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GROUPS SATISFYING CONDITIONS ON 2-GENERATOR SUBGROUPS. *

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1 Introduction

Our aim is to illustrate well-known and very recent results which show how the structure of a group can be influenced by properties satisfied by all or "many " of its 2-generator subgroups.

A class C of groups is called S-closed if from H subgroup of a group G in C it follows that also H is in C.

Examples of S-closed classes are: the class $\mathcal{F}$ of all finite groups, the class $\mathcal{A}$ of all abelian groups, the class $\check{C}$ of all cyclic groups, the class $S=\mathcal{PA}$ of all soluble groups.

The class $\mathcal{J}$ of all finitely generated groups is not S-closed (see for instance [21, Part 1, p.30]).

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Other interesting classes we will consider later on are: the class $\mathcal{N}_k$ (k a positive integer) of all nilpotent groups of class at most k, i.e. the groups G satisfying $[x_1,\dots,x_{k+1}] = 1$ for any $x_1,\dots,x_{k+1} \in G$; the class $\mathcal{N} = \bigcup_{k\geq 1} \mathcal{N}_k$ of all nilpotent groups; the class $\mathcal{E}_k$ (k a positive integer) of all k-Engel groups, i.e. the groups G satisfying $[x,_k y] = 1$ for any $x, y \in G$.

If C is any S-closed class of groups, we call $C(2)$ the class of all groups G such that every 2-generator subgroup H of G is in C; this means that $\langle x,y\rangle \in C$ for any $x,y \in G$.

Obviously C is contained in $C(2)$, since C is supposed S-closed. So a natural question is: is the class $C(2)$ contained in C, that is, are the classes C and $C(2)$ equal?

This means: is it true that a group G is in C whenever every 2-generator subgroup of G is in C ?

The answer depends on the class. Obviously the equality holds for the class $\mathcal{A}$, the class $\mathcal{E}_k$ ($k \geq 1$), more generally for every variety C of groups defined by a word in two letters. But the answer can be negative: for instance we have $\bar{C} \neq \bar{C}(2)$, since a locally cyclic group need not to be cyclic; again, $\mathcal{N}_2(2) \neq \mathcal{N}_2$ since there exist 2-Engel groups which are nilpotent of class 3.

Also for the class $\mathcal{N}$ we have $\mathcal{N}(2) \neq \mathcal{N}$: for M.F. Newmann (see [20]) constructed a 3-generator infinite p-group G such that $\langle x,y\rangle$ is (finite and) nilpotent for any $x,y \in G$. Moreover M.R. Vaughan-Lee and J.Wiegold (see [23]) exhibited an infinite perfect locally finite group G of exponent $p > 5$ such that the class of all $\langle x,y\rangle$, $x,y \in G$, is bounded by some fixed integer t.

Now consider the class $\bar{C}\mathcal{A}$ of all groups with cyclic commutator subgroup; therefore a group G is in $\bar{C}\mathcal{A}$ if and only if G' is cyclic.

It is easy to notice that $(\bar{C}\mathcal{A})(2)$ is different from $\bar{C}\mathcal{A}$. For, consider the group $G = (\langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle) \rtimes \langle x\rangle$ where $a^2 = b^2 = c^2 = d^2 = x^2 = 1$, $a^x = b$, $b^x = a$, $c^x = d$, $d^x = c$.

The group G is not in $\bar{C}\mathcal{A}$ since $G' = \langle ab\rangle \times \langle cd\rangle$ is a four group, but H' has order at most 2, for any 2-generator subgroup H of G.

If C is any S-closed class such that $C(2) \neq C$, we could try to determine some interesting class $\mathcal{D}$ such that $C(2)$ is contained in $\mathcal{D}$, i.e. a class $\mathcal{D}$ such that G is in $\mathcal{D}$ whenever all 2-generator subgroups of G are in C. For instance, we have $\bar{C}(2) \subseteq \mathcal{A}$ since a locally cyclic group is abelian; also $\mathcal{N}_2(2)$ is contained in $\mathcal{N}_3$, since a 2-Engel group is nilpotent of class ≤ 3 (see for instance [21, Part 2, p.45]).

Another investigation one could carry out is the following: given any S-closed class C such that $C(2) \neq C$, to determine a sufficiently large class X such that $C(2) \cap X$ is contained in C or at least in some interesting class $\mathcal{D}$.

For instance R. Baer showed (see [11, p.722]) that a finite group whose 2-generator subgroups are supersoluble is itself supersoluble. Even more from a famous result due to J. Thompson (see [22, p.388]) it follows that a finite group is soluble whenever all its 2-generator subgroups are soluble.

