FULVIA SKOF*

Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10
I-10123 Torino, Italy
fulvia.skof@unito.it

ABOUT A FORTY YEARS OLD CONJECTURE BY P. ERDŐS

Conferenza tenuta il giorno 30 Novembre 1998

ABSTRACT. In connection with a characterization of $c \log n$ within the class of additive arithmetic functions, i.e. such that $f(mn) = f(m) + f(n)$, which has been conjectured by P. Erdős in 1946 and is still unproved, we outline some subsequent researches involving various tools and standpoints, and we report some expressive results either lying close to that conjecture or revealing a similar spirit.

1 Introduction

On the occasion of a lecture on additive arithmetic functions delivered at this Seminario forty years ago ([Er2]) Pal Erdős collected some conjectures by himself concerning characterizations of the logarithm within the class of such functions. One of them is still of actual interest, because it did not get until now a complete proof in spite of the number of subsequent papers and different kinds of approaches to

*Lavoro eseguito nell'ambito del Gruppo G.N.A.F.A. del C.N.R. e con il contributo del Ministero dell' Università e della Ricerca Scientifica e Tecnologica (ex 40%).

this problem. We purpose here to trow a bridge backwards, and summarize some partial or even collateral results, framed sometimes in more recent fields of research.

We shall consider the class of functions $f : \mathbb{N}_1 \to \mathbb{R}$, where $\mathbb{N}_1 := \mathbb{N} \setminus \{0\}$, satisfying the equation

$$f(mn) = f(m) + f(n)\,, \qquad m, n \in \mathbb{N}_1\,; \tag{1.1}$$

f is said to be *additive* if (1.1) is satisfied for $(m, n) = 1$; f is *completely additive* (c. a.) when (1.1) holds for every $m, n \in \mathbb{N}_1$.

The logarithm, as the most important c. a. function, got classical characterizations. We report here the following one, by Erdős, which is the starting point of our talk:

Theorem 1.1 (P. Erdős) *If f is additive and satisfies*

$$f(n+1) - f(n) \to 0\,, \qquad \text{as } n \to \infty\,, \tag{1.2}$$

then $f(n) = c \log n$, $n \geq 1$ ($c = f(2)/\log 2$).

Two similar propositions, containing each a weakened form of (1.2), were proposed as conjectures by Erdős and mentioned in the abovesaid lecture ([Er2]). We shall treat very shortly the first one, namely the

Conjecture 1.1 ([Er1]) *If f is additive and $|f(n+1) - f(n)| < A$ then $f(n) = c \log n + g(n)$ with $|g(n)| < B$.*

because it has been completely proved, on the ground of some results stated by E. Maté (1969) and E. Wirsing (1970,1981):

([Ma]) *If f is additive and $f(n+1) - f(n) = O(1)$ as $n \to \infty$ then there exists a completely additive function g such that $f(n) = g(n) + O(1)$.*

([Wi1]) *If g is c. a. and $g(n+1) - g(n) < K$ then $g(n) = c \log n$.*

([Wi2]) *If g is c. a. and $g(n+1) - g(n) = o(\log n)$ then $g(n) = c \log n$.*

In this connection it has to be only noticed that, by resort to a new point of view from the recently developed theory of "stability" of functional equations initiated by S. Ulam and D.H. Hyers [Hy] in 1941, the function $f(n) = c \log n$ may be characterized within a larger class of functions, including the additive ones, with the advantage that we do not force the $O(1)$ term to be additive too:

([Sk6]) *The function* $f : \mathbb{N}_1 \to \mathbb{R}$ *satisfies* $f(n) = c\ \log n + O(1)$ *with* $c = \lim_{r\to\infty} f(2^r)/\log 2^r$*, if and only if the following conditions are both fulfilled:*

i) $|f(mn) - f(m) - f(n)| < A$

ii) $|f(n+1) - f(n)| < B$.

In virtue of the Wirsing's result [Wi2] the condition ii) may be equivalently replaced by

$$f(n+1) - f(n) = o(\log n) \qquad \text{as } n \to \infty\,.$$

Now let us proceed to discuss the second weakened form of (1.2) and the corresponding conjecture, which are the real object of this talk.

