ROSA M. MIRÓ-ROIG

Dept. Algebra y Geometría
Facultad de Matemáticas
Universidad de Barcelona
08007 Barcelona, Spain
miro@mat.ub.es

GORENSTEIN LIAISON *

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ABSTRACT. The purpose of the talk is to review some of the recent results on Gorenstein liaison confronting them with classical results in complete intersection liaison theory.

1 Introduction

This expository paper is a slightly modified version of a Colloquium talk I gave at the "Seminario Matematico e Fisico di Milano" on November 15, 1999. The purpose of the talk was to review some of the recent results on Gorenstein liaison (simply, G-liaison) confronting them with classical results in complete intersection liaison theory (simply, CI-liaison).

The notion of using complete intersections to link varieties has been used for a long time ago, going back at least to work of Noether, Macaulay and Severi. The development in the last four decades has

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been explosive. Many people has contributed to it and in the codimension 2 case the picture is complete. It is impossible to make a complete survey in one hour. I will make no attempt to do so. Instead I will try to convince you that Gorenstein liaison is a more natural approach if we want to carry out a program in higher codimension and I refer to the monograph [5] for a more detailed treatment.

In 1948, Gaeta proved that there is only one CI-liaison class containing arithmetically Cohen-Macaulay (briefly, ACM) curves $C \subset \mathbb{P}^3$ or, equivalently, all ACM curves $C \subset \mathbb{P}^3$ are licci [3]. The first goal of this work is to see that in the CI-liaison context Gaeta's Theorem does not generalize well to higher codimension. More precisely, I will prove the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$. I will give two kind of examples: (1) I will see that many ACM curves on a Castelnuovo (resp. Bordiga) surface give rise to an infinite number of CI-liaison classes containing ACM curves by just adding different number of hyperplane sections (Example 3.1) and, (2) ACM curves $C_t \subset \mathbb{P}^4$ with a t-linear resolution belong to different CI-liaison classes provided $t \neq t'$ (Corollary 3.3). The second goal is to convince the reader that G-liaison is in many ways more natural than CI-liaison and among other results I will state that ACM curves $C \subset \mathbb{P}^4$ lying on a general smooth, rational, ACM surface are glicci, i.e., they belong to the G-liaison class of a complete intersection (Theorem 4.1). The last goal is to generalize Gaeta's Theorem and prove that standard determinantal schemes are glicci. Since in codimension 2, ACM schemes are standard determinantal and since in codimension 2, arithmetically Gorenstein schemes and complete intersection schemes coincide, this result is indeed a full generalization of Gaeta's Theorem.

Next we outline the structure of the paper. In section 2, we collect the main definitions of this paper. In section 3, we introduce some graded modules which are liaison invariants under CI-liaison but not under G-liaison (Theorem 3.1 and Theorem 3.2) and we will use them to prove the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$. In section 4, we determine huge families of ACM curves $C \subset \mathbb{P}^4$ which are glicci (Theorem 4.1 and Theorem 4.3) and; in section 5, we generalize Gaeta's Theorem

(Theorem 5.2).

In view of the already vast literature I have only included the references that are directly related to the topics discussed here. I apologize to the many whose beautiful and deep contributions could not even be mentioned without overly enlarging the perspective of this note.

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NOTATION. Throughout this paper we work over an algebraically closed field k of characteristic 0. By $\mathbb{P}^N$ we denote the N-dimensional projective space over k, by R the polynomial ring $k[X_0, \dots, X_N]$ and $m = (X_0, \dots, X_N)$. For any closed subscheme V of $\mathbb{P}^N$ we denote by I_V its ideal sheaf, $I(V)$ its saturated homogeneous ideal (note that $I(V) = H^0_*(I_V) := \bigoplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^n, I_V(t))$), $A(V) = R/I(V)$ the homogeneous coordinate ring, $N_V = \mathcal{H}om(I_V, \mathcal{O}_V)$ the normal sheaf of V and $M_i(V) = H^i_*(I_V) := \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, I_V(t))$, $i = 1, \dots, \dim(V)$, the i-th Rao module of V.

Let $X \subset \mathbb{P}^N$ be a locally Cohen-Macaulay and equidimensional scheme of codimension c. X is said to be arithmetically Cohen-Macaulay (briefly, ACM) if and only if $M_i(X) = 0$ for $1 \leq i \leq N - c$ or, equivalently, $A(X)$ is a Cohen-Macaulay ring. X is said to be arithmetically Gorenstein (briefly, AG) if and only if $I(X)$ has a resolution

$$0 \longrightarrow R(-t) \longrightarrow \oplus_{i=1}^{\alpha_{c-1}} R(-n_i^{c-1}) \longrightarrow \cdots \longrightarrow \oplus_{i=1}^{\alpha_1} R(-n_i^1) \longrightarrow I(X) \longrightarrow 0.$$

In particular, X is arithmetically Cohen-Macaulay. It is well known that in codimension two AG subschemes and complete intersection subschemes coincide. In higher codimension, any complete intersection subscheme is AG but not vice versa (indeed, a set of $n+2$ points in $\mathbb{P}^n$ in linear general position is AG but not complete intersection).