In 1973 (see [14]) J.C. Lennox proved that soluble finitely generated groups in which every 2-generator subgroup is polycyclic are polycyclic. Key results for this are the following:

THEOREM 1.1 *Let G be a finitely generated soluble group and let A be an abelian normal subgroup of G such that G/A is polycyclic. Assume that $\langle a, x\rangle$ is polycyclic for any $a \in A$, $x \in G$. Then G is polycyclic.*

THEOREM 1.2 *Let G be a finitely generated group and let H be a normal subgroup of G such that G/H is cyclic. Assume that $\langle a\rangle^{\langle b\rangle}$ is finitely generated for any $a \in H$, $b \in G$. Then H is finitely generated.*

PROOF. There exists $g \in G$ such that $G = H\langle g\rangle$. From G finitely generated it follows that $G = \langle h_1, \dots, h_r, g\rangle$ and $H = \langle h_1, \dots, h_r\rangle^G$ for suitable $h_1, \dots, h_r \in H$.

For any $i = 1, \dots, r$, the group $\langle h_i\rangle^{\langle g\rangle}$ is finitely generated, then there exist $h_{i1}, \dots, h_{id(i)} \in H$ such that $\langle h_i\rangle^{\langle g\rangle} = \langle h_{i1}, \dots, h_{id(i)}\rangle$.

Therefore $H = \langle h_{is(i)} \mid 1 \leq i \leq r, 1 \leq s(i) \leq d(i)\rangle$ is finitely generated. □

More recently J.S. Wilson (see [24]) considered finitely generated residually finite groups. Among other results he proved:

THEOREM 1.3 *If G is a finitely generated residually finite group and if for some integer r each 2-generator subgroup of G has rank at most r, then G has finite rank.*

THEOREM 1.4 *If G is a finitely generated residually finite group and if there is an integer r such that every 2-generator subgroup of G has series of length at most r with cyclic factors, then G is polycyclic.*

THEOREM 1.5 *If G is a finitely generated residually finite group whose 2-generator subgroups are nilpotent of bounded class, then G is nilpotent.*

More generally:

THEOREM 1.6 *If G is a finitely generated residually finite group such that, for some integer k, $[x,_k y] = 1$ for all $x, y \in G$, then G is nilpotent.*

Therefore

THEOREM 1.7 *A finitely generated residually finite group in $\mathcal{E}_k$ is nilpotent.*

Notice that in 1.3 the uniform bound t is necessary: E.S. Golod (see [10]) showed that for each integer $d > 1$ and each prime p there exist infinite d-generator residually finite groups all of whose $(d-1)$-generator subgroups are finite p-groups. A similar remark holds for the other results.

In 1995 Y.K. Kim and A.H. Rhemtulla (see [12]) extended the result 1.7 to a very interesting class of groups, the class of locally graded groups.

Recall that a group G is said to be locally graded if every finitely generated non-trivial subgroup of G has a finite non-trivial quotient.

They showed that:

THEOREM 1.8 *A finitely generated locally graded k-Engel group is nilpotent.*

As we mentioned, J. Wilson proved that a finitely generated residually finite group G whose 2-generator subgroups are nilpotent of class at most k, for some fixed k, is nilpotent. So a natural problem is to investigate if there exists a function f of k such that the class of G

is bounded by $f(k)$, at least in the case G torsion-free; and possibly to determine this function.

Obviously groups in $\mathcal{N}_k(2)$ are k-Engel groups and one of the very strong results due to E.I. Zel'manov (see [25, 26, 27, 28]) ensures that for a torsion-free k-Engel group G there exists a function $f(k)$ such that G is nilpotent of class at most $f(k)$.

Therefore assume that G is a torsion-free nilpotent group in $\mathcal{N}_k(2)$ and consider the problem to determine an integer $n(k)$ such that G is in $\mathcal{N}_{n(k)}$.

Since torsion-free 2-Engel groups are in $\mathcal{N}_2$ and torsion-free 3-Engel groups belongs to $\mathcal{N}_4$, we have the following results:

THEOREM 1.9 *A torsion-free nilpotent group in $\mathcal{N}_2(2)$ is of class at most* 2.

THEOREM 1.10 *A torsion-free nilpotent group in $\mathcal{N}_3(2)$ is of class at most* 4.

Recently C. Delizia (see [6]) has shown that:

THEOREM 1.11 *A torsion-free nilpotent group in $\mathcal{N}_4(2)$ is of class at most* 9.

2 Groups with cyclic commutator subgroup.

Now come back to the class $\bar{C}\mathcal{A}$ of groups with cyclic commutator subgroup.

As already noticed, we have $(\bar{C}\mathcal{A})(2) \neq \bar{C}\mathcal{A}$, that means that a group G in $(\bar{C}\mathcal{A})(2)$ need not to have cyclic commutator subgroup. Therefore one could ask if G' is at least abelian.