2 The Conjecture E and Its Formulations

Conjecture E. ([Er1, Er2]) *If* f *is additive and* $f(n+1) - f(n) \to 0$ *as* $n \to \infty$ *when we neglect a sequence of density 0, then* $f(n) = c \log n$*,* $n > 0$ $(c = f(2)/\log 2)$.

This proposition contains two elements allowing us to intend the statement in different ways.

Firstly, it refers to a "density" condition, and it is well known that density of an increasing sequence $Z = \{z_k\} \subset \mathbb{N}$ may be defined in many ways. It is possible that the Author was referring to the *asymptotic* (or *natural*) *density* δ, which is defined as the common value, if existing, of *lower density* δ_* and *upper density* δ^* of the sequence, where

$$0 \le \delta_* := \liminf_{x\to\infty} \Big(\sum_{z_k \le x} 1 \Big) \Big/ x \le \limsup_{x\to\infty} \Big(\sum_{z_k \le x} 1 \Big) \Big/ x =: \delta^* \le 1\,.$$

However, other kinds of "density" existing in the literature may be suitably used in this context, as we shall observe later.

In the second place, when we neglect a sequence of density 0, the limit condition restricted to the complementary sequence $U = \{u_i\}$ having density 1 may be intended in different ways, namely either as

$$f(u_i + 1) - f(u_i) \to 0 \tag{I}$$

or as

$$f(u_{i+1}) - f(u) \to 0\,. \tag{II}$$

Obviously, if (II) is satisfied along a sequence U with density 1, there exists a sequence $U_0 \subset U$ with $\delta(U_0) = 1$ such that (I) holds; but the inverse implication has still to be proved.

Therefore, referring initially to asymptotic density, we shall describe the evolution of researches about the Conjecture E, following two different directions originated by (I) or (II) and, correspondently, we shall speak about "Conjecture (E,I)" or "Conjecture (E,II)".

3 About Conjecture (E,I)

CONJECTURE (E,I). *If $f : \mathbb{N}_1 \to \mathbb{R}$ is additive and satisfies the condition $f(u_i + 1) - f(u_i) \to 0$ as $i \to \infty$ along an increasing sequence $\{u_i\}$ ($i \in \mathbb{N}$) with density 1, then $f(n) = c \log n$, $n > 1$ ($c = f(2)/\log 2$).*

There are rather few papers devoted to the proof of this proposition, which still remains an open problem. It has been proved that the assumptions, even in a weaker form, imply complete additivity of f:

([Sk1]) *If $f : \mathbb{N}_1 \to \mathbb{R}$ is additive and satisfies the condition $f(v_i) - f(v_i) \to 0$ as $i \to \infty$ along an increasing sequence $\{v_i\}$ ($i \in \mathbb{N}$) having upper density $\delta^* = 1$, then f is completely additive.*

But neither the logarithmic form is proved nor counterexamples have been found. It seems to be a crucial point in this context to find a linkage between the values $f(u_i)$ and $f(u_{i+1})$, in consequence of the fact that the sequence $\{u_i\}$, even if density is 1, may have

$$\text{l.u.b.}\ (u_{i+1} - u_i) = +\infty\,.$$

On the contrary, many papers involving various and interesting topics are devoted to the condition (II), which will be the subject of the next section.

4 About Conjecture (E,II)

CONJECTURE (E,II). *If* $f : \mathbb{N}_1 \to \mathbb{R}$ *is additive and satisfies the condition (II)* $f(u_{i+1}) - f(u_i) \to 0$ *for* $i \to \infty$ *along an increasing sequence* $\{u_i\}$ *(*$i \in \mathbb{N}$*) with density 1, then* $f(n) = c \log n$*,* $n \geq 1$ *(*$c = f(2)/\log 2$*).*

4.a) The most direct and natural approach to the problem consists in searching for some connections between density properties of the sequence $U = \{u_i\}$ and arithmetic properties of subsequences $\{t_k\} \subset \mathbb{N}$ such that the equation $f(t_k) = c \ \log t_k$ holds for some real constant c. In this concern, let us report the following proposition involving, more in general, increasing sequences $\{u_i\}$ with a rather irregular distribution, in that lower density δ_* can be less than upper density δ^*: however a new condition, namely $u_{i+1} - u_i = O(1)$, is now required.