2 Background material

In this section, we collect the main definitions of this paper.

DEFINITION 2.1 *(See also* [5, Definitions 2.3, 2.4 and 2.10]). *Let* V_1 *and* $V_2 \subset \mathbb{P}^N$ *be two equidimensional schemes without embedded components. We say that* V_1 *and* V_2 *are* directly CI-linked *(resp.* directly G-linked*) if there exists a complete intersection scheme (resp. an AG scheme)* X *such that* $I_{V_1}/I_X \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_{V_2}, \mathcal{O}_X)$ *and* $I_{V_2}/I_X \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_{V_1}, \mathcal{O}_X)$. *If* V_1 *and* V_2 *do not share any common component then this is equivalent to* $X = V_1 \cup V_2$.

EXAMPLE 2.1 A simple example of schemes directly CI-linked is the following one: Let C_1 be a twisted cubic in $\mathbb{P}^3$ and let C_2 be a secant line to C_1. The union of C_1 and C_2 is a degree 4 curve which is the complete intersection X of two quadrics Q_1 and Q_2. So C_1 and C_2 are directly CI-linked by the complete intersection X.

As a simple example of schemes directly G-linked we have: We consider a set $Y_1 \subset \mathbb{P}^3$ of four points in linear general position and a sufficiently general point Y_2. Since $X = Y_1 \cup Y_2$ is an AG scheme, Y_1 and Y_2 are directly G-linked.

DEFINITION 2.2 *Let* V_1 *and* $V_2 \subset \mathbb{P}^N$ *be two equidimensional schemes without embedded components. We say that* V_1 *and* V_2 *are in the same* CI-liaison class *(resp.* G-liaison class*) if and only if there exists a sequence of schemes* $Y_1, \ldots, Y_r$ *such that* Y_i *is* directly CI-linked *(resp.* directly G-linked*) to* Y_{i+1} *and such that* $Y_1 = V_1$ *and* $Y_r = V_2$. *If* V_1 *is linked to* V_2 *in two steps by complete intersection (resp. AG) schemes we say that they are* CI-bilinked *(resp.* G-bilinked*).*

In other words *CI-liaison* (resp. *G-liaison*) is the equivalence relation generated by directly CI-linkage (resp. directly G-linkage) and roughly speaking liaison theory is the study of these equivalence relations and the corresponding equivalence classes.

DEFINITION 2.3 *A scheme* $X \subset \mathbb{P}^N$ *is said to be* licci *if it is in the CI-Liaison class of a complete intersection. Analogously, we say that a scheme* $X \subset \mathbb{P}^N$ *is* glicci *if it is in the Gorenstein Liaison class of a complete intersection.*

We are led to pose the following natural question:

Do CI-Liaison and G-Liaison generate the same equivalence relation on codimension c subschemes of $\mathbb{P}^n$?

In codimension two the answer is yes, since complete intersections and AG schemes coincide. In higher codimension the answer is no. Indeed, a simple counterexample is the following: Consider a set X of four points in $\mathbb{P}^3$ in linear general position. By Example 2.1 we can G-link X to a single point. Therefore, X is glicci. On the other hand, it follows from [4, Corollary 5.13] that X is not licci.

Although the goal of my talk was to show the merits of studying Gorenstein liaison, it is worth to mention some disadvantages: (1) It is easy to check that both CI-links and G-links are preserved under hyperplane sections. Nevertheless, CI-links lift and G-links do not lift, in general. (2) Given a scheme $V \subset \mathbb{P}^N$ it is, in general, very difficult to find "good" G-links, i.e., "good" Gorenstein ideals $I_X \subset I_V$ of the same high ("good" often means "small")

DEFINITION 2.4 *Let $X \subset \mathbb{P}^N$ be a locally Cohen-Macaulay equidimensional scheme. A graded R-module $C(X)$ which depends only on X is a CI-liaison (resp. G-liaison) invariant as an R-module (resp., k-module) if there exists a homogeneous R (resp. k)-module isomorphism $C(X) \cong C(X')$ for any X' in the CI-liaison (resp. G-liaison) class of X.*

It is well known that for equidimensional, locally Cohen-Macaulay schemes $X \subset \mathbb{P}^N$, the i-th module of Rao $M_i(X) := \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, I_V(t))$, $1 \leq i \leq \dim(X)$, are CI-liaison invariants (up to shifts and duals). Even more they are G-liaison invariants. In next section, we describe other CI-liaison invariants which allow us to distinguish between many CI-liaison classes which cannot be distinguished by Rao modules alone.

3 Liaison invariants and applications

Let $X \subset \mathbb{P}^{n+c}$ be a closed subscheme, locally CM, equidimensional of $\dim n > 0$*. If X is ACM all the CI-liaison invariants $M_i(X)$, $1 \leq$

*Throughout this paper we work with schemes of dimension $n > 0$. We want to point out that the results we give generalize to 0-dimensional schemes and we assume $n > 0$ for avoiding technical complications.