In 1963 J.L. Alperin (see [2]) answered positively the question assuming that the group is either finite nilpotent of odd order or torsion-free nilpotent. Moreover Alperin proved that a finite group in $(\bar{C}\mathcal{A})(2)$ is soluble, so supersoluble by Baer's result.

W. Dirscherl and H. Heineken continued in 1994 the study started by Alperin; they considered again finite groups in $(\bar{C}\mathcal{A})(2)$ and proved (see [8]) that for such a group G the quotient $G/\zeta_\infty(G)$ is metabelian and W is normal in G for any subgroup W of G satisfying $\zeta_\infty(G) \leq W \leq G'\zeta_\infty(G)$ (where $\zeta_\infty(G)$ is the hypercentre of G).

Very recently it has been shown (see [17]) that :

THEOREM 2.1 *A finite group of odd order in* $(\bar{C}\mathcal{A})(2)$ *is metabelian.*

THEOREM 2.2 *A torsion-free group in* $(\bar{C}\mathcal{A})(2)$ *need not to be metabelian.*

THEOREM 2.3 *A torsion-free group in* $(\bar{C}\mathcal{A})(2)$ *has commutator subgroup nilpotent of class at most* 2.

Therefore 2.1 shows that the hypothesis "nilpotent" in the first of Alperin's results can be removed, but the same is not true for the second result, as 2.2 points out.

To show 2.1, we need the following lemma.

LEMMA 2.1 *Let G be a finite group of odd order in* $(\bar{C}\mathcal{A})(2)$.

1. $[a,b]^a$, $[a,b]^b$ *are in* $\langle[a,b]\rangle$, *for all* $a, b \in G$.
2. *If* $G = P \rtimes \langle x\rangle$, *where P is a p-group, x a p'-element, then* $[P, \langle x\rangle]$ *is abelian.*
3. *If* $G = P \rtimes \langle x\rangle$, *where P is a p-group, x a p'-element, then G is metabelian.*

PROOF OF ITEM 2. It suffices to show that $[[g,x],[y,x]] = 1$ for all $g, y \in P$. Since G is supersoluble, there is a normal subgroup N of G of order p. By induction on $|G|$ we may assume that $([P,\langle x\rangle]N)/N$ is abelian. Assume by contradiction that there are $a, b \in P$ such that $[[a,x],[b,x]] \neq 1$. We get $[a,x]^x = [a,x]^r$, $[b,x]^x = [b,x]^s$ where $r \equiv s \pmod p$ and $r \not\equiv 1 \pmod p$. Moreover $[[a,x],[b,x]] \in N$.

Now, if $N \subseteq C_G(x)$, then $1 \neq [[a,x],[b,x]] = [[a,x],[b,x]]^x = [[a,x],[b,x]]^{rs}$, from which $r^2 \equiv 1 \pmod p$, a contradiction.

If $N \not\subseteq C_G(x)$, then $1 \neq [[a,x],[b,x]]^t = [[a,x],[b,x]]^x$ from which $r^2 \equiv t \equiv r \pmod p$, the final contradiction. □

SKETCH OF THE PROOF OF THEOREM 2.1. Let G be a minimal counterexample. Then G' is a non-abelian nilpotent group, so G' is a p-group for some p.

Therefore we have $G = P \rtimes X$ where $P \geq G'$ and X is an abelian p'-group.

Using Alperin's result we get $G' = P'[P,X]$ where P' is abelian and $[P,X]$ is non-abelian. So we may assume that $G = P\langle x,y\rangle$ for some $x, y \in G$ and get the final contradiction arguing in these hypothesis.

□

To show 2.2, we can consider the following group:
$G = ((\langle a\rangle \times \langle w_1\rangle \times \langle w_2\rangle) \rtimes \langle x,y\rangle$ where $a^x = a^{-1}w_1$ $a^y = a^{-1}w_2$, $1 = [w_1,x] = [w_1,y] = [w_2,x] = [w_2,y]$, $[x,y]^x = [y,x] = [x,y]^y$.

Then G' is non-abelian, since $G' \geq \langle a^{-2}w_1, [x,y]\rangle$, but G is in $(\check{C}\mathcal{A})(2)$ since $\langle [g_1,g_2]\rangle \lhd \langle g_1,g_2\rangle$ for all $g_1, g_2 \in G$.

Finally we consider torsion-free groups G in $(\check{C}\mathcal{A})(2)$.

First of all we show that if $g \in G$, then either $g^2 \in Z(G)$ or $[x,g,g] = 1$ for all $x \in G$.

For, let $x \in G$. From $\langle [x,g]\rangle \lhd \langle x,g\rangle$, it follows that either $[x,g,g] = 1$ or $[x,g]^g = [x,g]^{-1}$, so either $[x,g,g] = 1$ or $[x,g^2] = 1$. Thus we get $G = C_G(g^2) \cup \{x \in G \mid [x,g,g] = 1\}$.

