^2 - T_1T_3 = T_3^2 - T_1T_2 = 0$$

is (at least set-theoretically) the line

$$L := \{[\lambda, \mu, \mu, \mu] \in \mathbb{P}^3 \mid [\lambda, \mu] \in \mathbb{P}^1\},$$

as one can easily see. Thus $\mathrm{Sing}(X)$ is the point $A = [0, 1, 1, 1]$ and all other points of $L \setminus \{A\}$ are strange points of X (i.e. points in the intersection of all projective tangent spaces at all smooth points of X). In particular, X is a normal surface. Observe that $B := [0, 0, 0, 1] \in X$, but the line AB has in common with X only the points A and B. In particular, it follows that X cannot be a cone.

On the other hand, as in example 4.1, since $\frac{\partial f}{\partial T_0} = 0$, we have $\frac{\partial (fg)}{\partial T_0} = f\frac{\partial g}{\partial T_0}$, whence the derivation

$$\frac{\partial}{\partial T_0} : k[T_0, T_1, T_2, T_3] \to k[T_0, T_1, T_2, T_3]$$

yields a non-zero k-derivation $D : R \to R$ of weight -1, where

$$R = \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i)) \cong k[T_0, T_1, T_2, T_3]/(f).$$

As above we get that $H^0(X, T_X \otimes L^{-1}) \neq 0$, where $L := \mathcal{O}_X(1)$. In other words, the normal polarized surface (X, L) is a counter-example to theorem 2.2 in characteristic 3.

Here is another consequence of theorem 2.2.

PROPOSITION 4.1 *Let Y be a smooth, connected, non-degenerate closed subvariety of codimension ≥ 2 of $\mathbb{P}^n$. Assume* $\dim(Y) \geq 1$ *and* $\mathrm{char}(k) = 0$. *Then the Zak map $z : H^0(Y, (n+1)\mathcal{O}_Y) \to H^0(Y, N_{Y|\mathbb{P}^n}(-1))$ is injective. In particular,* $\dim_k(H^0(Y, N_{Y|\mathbb{P}^n}(-1))) \geq n+1$, *with equality if and only if the map z is surjective.*

PROOF. Consider the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_Y(-1) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{O}_Y(-1) \arrow[d] & & \\
0 \arrow[r] & F(-1) \arrow[r] \arrow[d] & (n+1)\mathcal{O}_Y \arrow[r] \arrow[d] & N_{Y|\mathbb{P}^n}(-1) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & T_Y(-1) \arrow[r] \arrow[d] & T_{\mathbb{P}^n}(-1)|Y \arrow[r] \arrow[d] & N_{Y|\mathbb{P}^n}(-1) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

(which is an analogue of the diagram used in the proof of theorem 2.1). The first long row implies that $\mathrm{Ker}(z) = H^0(Y, F(-1))$ (the functor H^0 is left exact!). On the other hand, the first column yields the exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y(-1)) \longrightarrow H^0(Y, F(-1)) \longrightarrow H^0(Y, T_Y(-1)).$$

Since $\mathcal{O}_Y(1)$ is ample and $\dim(Y) \geq 1$, $H^0(Y, \mathcal{O}_Y(-1)) = 0$. We claim that $H^0(Y, T_Y(-1)) = 0$. Indeed, assume first $\dim(Y) \geq 2$; if this space is $\neq 0$, by theorem 2.2 Y would be a cone. But since Y is smooth, this is possible only if Y is a linear subspace, and by non-degeneratedness, $Y = \mathbb{P}^n$, contradicting $\mathrm{codim}_{\mathbb{P}^n}(Y) \geq 2$. If $\dim(Y) = 1$, $H^0(Y, T_Y(-1)) \neq 0$ immediately implies that Y is a line or a conic, which again contradicts $\mathrm{codim}_{\mathbb{P}^n}(Y) \geq 2$. Therefore $H^0(Y, T_Y(-1)) = 0$, which implies $\mathrm{Ker}(z) = H^0(Y, F(-1)) = 0$. $\square$

NOTE. Proposition 4.1 is in general false in positive characteristic. The examples 4.1 and 4.2 above also yield counterexamples for proposition 4.1 in positive characteristic.

COROLLARY 4.1 *In the hypotheses of proposition* 4.1 *(but here the characteristic of k can be arbitrary), assume furthermore that* $\dim(Y) \geq 2$, $H^1(Y, \mathcal{O}_Y(-1)) = H^1(Y, T_Y(-1)) = 0$ *(the first vanishing always holds in* char *zero by Kodaira vanishing theorem). Then the Zak map* $z : H^0(Y, (n+1)\mathcal{O}_Y) \to H^0(Y, N_{Y|\mathbb{P}^n}(-1))$ *is surjective.*

PROOF. Indeed, the first column of the diagram from the proof of proposition 4.1 yields the exact sequence

$$0 = H^1(Y, \mathcal{O}_Y(-1)) \to H^1(Y, F(-1)) \to H^1(Y, T_Y(-1)) = 0.$$

It follows that $H^1(Y, F(-1)) = 0$. On the other hand, from the cohomology sequence associated to the first row of the same diagram we get

$$H^0(Y, (n+1)\mathcal{O}_Y) \xrightarrow{z} H^0(Y, N_{Y|\mathbb{P}^n}(-1)) \longrightarrow H^1(Y, F(-1)) = 0,$$

i.e. z is surjective. □

EXAMPLE 4.1 Let $v_s : \mathbb{P}^r \hookrightarrow \mathbb{P}^{n(r,s)}$ be the s-fold Veronese embedding of $\mathbb{P}^r$, with $n(r,s) = \binom{r+s}{r} - 1$. Set $Y := v_s(\mathbb{P}^r)$, and assume $r, s \geq 2$. Then $\mathcal{O}_Y(1) := \mathcal{O}_{\mathbb{P}^{n(r,s)}}(1)|Y$ coincides to $\mathcal{O}_{\mathbb{P}^r}(s)$.

CLAIM 4.1 $H^1(Y, T_Y(-1)) = 0$ *for every* $r \geq 3$ *or for* $r = s = 2$. *Moreover,* $H^1(Y, \mathcal{O}_Y(-1)) = 0$ *for every* $r \geq 2$.

Indeed, the second statement comes from the explicit cohomology of the projective space. For the first, the Euler sequence for $\mathbb{P}^r$ yields the cohomology sequence

$$H^1(\mathbb{P}^r, (r+1)\mathcal{O}_{\mathbb{P}^r}(1-s)) \to$$
$$\to H^1(\mathbb{P}^r, T_{\mathbb{P}^r}(-s)) \cong H^1(Y, T_Y(-1)) \to H^2(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-s)).$$

The first space is zero because $r \geq 2$, while the last one is zero either if $r \geq 3$, or if $r = s = 2$.

Therefore by corollary 4.1, the Zak map of Y in $\mathbb{P}^{n(r,s)}$ is surjective.

REMARK 4.1 If in example 4.1 we assume $r \geq 3$, we actually have $H^1(Y, \mathcal{O}_Y(i)) = H^1(Y, T_Y(i)) = 0$ for all $i \in \mathbb{Z}$.

EXAMPLE 4.2 Let $i : \mathbb{P}^1 \times \mathbb{P}^r \hookrightarrow \mathbb{P}^{2r+1}$, with $r \geq 2$, be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^r$. Set $Y = i(\mathbb{P}^1 \times \mathbb{P}^r)$. Then $\mathcal{O}_Y(1) = \mathcal{O}(1,1)$, where

$$\mathcal{O}(1,1) := p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^r}(1)),$$

where p_1 and p_2 are the first and the second projections of $\mathbb{P}^1 \times \mathbb{P}^r$ respectively.

CLAIM 4.2 *$H^1(Y, \mathcal{O}_Y(i)) = H^1(Y, T_Y(i)) = 0$ for every $i \in \mathbb{Z}$.*

Again the first statement is trivial. For the second we have $T_Y \cong p_1^*(T_{\mathbb{P}^1}) \oplus p_2^*(T_{\mathbb{P}^r}) = p_1^*(\mathcal{O}_{\mathbb{P}^1}(2)) \oplus p_2^*(T_{\mathbb{P}^r})$. Then the conclusion follows easily from Künneth's formula and from the Euler sequence of $\mathbb{P}^r$.

In particular, by corollary 4.1 the Zak map of Y in $\mathbb{P}^{2r+1}$ is surjective.

Another application of theorem 2.2 is the following result.

THEOREM 4.1 (FUJITA [19]) *Let Y be a smooth projective variety over $\mathbb{C}$ wich is embedded in the normal projective variety X as an ample Cartier divisor. Assume that* $\dim(Y) \geq 2$ *and $H^1(Y, T_Y \otimes N_{Y|X}^{-i}) = 0$ for every $i \geq 1$, where $N_{Y|X}$ is the normal bundle of Y in X. Then X is isomorphic to the cone over the polarized variety $(Y, N_{Y|X})$.*

PROOF. Set $L := N_{Y|X}$. The normal sequence of Y in X

$$0 \to T_Y \to T_X|Y \to L \to 0$$

yields the cohomology sequence ($i \geq 1$)

$$H^1(Y, T_Y \otimes L^{-i}) \to H^1(Y, T_X|Y \otimes L^{-i}) \to H^1(Y, L^{1-i}). \tag{4.1}$$

For every $i \geq 2$ the first space is zero by hypotheses, and the third space is also zero by Kodaira vanishing theorem ($\dim(Y) \geq 2$). It follows that $H^1(Y, T_X|Y \otimes L^{-i}) = 0$ for every $i \geq 2$. Therefore the exact sequence

$$0 \to T_X \otimes \mathcal{O}_X(-(i+1)Y) \to T_X \otimes \mathcal{O}_X(-iY) \to T_X|Y \otimes L^{-i} \to 0 \quad (4.2)$$

yields for every $i \geq 2$ a surjection

$$H^1(X, T_X \otimes \mathcal{O}_X(-(i+1)Y)) \to H^1(X, T_X \otimes \mathcal{O}_X(-iY)). \quad (4.3)$$

On the other hand, since X is normal of dimension ≥ 2, Y is an ample Cartier divisor on X, and T_X is a reflexive sheaf, a lemma of Enriques-Severi-Zariski-Serre shows that

$$H^1(X, T_X \otimes \mathcal{O}_X(-iY)) = 0 \text{ for every } i >> 0. \quad (4.4)$$

Then (4.3) and (4.4) and an induction on i yield $H^1(T_X \otimes \mathcal{O}_X(-2Y)) = 0$. Therefore the exact sequence (4.2) (for $i = 1$) yields the surjection

$$H^0(X, T_X \otimes \mathcal{O}_X(-Y)) \to H^0(T_X|Y \otimes L^{-1}) \to 0. \quad (4.5)$$

Finally, the exact sequence (4.1) (for $i = 1$) yields the cohomology sequence

$$H^0(Y, T_X|Y \otimes L^{-1}) \to H^0(Y, \mathcal{O}_Y) \to H^1(Y, T_Y \otimes L^{-1}), \quad (4.6)$$

in which the last space is zero by hypothesis. Therefore the first map of (4.6) is surjective. Recalling also the surjection (4.5), we get a surjection

$$H^0(X, T_X \otimes \mathcal{O}_X(-Y)) \to H^0(Y, \mathcal{O}_Y) \neq 0,$$

and in particular, $H^0(X, T_X \otimes \mathcal{O}_X(-Y)) \neq 0$. At this point we can apply theorem 2.2 to the normal polarized variety $(X, \mathcal{O}_X(Y))$ to get the conclusion. □

In all these examples we shall assume that $\mathrm{char}(k) = 0$.

EXAMPLE 4.3 $(Y, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(s))$, with $n \geq 3$ and $s \geq 1$. Then by the remark following 1, $H^1(Y, T_Y \otimes L^i) = 0$ for every $i \in \mathbb{Z}$.

EXAMPLE 4.4 Let Y be an abelian variety of dimension $d \geq 2$. Then for every ample line bundle L on Y, we have $H^1(Y, T_Y \otimes L^{-i}) = 0$ for every $i \geq 1$. Indeed, in this case T_Y is isomorphic to the trivial bundle of rank d, and the assertion follows from the Kodaira vanishing theorem for example.

EXAMPLE 4.5 Let $Y = Y_1 \times Y_2$ be a product of two smooth projective varieties Y_i with $\dim(Y_i) \geq 2$, $i = 1, 2$. Let L_i be an ample line bundle on Y_i, $i = 1, 2$ and set $L := p_1^*(L_1) \otimes p_2^*(L_2)$, where $p_i : Y \to Y_i$ is the canonical projection on Y_i, $i = 1, 2$. Then L is ample on Y. Moreover, $T_Y = p_1^*(T_{Y_1}) \oplus p_2^*(T_{Y_2})$. Then for every $i > 0$,

$$H^1(Y, T_Y \otimes L^{-i}) = H^1(Y, p_1^*(T_{Y_1} \otimes L_1^{-i}) \otimes p_2^*(L_2^{-i})) \oplus$$
$$\oplus H^1(Y, p_1^*(L_1^{-i}) \otimes p_2^*(T_{Y_2} \otimes L_2^{-i})),$$

and using Künneth's formulae we get $H^1(Y, T_Y \otimes L^{-i}) = 0$ for every $i \geq 1$.

EXAMPLE 4.6 Let $Y = Y_1 \times Y_2 \times Y_3$ be a product of three smooth projective varieties Y_1, Y_2 and Y_3 each of dimension ≥ 1, and let $L := p_1^*(L_1) \otimes p_2^*(L_2) \otimes p_3^*(L_3)$, with L_i an ample line bundle on Y_i, $i = 1, 2, 3$. Then by arguments similar to those in the previous example we have $H^1(Y, T_Y \otimes L^{-i}) = 0$ for all $i \geq 1$.

EXAMPLE 4.7 Let Y be a hyperelliptic surface. This is a surface with invariants $b_2 = 2$, $p_g = 0$, $q = 1$ and $\chi(\mathcal{O}_Y) = 0$. Moreover, there are two elliptic curves B_0 and B_1, a finite subgroup $A \subset B_1$, an injective homomorphism $\alpha : A \to \mathrm{Aut}(B_0)$, and a free action of A on $B_1 \times B_0$ of the form $a(b_1, b_0) = (b_1 + a, \alpha(a)(b_0))$ (see e.g. [7, 10.25]). Then $Y = (B_1 \times B_0)/A$. Let $f : Z := B_1 \times B_0 \to Y$ be the canonical étale morphism. Then $f^*(T_Y) = T_Z = 2\mathcal{O}_Z$ (since Z is an abelian surface), and $f^*(L)$ is ample on Z. Since $\mathrm{char}(k) = 0$ the vanishing of $H^1(T_Y \otimes L^{-i})$ (for $i \geq 1$) follows from the vanishing $H^1(f^*(T_Y) \otimes f^*(L^{-i})) = H^1(Z, 2f^*(L^{-i})) = 0$ (Z is an abelian surface and $f^*(L)$ is ample on Z, see example 4.4).

To give another application of theorem 4.1 we need the following result (see [29, page 110]):

THEOREM 4.2 *Let Y be an effective Cartier divisor on a complete algebraic variety X such that the normal bundle $N_{Y|X}$ is ample. Then there exists a birational projective morphism $f : X \to Z$ such that there exists an open subset $U \subseteq X$ with the property that $f|U : U \to f(U)$ is an isomorphism and $Y' = f(Y)$ is an ample Cartier divisor on Z. In particular, X is a projective variety.*

PROOF. We claim that the following three statements hold for every $n >> 0$:

1. $N^n_{Y|X} = \mathcal{O}_Y(nY)$ is very ample,
2. The restriction map $H^0(X, \mathcal{O}_X(Y)) \to H^0(Y, N^n_{Y|X})$ is surjective,
3. The complete linear system $|nY|$ has no base points.

(1) is obvious. For (2) and (3) we adjust part of the proof (due to Kleiman) the Nakai-Moishezon criterion, see e.g. [29]. Since $N_{Y|X}$ is ample and Y is a complete scheme, $H^1(Y, N^n_{Y|X}) = 0$ for $n >> 0$. Therefore from the cohomology exact sequence

$$H^1(X, \mathcal{O}_X((n-1)Y)) \xrightarrow{\alpha_n} H^1(X, \mathcal{O}_X(nY)) \longrightarrow H^1(Y, N^n_{Y|X})$$

associated to the exact sequence

$$0 \to \mathcal{O}_X((n-1)Y) \to \mathcal{O}_X(nY) \to N^n_{Y|X} \to 0,$$

the map α_n is surjective for $n >> 0$. Since X is complete, $H^1(X, \mathcal{O}_X((n-1)Y))$ is a finite dimensional vector space, it follows that the maps α_n actually become isomorphisms for $n >> 0$. Therefore the cohomology exact sequence of the above short exact sequence yields (2). Then (1) and (2) immediately imply (3).

Now let $\varphi = \varphi_{|nY|} : X \to P = \mathbb{P}(H^0(X, \mathcal{O}_X(nY))^*)$ be the morphism defined by the base-point-free complete linear system $|nY|$ for $n >> 0$. Set $Z_1 = \varphi(X)$. Then $Y = \varphi^*(Y_1)$, with Y_1 a very ample divisor on Z_1. By the above claim it follows that $\varphi|Y : Y \cong Y_1$ and $Y_{\mathrm{red}} = \varphi^{-1}((Y_1)_{\mathrm{red}})$. Let $\varphi = g \circ f$ be the Stein factorization of the projective morphism φ, with $f : X \to Z$ a proper morphism with $f_*(\mathcal{O}_X) = \mathcal{O}_Z$, and $g : Z \to Z_1$ a finite surjective morphism (see [26, 4.3.3]). Notice that $f|Y : Y \to Y' := g^*(Y_1)$ is an isomorphism and $Y_{\mathrm{red}} = f^{-1}(Y'_{\mathrm{red}})$, so by Zariski's Main Theorem (see [26, 4.4.1]), f is an isomorphism in a neighbourhood of Y onto an open subset of Z. Moreover since $\varphi = g \circ f$ is a projective morphism, f is also projective by a simple general property (see [26, 5.5.5]). Finally, $Y' = g^*(Y_1)$ is an ample Cartier divisor on Z because Y_1 is ample on Z_1 and g is finite. □

DEFINITION 4.1 *Let $i : Y \hookrightarrow X$ and $i' : Y' \hookrightarrow X'$ be two closed embeddings of the projective varieties Y and Y' into the projective varieties X and X' respectively. We shall say that i and i' are Zariski equivalent if there exist two Zariski open subsets $U \subseteq X$ and $U' \subseteq X'$ containing Y and Y' respectively, and an isomorphism $\varphi : U \cong U'$ such that $\varphi(Y) = Y'$.*

Now theorems 4.1 and 4.2 together imply the following:

COROLLARY 4.2 *Let (Y, L) be a smooth polarized variety over $\mathbb{C}$ of dimension ≥ 2 such that $H^1(Y, T_Y \otimes L^{-i}) = 0$ for every $i \geq 1$. Assume that Y admits two embeddings $Y \hookrightarrow X$ and $Y \hookrightarrow X'$ in the normal projective varieties X and X' as Cartier divisors such that $N_{Y|X} \cong L \cong N_{Y|X'}$. Then the embeddings $Y \hookrightarrow X$ and $Y \hookrightarrow X'$ are Zariski equivalent.*

5 The Zak map of a curve. Gaussian maps

We note that the condition $H^1(T_Y \otimes L^{-1}) = 0$ in corollary (3.4) is never fulfilled for a smooth projective curve Y embedded in $\mathbb{P}^n$ such that Y is non-degenerate and of codimension ≥ 2. The aim of this section is to provide an interpretation of the Zak map of a linearly normal smooth curve $Y \subset \mathbb{P}^n$ in terms of the so-called Gaussian maps. The advantage of the Gaussian maps comes from the fact that in certain cases there are methods to check their surjectivity, see [53], [54], [14] and the references therein.