$i \leq \dim(X)$, vanish. Our first goal is to describe non-trivial CI-liaison invariants of ACM schemes. To this end, we consider a graded R-free resolution of $I = I(X)$:

$$\cdots \oplus_i R(-n_i^2) \longrightarrow \oplus_i R(-n_i^1) \longrightarrow I \longrightarrow 0. \tag{3.1}$$

We apply $\mathcal{H}om(-, \mathcal{O}_X)$ to the exact sequence (3.1) and we obtain

$$0 \longrightarrow N_X \longrightarrow \oplus_i \mathcal{O}_X(-n_i^1) \longrightarrow \oplus_i \mathcal{O}_X(-n_i^2).$$

We take cohomology ($H_*^n \mathcal{O}_X \cong H_m^{n+1}(R/I)$, A=R/I); and we get a natural map

$$\delta_X : H_*^n N_X \longrightarrow Hom_R(I, H_m^{n+1}(A)) \cong Hom_R(I, H_m^{n+2}(I)).$$

This map δ_X plays an important role; in particular, its kernel and cokernel are CI-liaison invariants (See Theorem 3.1).

REMARK 3.1 If I/I^2 is a free R/I-module, then δ_X is an isomorphism. Thus, if $X \subset \mathbb{P}^{n+c}$ is a global complete intersection, then δ_X is an isomorphism.

THEOREM 3.1 *Let $X, X' \subset \mathbb{P}^{n+c}$ be ACM subschemes of dimension $n > 0$ algebraically linked by a complete intersection $Y \subset \mathbb{P}^{n+c}$. Then:*

1. *As graded R-modules: $H_*^i N_X \cong H_*^i N_{X'}$ for $1 \leq i \leq n-1$, $ker(\delta_X) \cong ker(\delta_{X'})$*

2. *As graded* $\mathbf{k}$*-modules: $Coker(\delta_X) \cong Coker(\delta_{X'})$*

PROOF. See [5, Theorem 6.1]. □

As application, we get the following criterion to check if an ACM scheme is licci.

COROLLARY 3.1 *Let $X \subset \mathbb{P}^{n+c}$ be a closed subscheme of dimension $n > 0$. If X is licci, then:*

1. $H_*^i N_X = 0$ *for* $1 \leq i \leq n-1$,

2. δ_X *is an isomorphism.*

PROOF. It follows from Theorem 3.1 and the fact that for complete intersections $Y \subset \mathbb{P}^{n+c}$, $H^i_* N_Y = 0$ for $1 \le i \le n-1$ and δ_Y is an isomorphism (Remark 3.1). □

From now until the end of this section, we will restrict our attention to closed subschemes $X \subset \mathbb{P}^{n+3}$, $n > 0$, of codimension 3 and we will deduce from the previous results the CI-liaison invariance of

$$H^i_m(K_{R/I} \otimes_R I) \quad i = 0, \dots, n$$

being $K_{R/I} = Ext^3_R(R/I, R)(-n-4)$ the canonical module of X.

Indeed, using basic facts on local cohomology, the spectral sequence relating local and global Ext:

$$E_2^{pq} := H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow Ext^{p+q}(\mathcal{F}, \mathcal{G}),$$

and the spectral sequence:

$$E_2^{pq} :=_\mu Ext^p_R(M_1, H^q_m(M_2)) \Rightarrow_\mu Ext^{p+q}_m(M_1, M_2),$$

we obtain

THEOREM 3.2 *Let $X \subset \mathbb{P}^{n+3}$ be an ACM subscheme of codimension 3 ($n > 0$) and $K := Ext^3_R(A, R)(-n-4)$ its canonical module. Then, we have*

1. $H^{i+1}_* N_X \cong H^i_m(K \otimes_R I)(n+4)$, $0 \le i \le n-2$, *as graded R-modules.*

2. *There exists an exact sequence:*

$$0 \to H^{n-1}_m(K \otimes_R I)(n+4) \to H^n_* N_X \xrightarrow{\delta_X} Hom(I, H^{n+1}_m(A))$$
$$\to H^n_m(K \otimes_R I)(n+4) \to 0.$$

In particular,

3. *$H^i_m(K \otimes_R I)$ are CI-liaison invariant as graded R (resp. $\mathbf{k}$)-modules, $0 \leq i < n$ (resp. $0 \leq i \leq n$). Moreover, if X is locally complete intersection then*

$$H^i_m(K \otimes_R I)(n+4) \cong H^{n-i}_m(K \otimes_R I)^v \quad i = 0, \dots, n$$

as R-modules.

PROOF. See [5, Proposition 6.8]. □

As application we get another criterion to check if an ACM subscheme X of $\mathbb{P}^N$ is licci.