Now it is not difficult to show that $\{x \in G \mid [x,g,g] = 1\}$ is a subgroup of G, and then the result will follow.

Sketch of the Proof of Theorem 2.3. Let G be a torsion-free group in $(\check{C}\mathcal{A})(2)$. We want to show that G' is nilpotent of class at most 2.

We know that if $g \in G$, then either $g^2 \in Z(G)$ or $[x,g,g] = 1$ for all $x \in G$.

If $[x,g,g] \in Z(G)$ for all $x, g \in G$, then $G/Z(G)$ is 2-Engel and we are done. So assume that there is $g \in G\backslash Z(G)$ such that $g^2 \in Z(G)$. We are able to show that:

- $\langle aZ(G)\rangle \lhd G/Z(G)$ for any $a \in G$ such that $a^2 \notin Z(G)$.
- $G'Z(G)/Z(G) \leq C_{G/Z(G)}(N/Z(G))$, where $N = \langle a \in G \mid a^2 \notin Z(G)\rangle$.

Therefore $(G/Z(G))/(N/Z(G))$ is abelian, so G' is contained in N, which yields that $G'Z(G)/Z(G)$ is abelian, and the result follows. □

3 A condition on infinite subsets.

Now some different problems.

Given any S-closed class of groups C, we can define the class $C(2)^*$ as follows: a group G is in $C(2)^*$ if any infinite subset X of G contains different elements x and y such that $\langle x, y\rangle \in C$.

Obviously $C(2)$ is contained in $C(2)^*$, moreover all finite groups are in $C(2)^*$.

As already noticed, the class $\mathcal{A}(2)$ coincides with $\mathcal{A}$. In 1976 B.H. Neumann (see [19]) considered the class $\mathcal{A}^*$, that is the class of all groups G in which every infinite subset contains two elements which commute. He proved that:

THEOREM 3.1 *A group G is in $\mathcal{A}^*$ if and only if its centre has finite index.*

The same result was obtained by Faber, Laver and McKenzie independently (see [9]).

In 1981 J.C. Lennox and J. Wiegold (see [15]) draw their attention on the class $P\bar{C}$ of all polycyclic groups. They showed that

THEOREM 3.2 *A finitely generated soluble group in $(P\bar{C})(2)^*$ is itself polycyclic.*

So the equality $(P\bar{C})(2) \cap S \cap \mathcal{J} = P\bar{C}$ proved by Lennox holds even if $(P\bar{C})(2)$ is substituted by $(P\bar{C})(2)^*$.

Lennox and Wiegold showed also that for a finitely generated soluble group G the following characterization holds: $G \in \mathcal{N}(2)^*$ if and only if G is finite-by-nilpotent.

In some papers (see [3, 4, 5]) C. Delizia studied groups in $\mathcal{N}_k(2)^*$. He proved the following results:

THEOREM 3.3 *Let G be a finitely generated soluble group. Then G is in $\mathcal{N}_2(2)^*$ if and only if the second centre $Z_2(G)$ has finite index.*

THEOREM 3.4 *Let G be a finitely generated residually finite group. Then G is in $\mathcal{N}_2(2)^*$ if and only if the second centre $Z_2(G)$ has finite index.*

THEOREM 3.5 *Let G be a finitely generated residually finite group. If G is in $\mathcal{N}_3(2)^*$, then the third centre $Z_3(G)$ need not to have finite index in G.*

THEOREM 3.6 *Let G be a finitely generated soluble group. If G is in $\mathcal{N}_k(2)^*$, then $G/Z_l(G)$ is finite for some l depending on k and on the derived length of G.*

Finitely generated soluble groups in $\mathcal{N}_k(2)^*$ have been studied also by A. Abdollahi and B. Taeri (see [1]). They showed that:

THEOREM 3.7 *Let G be a finitely generated soluble group. Then G is in $\mathcal{N}_k(2)^*$ if and only if G has a normal finite subgroup N such that G/N belongs to $\mathcal{N}_k(2)$.*

More recently C. Delizia, A.H. Rhemtulla and H. Smith (see [7]) have proved that

THEOREM 3.8 *Let G be a finitely generated locally graded group. If G belongs to $\mathcal{N}_k(2)^*$, then $G/Z_{f(k)}(G)$ is finite for some $f(k)$ depending only on k.*

Finally P. Longobardi (see [16]) has shown that:

THEOREM 3.9 *Let G be a finitely generated locally graded group. If G is in $\mathcal{E}_k^*$, then G is finite-by -(k-Engel), and so G is (k-Engel)-by-finite.*

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