5.1 General Gaussian maps

According to [54], let Y be a smooth projective variety, and let L, M be two line bundles on Y. Consider the canonical map

$$\mu_{L,M} : H^0(Y, L) \otimes H^0(Y, M) \to H^0(Y, L \otimes M). \tag{5.1}$$

Set $\mathcal{R}(L, M) := \mathrm{Ker}(\mu_{L,M})$. We are going to define *the Gaussian map*

$$\varphi_{L,M} : \mathcal{R}(L, M) \to H^0(Y, \Omega^1_{Y|k} \otimes L \otimes M) \tag{5.2}$$

associated to (L, M) on Y in the following way. Let $\alpha = \sum_i l_i \otimes m_i \in \mathcal{R}(L, M)$. Let U be an arbitrary affine open subset of Y such that

$L|U \cong \mathcal{O}_U \cdot S$ and $M|U \cong \mathcal{O}_U \cdot T$, so that (over U), $l_i = a_i S$, $m_i = b_i T$, with $a_i, b_i \in \Gamma(U, \mathcal{O}_Y)$. Since $\alpha \in \mathcal{R}(L, M)$, we have $\sum_i a_i b_i = 0$.

Then we set

$$\varphi_{L,M}(\alpha)|U := \sum_i (a_i db_i - b_i da_i) S \otimes T \in H^0(U, \Omega^1_{X|k} \otimes L \otimes M). \tag{5.3}$$

Replacing S by $\tilde{S} := u^{-1}S$ and T by $\tilde{T} := v^{-1}T$, with u and v units in $\Gamma(U, \mathcal{O}_Y)$, we have $l_i = \tilde{a}_i \tilde{S}$ and $m_i = \tilde{b}_i \tilde{T}$, where $\tilde{a}_i := u a_i$ and $\tilde{b}_i := v b_i$. Then

$$\begin{aligned}
&\sum_i (\tilde{a}_i d\tilde{b}_i - \tilde{b}_i d\tilde{a}_i) \tilde{S} \otimes \tilde{T} = \\
&\quad = \sum_i (u a_i v db_i + u a_i b_i dv - v b_i u da_i - v b_i a_i du) u^{-1} v^{-1} S \otimes T = \\
&\quad = \sum_i (a_i db_i - b_i da_i) S \otimes T + (\sum_i a_i b_i)(u dv - v du) = \\
&\quad = \sum_i (a_i db_i - b_i da_i) S \otimes T,
\end{aligned}$$

because $\sum_i a_i b_i = 0$. It follows that $\varphi_{L,M}(\alpha)$ is independent of the choices made, whence is a well defined element in $H^0(Y, \Omega^1_{Y|k} \otimes L \otimes M)$ as soon as we have checked the following

$$\sum_i l_i \otimes m_i = 0 \ \Rightarrow \ \varphi_{L,M}(\alpha) = 0.$$

To this end, let $\{n_j\}$ be a basis of $H^0(Y, M)$. Then $m_i = \sum_j \beta_{ij} n_j$, with $\beta_{ij} \in k$. Then

$$\sum_i l_i \otimes m_i = \sum_i l_i \otimes (\sum_j \beta_{ij} n_j) = \sum_j (\sum_i \beta_{ij} l_i) \otimes n_j = 0,$$

whence (taking into account that $\{n_j\}$ is a basis of $H^0(Y, M)$ and of the properties of the tensor product), $\sum_i \beta_{ij} l_i = 0$. Writing $n_j = c_j T$, with $c_j \in \Gamma(U, \mathcal{O}_Y)$, we have $b_i = \sum_j \beta_{ij} c_j$ and $\sum_i \beta_{ij} a_i = 0$, so

$$\sum_i (a_i db_i - b_i da_i) = \sum_{i,j} a_i \beta_{ij} dc_j - \sum_{i,j} \beta_{ij} c_j da_i = \sum_j (\sum_i \beta_{ij} a_i) dc_j -$$

$$- \sum_{i,j} \beta_{ij} c_j da_i = - \sum_{i,j} \beta_{ij} c_j da_i = - \sum_j c_j d(\sum_i \beta_{ij} a_i) = - \sum_j c_j d0 = 0,$$

so that $\varphi_{L,M}(\alpha) = 0$, as required.

PROPERTY 5.1 Let $f : X \to Y$ be a morphism of smooth projective varieties, and let L, M be two line bundles on Y. Then there is a natural commutative diagram

$$\begin{array}{ccc} \mathcal{R}(L,M) & \xrightarrow{\varphi_{L,M}} & H^0(\Omega^1_{Y|k} \otimes L \otimes M) \\ \downarrow & & \downarrow \\ \mathcal{R}(f^*(L), f^*(M)) & \xrightarrow[\varphi_{f^*(L),f^*(M)}]{} & H^0(\Omega^1_{X|k} \otimes f^*(L) \otimes f^*(M)) \end{array}$$

PROPERTY 5.2 If N is a third line bundle on Y, there is a natural commutative diagram

$$\begin{array}{ccc} \mathcal{R}(L,M) \otimes H^0(Y,N) & \xrightarrow{\varphi_{L,M}\otimes \mathrm{id}} & H^0(\Omega^1_{Y|k} \otimes L \otimes M) \otimes H^0(Y,N) \\ \downarrow & & \downarrow \\ \mathcal{R}(L, M\otimes N) & \xrightarrow[\varphi_{L,M\otimes N}]{} & H^0(\Omega^1_{Y|k} \otimes L \otimes M \otimes N) \end{array}$$

PROPERTY 5.3 For every two line bundles L and M on a smooth projective variety Y we have the following anti-commutative square

$$\begin{array}{ccc} \mathcal{R}(L,M) & \xrightarrow{\varphi_{L,M}} & H^0(Y, \Omega^1_{Y|k} \otimes L \otimes M) \\ u\downarrow & & \downarrow v \\ \mathcal{R}(M,L) & \xrightarrow[\varphi_{M,L}]{} & H^0(Y, \Omega^1_{Y|k} \otimes M \otimes L) \end{array}$$

i.e. $\varphi_{M,L} \circ u = -v \circ \varphi_{L,M}$, where u and v are the isomorphisms given by the commutativity of the tensor product. In particular, the Gaussian map $\varphi_{L,M}$ is surjective if and only if $\varphi_{M,L}$ is surjective.

PROPERTY 5.4 Assume $L = M$ and consider the natural map $\omega_L : \wedge^2 H^0(Y,L) \to \mathcal{R}(L,L)$ defined by

$$\omega_L(l_1 \wedge l_2) := l_1 \otimes l_2 - l_2 \otimes l_1.$$

We get a composition

$$w_L := \varphi_{L,L} \circ \omega_L : \wedge^2 H^0(Y,L) \to H^0(Y, \Omega^1_{Y|k} \otimes L^2),$$

which is called *the Wahl map* associated to L on Y. Then we have the following:

CLAIM 5.1 *If* $\mathrm{char}(k) \neq 2$, $\mathrm{Im}(w_L) = \mathrm{Im}(\varphi_{L,L})$, *and in particular,* $\mathrm{Coker}(w_L) = \mathrm{Coker}(\varphi_{L,L})$.

Indeed, the inclusion $\mathrm{Im}(w_L) \subseteq \mathrm{Im}(\varphi_{L,L})$ is obvious. Conversely, let $\alpha = \sum_i l_i \otimes m_i \in \mathcal{R}(L,L)$. Then define $\beta := \frac{1}{2}[\sum_i l_i \otimes m_i + \sum_i(-m_i) \otimes l_i]$. Clearly, $\beta \in \mathrm{Im}(\omega_L)$ and $\varphi_{L,L}(\alpha) = \varphi_{L,L}(\beta)$.

Of particular interest is the Wahl map

$$w_Y := w_{\omega_Y} : \wedge^2 H^0(Y,\omega_Y) \to H^0(Y,\Omega^1_{Y|k} \otimes \omega_Y^2) = H^0(Y,\omega_Y^3)$$

associated to a smooth projective curve Y of genus $g \geq 2$. Its interest comes from the fact that w_Y is a map intrinsically associated to the curve Y. As we shall see below, one of the fundamental questions related to w_Y is whether w_Y is surjective. A necessary condition for the surjectivity of w_Y is obviously the following

$$\dim(\wedge^2 H^0(Y,\omega_Y)) \geq \dim(H^0(Y,\omega_Y^3)).$$

Using Riemann-Roch, this condition amounts to the following one

$$\frac{1}{2}g(g-1) \geq 5g-5,$$

or else, $g \geq 10$.

5.2 Gaussian maps for curves

Let Y be a smooth projective curve and L a very ample line bundle on Y. Let $i = i_{|L|} : Y \hookrightarrow P := \mathbb{P}(H^0(Y,L)^*)$ be the linearly normal embedding into the projective space P given by the complete linear system $|L|$. In particular, $i^*(\mathcal{O}_P(1)) \cong L$. Consider the evaluation map $e : H^0(Y,L) \otimes \mathcal{O}_Y \to L$. Since L is very ample, the map e is surjective. Moreover, its kernel is identified to $\Omega^1_{P|k}(1)|Y = \Omega^1_{P|k} \otimes \mathcal{O}_P(1) \otimes \mathcal{O}_Y$, whence we get the exact sequence

$$0 \to \Omega^1_{P|k}(1)|Y \to H^0(Y,L) \otimes \mathcal{O}_Y \to L = \mathcal{O}_Y(1) \to 0, \qquad (5.4)$$

which is nothing but the dual of the Euler sequence of P restricted to Y. Let M be another line bundle on Y. Tensoring (5.4) by M and taking cohomology we get the exact sequence

$$0 \longrightarrow H^0(Y,\Omega^1_{P|k}(1)|Y \otimes M) \longrightarrow H^0(Y,L) \otimes H^0(Y,M) \xrightarrow{\mu_{L,M}}$$

$$\xrightarrow{\mu_{L,M}} H^0(Y, L\otimes M) \longrightarrow H^1(Y, \Omega^1_{P|k}(1)|Y\otimes M) \xrightarrow{\alpha}$$

$$\xrightarrow{\alpha} H^0(Y,L)\otimes H^1(Y,M). \tag{5.5}$$

In particular, we get the identification

$$\mathcal{R}(L,M) = H^0(Y, \Omega^1_{P|k}(1)|Y\otimes M). \tag{5.6}$$

On the other hand, the dual normal sequence of Y in P yields the exact sequence

$$0 \to N^*_{Y|P}\otimes L\otimes M \xrightarrow{b} \Omega^1_{P|k}(1)|Y\otimes M \xrightarrow{a} \Omega^1_{Y|k}\otimes L\otimes M \to 0,$$

whence, taking into account of (5.6), the cohomology sequence

$$\mathcal{R}(L,M) \xrightarrow{\varphi_{L,M}=H^0(a)} H^0(Y, \Omega^1_{Y|k}\otimes L\otimes M) \to$$

$$\to H^1(Y, N^*_{Y|k}\otimes L\otimes M) \xrightarrow{H^1(b)} H^1(Y, \Omega^1_{P|k}(1)|Y\otimes M). \tag{5.7}$$

In particular from (5.7) we get

$$\mathrm{Coker}(\varphi_{L,M}) \cong \mathrm{Ker}(H^1(b)). \tag{5.8}$$

Therefore (5.8) implies the following:

LEMMA 5.1 *In the above hypotheses the Gaussian map $\varphi_{L,M}$ is surjective if $H^1(Y, N^*_{Y|P}\otimes L\otimes M) = 0$.*

COROLLARY 5.1 *Assume furthermore that $H^1(Y, L^{-1}\otimes M) = 0$ and that the very ample line bundle L is normally presented, i.e. the graded k-algebra $R(Y,L) := \oplus_{i=0}^{\infty} H^0(Y, L^i)$ is generated by its homogeneous part of degree one and the ideal*

$$\mathrm{Ker}(S(H^0(X,L)^*) = \oplus_{i=0}^{\infty} H^0(P, \mathcal{O}_P(i)) \to R(Y,L))$$

is generated by its homogeneous part of degree two, where $S(V)$ denotes the symmetric k-algebra associated to a k-vector space V. Then the Gaussian map $\varphi_{L,M}$ is surjective.

PROOF. Since L is normally presented there exists a surjection of the form

$$m\mathcal{O}_P(-2) \to \mathcal{I}_Y \to 0,$$

where $\mathcal{I}_Y$ is the ideal sheaf of Y in $\mathcal{O}_P$. This yields the surjection

$$(m\mathcal{O}_P(-2)) \otimes L \otimes M \cong m(L^{-1} \otimes M) \to \mathcal{I}_Y \otimes L \otimes M \cong N^*_{Y|P} \otimes L \otimes M \to 0.$$

Since Y is a curve we get therefore a surjection

$$H^1(Y, m(L^{-1} \otimes M)) \to H^1(Y, N^*_{Y|P} \otimes L \otimes M) \to 0.$$

Using the hypothesis that $H^1(Y, L^{-1} \otimes M) = 0$ and lemma 5.1 we get the conclusion of our corollary. □

EXAMPLE 5.1 In corollary 5.1 take Y a non-hyperelliptic curve of genus $g \geq 3$, which is neither trigonal, nor a plane quintic. By a theorem of Max Noether-Petri (see [1]), the canonical class $L = \omega_Y$ is very ample and normally presented. Take M of degree $\geq 4g - 3$. Then

$$\deg(L^{-1} \otimes M) = \deg(M) - \deg(L) \geq (4g - 3) - (2g - 2) = 2g - 1.$$

By Riemann-Roch, $H^1(Y, L^{-1} \otimes M) = 0$. Therefore corollary 5.1 yields:

THEOREM 5.1 (LAZARSFELD) *Let Y be a non-hyperelliptic curve of genus $g \geq 3$ which is neither trigonal, nor a plane quintic. Let M be a line bundle on Y of degree $\geq 4g - 3$. Then the Gaussian map $\varphi_{\omega_Y, M}$ is surjective.*

NOTE. The conclusion of theorem 5.1 still holds if we take M of degree $4g - 4$, provided $M \not\cong \omega_Y^2$.

COROLLARY 5.2 *In the hypotheses of theorem 5.1 the map φ_{M, ω_Y} is surjective.*

PROOF. The conclusion follows from theorem 5.1 and from property 5.3. □

THEOREM 5.2 (WAHL) *Let Y be a smooth projective curve of genus $g > 0$ and let L be a very ample line bundle on Y. Let $i : Y \hookrightarrow P = \mathbb{P}(H^0(Y,L)^*)$ be the linearly normal embedding given by $|L|$ into the projective space P. Then there is a canonical isomorphism*

$$\mathrm{Coker}(z) \cong \mathrm{Coker}(\varphi_{L,\omega_Y})^*,$$

where $z : H^0(Y,L)^* \to H^0(Y,N_{Y|P}(-1))$ is the Zak map of Y in P. In particular, the Zak map z is surjective if and only if the Gaussian map φ_{L,ω_Y} is surjective.

PROOF. Using (5.8) we get

$$\mathrm{Coker}(\varphi_{L,\omega_Y}) \cong \mathrm{Ker}(H^1(Y,N^*_{Y|P}\otimes L\otimes\omega_Y) \to H^1(Y,\Omega^1_{P|k}(1)|Y\otimes\omega_Y)),$$

or by duality on Y,

$$\mathrm{Coker}(\varphi_{L,\omega_Y})^* \cong \mathrm{Coker}(H^0(Y,T_P(-1)|Y) \to H^0(Y,N_{Y|P}(-1)). \tag{5.9}$$

On the other hand, by a general classical statement due to Petri (see [54], or [14]) the map μ_{L,ω_Y} is surjective (recall that L is very ample and $g > 0$ by hypotheses). If in (5.5) we take $M = \omega_Y$, we get that the map

$$\alpha : H^1(Y,\Omega^1_{P|k}(1) \otimes \omega_Y) \to H^0(Y,L) \otimes H^1(Y,\omega_Y)$$

is injective. Therefore by duality the map

$$\alpha^* : H^0(Y,L)^* \otimes H^0(Y,\mathcal{O}_Y) = H^0(Y,L)^* \to H^0(Y,T_P(-1)|Y) \tag{5.10}$$

is surjective. Then the definition of the Zak map, (5.9) and the surjectivity of the map (5.10) yield the conclusion of our theorem. □

COROLLARY 5.3 *Let Y be a smooth projective non-hyperelliptic curve of genus $g \geq 3$ which is neither trigonal, nor a plane quintic. Let L be a line bundle on Y of degree $\geq 4g-4$ such that $L \not\cong \omega_Y^2$. Then the Zak map associated to the linearly normal embedding $i = i_{|L|} : Y \hookrightarrow \mathbb{P}^n$, with $n = h^0(Y,L) - 1 = \deg(L) - g$, is surjective. In particular, every extension in $\mathbb{P}^{n+1}$ of the embedded curve $Y \subset \mathbb{P}^n$ is a cone.*

PROOF. Since $4g - 4 \geq 2g + 1$ for $g \geq 3$, L is very ample on Y. Then the corollary is a consequence of theorem 5.2 and of corollary 5.2. For the last statement apply theorem 2.1. □

Now we come back to the Wahl map

$$w_Y : \wedge^2 H^0(Y, \omega_Y) \to H^0(Y, \omega_Y^3)$$

of a curve Y of genus $g \geq 2$. As we saw above, the surjectivity of w_Y implies $g \geq 10$. Then we have the following fundamental result:

THEOREM 5.3 (CILIBERTO-HARRIS-MIRANDA [14]) *For the general curve Y of genus $g \geq 10$ and $g \neq 11$ the Wahl map w_Y is surjective.*

The conclusion of theorem 5.3 is false for $g = 11$ (see [37], via theorem 5.4 below). Note that the surjectivity of the Wahl map w_Y is an open condition in the moduli space $\mathcal{M}_g$ of isomorphism classes of curves of genus g. For some genera $g \geq 10$ Wahl produced for the first time explicit examples of curves Y with w_Y surjective (see [53]). Therefore for those genera theorem 5.3 is due to Wahl. The method of Ciliberto-Harris-Miranda is entirely different, the main idea being to study the Wahl map for certain degenerations of curves of genus g. A consequence of theorem 5.3 via theorem 5.2 and the remarks made at 5.4 is the following:

COROLLARY 5.4 *For a general projective curve Y of genus $g \geq 10$, $g \neq 11$, let $Y \hookrightarrow \mathbb{P}^{g-1}$ be the canonical embedding of Y. Then every extension of Y in $\mathbb{P}^g$ is a cone.*

PROOF. The first part follows from theorem 5.3, theorem 5.2 (via the remarks made after property 5.4), and from theorem 2.1. □

Finally, we have the following useful result:

THEOREM 5.4 (WAHL) *Let Y be a smooth non-hyperelliptic projective curve of genus $g \geq 3$ which is contained in a $K3$ surface S. Then the Wahl map w_Y of Y is not surjective.*

PROOF. The genus' formula and the fact that the canonical class K_S is trivial yield $(Y^2) > 0$, where $(Y^2) = (Y^2)_S$ denotes the self-intersection of Y on S. It follows that $(Y \cdot C) \geq 0$ for every irreducible curve C on S. Assume that there exists an irreducible curve C on S such that $(Y \cdot C) = 0$. Using the Hodge index theorem (see e.g. [7]) it follows that $(C^2) < 0$. Then from the genus' formula we get $C \cong \mathbb{P}^1$ and $(C^2) = -2$. The Hodge index theorem also implies that there are only finitely many such (-2)-curves. Then a projective contractibility criterion of M. Artin (see [2], or also [7, chapter 3]) shows that there exists a birational morphism $f : S \to X$, with the following properties: X is a normal projective surface having finitely many singular points (which are rational double points) such that the canonical divisor K_X of X is Cartier (in particular, all the singularities of X are Gorenstein), $f^*(K_X)$ is a canonical divisor K_S on S, and the canonical map $H^1(X, \mathcal{O}_X) \to H^1(S, \mathcal{O}_S)$ is an isomorphism. In our case, since S is a $K3$-surface, $K_S = 0$ and $q = h^1(S, \mathcal{O}_S) = 0$. It follows that $K_X = 0$ and $h^1(X, \mathcal{O}_X) = 0$. In other words, X is a singular $K3$-surface. Moreover, by construction, f defines an isomorphism $f|U : U \cong f(U)$ from a Zariski open neighbourhood U of Y in S (we can take $U = S \setminus E$, where E is the union of all irreducible (-2)-curves C such that $(Y \cdot C) = 0$). In particular, Y can also be embedded (via $f|Y$) in X as a Cartier divisor. Again by construction, $(Y \cdot D)_X > 0$ for every irreducible curve D on X. Since $f^*(Y) = Y$, it also follows that $(Y^2)_X = (Y^2)_S > 0$. Then by the Nakai-Moishezon criterion of ampleness (see e.g. [29], or also [7, chapter 1]) we infer that Y is an ample Cartier divisor on X.