COROLLARY 3.2 *Let $X \subset \mathbb{P}^{n+3}$ be a closed subscheme of dimension $n > 0$. If X is licci then $H^i_m(K \otimes_R I) = 0$, $0 \leq i \leq n$.*

PROOF. It follows from Theorem 3.2 and the fact that for complete intersections $Y \subset \mathbb{P}^{n+3}$, $H^i_m(K_{R/I(Y)} \otimes_R I(Y)) = 0$, $0 \leq i \leq n$. □

We are led to pose the following question which, to my knowledge, is still open:

QUESTION 3.1 Whether the converse of Corollary 3.2 is true, i.e., is a codimension 3 ACM scheme $X \subset \mathbb{P}^{n+3}$ licci if $H^i_m(K \otimes_R I) = 0$ for $0 \leq i \leq n$?

Now, we will illustrate with an example how to use Theorem 3.2

EXAMPLE 3.1 Let $C \subset \mathbb{P}^4$ be a smooth, connected curve of degree d and genus g with an "almost linear" resolution:

$$0 \to R(-s-3)^a \to R(-s-2)^b \to R(-s-1)^{c_1} \oplus R(-s)^{c_0} \to I(C) \to 0.$$

If $d + g - 1 - ac_0 \neq 0$ then C is not licci.

IDEA OF THE PROOF. We compute the dimension,

$$l(C)_\mu := \dim_{\mu+5} H^0_m(K_A \otimes_R I(C)),$$

of the CI-liaison invariants ${}_{\mu+5}H^0_m(K_A \otimes_R I)$. The exact sequence and the duality of Theorem 3.2 gives us (small letters mean dimension)

$$l(C)_\mu - l(C)_{-\mu-5} = h^1 N_C(\mu) -_\mu hom_R(I(C), H^2_m(A)).$$

Since ${}_{-2}hom_R(I, H^2_m(A)) = ac_0$ and $h^1 N_C(-2) = -\chi N_C(-2) = d + g - 1$ (Riemann-Roch's Theorem), we obtain

$$l(C)_{-2} - l(C)_{-3} = h^1 N_C(-2) - {}_{-2}hom_R(I(C), H^2_m(A)) = d + g - 1 - ac_0.$$

Therefore, by Corollary 3.2, if $d + g - 1 - ac_0 \neq 0$ then C is not licci.

□

REMARK 3.2

1. The only smooth connected curve in $\mathbb{P}^4$ with a linear resolution ($c_0 = 0$) which is licci is a line.
2. The smooth rational quartic is not licci. Indeed, $(a, b, c_1, c_0, s) = (3, 8, 6, 0, 1)$ and $d + g - 1 - ac_0 = 3 \neq 0$.

Recall that Gaeta's Theorem states the existence of a unique CI-liaison class containing ACM curves $C \subset \mathbb{P}^3$. We will now deduce the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$. So, in the context of CI-liaison, Gaeta's Theorem does not generalize well to higher codimension. In next sections, we will try to convince the reader that G-liaison is a more natural approach if we want to carry out a program in higher codimension.

COROLLARY 3.3 *Let $C_t \subset \mathbb{P}^4$ be an ACM curve with a linear resolution:*

$$0 \longrightarrow R(-t-2)^{\frac{t^2+t}{2}} \longrightarrow R(-t-1)^{t^2+2t} \longrightarrow R(-t)^{\frac{t^2+3t+2}{2}} \longrightarrow I(C_t) \longrightarrow 0.$$

For $t \neq q$, C_t and C_q belong to different CI-liaison classes.

PROOF. We have $d(C_t) = \binom{t+3}{4} - \binom{t+2}{4}$, $p_a(C_t) = (t-1)d(C_t) + 1 - \binom{t+3}{4}$ and $d(C_t) + p_a(C_t) - 1 \neq d(C_q) + p_a(C_q) - 1$ for $t \neq q$. Therefore, by Example 3.1, C_t and C_q belong to different liaison classes provided $t \neq q$. □

REMARK 3.3 Corollary 3.3 shows that in the context of CI-liaison Gaeta's Theorem does not generalize to higher codimension. To prove Corollary 3.3 we strongly use that the graded modules $H^i_m(K \otimes_R I)$ are CI-liaison invariants. So, if one wants to see that G-liaison is a more natural approach in higher codimension, we have to prove that $H^i_m(K \otimes_R I)$ are not G-liaison invariants. Indeed, they are not. As example we have the following one:

Denote by $C_t \subset \mathbb{P}^4$ the ACM curve defined by the maximal minors of a $t \times (t+2)$ matrix with linear entries. D_t has a t-linear resolution. According to Corollary 3.3, $H^0_m(K_{D_t} \otimes_R I(D_t))$ changes when t varies and it follows from Theorem 5.2 that D_t is glicci. Therefore, $H^0_m(K \otimes_R I)$ is not a G-liaison invariant.

As another example about the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$ we have the following one

EXAMPLE 3.2 Let $S \subset \mathbb{P}^4$ be a Castelnuovo (resp. Bordiga) surface and let $C \subset S$ be a rational, normal quartic. Consider an effective divisor $C_t \in |C + tH|$, where H is a hyperplane section of S and $0 \le t \in \mathbb{Z}$. It holds:

- C_t is not licci, $\forall t \ge 0$;
- C_t and $C_{t'}$ belong to different CI-liaison classes provided $0 \le t < t'$.