Now the adjunction formula together with the fact that $\omega_X = \mathcal{O}_X(K_X) \cong \mathcal{O}_X$ yield $\mathcal{O}_X(Y)|Y \cong \omega_Y$. Therefore for every $n \geq 0$ we get the exact sequence

$$0 \to \mathcal{O}_X((n-1)Y) \to \mathcal{O}_X(nY) \to \omega_Y^n \to 0,$$

whence the cohomology sequence

$$\begin{aligned}0 \to H^0(X, \mathcal{O}_X((n-1)Y)) \to H^0(X, \mathcal{O}_X(nY)) \to H^0(Y, \omega_Y^n) \to \\ \to H^1(X, \mathcal{O}_X((n-1)Y)) \to H^1(X, \mathcal{O}_X(nY)) \to H^1(Y, \omega_Y^n) \to \\ \to H^2(X, \mathcal{O}_X((n-1)Y)) \to H^2(X, \mathcal{O}_X(nY)) \to 0.\end{aligned}$$

I claim that $H^1(\mathcal{O}_X(nY)) = 0$ for every $n \geq 0$. Indeed, since for $n = 0$ we already know this, we may assume $n > 0$. In characteristic zero, by duality this amounts to $H^1(X, \mathcal{O}_X(-nY)) = 0$ which holds by Kodaira-Mumford vanishing theorem. However, the assertion is valid in arbitrary characteristic, as one can immediately see by examining the above exact sequence (taking into account that $h^2(X, \mathcal{O}_X) = 1$, $H^2(X, \mathcal{O}_X(nY)) \cong H^0(X, \mathcal{O}_X(-nY)) = 0$ for all $n > 0$, $h^1(Y, \omega_Y) = 1$ and $H^1(Y, \omega_Y^n) = 0$ for all $n > 1$).

Therefore for every $n \geq 0$ we get the exact sequence

$$0 \to H^0(X, \mathcal{O}_X((n-1)Y)) \to H^0(X, \mathcal{O}_X(nY)) \to H^0(Y, \omega_Y^n) \to 0.$$

In particular, $X \subset \mathbb{P}^g$ and $A/tA \cong R$, where $A := \oplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nY))$, and $R := \oplus_{n=0}^{\infty} H^0(Y, \omega_Y^n)$, and $t \in A_1 = H^0(X, \mathcal{O}_X(Y))$ is a global equation of Y in X. Now, by a classical theorem of Max Noether, the canonical ring R of a non-hyperelliptic curve Y of genus $g \geq 3$ is generated by R_1 (see [1]). Therefore A is also generated by A_1 (because $A/tA \cong R$ and $\deg(t) = 1$).

In particular, Y is a very ample divisor on X. Recalling that $Y \subset \mathbb{P}^{g-1}$ and $X \subset \mathbb{P}^g$, we infer that X is an extension of Y in $\mathbb{P}^g$ in the sense of section 2. This extension cannot be trivial because otherwise X (and hence also S) would be birationally equivalent to $Y \times \mathbb{P}^1$. At this point we can apply theorem 2.1 to deduce that the Zak map of Y in $\mathbb{P}^{g-1}$ cannot be surjective. Therefore by theorem 5.2, the Wahl map w_Y also cannot be surjective. □

REMARK 5.1 In theorem 5.4 we could have used theorem 4.2 instead of the more delicate contractibility theorem of M. Artin.

REMARK 5.2 The major interest of theorem 2.1 consists in the fact that it holds also for the case when Y is a curve, in which case the cokernel of the Zak map can be interpreted (via theorem 5.2 above) in terms of the cokernel of a certain Gaussian map. Moreover, the theorems 5.1 and 5.3 above produce very interesting examples of surjective Gaussian maps.

6 Deformations of quasi-homogeneous singularities and projective geometry

Let X be a smooth variety and Y a closed subvariety having only one singularity $y \in Y$. Then we may consider the normal sequence of Y in X

$$0 \longrightarrow T_Y \longrightarrow T_X|Y \xrightarrow{\varphi} N_{Y|X}.$$

Since Y is smooth outside y and X is smooth, φ is surjective outside the point y. Therefore

$$T^1_{Y,y} := \mathrm{Coker}(\varphi)$$

is a coherent sheaf concentrated at the point y. In particular, $T^1_{Y,y}$ (sometimes also denoted by T^1_Y) is a finitely dimensional vector space over k (because it is a coherent sheaf concentrated at one point), which is called *the space of first order infinitesimal deformations* of the isolated singularity. This space turns out to be an extremely important intrinsic invariant of the isolated singularity (Y, y). In fact, $T^1_{Y,y}$ depends only on the singularity (Y, y), and not on the choice of the embedding $Y \hookrightarrow X$ into the smooth variety X (see corollary 6.1 and remark 6.1 below).

The definition of $T^1_{Y,y} = T^1_Y$ being local, we may assume that $Y = \mathrm{Spec}(A)$, with A a finitely generated k-algebra. Then we may write $A = k[T_1, ..., T_n]/I$, where I is an ideal of the polynomial k-algebra $P := k[T_1, ..., T_n]$, and therefore we may take $X = \mathrm{Spec}(k[T_1, ..., T_n]) = \mathbb{A}^n$, and as $Y \hookrightarrow X$ the subvariety $V(I)$ of X defined by the ideal I. The canonical exact sequence

$$I/I^2 \to \Omega^1_{P|k} \otimes_P A \to \Omega^1_{A|k} \to 0$$

yields for every A-module M the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(\Omega^1_{A|k}, M) = \mathrm{Der}_k(A, M) \longrightarrow \mathrm{Hom}_A(\Omega^1_{P|k} \otimes_P A, M) =$$

$$= \mathrm{Der}_k(P, M) \xrightarrow{\varphi_M} \mathrm{Hom}_A(I/I^2, M) \longrightarrow T^1(A|k, M) \longrightarrow 0$$

where by definition $T^1(A|k, M) := \mathrm{Coker}(\varphi_M)$. Taking $M = A$ we get that $T^1_{Y,y}$ is the sheaf associated to the A-module $T^1(A|k, A)$.

DEFINITION 6.1 *Let A be a commutative k-algebra of finite type, and M an A-module. An extension of A over k (or of A/k) by M is an exact sequence*

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{j} A \longrightarrow 0, \tag{6.1}$$

where E is a commutative k-algebra, j is a surjective homomorphism of k-algebras, $i(M) = \mathrm{Ker}(j)$ is a square-zero ideal of E (in particular, $i(M)$ becomes an A-module), and i defines an isomorphism of A-modules between M and $i(M)$.

Two extensions (E) and (E′) of A/k by M are said to be *equivalent* if there exists a k-algebra homomorphism $u : E \to E'$ inducing a commutative diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "\mathrm{id}_M"'] & E \arrow[r, "j"] \arrow[d, "u"'] & A \arrow[d, "\mathrm{id}_A"] \\ M \arrow[r, "i'"'] & E' \arrow[r, "j'"'] & A \end{tikzcd}$$

It follows easily that u must be an isomorphism of k-algebras. The set of equivalence classes of extensions of A/k by M will be denoted by $\mathrm{Ex}^1(A|k, M)$.

THEOREM 6.1 *With the above definitions and notations, there is a natural bijection*

$$\alpha : \mathrm{Ex}^1(A|k, M) \to T^1(A|k, M).$$

PROOF. Consider an extension

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{j} A \longrightarrow 0 \tag{E}$$

Since P is a polynomial k-algebra the canonical surjection $g : P \to A = P/I$ factors through a (not necessarily unique) k-algebra homomorphism $f : P \to E$, i.e. $j \circ f = g$. Since $g(I) = 0$, $f(I) \subseteq \mathrm{Ker}(j) = M$. However, $i(M)$ is a square-zero ideal of E, hence f induces an homomorphism of A-modules $h : I/I^2 \to M$, and thus, an element of

$T^1(A|k,M)$. If $f' : P \to E$ is another lifting of g, then $i^{-1}(f - f') : P \to M$ is a k-derivation, thus the map

$$\alpha : \mathrm{Ex}^1(A|k,M) \to T^1(A|k,M)$$

given by (E)$\to \alpha(E) := \hat{h}$ is well defined.

Conversely, let $\hat{h}$ be an arbitrary element of $T^1(A|k,M)$, with h an element of $\mathrm{Hom}_A(I/I^2,M)$, i.e. $h : I \to M$ a k-homomorphism which vanishes on I^2. On the product $P \times M$ consider the usual addition and the multiplication defined by

$$(x,m)(x',m') := (xx', x'm + xm'), \quad \forall (x,m),\ (x',m') \in P \times M,$$

where M is regarded as a P-module by the restriction of scalars via the canonical map of k-algebras $g : P \to A$. With this multiplication, it is easy to see that $\{(x,-h(x)) \mid x \in I\}$ is an ideal of $P \times M$. Set

$$E_h := (P \times M)/\{(x,-h(x)) \mid x \in I\}.$$

Let $j_h : E_h \to A$ be the map defined by $j_h(\widehat{x,m}) = x \bmod I \in A$, $\forall (\widehat{x,m}) \in E_h$. Clearly, j_h is a well defined surjective homomorphism of k-algebras. For every $(\widehat{x,m})$, $(\widehat{x',m'}) \in \mathrm{Ker}(j_h)$ we have

$$\begin{aligned}(\widehat{x,m})(\widehat{x',m'}) &= (\widehat{xx', x'm + xm'}) = (\widehat{xx', 0}) \\ &= (\widehat{xx', -h(xx')}) = (\widehat{0,0}),\end{aligned}$$

because $x, x' \in I$ and h vanishes on I^2. In other words, $\mathrm{Ker}(j_h)$ is a square-zero ideal of E_h. Moreover, it is easily checked that the map $(\widehat{x,m}) \to m + h(x)$ yields a well defined isomorphism of A-modules $\mathrm{Ker}(j_h) \cong M$. In other words, in this way we get an extension (E_h) of A/k by M.

We prove now that if $\hat{h} = \hat{h}'$ (with $h' \in \mathrm{Hom}_A(I/I^2,M)$), the extensions (E_h) and $(E_{h'})$ (of A/k by M) are equivalent. The equality $\hat{h} = \hat{h}'$ means that there is a derivation $d \in \mathrm{Der}_k(P,M)$ such that $h' := h + d|I : I \to M$ (h and h' are regarded as k-linear maps which vanish on I^2). Then it is easily checked that the function $\varphi : E_h \to E_{h'}$ defined by $\varphi((\widehat{x,m})) := (\widehat{x, m + d(x)})$ yields an isomorphism of extensions $(E_h) \cong (E_{h'})$.

Therefore we get a well defined map $\beta : T^1(A|k,M) \to \mathrm{Ex}^1(A|k,M)$ given by $\beta(\hat{h}) :=$ class of (E_h), which is easily checked to be the inverse of the map α. □

COROLLARY 6.1 *$T^1(A/k,M)$ is independent of the choice of the presentation $A = P/I$. In particular, if (Y,y) is an isolated singularity, $T^1_{Y,y}$ is independent on the embedding $Y \hookrightarrow \mathbb{A}^n$.*

PROOF. The corollary follows from theorem 6.1 because the set $\mathrm{Ex}^1(A|k,M)$ is independent of the presentation $A = P/I$. □

REMARK 6.1 One can prove that the structure of A-module of $T^1(A|k,M)$ is also independent on the choice of the presentation. Roughly speaking, this is done in the following way: one can intrinsically define an addition on $\mathrm{Ex}^1(A|k,M)$ and a scalar multiplication $A \times \mathrm{Ex}^1(A|k,M) \to \mathrm{Ex}^1(A|k,M)$ such that $\mathrm{Ex}^1(A|k,M)$ becomes an A-module (i.e. depending only of A/k and on M). Then one checks that the bijective map α is in fact an isomorphism of A-modules (see [44] for details).

6.1 Quasi-homogeneous singularities

Assume now that $A = \oplus_{i=0}^{\infty} A_i$ is a finitely generated graded k-algebra such that $Y := \mathrm{Spec}(A)$ has an isolated singularity y at the maximal irrelevant ideal $m_A := \oplus_{i=1}^{\infty} A_i$. The singularities (Y,y) of the form $Y = \mathrm{Spec}(A)$ and $y = m_A$, where A is a finitely generated graded k-algebra are called *quasi-homogeneous*. Since A is finitely generated there are homogeneous elements $a_1, ..., a_n \in A$ of positive degrees $d_1, ..., d_n$ respectively such that $A = k[a_1, ..., a_n]$. Consider the polynomial k-algebra $P := k[T_1, ..., T_n]$ graded by the conditions that $\deg(T_i) = d_i$, $\forall i = 1, ..., n$. Then the map $\varphi : P \to A$ of graded k-algebras such that $\varphi(T_i) = a_i$, $\forall i = 1, ..., n$, becomes a surjective homomorphism of graded k-algebras. If $I := \mathrm{Ker}(\varphi)$, then I is a homogeneous ideal of P such that $A = P/I$. Let M be a graded A-module. By the definition of $T^1(A|k,M)$ we have the exact sequence

$$0 \longrightarrow \mathrm{Der}_k(A,M) \xrightarrow{u} \mathrm{Der}_k(P,M) \xrightarrow{v=\varphi_M}$$

$$\xrightarrow{v} \mathrm{Hom}_A(I/I^2, M) \longrightarrow T^1(A|k, M) \longrightarrow 0,$$

in which the first three A-modules are graded and u and v are homomorphisms of graded A-modules. It follows that $T^1(A|k,M)$ becomes a graded A-module.

On the other hand, by corollary 6.1 and remark 6.1, the A-module $T^1(A|k,M)$ is independent on the presentation $A = P/I$, i.e. $T^1(A|k, M)$ is independent on the choice of the system of homogeneous generators $a_1, ..., a_n$ of the graded k-algebra A. One can prove that the structure of *graded* A-module of $T^1(A|k,M)$ thus obtained is also independent on the choice of homogeneous presentation $A = P/I$ (see [42]). In particular, the space $T^1_{Y,y} = T^1(A|k,A)$ of first order infinitesimal deformations of the quasi-homogeneous isolated singularity (Y,y) (corresponding to the irrelevant ideal m_A) has an intrinsic decomposition

$$T^1_{Y,y} = \bigoplus_{i \in \mathbb{Z}} T^1_{Y,y}(i)$$

which corresponds to the structure of the graded A-module of $T^1(A|k, A)$. The subspace $T^1_{Y,y}(i)$ is called *the space of first order infinitesimal deformations of weight* i of the quasi-homogeneous isolated singularity (Y,y). Since the singularity (Y,y) is isolated, $T^1_{Y,y}$ is a finite dimensional k-vector space, and in particular $T^1_{Y,y}(i) \neq 0$ only for finitely many values of $i \in \mathbb{Z}$.

6.2 Cones over projectively normal varieties

Let Y be a closed smooth subvariety of the n-dimensional projective space $\mathbb{P}^n$ of dimension ≥ 1. We recall that Y is called *projectively normal* in $\mathbb{P}^n$ if the Serre map (see [28])

$$\alpha : k[Y] \to R(Y) := \bigoplus_{i \in \mathbb{Z}} H^0(Y, \mathcal{O}_Y(i))$$

is an isomorphism, where $k[Y] := k[T_0, ..., T_n]/I(Y)$ is the homogeneous coordinate k-algebra of Y in $\mathbb{P}^n$ (with $T_0, ..., T_n$ $n+1$ variables over k and $I(Y)$ the saturated ideal of Y in $\mathbb{P}^n$). Geometrically, this means that Y is irreducible and for every $i \geq 1$ the linear system cut out on Y by all hypersurfaces of degree i is complete. The ring $R(Y)$

is normal since Y is smooth. On the other hand, the Serre map α of any closed irreducible subvariety $Y \subset \mathbb{P}^n$ is TN-surjective, i.e. the maps $\alpha_i : k[Y]_i \to H^0(Y, \mathcal{O}_Y(i))$ are isomorphisms for all $i >> 0$. It follows that (for a non necessarily projectively normal subvariety $Y \subseteq \mathbb{P}^n$) the ring $R(Y)$ is the normalization of $k[Y]$ in its field of quotients. Therefore Y is projectively normal if and only if the ring $R := k[Y]$ is normal.

Set $C := \mathrm{Spec}(k[T_0, ..., T_n]) = \mathbb{A}^{n+1}$. Inside C we have the affine cone $C_Y := \mathrm{Spec}(R) = \mathrm{Spec}(k[Y])$ with vertex $x = (0, 0, ..., 0)$ (which corresponds to the maximal irrelevant ideal m_R). Since Y is smooth and projectively normal in $\mathbb{P}^n$, C_Y is a closed normal subvariety of C having an isolated singularity at x. Moreover the gradings of R and of $k[T_0, ..., T_n]$ ($\deg(T_i) = 1$, $\forall i = 0, ..., n$) yield obvious $\mathbb{G}_m$-actions on C_Y and on C such that the closed embedding $C_Y \hookrightarrow C$ becames $\mathbb{G}_m$-invariant. Set $U_Y := C_Y \setminus \{x\}$ and $U := C \setminus \{x\}$. Then U_Y and U are $\mathbb{G}_m$-invariant open subsets of C_Y and of C respectively, such that the closed embedding $U_Y \hookrightarrow U$ is also $\mathbb{G}_m$-invariant. We have natural projections $\pi_Y : U_Y \to Y$ and $\pi : U \to \mathbb{P}^n$ such that the following diagram is cartesian

$$\begin{tikzcd} \pi^{-1}(Y) = U_Y \arrow[r] \arrow[d, "\pi_Y"'] & U \arrow[d, "\pi"] \\ Y \arrow[r] & \mathbb{P}^n \end{tikzcd} \tag{6.2}$$

in which the horizontal arrows are the natural inclusions.

Our aim is to compute in geometric terms the space $T^1_{C_Y} = T^1_{C_Y,x}$ of first order infinitesimal deformations of the isolated singularity (C_Y, x). By the definition of $T^1_{C_Y}$ we have the exact sequence

$$0 \to T_{C_Y} \to T_C|C_Y \to N_{C_Y|C} \to T^1_{C_Y} \to 0.$$

Since C_Y is an affine variety, by passing to global sections we get the exact sequence

$$0 \to H^0(T_{C_Y}) \to H^0(T_C|C_Y) \to H^0(N_{C_Y|C}) \to H^0(T^1_{C_Y}) = T^1(R|k, R) \to 0$$

On the other hand, since T_{C_Y}, $T_C|C_Y$ and $N_{C_Y|C}$ are reflexive sheaves on the normal variety C_Y of dimension ≥ 2, and since x is a point of C_Y, by proposition 2.1 we get

$$H^0(T_{C_Y}) \cong H^0(U_Y, T_{C_Y}),\ \ H^0(T_C|C_Y) \cong H^0(U_Y, T_C|C_Y),$$
$$H^0(N_{C_Y|C}) \cong H^0(U_Y, N_{C_Y|C}).$$

Therefore the above exact sequence becomes

$$0 \to H^0(U_Y, T_{C_Y}) \to H^0(U_Y, T_C|C_Y) \to H^0(U_Y, N_{C_Y|C}) \to$$
$$\to H^0(T^1_{C_Y}) = T^1(R|k, R) \to 0. \tag{6.3}$$

Since $U_Y \subset U$ is a closed immersion of smooth varieties we have the normal exact sequence

$$0 \to T_{U_Y} \to T_U|U_Y \to N_{U_Y|U} \to 0, \tag{6.4}$$

whose cohomology sequence together with the exact sequence (6.3) yield in particular the inclusion

$$T^1_{C_Y} = H^0(T^1_{C_Y}) \subseteq H^1(U_Y, T_{U_Y}). \tag{6.5}$$

Now look at the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{U_Y} \arrow[r] \arrow[d, "\mathrm{id}"'] & T_{U_Y} \arrow[r] \arrow[d] & \pi_Y^*(T_Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{U_Y} \arrow[r] & T_U|U_Y \arrow[r] \arrow[d] & \pi^*(T_{\mathbb{P}^n})|U_Y = \pi_Y^*(T_{\mathbb{P}^n}|Y) \arrow[r] \arrow[d] & 0 \\
 & & N_{U_Y|U} \arrow[r, "\mathrm{id}"'] \arrow[d] & \pi_Y^*(N_{Y|\mathbb{P}^n}) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The fact that the first two rows are exact comes from lemma 3.1 (applied to π_Y and to π), the equality in the second row from the commutativity of the diagram (6.2), the last column is the pull back of the normal sequence of Y in $\mathbb{P}^n$, and the middle column is just (6.4). In particular, $N_{U_Y|U} \cong \pi_Y^*(N_{Y|\mathbb{P}^n})$. Then (6.3) yields

$$H^0(U_Y, T_U|U_Y) \xrightarrow{\alpha} H^0(U_Y, \pi_Y^*(N_{Y|\mathbb{P}^n})) \longrightarrow T^1_{C_Y} \to 0. \quad (6.6)$$

Now we have to understand the map α more closely.

CLAIM 6.1 *There is a canonical identification* $T_U \cong \pi^*((n+1)\mathcal{O}_{\mathbb{P}^n}(1))$ *such that the map*

$$\alpha_i : H^0(U_Y, T_U|U_Y)_i \to H^0(U_Y, \pi_Y^*(N_{Y|\mathbb{P}^n}))_i$$

is identified (up to multiplication by a non-zero constant) to the Zak map

$$z(i+1) : H^0(Y, (n+1)\mathcal{O}_Y(i+1)) \to H^0(Y, N_{Y|\mathbb{P}^n}(i))$$

twisted by $i+1$.

Then (6.6) and claim 6.1 yield the following result:

THEOREM 6.2 (SCHLESSINGER [46], [42]) *Let* $Y \subset \mathbb{P}^n$ *be a closed smooth projectively normal subvariety of* $\mathbb{P}^n$ *of dimension* ≥ 1, *and let* C_Y *be the affine cone over* Y *in* $\mathbb{P}^n$. *Then there is a natural identification*

$$T^1_{C_Y}(i) \cong \mathrm{Coker}(z(i+1)),$$

where $z(i+1) : H^0((n+1)\mathcal{O}_Y(i+1)) \to H^0(N_{Y|\mathbb{P}^n}(i))$ *is the Zak map of* Y *in* $\mathbb{P}^n$ *twisted by* $i+1$.

It remains to prove the claim 6.1. To do this we shall make use of the following special cases of well known Bott's formulae:

$$H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) = 0,\ H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(i)) = 0,\ \forall i \neq 0, \text{ and } H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \cong k. \quad (6.7)$$

First observe that the isomorphism classes of vector bundle extensions

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to E \to T_{\mathbb{P}^n} \to 0$$

of $T_{\mathbb{P}^n}$ by $\mathcal{O}_{\mathbb{P}^n}$ are classified by $H^1(P^n, \Omega^1_{\mathbb{P}^n})$ which by (6.7) is a one-dimensional vector space. Therefore either $E \cong \mathcal{O}_{\mathbb{P}^n} \oplus T_{\mathbb{P}^n}$, or the

above exact sequence is isomorphic (up to multiplication of the second map by a non-zero constant) to the Euler sequence of $\mathbb{P}^n$.
On the other hand, the isomorphism classes of vector bundle extensions of $\pi^*(T_{\mathbb{P}^n})$ by $\pi^*(\mathcal{O}_{\mathbb{P}^n}) = \mathcal{O}_U$ are classified by $H^1(U, \pi^*(\Omega^1_U))$, which by lemma 3.2 and (6.7) is isomorphic to $\oplus_{i\in\mathbb{Z}} H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(i)) \cong H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \cong k$. We infer that every vector bundle extension of $\pi^*(T_{\mathbb{P}^n})$ by $\pi^*(\mathcal{O}_{\mathbb{P}^n}) = \mathcal{O}_U$ is the pull back via π of a vector bundle extension of $T_{\mathbb{P}^n}$ by $\mathcal{O}_{\mathbb{P}^n}$. In particular, considering the exact sequence (given by lemma 3.1)

$$0 \to \mathcal{O}_U \to T_U \to \pi^*(T_{\mathbb{P}^n}) \to 0,$$

it follows that either $T_U \cong \mathcal{O}_U \oplus \pi^*(T_{\mathbb{P}^n})$, or

$$T_U \cong (n+1)\pi^*(\mathcal{O}_{\mathbb{P}^n}(1)). \tag{6.8}$$

Notice that $T_U \cong (n+1)\mathcal{O}_U$, or else $\Omega^1_U \cong (n+1)\mathcal{O}_U$. Then the first possibility is ruled out because otherwise lemma 3.2 would imply $H^0(\mathbb{P}^n, (n+1)\mathcal{O}_{\mathbb{P}^n}) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \oplus H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n})$, which contradicts the first equality of (6.7).

Therefore there is an isomorphism of the form (6.8). This fact together with the above remarks prove claim 6.1, and thereby theorem 6.2 completely.

In view of theorem 6.2, theorem 2.1 implies:

COROLLARY 6.2 *In the hypotheses of theorem* 2.1 *assume furthermore that Y is smooth and projectively normal in $\mathbb{P}^n$ and $T^1_{C_Y}(-1) = 0$, where C_Y is the affine cone over Y in $\mathbb{P}^n$. Then every extension of Y in $\mathbb{P}^{n+1}$ is trivial.*

Corollary 6.2 is especially interesting because it makes a connection between two completely different things: the deformation theory of the vertex of the cone over Y in $\mathbb{P}^n$ (which is of local nature), and a problem of global projective geometry (i.e. the classification of extensions of Y in $\mathbb{P}^{n+1}$).

DEFINITION 6.2 *Let (Y, y) be an isolated singularity of the affine variety Y (with Y smooth outside y). Then (Y, y) is called a rigid singularity if $T^1_{Y,y} = 0$, i.e. if the space of first order infinitesimal deformations of (Y, y) is zero. Clearly, (Y, y) is rigid if y is a smooth point of Y.*

The following result provides a criterion for the vertex x of the affine cone C_Y over a smooth projectively normal closed subvariety Y in $\mathbb{P}^n$ to be rigid.

PROPOSITION 6.1 *Let Y be a smooth projectively normal closed subvariety of $\mathbb{P}^n$ of dimension ≥ 1 such that $H^1(Y, \mathcal{O}_Y(i)) = H^1(Y, T_Y(i)) = 0$ for all $i \in \mathbb{Z}$. Then $T^1_{C_Y} = 0$, i.e. the vertex x of C_Y is a rigid singularity.*

PROOF. By (6.5) it will be sufficient to prove that $H^1(U_Y, T_{U_Y}) = 0$. To do this consider the following cohomology sequence associated to the first row of the diagram of the proof of theorem 6.2:

$$H^1(U_Y, \mathcal{O}_{U_Y}) \to H^1(U_Y, T_{U_Y}) \to H^1(U_Y, \pi_Y^*(T_Y)).$$

Therefore the middle space is zero if the first and the last spaces are both zero. But by lemma 3.2 and our assumptions we have

$$H^1(U_Y, \mathcal{O}_{U_Y}) = \bigoplus_{i \in \mathbb{Z}} H^1(Y, \mathcal{O}_Y(i)) = 0, \text{ and}$$

$$H^1(U_Y, \pi_Y^*(T_Y)) = \bigoplus_{i \in \mathbb{Z}} H^1(Y, T_Y(i)) = 0.$$

□

By proposition 6.1 and the examples 4.1 (and the remark following it) and 4.2, the vertex of the affine cone over the Veronese variety $Y = v_s(\mathbb{P}^r)$, with $r \geq 3$ and $s \geq 2$, or over the Segre variety $Y = i(\mathbb{P}^1 \times \mathbb{P}^r)$, with $r \geq 2$, is a rigid singularity.

We close this section by proving proposition 2.3.

PROOF OF PROPOSITION 2.3. Let I be the saturated homogeneous ideal of Y in the polynomial k-algebra $P := k[T_0, T_1, ..., T_n]$, and set $A = P/I$ (the homogeneous coordinate k-algebra of Y in $\mathbb{P}^n$). Since Y is projectively normal in $\mathbb{P}^n$ we can apply theorem 6.2 to get

$$T^1_{C_Y}(-2) = \mathrm{Coker}(H^0(Y, (n+1)\mathcal{O}_Y(-1)) \longrightarrow H^0(Y, N_{Y|\mathbb{P}^n}(-2))),$$

whence $T^1_{C_Y}(-2) = H^0(Y, N_{Y|\mathbb{P}^n}(-2))$ because $H^0(Y, (n+1)\mathcal{O}_Y(-1)) = 0$. Therefore we have to check that $T^1_{C_Y}(-2) = T^1(A|k, A)(-2) = 0$.

Every element $t \in T^1(A|k,A)(-2)$ is represented by a homomorphism $\varphi \in \mathrm{Hom}_A(I/I^2,A)(-2)$ of degree -2. To prove the proposition it will be enough to show that under its hypotheses we have $\varphi = 0$.

By our hypotheses, there is a system of generators $f_1,...,f_s \in I_2$ of I of degree 2. We may assume that $f_1,...,f_s$ is minimal. A relation r among $f_1,...,f_s$ is a system of homogeneous polynomials $(r_1,...,r_s)$ of the same degree such that $r_1f_1+\ldots+r_sf_s = 0$. The hypotheses also say that there is a generating set $r^1,...,r^u$ of independent relations $r^i = (r_1^i,...,r_s^i)$, with $\deg(r_j^i) = 1$.

Then the classes $\overline{f}_i$, $\forall i = 1,...,s$ of f_i in I/I^2 form a system of generators of I/I^2. Therefore φ is perfectly determined by the constants $\varphi(\overline{f}_i)$, $\forall i = 1,...,s$. Now, assume that $\varphi \neq 0$. Then there is an i such that $\varphi(\overline{f}_i) \neq 0$ (in k). We may assume $\varphi(\overline{f}_1) \neq 0$. Then for every $i = 2,...,s$ we have $\varphi(\overline{f}_i) = a_i\varphi(\overline{f}_1)$, with $a_i \in k$. Replacing $f_1,...,f_s$ by f_1, $f_2 - a_1f_1$, $f_s - a_sf_1$, we may therefore assume that $\varphi(\overline{f}_1) \neq 0$ and $\varphi(\overline{f}_i) = 0$, $\forall i = 2,...,s$.

On the other hand, we claim that there is a linear relation $r = (r_1,...,r_s)$ among $f_1,...,f_s$ such that $r_1 \neq 0$. Indeed, considering the obvious relation $f = (f_2,-f_1,0,...,0)$ among $f_1,...,f_s$, by hypothesis we know that there are linear forms $c_1,...,c_u \in P_1$ such that $f = c_1r^1 + ... + c_ur^u$. In particular, $f_2 = c_1r_1^1 + ... + c_ur_1^u$, which forces $r_1^i \neq 0$ for at least one $i \in \{1,...,u\}$ (because $f_2 \neq 0$).

Now, fixing a relation $r = (r_1,...,r_s)$ among $f_1,...,f_s$ with $r_1 \neq 0$, we have $0 = \varphi(r_1f_1 + ... + r_sf_s) = r_1\varphi(f_1)$. Since $\varphi(f_1)$ is a non zero constant, it follows $r_1 = 0$, a contradiction. In this way we have proved that $\mathrm{Hom}_A(I/I^2,A)_{-2} = 0$, which implies $T^1(A|k,A)(-2) = 0$.

7 A characterization of linear subspaces

In this section we prove the following characterization of linear subspaces:

THEOREM 7.1 (VAN DE VEN [51]) *Let Y be a smooth closed irreducible subvariety of $\mathbb{P}^n$ of dimension $d \geq 1$ over the field $\mathbb{C}$ of complex num-*

bers. Then the normal sequence

$$0 \to T_Y \to T_{\mathbb{P}^n}|Y \to N_{Y|\mathbb{P}^n} \to 0$$

splits if and only if Y is a linear subspace of $\mathbb{P}^n$.

One implication is easy. Namely, assume that Y is a linear subspace of $\mathbb{P}^n$. Let L be a linear subspace of $\mathbb{P}^n$ of dimension $n-d-1$ such that $Y \cap L = \emptyset$. Projecting from L we get a morphism $\pi : \mathbb{P}^n \setminus L \to Y = \mathbb{P}^d$ such that $\pi \circ i = \mathrm{id}_Y$, where $i : Y \hookrightarrow \mathbb{P}^n$ is the natural inclusion. Therefore we get a canonical map $T_{\mathbb{P}^n \setminus L} \to \pi^*(T_Y)$, which restricted to Y yields a map $T_{\mathbb{P}^n}|Y \to i^*\pi^*(T_Y) = T_Y$. This is the desired splitting of the normal sequence.

The other implication is non-trivial. The main idea of the proof below (which is due to Mustaţă-Popa [41]) is to make use of the first infinitesimal neighbourhood $Y(1) = (Y, \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}^2)$ of Y in $\mathbb{P}^n$, where $\mathcal{I}$ is the ideal sheaf of Y in $\mathcal{O}_{\mathbb{P}^n}$. The first step is to interpret the splitting of the normal bundle in terms of $Y(1)$. Precisely, we have the following general result:

LEMMA 7.1 *Let Y be a closed subvariety of an algebraic variety X, and let $\mathcal{I}$ be the ideal sheaf of Y in $\mathcal{O}_X$. Let*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\ \delta\ } \Omega^1_X|Y \longrightarrow \Omega^1_Y \longrightarrow 0$$

be the dual of the normal sequence of Y in X. Then δ admits a left inverse (i.e. a map $s : \Omega^1_X|Y \to \mathcal{I}/\mathcal{I}^2$ of coherent $\mathcal{O}_Y$-modules such that $s \circ \delta = \mathrm{id}$) if and only if the canonical inclusion $i : Y \to Y(1)$ (with $Y(1)$ the first infinitesimal neighbourhood of Y in X) admits a retraction, i.e. there exists a morphism of schemes $r : Y(1) \to Y$ such that $r \circ i = \mathrm{id}_Y$.

PROOF OF LEMMA 7.1. The existence of a retraction π is equivalent to the existence of a map of sheaves of rings $\pi' : \mathcal{O}_Y = \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I}^2$ such that the composition

$$\mathcal{O}_X/\mathcal{I} \xrightarrow{\ \pi'\ } \mathcal{O}_X/\mathcal{I}^2 \xrightarrow{\text{canonical}} \mathcal{O}_X/\mathcal{I}$$

is the identity. On the other hand, the existence of a map $s : \Omega^1_X|Y \to \mathcal{I}/\mathcal{I}^2$ such that $s \circ \delta = \mathrm{id}$ is equivalent to the existence of a derivation $D : \mathcal{O}_X \to \mathcal{I}/\mathcal{I}^2$ such that $D|\mathcal{I}$ coincides to the canonical map $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2$.

Given $D \in \operatorname{Der}(\mathcal{O}_X, \mathcal{I}/\mathcal{I}^2)$ such that $D|\mathcal{I} : \mathcal{I} \to \mathcal{I}/\mathcal{I}^2$ is the canonical map, we define $\pi' : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}^2$ by $\pi' = \varphi - D$, where $\varphi : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}^2$ is the canonical map. One checks easily that π' is a map of rings and $\pi'|\mathcal{I} = 0$. Therefore one obtains a map $\pi' : \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I}^2$ which composed with the canonical map $\mathcal{O}_X/\mathcal{I}^2 \to \mathcal{O}_X/\mathcal{I}$ is the identity. Conversely, given π' one easily gets back D by $D = \varphi - \pi'$.

PROOF OF THEOREM 7.1.

Step 1. We may assume that Y is non-degenerate in $\mathbb{P}^n$.

Indeed, let L be the linear subspace of $\mathbb{P}^n$ generated by Y. Then we have the commutative diagram of dual normal sequences

$$\begin{tikzcd}
0 \arrow[r] & N^*_{Y|\mathbb{P}^n} \arrow[r] \arrow[d] & \Omega^1_{\mathbb{P}^n}|Y \arrow[r, "u"] \arrow[d, "\alpha"] & \Omega^1_Y \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & N^*_{Y|L} \arrow[r] & \Omega^1_L|Y \arrow[r, "v"'] & \Omega^1_Y \arrow[r] & 0
\end{tikzcd}$$

If $s : \Omega^1_Y \to \Omega^1_{\mathbb{P}^n}|Y$ is a map such that $u \circ s = \mathrm{id}$, then $v \circ (\alpha \circ s) = u \circ s = \mathrm{id}$. In other words, if the top sequence splits so does the bottom one.

Henceforth we may assume that Y is non-degenerate and we have to prove that $Y = \mathbb{P}^n$. We shall first consider the case $d = \dim(Y) \geq 2$.

Step 2. $H^1(Y, N^*_{Y|\mathbb{P}^n}) = 0$.