This last example is a particular case of a much more general result that I will state after fixing some extra notation:

We consider a Cartier divisor C on $S \subset \mathbb{P}^{n+3}$ where $\dim C = n$ and $C, S \subset \mathbb{P}^{n+3}$ are ACM subschemes generically complete intersection.

We take a free resolution of $I(S)$:

$$0 \to \oplus_i R(-q_i) \to \oplus_i R(-p_i) \to I(S) \to 0.$$

Since $\mathcal{E}xt^1(I_S, I_{C/S}) \cong \omega_S(n+4) \otimes I_{C/S}$, applying $Hom(., I_{C/S})$ to the above exact sequence we get, for any integer μ, the complex:

$$\oplus_i H^0 I_{C/S}(p_i + \mu) \to \oplus_i H^0 I_{C/S}(q_i + \mu) \xrightarrow{\varphi_\mu} H^0(\omega_S(n+4) \otimes I_{C/S}(\mu)).$$

We define:

1. $L^0(C)_\mu = Coker\varphi_\mu$,

2. $L^j(C)_\mu = H^j(\omega_S(n+4) \otimes I_{C/S}(\mu))$, $j \geq 1$.

Notice that if H is a hyperplane section of $S \subset \mathbb{P}^{n+3}$, we have an isomorphism $L^0(C)_\mu \cong L^0(C_t)_{\mu+t}$ for any $C_t \in |C + tH|$.

PROPOSITION 3.1 *Let $C \subset S \subset \mathbb{P}^{n+3}$ be two ACM subschemes, S generically complete intersection in $\mathbb{P}^{n+3}$ and suppose C is a Cartier divisor on S of* $\dim C = n > 0$. *We take an effective divisor $C_t \in |C + tH|$, being H a hyperplane section of S, and we assume $L^{n-1}(C)_{\mu_0} \neq 0$ for some integer μ_0. It holds:*

1. *C_t is not licci, $\forall t >> 0$;*

2. *C_t and $C_{t'}$ belong to different liaison classes for any $t > t' >> 0$.*

PROOF. See [5, Corollary 7.5]. □

4 Glicci curves in $\mathbb{P}^4$

In this section, using the fact that the Picard group of a "general" ACM surface $X \subset \mathbb{P}^4$ is well known together with the fact that roughly speaking Gorenstein liaison is a theory about divisors on ACM schemes, we will see that there is only one G-liaison class containing ACM curves $C \subset \mathbb{P}^4$ lying on a smooth, rational, ACM surface $S \subset \mathbb{P}^4$. More precisely, we will see that all ACM curves $C \subset \mathbb{P}^4$ lying on a smooth, rational, ACM surface $S \subset \mathbb{P}^4$ are glicci (Theorem 4.1). We will also prove that ACM curves $C \subset \mathbb{P}^4$ lying on a "general" ACM surface $X \subset \mathbb{P}^4$ with degree matrix $[u_{i,j}]$, $u_{i,j} > 0$, are glicci provided $16((KH)^2 - K^2H^2) - H^2[H^2 - K^2 + 8(1 + p_a)] \geq 0$; being K the canonical divisor on X and H the hyperplane section of X (Theorem 4.3). See Example 4.1 and Corollary 4.2 for examples of ACM surfaces $X \subset \mathbb{P}^4$ verifying the above numerical condition and [1] for further generalizations of Theorem 4.3. Notice that these results drastically differ from the one obtained in Example 3.2.

We start with some preliminary results.

DEFINITION 4.1 *A noetherian ring A (resp. a noetherian scheme X) satisfies the condition G_1, "Gorenstein in codimension ≤ 1", if every localization A_p (resp. every local ring $\mathcal{O}_X$) of dimension ≤ 1 is a Gorenstein local ring.*

LEMMA 4.1 *Let $X \subset \mathbb{P}^n$ be an ACM subscheme satisfying the property G_1, and let C be a subcanonical divisor on X. Let $F \in I(C)$ be a homogeneous polynomial of degree d such that F does not vanish on any component of X. Let H_F be the divisor cut out on X by F. Then the effective divisor $H_F - C$ on X, viewed as a subscheme of $\mathbb{P}^n$, is AG. In fact, any effective divisor in the linear system $|H_F - C|$ is AG.*

SKETCH OF THE PROOF. We are assuming that C is the divisor associated to a regular section of $\omega_X(l)$ for some $l \in \mathbb{Z}$. Let Y be the residual divisor, $Y \in |H_F - C|$. We have $I_{Y|X}(d) \cong \mathcal{O}_X(dH - Y) \cong \mathcal{O}_X(C) \cong \omega_X(l)$ and the exact sequence

$$0 \longrightarrow I(X) \longrightarrow I(Y) \longrightarrow H^0_*(\omega_X)(l-d) \longrightarrow 0.$$

Using the minimal free resolutions of $I(X)$ and $H^0_*(\omega_X)(l-d)$ together with the Horseshoe Lemma [10, 2.2.8, pag. 37] we deduce that Y is AG. □

In next Proposition we are going to prove that in contrast to the fact that adding hyperplane sections does not preserve the CI-liaison class (see Proposition 3.1), it preserves the G-liaison class.