Indeed, from the splitting of the dual of the normal sequence of Y in $\mathbb{P}^n$ we get

$$H^1(Y, \Omega^1_{\mathbb{P}^n}|Y) \cong H^1(Y, N^*_{Y|\mathbb{P}^n}) \oplus H^1(Y, \Omega^1_Y). \tag{7.1}$$

On the other hand, the restricted Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^n}|Y \to (n+1)\mathcal{O}_Y(-1) \to \mathcal{O}_Y \to 0$$

yields the cohomology sequence

$$0 = H^0(Y, (n+1)\mathcal{O}_Y(-1)) \to H^0(Y, \mathcal{O}_Y) \to$$

$$\to H^1(Y, \Omega^1_{\mathbb{P}^n}|Y) \to H^1(Y, (n+1)\mathcal{O}_Y(-1)).$$

Since $k = \mathbb{C}$ and $\dim(Y) \geq 2$, by the Kodaira vanishing theorem the last cohomology space is zero (because $d \geq 2$). Therefore

$$\dim(H^1(Y, \Omega^1_{\mathbb{P}^n}|Y)) = 1.$$

On the other hand, $\dim(H^1(Y, \Omega^1_Y)) \geq 1$ because by a theorem of Néron-Severi $H^1(Y, \Omega^1_Y)$ contains $NS(Y) \otimes_{\mathbb{Z}} \mathbb{C}$, where $NS(Y)$ is the Néron-Severi group of Y. These two observations together with (7.1) prove step 2.

Step 3. $H^0(Y, N^*_{Y|\mathbb{P}^n}(1)) = 0$.

Indeed, from the restricted Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n}|Y(1) \longrightarrow (n+1)\mathcal{O}_Y \xrightarrow{\beta} \mathcal{O}_Y(1) \longrightarrow 0$$

we get

$$H^0(Y, \Omega^1_{\mathbb{P}^n}|Y(1)) = \operatorname{Ker}(\zeta), \tag{7.2}$$

where $\zeta := H^0(\beta) : H^0(Y, (n+1)\mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y(1))$. On the other hand, in the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^n, (n+1)\mathcal{O}_{\mathbb{P}^n}) & \xrightarrow{\text{iso}} & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \\ {\scriptstyle\text{iso}}\downarrow & & \downarrow{\scriptstyle\text{inj}} \\ H^0(Y, (n+1)\mathcal{O}_Y) & \xrightarrow[\zeta]{} & H^0(Y, \mathcal{O}_Y(1)) \end{array}$$

the first vertical map is an isomorphism, the first horizontal map is also an isomorphism, and the second vertical map is injective (because Y is non-degenerate in $\mathbb{P}^n$). It follows that the map ζ is injective. Recalling (7.2), this implies that $H^0(Y, \Omega^1_{\mathbb{P}^n}|Y(1)) = 0$. Using the splitting of the dual normal sequence we get

$$0 = H^0(Y, \Omega^1_{\mathbb{P}^n}|Y(1)) \cong H^0(Y, N^*_{Y|\mathbb{P}^n}(1)) \oplus H^0(Y, \Omega^1_Y(1)).$$

This proves step 3.

Step 4. Let $i : Y \to Y(1)$ be the natural inclusion of Y in its first infinitesimal neighbourhood in $\mathbb{P}^n$. Then the map $i^* : \operatorname{Pic}(Y(1)) \to \operatorname{Pic}(Y)$ is injective.

Indeed, from the truncated exponential sequence

$$0 \to N^*_{Y|\mathbb{P}^n} = \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}^*_{Y(1)} \to \mathcal{O}^*_Y \to 0,$$

(where for any scheme Z, $\mathcal{O}^*_Z$ is the sheaf of multiplicative groups of germs of all nowhere zero functions of $\mathcal{O}_Z$, and where the map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}^*_{Y(1)}$ is given by $u \to 1+u$), we get the cohomology sequence

$$H^1(Y, N^*_{Y|\mathbb{P}^n}) \to \mathrm{Pic}(Y(1)) \to \mathrm{Pic}(Y).$$

Then step 4 follows from $H^1(Y, N^*_{Y|\mathbb{P}^n}) = 0$ (by step 2).

Step 5 (Conclusion in case $\dim(Y) \geq 2$).

By lemma 7.1 our hypothesis says that there is a retraction $r : Y(1) \to Y$ of the inclusion $i : Y \hookrightarrow Y(1)$. Set $L := r^*\mathcal{O}_Y(1)$. Since $r \circ i = \mathrm{id}$, $i^*(L) \cong \mathcal{O}_Y(1) = \mathcal{O}_{Y(1)}(1)|Y$. Since the map $i^* : \mathrm{Pic}(Y(1)) \to \mathrm{Pic}(Y)$ is injective (step 4), $L \cong \mathcal{O}_{Y(1)}(1)$, i.e. $r^*(\mathcal{O}_Y(1)) \cong \mathcal{O}_{Y(1)}(1)$.

On the other hand, in the exact sequence

$$0 \longrightarrow H^0(Y, N^*_{Y|\mathbb{P}^n}(1)) \longrightarrow H^0(Y(1), \mathcal{O}_{Y(1)}(1)) \xrightarrow{\ i^*\ } H^0(Y, \mathcal{O}_Y(1))$$

the first H^0 is zero by step 3. Therefore the map

$$i^* : H^0(Y(1), \mathcal{O}_{Y(1)}(1)) \to H^0(Y, \mathcal{O}_Y(1))$$

is injective. Now, let $t_0, \ldots, t_n$ be a basis of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and set $s_i := r^*(t_i|Y) \in H^0(Y(1), \mathcal{O}_{Y(1)}(1))$, $\forall i = 0, \ldots, n$. Since i^* is injective, it follows that $s_i = t_i|Y(1)$, $\forall i = 0, \ldots, n$. Since $\mathcal{O}_{Y(1)}(1)$ is very ample we get that $i \circ r = i(1)$, where $i(1) : Y(1) \hookrightarrow \mathbb{P}^n$ is the canonical inclusion. But this last fact is impossible, unless $Y = \mathbb{P}^n$. This proves theorem 7.1 in case $\dim(Y) \geq 2$.

Step 6. Theorem 7.1 holds true if $\dim(Y) = 1$.

The argument of this step is taken from [32]. The splitting of the normal bundle of Y in $\mathbb{P}^n$ implies that $T_{\mathbb{P}^n}|Y \cong T_Y \oplus N_{Y|\mathbb{P}^n}$. Since $T_{\mathbb{P}^n}$ is an ample vector bundle, its restriction $T_{\mathbb{P}^n}|Y$ is also ample, whence T_Y is ample. Since Y is a curve, this is possible only when $Y \cong \mathbb{P}^1$. Set $e := \deg(Y)$. Then $\mathcal{O}_Y(1) \cong \mathcal{O}_{\mathbb{P}^1}(e)$. It remains to prove that $e = 1$.

To this extent look at the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N^*_{Y|\mathbb{P}^n} \arrow[r] \arrow[d, "\mathrm{id}"] & \Omega^1_{\mathbb{P}^n}|Y \arrow[r, "a"] \arrow[d] & \Omega^1_Y = \mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N^*_{Y|\mathbb{P}^n} \arrow[r] & (n+1)\mathcal{O}_{\mathbb{P}^1}(-e) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_Y \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{O}_Y \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

in which the first row is the dual of the normal bundle of Y in $\mathbb{P}^n$, and the middle column is the dual of the Euler sequence of $\mathbb{P}^n$ restricted to Y. The splitting of the top row yields a map $s : \Omega^1_Y = \mathcal{O}_{\mathbb{P}^1}(-2) \to \Omega^1_{\mathbb{P}^n}|Y$ such that $a \circ s = \mathrm{id}$. In particular, s is injective. The injectivity of s and the above diagram yield an injective map $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow (n+1)\mathcal{O}_Y(-1) = (n+1)\mathcal{O}_{\mathbb{P}^1}(-e)$. This forces $e \leq 2$.

If $e = 2$, Y is a plane conic. Then the dual of the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^2}|Y(1) \to 3\mathcal{O}_Y \to \mathcal{O}_Y(1) = \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

yields the exact sequence

$$0 \to H^0(Y, \Omega^1_{\mathbb{P}^2}|Y(1)) \to H^0(Y, 3\mathcal{O}_Y) \to H^0(Y, \mathcal{O}_{\mathbb{P}^1}(2)).$$

Since Y is a plane conic, the last map is an isomorphism. Therefore

$$H^0(Y, \Omega^1_{\mathbb{P}^2}|Y(1)) = 0 = H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^2}|Y \otimes \mathcal{O}_{\mathbb{P}^1}(2)).$$

But this contradicts the existence of the injective map $s : \mathcal{O}_{\mathbb{P}^1}(-2) \to \Omega^1_{\mathbb{P}^2}|Y$. Therefore $e = 1$, so the proof of step 6 (and thereby the proof of theorem 7.1) is complete. $\square$

REMARK 7.1 The main reason why we included the present proof of theorem 7.1 in these notes consists primarily in the use of the first infinitesimal neighbourhood of a subvariety in an ambient variety. This illustrates very well the philosophy that using schemes with nilpotents in the study of some rather classical questions of projective geometry could sometimes provide natural proofs. In section 9 we shall apply again this method (of using the first infinitesimal neighbourhoods) in connection with a problem of complete intersection.

8 Cohomological dimension and connectivity theorems

DEFINITION 8.1 *Let Z be an irreducible algebraic variety over k. We shall denote by* $\mathrm{Coh}(Z)$ *the category of all coherent sheaves $\mathcal{O}_Z$-modules. According to Hartshorne [29] we define the cohomological dimension,* $\mathrm{cd}(Z)$, *of Z by the following*

$$\mathrm{cd}(Z) = \max\{n \geq 0 \mid \exists F \in \mathrm{Coh}(Z) \text{ such that } H^n(Z,F) \neq 0\}.$$

A general result says that $0 \leq \mathrm{cd}(Z) \leq \dim(Z)$; *moreover, a result of Serre asserts that* $\mathrm{cd}(Z) = 0$ *if and only if Z is affine (see e.g. [Ha1]).*

The aim of this section is to prove the following special case of a basic result due of Hartshorne-Lichtenbaum (see [25], or also [30]). This special form of the theorem will be sufficient to prove a generalization of the connectivity theorem of Fulton-Hansen (see theorem 8.4 below).

THEOREM 8.1 (HARTSHORNE-LICHTENBAUM) *Let X be an irreducible projective variety of dimension $n \geq 1$, let Y be a closed subset of X, and set $U := X \setminus Y$. Then* $\mathrm{cd}(U) = n$ *if and only if $Y = \emptyset$.*

Before proceeding to the proof (which follows Kleiman [30] closely) we need some preparation.

LEMMA 8.1 *For every irreducible quasi-projective variety V of dimension n the following two conditions are equivalent:*

1. *$H^n(V,F) = 0$ for all $F \in \mathrm{Coh}(V)$.*

2. *$H^n(V, L) = 0$ for all invertible sheaves L on V.*

PROOF. Clearly (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Since V is a quasi-projective variety, every $F \in \mathrm{Coh}(V)$ is a quotient of direct sum of invertible sheaves. Indeed, pick an ample line bundle L on V. Since L is ample, $F \otimes L^m$ is generated by its global sections for $m >> 0$. Since F is coherent it follows that there is a surjection $p\mathcal{O}_V \to F \otimes L^m \to 0$ for some $p \geq 1$, and therefore a surjection $E := pL^{-m} \to F \to 0$. In particular, E is a finite direct sum of line bundles. In this way we get an exact sequence of the form

$$0 \to G \to E \to F \to 0,$$

with G a coherent sheaf on V. This yields the cohomology sequence

$$H^n(V, E) \to H^n(V, F) \to H^{n+1}(V, G).$$

By (2) the first space is zero (the functor H^n commutes with the direct sums), and since $n + 1 > \dim(V) = n$ the third space is also zero, whence the middle space is zero as well. $\square$

LEMMA 8.2 *Let W be a closed subvariety of an algebraic variety V. For a fixed integer $n \geq 1$ the following two conditions are equivalent:*

1. *$H^n(W, F) = 0$ for all coherent $\mathcal{O}_W$-modules F.*

2. *$H^n(V, F) = 0$ for all quasi-coherent $\mathcal{O}_V$-modules F with* $\mathrm{Supp}(F) \subseteq W$.

PROOF. The implication (2) $\Rightarrow$ (1) is obvious. Conversely, let F be a quasi-coherent $\mathcal{O}_V$-module with support in W. F is the direct limit of all its coherent $\mathcal{O}_V$-submodules G. Since the functor H^n commutes with direct limits, it follows that $H^n(V, F)$ is the direct limit of the $H^n(V, G)$'s. Hence we may assume F coherent. Now, let Z be the closed subscheme of V defined by the annihilator ideal of F. By hypothesis, Z_{red} is a closed subvariety of W, where Z_{red} denotes the (reduced) subscheme of Z defined by the nilpotent radical I of $\mathcal{O}_Z$. Since Z is a noetherian quasi-compact scheme, there is an integer $m > 0$

such that $I^m = 0$. Then for any coherent $\mathcal{O}_Z$-module F consider the exact sequences

$$0 \to I^{i+1}F \to I^iF \to I^iF/I^{i+1}F \to 0, \quad \forall i = 0, 1, ..., m-1,$$

where $I^0 := \mathcal{O}_Z$. Therefore we get the cohomology sequences

$$H^n(V, I^{i+1}F) \to H^n(V, I^iF) \to H^n(V, I^iF/I^{i+1}F), \quad \forall i = 0, 1, ..., m-1. \tag{8.1}$$

Clearly the quotients $I^iF/I^{i+1}F$ are $\mathcal{O}_{Z_{\mathrm{red}}}$-modules, and hence also $\mathcal{O}_W$-modules. Therefore by (1), $H^n(V, I^iF/I^{i+1}F) = H^n(W, I^iF/I^{i+1}F) = 0$ for all $i = 0, 1, ..., m-1$. In particular,

$$H^n(V, I^{m-1}F) = H^n(V, I^{m-1}F/I^mF) = 0.$$

Using this, (8.1) and an obvious descending induction on i we get $H^n(V, F) = 0$, as desired. □

PROOF OF THEOREM 8.1. Assume first that $Y = \emptyset$, i.e. $U = X$ is projective. We have to find a coherent sheaf $F \in \mathrm{Coh}(U)$ such that $H^n(U, F) \neq 0$. Choose an ample line bundle L on U. Since U is projective, $H^n(U, L^{-m}) \neq 0$ for every $m >> 0$ (for example use duality on U).

Conversely, assume $Y \neq \emptyset$. In this case we proceed by induction on $n = \dim(X)$. If $n = 1$ then U is an affine curve and the conclusion follows from a well known vanishing result of Serre (see [28]). Assume therefore $n \geq 2$. By lemma 8.1 we may assume that F is invertible, and in particular, a torsion free sheaf.

Now we need the following:

CLAIM 8.1 *In the hypotheses of theorem 8.1 assume $n \geq 2$. Then there is a closed irreducible subscheme Z of U of dimension $n-1$ such that $U \setminus Z$ is affine. Moreover, the scheme Z is quasi-projective, but not projective.*

We first prove claim 8.1. Blow up X along Y to get the birational morphism $f : X' \to X$ with exceptional locus $Y' = f^{-1}(Y)$. Then Y' is an effective Cartier divisor on X'. Since X is projective and irreducible, X' is also projective and irreducible. Choose a projective embedding $X' \hookrightarrow \mathbb{P}^m$. Since $n \geq 2$, by Bertini we can find an irreducible hyperplane section T of X'. Because T is a very ample Cartier divisor on X' ($T \neq Y'$) and Y' is a Cartier divisor on X', $Y' + aT$ is very ample for $a >> 0$. Therefore $X' \setminus (T \cup Y')$ is an affine open subset of X'. Set $Z' := f(T \cup Y')$ and $Z := U \cap Z'$. Then Z is irreducible because $Z = f(T) \cap U$ and T is irreducible. Moreover, $U \setminus Z = X \setminus Z' \cong X' \setminus (T \cup Y')$, and in particular $U \setminus Z$ is affine. Finally, Z is not projective because $Z = f(T) \setminus Y$ and $Y \cap f(T) \neq \emptyset$ ($T \cap Y' \neq \emptyset$ because T is a hyperplane section on X' and $\dim(Y') = n - 1 > 0$). Thus the claim 8.1 is proved.

To conclude the proof of theorem 8.1 in case $n \geq 2$, apply the claim 8.1 to find Z as above. Let $i : U \setminus Z \hookrightarrow U$ be the canonical inclusion. Then i is an affine morphism because for every affine open subset V of U, $i^{-1}(V) = V \cap (U \setminus Z)$ (on a separated scheme the intersection of any two affine open subsets is again affine!). Now, starting with a torsion free coherent sheaf F on U, consider the canonical map $\alpha : F \to i_*i^*(F)$. Since $\mathrm{Supp}(\mathrm{Ker}(\alpha)) \subseteq Z$, $\mathrm{Ker}(\alpha)$ is a torsion subsheaf of F, and since F is torsion free, $\mathrm{Ker}(\alpha) = 0$. Therefore we get the exact sequence

$$0 \to F \to i_*i^*(F) \to G \to 0$$

(where $G := \mathrm{Coker}(F \to i_*i^*(F))$) which yields the cohomology sequence

$$H^{n-1}(U, G) \to H^n(U, F) \to H^n(U, i_*i^*(F)). \tag{8.2}$$

Note that $i_*i^*(F)$ (and hence also G) is only a quasi-coherent $\mathcal{O}_U$-module.

Since i is an affine morphism $H^n(U, i_*i^*(F)) \cong H^n(U \setminus Z, i^*(F))$, and the latter space is zero because $n > 0$ and $U \setminus Z$ is affine, by Serre's theorem. Therefore

$$H^n(U, i_*i^*(F)) = 0. \tag{8.3}$$

On the other hand, $\mathrm{Supp}(G) \subseteq Z$. Since Z is an irreducible quasi-projective (but not projective) variety of dimension $n - 1$, by the inductive hypothesis, $H^{n-1}(Z, H) = 0$ for every coherent $\mathcal{O}_Z$-module H.

Then by lemma 8.2 we get

$$H^{n-1}(U,G) = 0. \tag{8.4}$$

Then (8.2), (8.3) and (8.4) yield $H^n(U,F) = 0$.

REMARK 8.1 Theorem 8.1 can be restated by saying that $\mathrm{cd}(U) \leq n-1$ if and only if $Y \neq \emptyset$. The above inductive proof of theorem 8.1 required considering effectively the category of quasi-coherent (and not only of coherent) sheaves.

We illustrate now the use of the notion of cohomological dimension by proving the following useful result, which is going to play a crucial role in the proof of the connectivity theorem of Fulton-Hansen:

THEOREM 8.2 *In the hypotheses of theorem* 8.1 *assume* $n \geq 2$ *and* $\mathrm{cd}(U) \leq n-2$. *Then* Y *is connected.*

PROOF. Assume that Y is not connected, i.e. Y can be written as $Y = Y_1 \cup Y_2$, with Y_1 and Y_2 non-empty closed subsets of Y such that $Y_1 \cap Y_2 = \emptyset$. Because X is projective of dimension n, by theorem 8.1 and lemma 8.1, there is an invertible sheaf L on X such that $H^n(X,L) \neq 0$.

In the exact sequence

$$H^n_{Y_i}(X,L) \longrightarrow H^n(X,L) \longrightarrow H^n(X \setminus Y_i, L)$$

the last space is zero by theorem 8.1 because $Y_i \neq \emptyset$ for $i = 1,2$. It follows that $h^n_{Y_i}(X,L) \geq h^n(X,L)$ for $i = 1,2$, where $h^i_W(V,F) := \dim_k(H^i_W(V,F))$ for every coherent sheaf F on an algebraic variety V and for every closed subvariety W of V. Moreover, since $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$, by Mayer-Vietoris we have $H^n_Y(X,L) \cong H^n_{Y_1}(X,L) \oplus H^n_{Y_2}(X,L)$, and hence, $h^n_Y(X,L) \geq 2h^n(X,L)$.

On the other hand, in the exact sequence

$$H^{n-1}(X \setminus Y, L) \longrightarrow H^n_Y(X,L) \longrightarrow H^n(X,L) \longrightarrow H^n(X \setminus Y, L)$$

the the first and the last spaces are zero by $\mathrm{cd}(X \setminus Y) \leq n-2$, whence $h^n_Y(X,L) = h^n(X,L)$, contradicting the inequality $h^n_Y(X,L) \geq 2h^n(X,L)$, because $h^n(X,L) \geq 1$. □

We shall use theorem 8.2 to prove the following two results (corollaries 8.1 and 8.2) due to Grothendieck (see [27, éxposé XIII]).

COROLLARY 8.1 *Let X be an irreducible algebraic variety of dimension $n \geq 2$ over k, and let $f : X \to \mathrm{Proj}(S)$ be a finite morphism, where S is a finitely generated graded k-algebra. Let $t_1, ..., t_r \in S_+$ be homogeneous elements of positive degrees. If $r \leq n-1$ (resp. $r \leq n$) then $f^{-1}(V_+(t_1, ..., t_r))$ is connected (resp. non-empty), where $V_+(t_1, ..., t_r)$ is the locus of $\mathrm{Proj}(S)$ of equations $t_1 = ... = t_r = 0$.*

PROOF. In the standard notations of [26], set $Y := f^{-1}(V_+(t_1, ..., t_r))$ and $P := \mathrm{Proj}(S)$. Since $P \setminus V_+(t_1, ..., t_r) = D_+(t_1) \cup ... \cup D_+(t_r)$, $D_+(t_i) := \mathrm{Spec}(S_{t_i})$ is affine $\forall i = 1, ..., r$, and f is finite, it follows that $X \setminus Y$ is the union of the affine open subsets $f^{-1}(D_+(t_i))$, $i = 1, ..., r$. Therefore by Čech, $\mathrm{cd}(X \setminus Y) \leq r-1 \leq n-2$ (resp. $\mathrm{cd}(X \setminus Y) \leq n-1$ if $r \leq n$). The conclusion follows in this case from theorem 8.2 (resp. from theorem 8.1). □

The above corollary can be slightly refined. First we need the following:

DEFINITION 8.2 *Let V be an algebraic variety over k, and let $d \geq 0$ be a non-negative integer. V is said to be d-connected if every irreducible component of V is of dimension $\geq d+1$ and if $V \setminus W$ is connected for every closed subvariety W of V of dimension $< d$.*

For example, every irreducible variety X of dimension $n \geq 1$ is $(n-1)$-connected. An algebraic variety X is 0-connected if and only if X is connected and every irreducible component of X is of dimension ≥ 1.

EXAMPLE 8.1 Let X be the closed algebraic subvariety of the affine space $\mathbb{A}^4$ having two irreducible components X_1 and X_2, where X_1 is the plane of equations $x_1 = x_2 = 0$ and X_2 the plane of equations $x_3 = x_4 = 0$. Then $X_1 \cap X_2 = \{x\}$, where $x = (0, 0, 0, 0)$. Thus X is 0-connected. On the other hand, X is not 1-connected because $X \setminus \{x\}$ has two connected components $X_1 \setminus \{x\}$ and $X_2 \setminus \{x\}$.

DEFINITION 8.3 *A sequence $Z_0, Z_1, \ldots, Z_n$ of (not necessarily pairwise distinct) irreducible components of an algebraic variety V is called a d-join within V if $\dim(Z_i) \geq d+1$ for every $i = 0, 1, \ldots, n$ and if $\dim(Z_{j-1} \cap Z_j) \geq d$ for every $j = 2, \ldots, n$.*

The following elementary result (whose proof is left as an exercise to the reader) will not be used in the sequel, but it explains better the concept of d-connectedness.

PROPOSITION 8.1 *An algebraic variety X is d-connected if and only if $X = Z_0 \cup Z_1 \cup \ldots \cup Z_n$ for some d-join $Z_0, Z_1, \ldots, Z_n$ within X.*

COROLLARY 8.2 *In the hypotheses of corollary 8.1 assume that $r \leq n-1$. Then $f^{-1}(V_+(t_1, \ldots, t_r))$ is $(n-r-1)$-connected.*

PROOF. Since S is a finitely generated k-algebra, $\mathrm{Proj}(S)$ is a projective scheme over k, whence X is a projective variety because f is a finite morphism. Let $X \subset \mathbb{P}^N$ be an arbitrary projective embedding of X, and let A be a general linear subspace of $\mathbb{P}^N$ of dimension $N+r-n+1$. Since $\dim(X) = n$ and $Y := f^{-1}(V_+(t_1, \ldots, t_r))$ is locally given by r equations in X, every irreducible component Z of Y is of dimension $\geq n-r$. It follows that $\dim(Z \cap A) = \dim(Z) + \dim(A) - N \geq (n-r) + (N+r-n+1) - N = 1$, and in particular, A meets every irreducible component of Y.

Set $Y' := Y \cap A$. If A is defined by linear equations $s_{r+1} = \ldots = s_{n-1} = 0$ in $\mathbb{P}^N$ then $X \setminus Y'$ is the union of the $n-1$ affine open subsets $f^{-1}(D_+(t_1)), \ldots, f^{-1}(D_+(t_r)), U_{r+1}, \ldots, U_{n-1}$, where $U_i := \{x \in X \mid s_i(x) \neq 0\}$, $\forall i = r+1, \ldots, n-1$. It follows that $\mathrm{cd}(X \setminus Y') \leq n-2$, whence by theorem 8.2, Y' is connected.

We saw above that every irreducible component of Y is of dimension $\geq n-r$. Therefore to show that Y is $(n-r-1)$-connected it will be sufficient to check that $Y \setminus W$ is connected for every closed subset W of Y of dimension $< n-r-1$. Assume the contrary, i.e. there is a closed subscheme W of Y of dimension $< n-r-1$ such that $Y \setminus W$ is not connected. Since A is general, $\dim(W \cap A) = \dim(W) + \dim(A) - N < (n-r-1) + (N+r-n+1) - N = 0$, whence A does not meet W. Moreover, since A meets every irreducible component of Y, the fact that $Y \setminus W$ is not connected implies that $Y' = Y \cap A = (Y \setminus W) \cap A$ is also not connected, a contradiction. $\square$

In the sequel we shall need the following more general version of corollary 8.2:

THEOREM 8.3 *Let S be a finitely generated graded k-algebra, $t_1, ..., t_r \in S_+$ homogeneous elements of positive degrees, and U a Zariski open subset of* $\mathrm{Proj}(S)$ *containing $L := V_+(t_1, ..., t_r)$. Let $f : X \to U$ be a finite morphism, with X an irreducible algebraic variety of dimension n over k. If $r \le n - 1$ then $f^{-1}(L)$ is $(n-r-1)$-connected.*

PROOF. By passing to the normalization of X we may assume that X is normal. Let Z' be the closure of $X' := f(X)$ in $P := \mathrm{Proj}(S)$, and let $g : Z \to Z'$ be the normalization of Z' in the field $K(X)$ of rational functions of X (which makes sense because the dominant morphism $X \to Z'$ yields the finite field extension $K(Z') = K(X') \to K(Z)$). Then we get a commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[i']{} & Z' \end{array}$$

in which i and i' are open immersions (i is an open immersion because X is normal), and g is a finite morphism. Since $L \subset U$ and $Z' \cap U = X'$ (X' is closed in U because f is finite), then $X' \cap L = Z' \cap L$, whence $f^{-1}(L) = g^{-1}(L)$. Then theorem 8.3 follows from corollary 8.2 applied to the composition of the closed immersion $Z' \hookrightarrow P$ with the finite morphism $g : Z \to Z'$. $\square$

8.1 Weighted projective spaces

Let $k[T_0, T_1, ..., T_n]$ be the polynomial k-algebra in $n+1$ variables T_0, T_1,...,T_n (with $n \ge 1$). An $(n+1)$-uple $(e_0, e_1, ..., e_n) \in \mathbb{Z}^{n+1}$ of positive integers is called a *system of weights* if $e_i \ge 1$, $\forall i = 0, 1, ..., n$. Given a system of weights $e = (e_0, e_1, ..., e_n)$, grade $k[T_0, T_1, ..., T_n]$ by the conditions $\deg(T_i) = e_i$, $\forall i = 0, 1, ..., n$. In this way we get a finitely generated graded k-algebra (depending of $e = (e_o, e_1, ...e_n)$), and set

$$\mathbb{P}^n(e) = \mathbb{P}^n(e_0, e_1, ..., e_n) := \mathrm{Proj}(k[T_0, T_1, ..., T_n]).$$

Then $\mathbb{P}^n(e) = \mathbb{P}^n(e_0, e_1, ..., e_n)$ is a normal projective variety of dimension n which is called *the weighted projective space of weights* $e = (e_0, e_1, ..., e_n)$. $\mathbb{P}^n(1, 1, ..., 1)$ coincides of course with the usual projective space $\mathbb{P}^n$.

As a non-trivial example of weighted projective space, take $e = (1, ..., 1, s)$, with $s \geq 2$ and $n \geq 2$. Then $\mathbb{P}^n(1, 1, ..., 1, s)$ is isomorphic to the projective cone over the Veronese embedding $v_s : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N(n,s)}$, with $N(n, s) := \binom{n-1+s}{s} - 1$. Indeed,

$$\mathbb{P}(1, 1, ..., 1, s) = \mathrm{Proj}(k[T_0, ..., T_n]),$$

with $\deg(T_i) = 1$, $\forall i = 0, 1, ..., n-1$, and $\deg(T_n) = s$. Then using the general elementary properties of Proj (see e.g. [26]), we have canonical isomorphisms

$$\begin{aligned}\mathrm{Proj}(k[T_0, ..., T_n]) &\cong \mathrm{Proj}(k[T_0, ..., T_n])^{(s)} \\ &\cong \mathrm{Proj}(k[T_0, ..., T_{n-1}]^{(s)}[T]),\end{aligned}$$

with T a variable of degree one. We adopted the standard notation according to which $S^{(s)}$ denotes the graded k-algebra obtained from a graded k-algebra S by putting $S_m^{(s)} := S_{ms}$, $\forall m \geq 0$. The above isomorphisms and the definition of the projective cone over the Veronese embedding v_s yield the assertion.

An alternate description of the weighted projective space $\mathbb{P}^n(e)$ is the following. $\mathbb{P}^n(e)$ is the geometric quotient $(k^{n+1} \setminus \{(0, ..., 0)\})/\mathbb{G}_m$, where the action of the multiplicative group $\mathbb{G}_m = k \setminus \{0\}$ on $k^{n+1} \setminus \{(0, ..., 0)\}$ is given by $\lambda(t_0, ..., t_n) := (\lambda^{e_0} t_0, ..., \lambda^{e_n} t_n)$, for all $\lambda \in \mathbb{G}_m$ and $(t_0, ..., t_n) \in k^{n+1} \setminus \{(0, ..., 0)\}$. Then the orbit of $(t_0, t_1, ..., t_n) \in k^{n+1} \setminus \{(0, ..., 0)\}$ (regarded as a point of $\mathbb{P}^n(e)$) will be denoted by $[t_0, t_1, ..., t_n]$.

We refer the reader to [16] or to [11] for the basic properties of weighted projective spaces.

With these definitions we can prove the following generalization of a connectivity theorem of Fulton-Hansen (see [21], or also [22]):

THEOREM 8.4 *Let $f : X \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ be a finite morphism from the d-dimensional irreducible variety X to the product of the weighted projective space $\mathbb{P}^n(e)$ of weights $e = (e_0, e_1, ..., e_n)$ by itself. If $d > n$ then $f^{-1}(\Delta)$ is $(d-n-1)$-connected, where Δ is the diagonal of $\mathbb{P}^n(e) \times \mathbb{P}^n(e)$.*

PROOF. We shall show that a construction used by Deligne (see [15], or also [22]) to simplify the proof of Fulton-Hansen connectedness theorem can easily be generalized to weighted projective spaces. Having the system $e = (e_0, e_1, ..., e_n)$ of weights fixed, consider the weighted projective space

$$\mathbb{P}^{2n+1}(e,e) = \operatorname{Proj}(k[T_0, ..., T_n; U_0, ..., U_n]),$$

of weights $(e,e) := (e_0, e_1, ..., e_n, e_0, e_1, ..., e_n)$, where $T_0, ..., T_n, U_0, ..., U_n$ are $2n+2$ independent indeterminates over k such that $\deg(T_i) = \deg(U_i) = e_i$ for every $i = 0, 1, ..., n$. Consider the closed subschemes

$$L_1 = V_+(T_0, ..., T_n) \text{ and } L_2 = V_+(U_0, ..., U_n)$$

of $P := \mathbb{P}^{2n+1}(e,e)$. Then $L_1 \cap L_2 = \emptyset$. Set $U := P \setminus (L_1 \cup L_2)$. Since $T_i - U_i$ is a homogeneous element of degree e_i, it makes sense to consider also the closed subscheme $H := V_+(T_0 - U_0, ..., T_n - U_n)$ of P. Clearly, $H \subset U$. The two natural inclusions $k[T_0, ..., T_n] \subset k[T_0, ..., T_n; U_0, ..., U_n]$ and $k[U_0, ..., U_n] \subset k[T_0, ..., T_n; U_0, ..., U_n]$ yield two rational maps $g_i : \mathbb{P}^{2n+1}(e,e) \dashrightarrow \mathbb{P}^n(e)$, $i = 1, 2$, which give rise to the rational map

$$g : \mathbb{P}^{2n+1}(e,e) \dashrightarrow \mathbb{P}^n(e) \times \mathbb{P}^n(e).$$

Then g is defined precisely in the open subset U of $\mathbb{P}^{2n+1}(e,e)$. Alternatively, if we interpret $\mathbb{P}^n(e)$ as the geometric quotient $(k^{n+1} \setminus \{(0, ..., 0)\})/\mathbb{G}_m$ mentioned above, then the map g is defined by

$$g([t_0, ..., t_n; u_0, ..., u_n]) = ([t_0, ..., t_n], [u_0, ..., u_n]).$$

It is clear that g/H defines an isomorphism $H \cong \Delta$. Consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ U \arrow[r, "g"] & \mathbb{P}^n(e) \times \mathbb{P}^n(e) \\ H \arrow[r] \arrow[u] & \Delta \arrow[u] \end{tikzcd}$$

where the top square is cartesian, the vertical arrows of the bottom square are the canonical closed immersions, and the bottom horizontal arrow is an isomorphism. In our situation the variety $X' = X \times_{\mathbb{P}^n(e)\times\mathbb{P}^n(e)} U$ is irreducible of dimension $d + 1$ because X is irreducible and all the fibres of g (and hence also of g') are isomorphic to $\mathbb{G}_m$.

Therefore we can apply theorem 8.3 to the finite morphism $f' : X' \to U \subset \mathbb{P}^{2n+1}(e, e)$ and $L := H$, with $r = n + 1 < d + 1 = \dim(X')$, to deduce that $f'^{-1}(H)$ is $(d-n-1)$-connected. On the other hand, since $f^{-1}(\Delta) \cong f'^{-1}(H)$ we conclude the proof of our theorem. □

Corollary 8.3 *Let $f : X \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ be a proper morphism from an irreducible variety X such that* $\dim(f(X)) > n$. *Then $f^{-1}(\Delta)$ is connected.*

Proof. Let $f = g \circ h$ be the Stein decomposition of f, with $h : X \to X'$ a proper surjective morphism with connected fibres and $g : X' \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ a finite morphism. Then X' is irreducible of dimension equal to the dimension of $f(X)$. By theorem 8.4 $g^{-1}(\Delta)$ is $(\dim(X) - n - 1)$-connected, whence connected. Since h is proper with connected fibres it follows that $f^{-1}(\Delta) = h^{-1}(g^{-1}(\Delta))$ is also connected. □

Remark 8.2 In the case $e = (1, 1, ..., 1)$ (i.e. in the case of ordinary projective spaces) corollary 8.3 is just the Fulton-Hansen connectivity theorem (see [21], or also [22]). However, the above proof is substantially different from the proofs of [21] or [22]. In fact, the present proof allowed us to give this generalized version of the result of Fulton-Hansen.

At least in characteristic zero the weighted projective space $\mathbb{P}^n(e)$ appears as the quotient of $\mathbb{P}^n$ by the action of the finite group $G = \mu_{e_0} \times \mu_{e_1} \times ... \times \mu_{e_n}$ (where μ_m is the cyclic group of all roots of order m of 1) via the action given by

$$(\lambda_0, \lambda_1, ...\lambda_n) \cdot [t_0, t_1, ..., t_n] := [\lambda_0 t_0, \lambda_1 t_1, ..., \lambda_n t_n],$$

$\forall(\lambda_0, \lambda_1, ...\lambda_n) \in G$, and $\forall [t_0, t_1, ..., t_n] \in \mathbb{P}^n$. One may ask whether the connectivity theorem (theorem 8.4 or corollary 8.3) is valid for an arbitrary quotient $\mathbb{P}^n/G$ of $\mathbb{P}^n$ by the action of a finite group G. The following example shows that in general this is not the case.

EXAMPLE 8.2 Consider the action of the group $G = \mu_5$ of roots of order 5 of 1 on $\mathbb{P}^3$ ($\mathrm{char}(k) \neq 5$) given by

$$g \cdot [t_0, t_1, t_2, t_3] = [t_0, gt_1, g^2 t_2, g^3 t_3], \ \forall g \in G, \ \forall [t_0, t_1, t_2, t_3] \in \mathbb{P}^3.$$

Denote by P the quotient $\mathbb{P}^3/G$. Then G acts freely outside the four points $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$. Consider the Fermat surface Y of equation $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$. Then Y is a G-invariant smooth surface on which G acts freely, and the quotient $S := Y/G$ (the Godeaux surface) is embedded in P. Let $\pi : Y \to S$ denote the canonical morphism, and let $f : X := Y \times Y \to P \times P$ be the composition of the inclusion $S \times S \hookrightarrow P \times P$ with the morphism $\pi \times \pi : Y \times Y \to S \times S$. Then f is a finite morphism, X is irreducible of dimension 4 and $\dim(P) = 3$. However, as is easily checked, $f^{-1}(\Delta)$ has five connected components, where Δ is the diagonal of $P \times P$.

We conclude this section by indicating a few applications of the connectivity theorem 8.4.

COROLLARY 8.4 *Let Y and Z be two closed irreducible subvarieties of the weighted projective space $\mathbb{P}^n(e)$ such that* $\dim(Y) + \dim(Z) > n$. *Then $Y \cap Z$ is* $(\dim(Y) + \dim(Z) - n - 1)$*-connected. More generally, let $u : Y \to \mathbb{P}^n(e)$ be a finite morphism from the irreducible variety Y, and let Z be an irreducible closed subvariety of $\mathbb{P}^n(e)$ such that* $\dim(Y) > \mathrm{codim}_{\mathbb{P}^n(e)}(Z)$. *Then $u^{-1}(Z)$ is* $(\dim(Y) + \dim(Z) - n - 1)$*-connected.*

PROOF. Set $X := Y \times Z$ and take as $f : X \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ the product of the natural inclusions of Y and Z in $\mathbb{P}^n(e)$. Then apply theorem 8.4 to get that $f^{-1}(\Delta) \cong Y \cap Z$ is $(\dim(Y) + \dim(Z) - n - 1)$-connected.