PROPOSITION 4.1 *Let $X \subset \mathbb{P}^n$ be a smooth ACM subscheme and let $C \subset X$ be an effective divisor. Take any divisor C_t in the linear system $|C + tH|$ being H a hyperplane section of X and $t \in \mathbb{Z}$. Then, C and C_t are G-bilinked. (Notice that if $t = 0$ then C and C_t are linearly equivalent.)*

SKETCH OF THE PROOF. Let K be a subcanonical divisor of X. Take $A \in I(K)$ a form of degree $a >> 0$ not vanishing on any component of X. So $H_A - K$ is effective (We denote by H_A the codimension one subscheme of X cut out by A). Now we choose forms $F \in I(C)$ and $G \in I(C_t)$ with $\deg F + t = \deg G$ and a divisor D on X such that

$H_F - C = D = H_G - C_t$. By lemma 4.1, $H_{AF} - K$ and $H_{AG} - K$ are Gorenstein. Moreover, $H_{AF} - K - C = (H_A - K) + (H_F - C) = H_A - K + D$ and $H_{AG} - K - C_t = (H_A - K) + (H_G - C_t) = H_A - K + D$. So C and C_t are Gorenstein linked to $H_A - K + D$ as subschemes of $\mathbb{P}^n$ or, equivalently, C and C_t are G-bilinked. □

Proposition 4.1 motivates the following definition

DEFINITION 4.2 *Let $X \subset \mathbb{P}^n$ be a smooth scheme. We say that an* effective divisor C *on* X *is* minimal *if there is no effective divisor in the linear system $|C - H|$ being H a hyperplane section divisor of X.*

We are now ready to state one of the main results of this section.

THEOREM 4.1 *All ACM curves $C \subset \mathbb{P}^4$ lying on a general smooth, rational, ACM surface $S \subset \mathbb{P}^4$ are glicci, i.e., they belong to the Gorenstein liaison class of a complete intersection.*

SKETCH OF THE PROOF. According to the classification of general smooth, rational, ACM surfaces S is

1. A cubic scroll: $S = Bl_{\{p_1\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|2E_0 - E_1|$, $\deg(S) = 3$, and $Pic(S) \cong \mathbb{Z}^2 =< E_0; E_1 >$, or

2. A Del Pezzo surface: $S = Bl_{\{p_1,\ldots,p_5\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|3E_0 - \sum_{i=1}^{5} E_i|$, $\deg(S) = 4$, and $Pic(S) \cong \mathbb{Z}^6 =< E_0; E_1, \ldots, E_5 >$, or

3. A Castelnuovo surface: $S = Bl_{\{p_1,\ldots,p_8\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|4E_0 - 2E_1 - \sum_{i=2}^{8} E_i|$, $\deg(S) = 5$, and $Pic(S) \cong \mathbb{Z}^9 =< E_0; E_1, \ldots, E_8 >$, or

4. A Bordiga surface: $S = Bl_{\{p_1,\ldots,p_{10}\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|4E_0 - \sum_{i=1}^{10} E_i|$, $\deg(S) = 6$, and $Pic(S) \cong \mathbb{Z}^{11} =< E_0; E_1, \ldots, E_{10} >$.

For each general smooth, rational, ACM surface, we classify the minimal ACM curves C on S (see [5, §8]). Finally, we check that each minimal ACM curve C on S is glicci by direct examination. □

We are led to pose the following question which should be viewed as a generalization of Gaeta's Theorem (see section 5).

QUESTION 4.1 In codimension three, is there only one Gorenstein liaison class containing ACM schemes? or, equivalently, are all ACM subschemes glicci?

Although we do not fully answer this question, we make a substantial progress and we determine a huge family of ACM surfaces $S \subset \mathbb{P}^4$ such that all ACM curves C lying on S are glicci (see, Theorem 4.3). Hence, Theorem 4.1 and 4.3 suggest that the answer to question 4.1 should be "yes".

TERMINOLOGY 4.1 To say that a statement holds for a general point of a projective variety Y means that there exists a countable union Z of proper subvarieties of Y such that the statement holds for every $x \in Y \setminus Z$. In particular, we say that a statement holds for a *general* surface $X \subset \mathbb{P}^4$ with Hilbert polynomial $p(t)$ if the statement holds for a general point of an irreducible component of $Hilb^{\mathbb{P}^4}_{p(t)}$.

From now on, unless otherwise specified the word general, when referred to elements of projective varieties, will have this meaning. We have:

THEOREM 4.2 *Let $X \subset \mathbb{P}^4$ be a general ACM surface not complete intersection with degree matrix $[u_{i,j}]$, $u_{i,j} > 0$ for all i, j. Then, three cases are possible for the Picard group of X:*

1. *$Pic(X) \cong \mathbb{Z}^9$ and X is a Castelnuovo surface, or*
2. *$Pic(X) \cong \mathbb{Z}^{11}$ and X is a Bordiga surface, or*
3. *$Pic(X) \cong \mathbb{Z}^2$ if X is none of the above.*

PROOF. See [6, Theorem III.4.2]. □

REMARK 4.1 In the last case of Theorem 4.2, $Pic(X)$ is generated by $H = \mathcal{O}_X(1)$ and K, being K the canonical sheaf of X.