For the second part one takes $X := Y \times Z$ and $f := u \times i : X = Y \times Z \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$, with $i : Z \hookrightarrow \mathbb{P}^n(e)$ the natural inclusion. Then by theorem 8.4 $f^{-1}(\Delta) \cong u^{-1}(Z)$ is $(\dim(Y) + \dim(Z) - n - 1)$-connected. □

DEFINITION 8.4 *Let $f : X \to Y$ be a morphism of algebraic schemes over k. The morphism f is said to be unramified (resp. unramified at the point $x \in X$) if $\Omega^1_{X|Y} = 0$ (resp. if $(\Omega^1_{X|Y})_x = 0$). Since by definition $\Omega^1_{X|Y}$ is $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of the closed (diagonal) immersion $\Delta_{X|Y} : X \to X \times_Y X$, then one sees immediately that saying that f is unramified is the same as saying that the diagonal immersion $\Delta_{X|Y}$ is also an open immersion. In other words, if $f : X \to Y$ is unramified then $\Delta_{X|Y}(X)$ is a connected component of $X \times_Y X$. A morphism $f : X \to Y$ is said to be étale if f is unramified and flat. As a trivial observation, if $f : X \to Y$ is an étale morphism, with Y irreducible, and if Z is an irreducible component of X, then the restriction $f|Z : Z \to Y$ is unramified.*

COROLLARY 8.5 *Let $f : X \to \mathbb{P}^n(e)$ be a finite unramified morphism from an irreducible projective variety X such that* $\dim(X) > \frac{n}{2}$. *Then f is a closed embedding.*

PROOF. Apply theorem 8.4 to $f \times f : X \times X \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ to deduce that $X \times_{\mathbb{P}^n(e)} X$ is connected. On the other hand, since f is unramified, Δ_X is a connected component of $X \times_{\mathbb{P}^n(e)} X$, whence $\Delta_X = X \times_{\mathbb{P}^n(e)} X$. Therefore f is injective. But an injective unramified morphism is a closed embedding (see [26, IV 8.11.5 and 17.2.6]). □

COROLLARY 8.6 *Let Y be a closed irreducible subvariety of $\mathbb{P}^n(e)$ such that* $\dim(Y) > \frac{n}{2}$. *If Y is not normal, then the normalization morphism $f : Y' \to Y$ must be ramified.*

COROLLARY 8.7 *Let Y be a closed irreducible subvariety of $\mathbb{P}^n(e)$ such that* $\dim(Y) > \frac{n}{2}$. *Then Y is algebraically simply connected, i.e. every finite étale morphism $u : Y' \to Y$, with Y' connected, is an isomorphism.*

PROOF. Let $u : Y' \to Y$ be a connected finite étale morphism, and let Z be an arbitrary irreducible component of Y'. If we set $v := u|Z : Z \to Y$, then the morphism v is finite and unramified. By corollary 8.5, it follows that v is a closed embedding, i.e. v defines an isomorphism $Z \cong Y$. In particular, Z is a connected component of Y', whence $Z = Y'$ because Y' is connected. □

The last application of Fulton-Hansen connectivity theorem is the beautiful theorem of Zak on tangencies. Let $Y \subset \mathbb{P}^n$ be a smooth irreducible closed subvariety of dimension $d \geq 1$ of $\mathbb{P}^n$. Henceforth (until the end of this section) we shall assume that Y is *non-degenerate* in $\mathbb{P}^n$. For every point $y \in Y$ let us denote by T_y the projective tangent space to Y at y. Let L be a linear subspace of $\mathbb{P}^n$. One says that L is *tangent* to Y at y if $T_y \subseteq L$. It follows that L is tangent to Y at y if and only if y is a singular point of the (scheme-theoretic) intersection $Y \cap L$. Then Zak's theorem on tangencies is the following:

THEOREM 8.5 (ZAK) *In the above hypotheses, fix a linear subspace L of $\mathbb{P}^n$ of dimension e, with $d \leq e \leq n-1$. Then the closed subset $\{y \in Y \mid T_y \subseteq L\}$ has dimension $\leq e - d$.*

PROOF. Assume the contrary: there is an irreducible component $X \subseteq \{y \in Y \mid T_y \subseteq L\}$ of dimension $> e - d$. Then we claim that there exists a linear subspace $V \subseteq \mathbb{P}^n$ of codimension $e+1$ such that $V \cap (Y \cup L) = \emptyset$ and such that there exist two points $x \in X$ and $y \in Y$, $x \neq y$, with $\pi_V(x) = \pi_V(y)$ (i.e. the line xy intersects V). Indeed, since Y is non-degenerate in $\mathbb{P}^n$, $Y \nsubseteq L$, and in particular, there exists a point $y \in Y \setminus L$; if $x \in X$ is an arbitrary point of X then $x \in L$, whence $x \neq y$. Because T_x does not contain the line yx, yx cannot lie in Y. Then picking a point $z \in yx \setminus Y$, we may take as V a general linear subspace of $\mathbb{P}^n$ of dimension $n - e - 1$ through the point z.

Since $\dim(Y\times X) > e$ we may apply the connectedness theorem 8.4 to the finite morphism $f := (\pi_V|Y) \times (\pi_V|X) : Y \times X \to \mathbb{P}^e \times \mathbb{P}^e$ to get that $f^{-1}(\Delta) = Y \times_{\mathbb{P}^e} X$ is connected. By the choice of V, the diagonal $\Delta_X \subseteq Y \times X$ is strictly contained in $Y \times_{\mathbb{P}^e} X$. The connectedness of $Y \times_{\mathbb{P}^e} X$ implies then that there exists a smooth curve T and a morphism $T \to Y \times_{\mathbb{P}^e} X$ whose image meets, but is not contained in Δ_X. In particular (restricting T if necessary), we get a family of pairs $\{(y_t, x_t)\}_{t\in T} \subseteq Y \times_{\mathbb{P}^e} X$ parametrized by T and a point $t_0 \in T$ such that $y_t \neq x_t$ for all $t \in T \setminus \{t_0\}$, and $y_{t_0} = x_{t_0} =: u$. As $t \to t_0$ the secant lines $y_t x_t$ degenerate to a tangent line $\omega \subseteq T_u$ to Y. On the other hand, for $t \neq t_0$ the secant line $y_t x_t$ meets the center of projection V, and hence ω also meets V. But $\omega \subseteq T_u \subseteq L$ (because $u \in X$), and L was disjoint from V, a contradiction. □

Here are two immediate corollaries of theorem 8.5.

COROLLARY 8.8 *Under the hypotheses of* 8.5, *the map*

$$Y \to \mathrm{Grass}(\mathbb{P}^n, \mathbb{P}^d)$$

defined by $y \to T_y$ (which is called the Gauss map), is a finite morphism.

PROOF. Take $e = d$ in theorem 8.5. □

COROLLARY 8.9 *Under the hypotheses of theorem* 8.5, *let X be an arbitrary hyperplane section of Y. Then X is nonsingular in codimension* $2d - n - 1$. *If moreover $d \geq \frac{n+2}{2}$ then every hyperplane section of Y is irreducible and normal.*

PROOF. For the first part take $e = n - 1$ in theorem 8.5. If $d \geq \frac{n+2}{2}$ it follows that every hyperplane section X of Y is nonsingular in codimension 1, whence normal by Serre's criterion of normality (see [48]); in particular, being connected, X is also irreducible. □

The last consequence of theorem 8.5 is the following result which gives a lower bound for the dimension of the dual variety of a projective subvariety of $\mathbb{P}^n$.

COROLLARY 8.10 *Let* $Y^* \subseteq (\mathbb{P}^n)^*$ *be the dual variety of* $Y \subseteq \mathbb{P}^n$ *(with* Y *smooth, irreducible and non-degenerate in* $\mathbb{P}^n$*). Then* $\dim(Y^*) \geq d$.

PROOF. Consider the incidence correspondence

$$P := \{(y, L) \mid T_y \subseteq L\} \subseteq Y \times (\mathbb{P}^n)^*.$$

The first projection makes P a $\mathbb{P}^{n-d-1}$-bundle over Y, and in particular P is smooth, irreducible of dimension $n-1$. The dual variety Y^* is the image of P under the second projection. By theorem 8.5 all fibres of $P \to Y^*$ have dimension $\leq n-d-1$. Then the conclusion follows from the theorem of dimension of fibres (see e.g. [50, page 60, theorem 7]). □

9 A problem of complete intersection

9.1 Let X be a closed smooth irreducible subvariety of dimension ≥ 2 of the smooth irreducible algebraic variety P. Let Y be an effective Cartier divisor of X. In this section, roughly speaking, we want to study the following:

PROBLEM. Under which conditions there exists a hypersurface H of P such that the scheme Y coincides with the scheme-theoretical intersection $X \cap H$?

The main result proved here (theorem 9.1 below) can be found in [17] (see also [13]). We shall follow these two papers closely. We shall also apply the techniques of [17] to prove geometrically the following weaker form of a theorem of Barth (see theorem 9.3 below): $\mathrm{Pic}(X) \cong \mathbb{Z}$ for every closed smooth subvariety X of the complex projective space $\mathbb{P}^n$ with $\dim(X) \geq \frac{n+2}{2}$.

Coming back to the above problem, assume that such a hypersurface H of P does exist. First we want to find a (rather obvious) necessary condition in terms of the canonical exact sequence of normal bundles

$$0 \to N_{Y|X} \to N_{Y|P} \to N_{X|P}|Y \to 0. \tag{9.1}$$

From the equality $Y = X \cap H$ (in the scheme-theoretical sense) and from the above assumptions it follows that Y is a local complete intersection in H, hence we also have the canonical exact sequence of normal bundles

$$0 \to N_{Y|H} \to N_{Y|P} \to N_{H|P}|Y \to 0. \tag{9.2}$$

By general elementary statements, the fact that Y is a proper intersection of X with H implies that there are canonical isomorphisms

$$N_{Y|H} \cong N_{X|P}|Y \text{ and } N_{H|P}|Y \cong N_{Y|X}.$$

Therefore the exact sequence (9.2) yields a splitting of the exact sequence (9.1).

In other words, a necessary condition for the triple (Y, X, P) of varieties satisfying the hypotheses from the beginning for which there exists a hypersurface H of P such that $Y = X \cap H$ (scheme-theoretically), is the splitting of the exact sequence (9.1).

This is why we begin this section by trying to express the splitting of (9.1) in terms of the first infinitesimal neighbourhood $X(1)$ of X in P. Therefore we are going to study a question somewhat similar to the splitting condition of the normal sequence of a smooth subvariety of a smooth variety (see lemma 7.1).

The dual of (9.1) is the following exact sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{I}\mathcal{J} \xrightarrow{\beta} \mathcal{I}/\mathcal{I}^2 \xrightarrow{\alpha} \mathcal{I}/(\mathcal{J}+\mathcal{I}^2) \longrightarrow 0, \tag{9.3}$$

where $\mathcal{I} = \mathcal{I}_Y$ (resp. $\mathcal{J} = \mathcal{I}_X$) is the ideal sheaf of Y (resp. of X) in $\mathcal{O}_P$. Since $Y \subset X$, $\mathcal{J} \subset \mathcal{I}$, and the maps α and β are defined as follows: α is the canonical map induced by the identity of $\mathcal{I}$ (taking into account that $\mathcal{I}^2 \subseteq \mathcal{J} + \mathcal{I}^2$), while β is the map induced by the inclusion $\mathcal{J} \subset \mathcal{I}$ (taking into account that $\mathcal{I}\mathcal{J} \subseteq \mathcal{I}^2$).

LEMMA 9.1 *Under the hypotheses of paragraph* 9.1, *the exact sequence* (9.1) *splits if and only if there exists an effective Cartier divisor Y' on the first infinitesimal neighbourhood $X(1)$ of X in P such that $Y' \cap X = Y$ (scheme-theoretically).*

REMARK 9.1 If an effective Cartier divisor Y' on $X(1)$ as in lemma 9.1 does exist one gets the cartesian diagram of natural inclusions

$$\begin{tikzcd} Y \arrow[r, "\mathrm{incl}"] \arrow[d, "\mathrm{incl}"'] & X \arrow[d, "\mathrm{incl}"] \\ Y' \arrow[r, "\mathrm{incl}"'] & X(1) \end{tikzcd}$$

PROOF. Clearly (9.1) splits if and only if (9.3) does. So, from now on it will be more convenient to work with the exact sequence (9.3). Assume first that (9.3) splits, i.e. there exists a map $\sigma : \mathcal{I}/\mathcal{I}^2 \to \mathcal{J}/\mathcal{I}\mathcal{J}$ of $\mathcal{O}_Y$-modules such that $\sigma \circ \beta = \mathrm{id}$. If $\pi : \mathcal{I} \to \mathcal{I}/\mathcal{I}^2$ is the canonical surjection, set $\gamma := \sigma \circ \pi : \mathcal{I} \to \mathcal{J}/\mathcal{I}\mathcal{J}$ and $\mathcal{I}' := \mathrm{Ker}(\gamma)$. Clearly $\mathcal{J}^2 \subseteq \mathcal{I}^2 \subseteq \mathcal{I}' \subseteq \mathcal{I}$, whence $\mathcal{I}'/\mathcal{J}^2 \subseteq \mathcal{O}_P/\mathcal{J}^2 = \mathcal{O}_{X(1)}$ defines a subscheme $Y' \subseteq X(1)$. We shall prove that Y' is an effective Cartier divisor on $X(1)$ and that $Y' \cap X = Y$ (scheme-theoretically). The latter property is equivalent to

$$\mathcal{I}' + \mathcal{J} = \mathcal{I},$$

which is a consequence of the definition of $\mathcal{I}'$ plus the equality $\sigma \circ \beta = \mathrm{id}$. It remains therefore to check that Y' is a Cartier divisor on $X(1)$. This is a local calculation. Let $x \in Y$ be an arbitrary point and set $R := \mathcal{O}_{P,x}$, $I := \mathcal{I}_x$ and $J := \mathcal{J}_x$. Then R is a regular local ring, and from our hypotheses it follows that there exists an R-sequence $f_1, ..., f_{n-d}, f_{n-d+1}$ such that $J = Rf_1 + ... + Rf_{n-d}$ and $I = Rf_1 + ... + Rf_{n-d} + Rf_{n-d+1}$, with $n := \dim(P) = \dim(R)$ and $d := \dim(X)$. Then

$$J^2 = \sum_{i,j=1}^{n-d} Rf_i f_j, \quad I^2 = J^2 + Jf_{n-d+1} + Rf_{n-d+1}^2, \text{ and } IJ = J^2 + Jf_{n-d+1}$$

Set $\overline{f}_i := f_i \bmod I^2$, $\forall i = 1, ..., n-d+1$ and $\tilde{f}_i := f_i \bmod IJ$, $\forall i = 1, ..., n-d$. Since $f_1, ..., f_{n-d+1}$ is an R-sequence, $\overline{f}_1, ..., \overline{f}_{n-d+1}$ is a basis of the R/I-module I/I^2, and from the hypotheses, $\tilde{f}_1, ..., \tilde{f}_{n-d}$ is a basis of the R/I-module J/IJ.

Since $\beta(\tilde{f}_i) = \overline{f}_i$, $\forall i = 1, ..., n-d$ and $\sigma \circ \beta = \mathrm{id}$, $\sigma(\overline{f}_i) = \tilde{f}_i$, $\forall i = 1, ..., n-d$. Moreover,

$$\sigma(\overline{f}_{n-d+1}) = \sum_{i=1}^{n-d} G_i \tilde{f}_i, \text{ with } G_i \in R/I.$$

CLAIM 9.1 *$\mathrm{Ker}(\sigma) = (R/I)\overline{F}$, where $\overline{F} := \overline{f}_{n-d+1} - \sum_{i=1}^{n-d} G_i\overline{f}_i \in I/I^2$ (with $F \in I$).*

Clearly, $\overline{F} \in \mathrm{Ker}(\sigma)$. Conversely, let $\overline{H} \in \mathrm{Ker}(\sigma)$, with $\overline{H} = \sum_{i=1}^{n-d} F_i\overline{f}_i$, with $F_i \in R/I$. Then

$$0 = \sigma(\overline{H}) = \sum_{i=1}^{n-d} F_i\sigma(\overline{f}_i) = \sum_{i=1}^{n-d} F_i\tilde{f}_i + F_{n-d+1}\sum_{i=1}^{n-d} G_i\tilde{f}_i$$
$$= \sum_{i=1}^{n-d}(F_i + F_{n-d+1}G_i)\tilde{f}_i.$$

Since $\tilde{f}_1, \ldots, \tilde{f}_{n-d}$ is a basis of the R/I-module J/IJ we get $F_i + F_{n-d+1}G_i = 0$, $\forall i = 1, \ldots, n-d$, and consequently

$$\overline{H} = \sum_{i=1}^{n-d+1} F_i\overline{f}_i = F_{n-d+1}(\overline{f}_{n-d+1} - \sum_{i=1}^{n-d} G_i\overline{f}_i) = F_{n-d+1}\overline{F},$$

and claim 9.1 is proved.

CLAIM 9.2 *$I' = I^2 + RF$.*

This is a direct consequence of claim 9.1 taking into account that $I' = \mathrm{Ker}(\gamma)$.

CLAIM 9.3 *$I^2 + RF = J^2 + RF$.*

Since $I^2 = J^2 + Jf_{n-d+1} + Rf_{n-d+1}^2$ the claim is equivalent to

$$f_i f_{n-d+1} \in J^2 + RF, \quad \forall i = 1, \ldots, n-d+1. \tag{9.4}$$

Taking into account of the formula defining $\overline{F}$ we have $F = f_{n-d+1} - \sum_{i=1}^{n-d} g_i f_i$, where $g_i \in R$ such that $G_i = g_i \bmod I$. Then

$$f_j F = f_j(f_{n-d+1} - \sum_{i=1}^{n-d} g_i f_i) = f_j f_{n-d+1} - \sum_{i=1}^{n-d} g_i f_i f_j, \tag{9.5}$$

for all $j = 1, \ldots, n-d+1$. In particular,

$$f_j f_{n-d+1} \in J^2 + RF, \ \forall j = 1, \ldots, n-d. \tag{9.6}$$

Taking $j = n - d + 1$ in (9.5) we get

$$f_{n-d+1}F = f_{n-d+1}^2 - \sum_{i=1}^{n-d} g_i f_{n-d+1} f_i,$$

whence, using also (9.6) we get $f_{n-d+1}^2 \in J^2 + RF$. Claim 9.3 is proved.

By the above three claims it follows that $I' = J^2 + RF = I^2 + RF$, where $I' = \mathcal{I}'_x$. In particular, the subscheme Y' of $X(1)$ is locally given by one equation. It remains to prove that this equation is a non-zero divisor. Again the verification is local, so that we have to prove that F is not a zero divisor in R/J^2. This can be done in the following way. Since $Y' \cap X = Y$ it follows that $F \mathrm{mod} J$ is a local equation of Y in X, and in particular, $F \mathrm{mod} J$ is not a zero divisor in R/J. If $F \mathrm{mod} J^2$ would be a zero divisor in R/J^2, then $F \mathrm{mod} J^2 \in J/J^2$ (R/J is a domain) which is not possible because we just remarked that $F \mathrm{mod} J$ was not a zero divisor in R/J.

Conversely, assume now the existence of the effective Cartier divisor Y' on $X(1)$ such that $Y' \cap X = Y$ (scheme-theoretically). Then we have to find a splitting of (9.3). This means that we have an ideal $\mathcal{I}'$ containing $\mathcal{J}^2$ such that $\mathcal{I}' + \mathcal{J} = \mathcal{I}$, and in particular, $\mathcal{J}^2 \subseteq \mathcal{I}' \subseteq \mathcal{I}$.

CLAIM 9.4 $\mathcal{I}^2 \subseteq \mathcal{I}'$.

Again the verification is local. In the above notations, $I' = J^2 + RF$, with F a non-zero divisor modulo J^2. The equality $I' + J = I$ implies $J + RF = I$. Therefore we may assume that $F \equiv f_{n-d+1} \bmod J$, i.e.

$$F = \sum_{i=1}^{n-d} g_i f_i + f_{n-d+1}. \tag{9.7}$$

Then proving claim 9.4 is equivalent to checking that

$$f_j f_{n-d+1} \in J^2 + RF, \ \forall j = 1, \ldots, n - d + 1. \tag{9.8}$$

Multiplying (9.7) by f_j we get (9.8).

CLAIM 9.5 *The exact sequence* (9.3) *splits.*

Indeed, using claim 9.4 we may consider the map

$$\mu : \mathcal{I}'/\mathcal{I}^2 \oplus \mathcal{J}/\mathcal{I}\mathcal{J} \to \mathcal{I}/\mathcal{I}^2$$

defined by $\mu(f \bmod \mathcal{I}^2, g \bmod \mathcal{I}\mathcal{J}) = f + g \bmod \mathcal{I}^2$. We claim that μ is an isomorphism. The surjectivity of μ comes from $\mathcal{I}' + \mathcal{J} = \mathcal{I}$. The verification of the injectivity of μ is local (along the same lines as above) and is left to the reader. The splitting $\sigma : \mathcal{I}/\mathcal{I}^2 \to \mathcal{J}/\mathcal{I}\mathcal{J}$ of (9.3) is then given by the second projection of the direct sum composed with μ^{-1}. □

If Y is a projective variety we shall denote by $\mathrm{Pic}^0(Y)$ (resp. by $\mathrm{Pic}^\tau(Y)$) the subgroup of $\mathrm{Pic}(Y)$ consisting of all isomorphism classes of line bundles which are algebraically (resp. numerically) equivalent to zero. Clearly, $\mathrm{Pic}^0(Y) \subseteq \mathrm{Pic}^\tau(Y)$, and a theorem of Matsusaka asserts that $\mathrm{Pic}^\tau(Y)/\mathrm{Pic}^0(Y)$ is a finite group. The Néron-Severi group of Y is by definition $\mathrm{NS}(Y) := \mathrm{Pic}(Y)/\mathrm{Pic}^0(Y)$. By a result of Néron-Severi, $\mathrm{NS}(Y)$ is a finitely generated abelian group. Moreover, we also set $\mathrm{Num}(Y) := \mathrm{Pic}(Y)/\mathrm{Pic}^\tau(Y)$. By the definition of $\mathrm{Num}(Y)$ it follows that $\mathrm{Num}(Y)$ is torsion free, whence $\mathrm{Num}(Y)$ is a free abelian group of finite rank since $\mathrm{NS}(Y)$ is a finitely generated abelian group by Néron-Severi's result.

LEMMA 9.2 *Let X be a closed smooth irreducible subvariety of $\mathbb{P}^n$ over $\mathbb{C}$ of dimension ≥ 2. Let $X(1)$ be the first infinitesimal neighborhood of X in $\mathbb{P}^n$. Then the image of the composition of natural maps*

$$\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X) \to \mathrm{Num}(X)$$

is isomorphic to $\mathbb{Z}$.

PROOF. Consider the (logarithmic derivative) map $\mathrm{dlog} : \mathrm{Pic}(X) \to H^1(X, \Omega^1_X)$ defined in the following way. If $[L] \in \mathrm{Pic}(X)$ is represented by the 1-cocycle $\{\xi_{ij}\}_{i,j}$ of $\mathcal{O}^*_X$ with respect to the affine cover $\{U_i\}$ of X (with $\xi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}^*_X)$), then $\mathrm{dlog}(\{\xi_{ij}\})$ is by definition the cohomology class of the 1-cocycle $\{\frac{d\xi_{ij}}{\xi_{ij}}\}_{i,j}$ of Ω^1_X. Since $\mathrm{dlog}(\mathrm{Pic}^0(X)) = 0$ the map dlog factors to

$$\mathrm{dlog} : \mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X) \to H^1(X, \Omega^1_X).$$

Moreover, since $\mathrm{Pic}^{\tau}(X)/\mathrm{Pic}^{0}(X)$ is a finite subgroup of $\mathrm{NS}(X)$ and the underlying abelian group of the $\mathbb{C}$-vector space $H^1(X, \Omega_X^1)$ is torsion free, we infer that $\mathrm{dlog}(\mathrm{Pic}^{\tau}(X)) = 0$, i.e. there is a unique map $\alpha : \mathrm{Num}(X) \to H^1(X, \Omega_X^1)$ such that $\mathrm{dlog} = \alpha \circ \beta$, where $\beta : \mathrm{Pic}(X) \to \mathrm{Num}(X)$ is the canonical surjection. Then it is a general fact (see [24, page 163]) that α induces an injective map $\alpha' := \alpha_{\mathbb{C}} : \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{C} \to H^1(X, \Omega_X^1)$. Using this, to prove the lemma it will be sufficient to show that the image of $\mathrm{Pic}(X(1))$ in $H^1(X, \Omega_X^1)$ (via the map α composed with $\mathrm{Pic}(X(1)) \to \mathrm{Num}(X)$) is contained in an one-dimensional complex vector subspace of $H^1(X, \Omega_X^1)$.

To do this we need the following three facts:

1. The canonical surjective map $\Omega_{\mathbb{P}^n}^1 | X(1) \to \Omega_{X(1)}^1$ yields by restriction to X an isomorphism $\Omega_{\mathbb{P}^n}^1 | X \cong \Omega_{X(1)}^1 | X$.
2. There is a natural map $H^1(X, \Omega_{\mathbb{P}^n}^1 | X) \to H^1(X, \Omega_X^1)$.
3. The $\mathbb{C}$-vector space $H^1(X, \Omega_{\mathbb{P}^n}^1 | X)$ is one dimensional.

2 is obvious because the map in question is induced by the canonical (surjective) map $\Omega_{\mathbb{P}^n}^1 | X \to \Omega_X^1$, while 3 is just step 2 of the proof of theorem 7.1. To prove 1 consider the canonical exact sequence

$$\mathcal{I}_X^2 / \mathcal{I}_X^4 \to \Omega_{\mathbb{P}^n}^1 | X(1) \to \Omega_{X(1)}^1 \to 0,$$

and observe that after restricted to X the first map becomes zero.

Using 1–3, the fact that the image of $\mathrm{Pic}(X(1))$ in $H^1(X, \Omega_X^1)$ is contained in an one-dimensional complex vector subspace follows from the injectivity of $\alpha_{\mathbb{C}}$ and the following commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(X(1)) \arrow[r] \arrow[d, "\mathrm{dlog}"'] & \mathrm{Pic}(X) \arrow[d] \\
H^1(X(1), \Omega_{X(1)}^1) \arrow[d] & \mathrm{Num}(X) \arrow[d, "\mathrm{incl}"] \\
H^1(X, \Omega_{X(1)}^1 | X) & \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d, "\alpha'"] \\
H^1(X, \Omega_{\mathbb{P}^n}^1 | X) \arrow[r] \arrow[u, "\mathrm{isom}"] & H^1(X, \Omega_X^1)
\end{tikzcd}$$

Lemma 9.2 is proved. $\square$

THEOREM 9.1 (ELLINGSRUD-GRUSON-PESKINE-STROMME [17], [13]) *Let X be a smooth projective complex surface embedded in $\mathbb{P}^n$ ($n \geq 3$) as a complete intersection. Let Y be a smooth connected curve in X such that the exact sequence of normal bundles*

$$0 \to N_{Y|X} \to N_{Y|\mathbb{P}^n} \to N_{X|\mathbb{P}^n}|Y \to 0$$

splits. Then there exists a hypersurface H of $\mathbb{P}^n$ such that $Y = X \cap H$ (scheme-theoretically).

PROOF. Since X is a complete intersection in $\mathbb{P}^n$ the Lefschetz theorem on hyperplane sections (see e.g. [27], or [29], or also appendix A below) implies that the restriction map $\mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X)$ is injective, $\mathrm{Pic}^{\tau}(X) = 0$, and the class of $\mathcal{O}_X(1)$ is not divisible in $\mathrm{Pic}(X)$. In particular, the canonical map $\mathrm{Pic}(X) \to \mathrm{Num}(X)$ is an isomorphism.

By lemma 9.1 the splitting of the above sequence implies that there is an effective Cartier divisor Y' on $X(1)$ such that $Y' \cap X = Y$ (scheme-theoretically). In particular, the class of $\mathcal{O}_X(Y)$ is in the image of $\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X) \cong \mathrm{Num}(X)$, which by lemma 9.2 is isomorphic to $\mathbb{Z}$. It follows that there are two non-zero integers s and t such that $\mathcal{O}_X(sY) \cong \mathcal{O}_X(t)$. Since $\mathcal{O}_X(1)$ is not divisible in $\mathrm{Pic}(X)$ this implies that $\mathcal{O}_X(Y) \cong \mathcal{O}_X(d)$ for some $d > 0$.

On the other hand, X being a complete intersection in $\mathbb{P}^n$, X is projectively normal in $\mathbb{P}^n$. This implies that the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(X, \mathcal{O}_X(d)) = H^0(X, \mathcal{O}_X(Y))$ is surjective. In particular, there exists a section $a \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ such that $a|Y \in H^0(X, \mathcal{O}_X(Y))$ is a global equation of Y. In other words, there is a hypersurface H of degree d such that $X \cap H = Y$ (scheme-theoretically).

$\square$

Another application of lemma 9.2 is a weak form of a theorem of Barth concerning the Picard group of small-codimensional smooth subvarieties of the complex projective space. In this application we shall use the following generalization of the Kodaira vanishing theorem due to Le Potier ([34], or also [47]):

THEOREM 9.2 (LE POTIER VANISHING THEOREM) *Let X be a smooth complex projective variety of dimension $d \geq 2$, and let E be an ample vector bundle of rank r on X. Then $H^i(X, E^*) = 0$ for every $i \leq d - r$.*

THEOREM 9.3 (BARTH) *Let X be a closed smooth subvariety of the complex projective space $\mathbb{P}^n$ of dimension $\geq \frac{n+2}{2}$ ($n \geq 4$). Then $\mathrm{Pic}(X) \cong \mathbb{Z}$.*

REMARK 9.2 The Barth theorem on the Picard group also states that (under the hypotheses of theorem 9.3) $\mathcal{O}_X(1)$ generates $\mathrm{Pic}(X)$.

PROOF OF THEOREM 9.3. Let $\mathcal{I}$ be the ideal sheaf of X in $\mathcal{O}_{\mathbb{P}^n}$. Then the truncated exponential sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 = N^*_{X|\mathbb{P}^n} \to \mathcal{O}^*_{X(1)} \to \mathcal{O}^*_X \to 0$$

yields the cohomology sequence

$$H^1(X, N^*_{X|\mathbb{P}^n}) \to \mathrm{Pic}(X(1)) \to \mathrm{Pic}(X) \to H^2(X, N^*_{X|\mathbb{P}^n}).$$

Since the normal bundle $N_{X|\mathbb{P}^n}$ of X in $\mathbb{P}^n$ is a quotient of $T_{\mathbb{P}^n}|X$, $N_{X|\mathbb{P}^n}$ is ample of rank $= \mathrm{codim}_{\mathbb{P}^n}(X)$. The hypothesis $\dim(X) \geq \frac{n+2}{2}$ is equivalent to $\dim(X) - \mathrm{codim}_{\mathbb{P}^n}(X) \geq 2$, whence by theorem 9.2 the first and the last cohomology groups are zero. It follows that the restriction map $\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X)$ is an isomorphism.

The hypothesis $\dim(X) \geq \frac{n+2}{2}$ also implies $\dim(X) > \frac{n}{2}$, whence by corollary 8.5, X is algebraically simply connected. In particular, $q = \dim(H^1(X, \mathcal{O}_X)) = 0$. Indeed, otherwise $\mathrm{Pic}^0(X) \neq 0$, and since $\mathrm{Pic}^0(X)$ is the underlying group of a complex torus, there exists a non-trivial line bundle L of finite order $m \geq 2$. Then L produces a connected non-trivial étale cover $X' \to X$, with $X' = \mathrm{Spec}(\bigoplus_{i=0}^{m-1} L^i)$, contradicting the fact that X is algebraically simply connected.

Since $\mathrm{Pic}^0(X) = 0$, by Matsusaka's theorem $\mathrm{Pic}^\tau(X)$ is a finite subgroup of $\mathrm{Pic}(X)$. Again if there is a non-trivial $L \in \mathrm{Pic}^\tau(X)$, one gets a connected non-trivial étale cover of X as above. Therefore $\mathrm{Pic}^\tau(X) = 0$, i.e. $\mathrm{Pic}(X) \cong \mathrm{Num}(X)$.

At this point, using the bijectivity of the restriction maps $\mathrm{Pic}(X(1)) \to \mathrm{Pic}(X)$ and $\mathrm{Pic}(X) \to \mathrm{Num}(X)$, we can conclude by applying lemma 9.2. □

REMARK 9.3 Barth's theorem asserts that (under the hypotheses of theorem 9.3) $\mathrm{Pic}(X)$ is the (infinite) cyclic group generated by the class of $\mathcal{O}_X(1)$. It would be also interesting to prove geometrically that the class of $\mathcal{O}_X(1)$ generates $\mathrm{Pic}(X)$.

9.1 Appendix A

In this appendix we shall show how the following Lefschetz theorem for the Picard group (which was used in the proof of theorem 9.1):

THEOREM 9.4 *Let X be a smooth projective complex surface embedded in $\mathbb{P}^n$ ($n \geq 3$) as a complete intersection. Then the natural restriction map $a : \mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X)$ is injective and* $\mathrm{Coker}(a)$ *is torsion free. In other words, a is injective and the class of $\mathcal{O}_X(1)$ is not divisible in* $\mathrm{Pic}(X)$.

can be deduced from the following special case of the topological Lefschetz theorem for hyperplane sections (see e.g. Milnor [36]):

THEOREM 9.5 *Let X be a smooth projective complex surface embedded in $\mathbb{P}^n$ ($n \geq 3$) as a complete intersection. Then the natural maps of singular integral cohomology $H^i(\mathbb{P}^n, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ are isomorphisms for $i < 2$ and injective with torsion free cokernel for $i = 2$.*

PROOF OF THEOREM 9.4. For a complex algebraic variety V denote by $\mathcal{O}_V^{\mathrm{an}}$ (resp. by $(\mathcal{O}_V^{\mathrm{an}})^*$, resp. by $\mathbb{Z}_V$) the sheaf of holomorphic functions on V (resp. the sheaf of nowhere vanishing holomorphic functions on V, resp. the constant sheaf $\mathbb{Z}$ on V). Then the commutative diagram of exponential sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_{\mathbb{P}^n} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{\mathrm{an}} & \xrightarrow{\exp} & (\mathcal{O}_{\mathbb{P}^n}^{\mathrm{an}})^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \mathcal{O}_X^{\mathrm{an}} & \xrightarrow{\exp} & (\mathcal{O}_X^{\mathrm{an}})^* & \longrightarrow & 0
\end{array}$$

yields the following commutative diagram with exact cohomology sequences

$$\begin{array}{ccccccc}
H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{\mathrm{an}}) & \longrightarrow & H^1(\mathbb{P}^n, (\mathcal{O}_{\mathbb{P}^n}^{\mathrm{an}})^*) & \longrightarrow & H^2(\mathbb{P}^n, \mathbb{Z}) & \longrightarrow & H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{\mathrm{an}}) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(X, \mathcal{O}_X^{\mathrm{an}}) & \longrightarrow & H^1(X, (\mathcal{O}_X^{\mathrm{an}})^*) & \longrightarrow & H^2(X, \mathbb{Z}) & &
\end{array}$$

By the GAGA results of Serre [49], $H^i(V, \mathcal{O}_V) \cong H^i(V, \mathcal{O}_V^{\mathrm{an}})$ for all $i \geq 0$ and $\mathrm{Pic}(V) = H^1(V, \mathcal{O}_V^*) \cong H^1(V, (\mathcal{O}_V^{\mathrm{an}})^*)$ for every complex projective variety V. Moreover, $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $i = 1, 2$, and $H^1(X, \mathcal{O}_X) = 0$ because X is a complete intersection surface in $\mathbb{P}^n$. Therefore the last commutative diagram becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(\mathbb{P}^n) & \longrightarrow & H^2(\mathbb{P}^n, \mathbb{Z}) & \longrightarrow & 0 \\ & & \alpha \downarrow & & \downarrow \beta & & \\ 0 & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}) & & \end{array}$$

By theorem 9.5, the map β is injective and $\mathrm{Coker}(\beta)$ has no torsion. This fact together with the last commutative diagram with exact rows imply that α is injective and $\mathrm{Coker}(\alpha)$ has no torsion.

9.2 Appendix B

In this short appendix we recall briefly some basic facts about cyclic covers. Let X be an irreducible projective variety of dimension ≥ 1, and let $L \in \mathrm{Pic}(X)$ be a line bundle of finite order $n \geq 2$. In particular, there is an isomorphism

$$L^n \cong \mathcal{O}_X. \tag{9.9}$$

We shall assume that $\mathrm{char}(k)$ does not divide n. Using this isomorphism, we can endow the $\mathcal{O}_X$-module $\mathcal{A} := \oplus_{i=0}^{n-1} L^i$ with a structure of commutative $\mathcal{O}_X$-algebra in the following way. For any two local sections s of L^i and t of L^j we define the product st as follows:

- $st := s \otimes t$ which is a local section of L^{i+j}, if $i + j \leq n - 1$, and
- st is the image of $s \otimes t$ (which is a section of L^{i+j}) under the isomorphism $L^{i+j} \to L^{i+j-n}$ deduced from (9.9), if $i + j \geq n$.

Then taking $X' := \underline{\mathrm{Spec}}(\mathcal{A})$ and $f : X' \to X$ the structural morphism of $\underline{\mathrm{Spec}}(\mathcal{A})$, we get an irreducible projective variety X' together with a canonical finite étale morphism $f : X' \to X$ of degree n (here the fact that the characteristic of k does not divide the order n of L is essential). By construction, $f^*(L) \cong \mathcal{O}_{X'}$.

Then the morphism f is called *the cyclic étale cover* of X associated to the line bundle $L \in \operatorname{Pic}(X)$ of order n.

More explicitly, X' is obtained as follows. Let $(\xi_{ij})_{i,j} \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$ be a 1-cocycle of $\mathcal{O}_X^*$ with respect to a finite affine cover $\mathcal{U} = (U_i)_i$ of X which represents $L \in \operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. Since $L^n \cong \mathcal{O}_X$ the 1-cocycle $(\xi_{ij}^n)_{i,j}$ is a 1-boundary, i.e. we can write

$$\xi_{ij}^n = g_i/g_j, \text{ on } U_i \cap U_j, \text{ with } g_i \in \Gamma(U_i, \mathcal{O}_X^*).$$

Then define $f : X' \to X$ locally on U_i by taking f_i the restriction to the open subset

$$X_i := \{(x, z_i) \in U_i \times \mathbb{A}^1 \mid z_i(x)^n = g_i(x)\}$$

of the second projection of $U_i \times \mathbb{A}^1$. Then the morphisms f_i patch together to yield the étale morphism f with the above properties.

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