REMARK 4.2 Let $X \subset \mathbb{P}^4$ be a smooth general ACM surface. Assume that either X is a complete intersection or X is rational. Then, any ACM curve C on X is glicci. Indeed, either X is rational and the result follows Theorem 4.1, or X is a complete intersection, $\deg(X) > 4$ and $Pic(X) \cong \mathbb{Z} =< H >$. In this last case, the result follows from Proposition 4.1 and the fact that the hyperplane section H of X is an ACM curve C contained in $\mathbb{P}^3$, and according to Gaeta's Theorem [3], H is licci.

From now on, we restrict our attention to general ACM surfaces $X \subset \mathbb{P}^4$ which are neither rational, nor complete intersection. We will also assume that the degree matrix $[u_{i,j}]$ of X verifies $u_{i,j} > 0$ for all i, j. According to Theorem 4.2, $Pic(X) \cong \mathbb{Z}H \oplus \mathbb{Z}K$. Set $d = H^2$ the degree of X, $\pi = \frac{H(H+K)}{2} + 1$ the sectional genus of X and $p_a = \chi\mathcal{O}_X - 1$ the arithmetic genus of X. Define

$$\begin{aligned} m_0 := \min\{0 \le m \in \mathbb{Z} \mid & H^2[H^2 - K^2 + 8(1 + p_a)] \\ & \le 4m^2((KH)^2 - K^2H^2)\}. \end{aligned}$$

REMARK 4.3 Using the double point formula $2K^2 = d^2 - 5d - 10\pi + 12p_a + 22$, we can write m_0 in terms of the degree of X, the arithmetic genus of X and the sectional genus of X:

$$\begin{aligned} m_0 = \min\{0 \le m \in \mathbb{Z} \mid & 10\pi d - 6d + 7d^2 - d^3 + 4p_a d \\ & \le 4m^2(8\pi^2 - 16\pi + 8 + 2\pi d - 14d + 7d^2 - d^3 - 12p_a)\}. \end{aligned}$$

EXAMPLES 4.1 (i) Let $X \subset \mathbb{P}^4$ be an ACM surface defined by the maximal minors of a matrix A with entries homogeneous forms of fixed degree n. X has a graded minimal free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p-n)^{\frac{p+n}{n}-1} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p)^{\frac{p+n}{n}} \longrightarrow I_X \longrightarrow 0,$$

where $p \in \mathbb{N}$ is a multiple of n.

We have

$$(KH)^2 - K^2H^2 = \frac{p^2(p+n)^2(p^2+np-2n^2)}{144}$$

and

$$H^2[H^2 - K^2 + 8(1+p_a)] = \frac{p^2(p+n)^2(3p^2+3np+2n^2-8)}{48}.$$

Therefore, $m_0 = 2$ for all p multiple of n, $p \geq 2n$, and for all $n \in \mathbb{N}$.

(ii) Let $X \subset \mathbb{P}^4$ be the ACM surface defined by the maximal minors of a matrix $[A, B]$ where A is an $n \times n$ matrix with linear entries and B is a column with entries of degree n. Then, I_X has a graded minimal free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2n)^n \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2n+1)^n \oplus \mathcal{O}_{\mathbb{P}^4}(-n) \longrightarrow I_X \longrightarrow 0.$$

In this case, we get that for $n \geq 11$, $m_0 > 2$.

We are now ready to state the main result of [2].

THEOREM 4.3 *Let $X \subset \mathbb{P}^4$ be a general ACM surface with degree matrix $[u_{i,j}]$, $u_{i,j} > 0 \quad \forall i, j$. Then, there are at most $m_0 - 1$ G-liaison classes containing ACM curves C on X.*

SKETCH OF THE PROOF. We first prove, using Proposition 4.1 and Theorem 4.2, that the only G-Liaison classes which may contain ACM curves are those determined by aK with $0 \leq a \leq m_0 - 1$.

Now, we will check that the ones determined by H and K coincide. In fact, we know that H is licci (indeed, H is an ACM curve contained in $\mathbb{P}^3$ and, by Gaeta's Theorem, H is licci). Therefore, any effective divisor in the linear system $\mid nH \mid$ is glicci. Now we are going to prove that also any effective divisor in the linear system $\mid K + lH \mid$ is glicci:

Let L be the $(n+1) \times (n+2)$ matrix defining the surface X and let $A = [L, M]$ be the matrix obtained adding to L a column M. Thus, A defines a codimension 3 standard determinantal scheme $D \subset X \subset \mathbb{P}^4$. By Theorem 5.2 (see below), D is glicci. Moreover, $\mathcal{O}_X(D) \cong \omega_X(t)$ for some $t \in \mathbb{Z}$, i.e., $D \in |K + tH|$ (see [5, Proposition 10.7]). Hence, K

and D are G-bilinked (Proposition 4.1). So K is glicci and it is in the same G-Liaison class of H.

Therefore the number of G-Liaison classes containing ACM curves on X is at most $m_0 - 1$. □

COROLLARY 4.1 *Using the notation above, let $X \subset \mathbb{P}^4$ be a general ACM surface with a graded minimal free resolution*

$$0 \longrightarrow \oplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^4}(-m_i) \longrightarrow \oplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^4}(-d_j) \longrightarrow I_X \longrightarrow 0.$$

Assume that $m_0 = 2$ and $m_i - d_j > 0 \quad \forall i, j$. Then every ACM curve $C \subset X$ is glicci.

PROOF. It follows directly from Theorem 4.3. □

COROLLARY 4.2 *Let $p \in \mathbb{N}$ be a multiple of $n \in \mathbb{N}$ and let $X_{p,n} \subset \mathbb{P}^4$ be a general ACM surface with a graded minimal free resolution:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p-n)^{\frac{p+n}{n}-1} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p)^{\frac{p+n}{n}} \longrightarrow I_{X_{p,n}} \longrightarrow 0.$$

Then, $m_0 = 2$, $\forall\ p$ and every ACM curve $C \subset X_{p,n}$ is glicci.

PROOF. We may assume that $p \geq 2n$, because in the case $p = n$ $X_{p,n}$ is a complete intersection, and we may assume that $Pic(X_{p,n}) \cong \mathbb{Z}H \oplus \mathbb{Z}K$ (Theorem 4.2 and Remark 4.2).

As we have seen in Example 4.1 (i), $m_0 = 2$ for all p multiple of n, so we conclude by Corollary 4.1. □

5 Generalization of Gaeta's Theorem

In this section, we generalize Gaeta's theorem and we prove that any standard determinantal subscheme $X \subset \mathbb{P}^n$ is in the G-liaison class of a complete intersection. We start fixing some notation.

DEFINITION 5.1 *A subscheme $X \subset \mathbb{P}^n$ of codimension $c+1$ is said to be standard determinantal if $I(V)$ is defined by the maximal minors of a $t \times (t+c)$ homogeneous matrix A. To simplify, we will often write $I(X) = I(A)$.*

If $X \subset \mathbb{P}^n$ is standard determinantal then X is ACM. Moreover, the Hilbert-Burch Theorem state that, in codimension 2, the converse is also true.

In section 3, we have pointed out that if $X \subset \mathbb{P}^n$ is licci then it is ACM, and hence, if we also have $\operatorname{codim}(X) = 2$, then X is standard determinantal. The important contribution to liaison theory of Gaeta's theorem (See [9] for a rigorous, modern proof of Gaeta's theorem) is the converse:

THEOREM 5.1 *Let $V \subset \mathbb{P}^n$ be a pure codimension 2 subscheme defined by the maximal minors of a $t \times (t+1)$ homogeneous matrix A. Then, V is licci.*

SKETCH OF THE PROOF. We link V to a scheme V_1 by means of a complete intersection X defined by two minimal generators of V. V_1 is ACM and, hence, standard determinantal. Gaeta proved that the matrix A_1 defining $I(V_1)$ is obtained from A deleting two columns and transposing. Going on, in a finite number of steps, we reach a 1×2 matrix, i. e. a complete intersection. □

THEOREM 5.2 *Let $V \subset \mathbb{P}^n$ be a pure codimension c subscheme defined by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix A. Then, V is glicci.*

IDEA OF THE PROOF. The proof is rather technical and the main idea is the following one:

We denote by B the matrix obtained deleting a "suitable" column of A and we call X the subscheme defined by the maximal minors of B. ("Suitable" means that $\operatorname{codim}(X) = c-1$. First take, if necessary, a general linear combination of the rows and columns of A.) We denote by A' the matrix obtained deleting a "suitable" row of B and we call V' the subscheme defined by the maximal minors of A'. ("Suitable" means that $\operatorname{codim}(V') = c$. First take, if necessary, a general linear combination of the rows and columns of B.)

We consider V and V' as divisors on X, we show that V and V' are G-bilinked. Hence in $2t-2$ steps we reach a scheme defined by a 1×3 matrix, i.e., we arrive at a complete intersection. □

REMARK 5.1 Gaeta's original theorem says that all ACM subschemes of codimension 2 are licci. Since it is well known for subschemes of codimension two that ACM subschemes are standard determinantal and that AG subschemes and complete intersections coincide, Theorem 5.2 is a full generalization of Gaeta's Theorem.

Finally, we want to stress that this last result drastically differs from the one we obtain when we link by means of complete intersection schemes. Indeed, since any ACM curve D_p in $\mathbb{P}^4$ defined by the maximal minors of a $p \times (p+2)$ matrix with linear entries has a linear resolution, we have that D_p and $D_{p'}$ belong to different CI-Liaison classes provided $p \neq p'$ (See Corollary 3.3) and, by Theorem 5.2. they belong to the same G-liaison class.

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