([Sk2]) *Let* $U = \{u_i\} \subset \mathbb{N}$ *be an increasing sequence such that* $2/3 < A = \delta_*(U) \leq \delta^*(U) = 1$ *and* $u_{i+1} - u_i = O(1)$ *as* $i \to \infty$*. If* $f : \mathbb{N}_1 \to \mathbb{R}$ *is additive and satisfies (II)* $f(u_{i+1}) - f(u_i) \to 0$ *as* $i \to \infty$*, then* $f(t) = c \log t$ *(*c *independent of* t*) for every natural* t *composed only by prime factors* p *such that* $(1 - A)p < A$*.*

The proof consists in the direct calculation of $f(t)$ through an iterative process involving a suitable sequence of numbers less than t. The first idea of such an iterative process may be found in a paper by A. Rényi ([Re]) where a new proof of the classical Erdős theorem mentioned in Section 1 is given.

Now, rewriting the above proposition in the special case of $A = \delta_*(U) = \delta^*(U) = 1$, we get

COROLLARY 4.1 ([Sk2]) *Let* $U = \{u_i\} \subset \mathbb{N}_1$ *be an increasing sequence with density* $\delta(U) = 1$ *and* $u_{i+1} - u_i = O(1)$ *as* $i \to \infty$*. If* f *is additive and* $f(u_{i+1}) - f(u_i) \to 0$ *when* $i \to \infty$*, then* $f(n) = c \log n, n \geq 1$ *(*$c = f(2)/\log 2$*).*

This proposition looks rather close to the Conjecture (E,II); it will be a helpful tool for further results in next sections.

4.b) The just considered condition $u_{i+1}-u_i = O(1)$ for the distribution of the elements within the sequence $\{u_i\}$ does not follow from density 1, which implies uniquely $u_{i+1}-u_i = o(u_i)$ as $i \to \infty$. On the other hand, it seems to be reasonable that the arithmetic composition of numbers u_i and the behaviour of their distribution within $\mathbb{N}$ have to play some role when the sequence $\{u_i\}$ is asked to have influence on the value $c \log n$ of f for every natural n. So, by analogy to the similar problem of characterizing the regular solution $F(x) = ax$ for the Cauchy equation $F(x+y) = F(x)+F(y)$ when $F : \mathbb{R} \to \mathbb{R}$ and to the related well known measure-theoretic responses, one may purpose to look for a definition of density fulfilling the fundamental properties of a measure. The asymptotic density does not satisfy this requisite, but a measure-theoretic approach to "density" of sequences in $\mathbb{N}$ does exist; it has been developed by R.C.Buck in 1946 ([Bu1, Bu2]) starting from an idea by S. Banach ([Ba]).

So, according to Banach and Buck, let $\mathcal{D}_0$ be the algebra generated by finite sets in $\mathbb{N}$ and by the arithmetic progressions $\{a+nd\}$, and $\lambda : \mathcal{D}_0 \to \mathbb{R}$ a set-function defined as follows:

$\lambda(E) = 0$ if E is a finite set in $\mathbb{N}$;

$\lambda(E) = 1/d$ if E is an arithmetic progression $\{a+nd\}$, $n \in \mathbb{N}$;

$\lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2)$ if $E_1, E_2 \in \mathcal{D}_0$ and $E_1 \cap E_2$ a finite set.

Denoting by S the set of parts of $\mathbb{N}$, we define in S the *outer measure* function μ^*, as an extension of λ to S from $\mathcal{D}_0$, in the following way:

$$\mu^*(E) = \text{g. l. b. } \{\lambda(A) : A \in \mathcal{D}_0,\ E \mathring{\subset} A\},$$

where $E \mathring{\subset} A$ means $E \setminus H \subset A \setminus K$ with H, K finite sets in $\mathbb{N}$.

An increasing sequence E is said to be *measurable* if and only if $\mu^*(E) + \mu^*(\mathbb{N} \setminus E) = 1$. This infers that for every $\epsilon > 0$ there exist $A, B \in \mathcal{D}_0$ with $A \mathring{\subset} E \mathring{\subset} B$, such that $\lambda(B) - \lambda(A) < \epsilon$.

Within the class $\mathcal{D}_\mu$ of measurable sequences, μ^* is a finitely additive "measure", which will be denoted by μ.

In view of application to the Erdős conjecture, let us point out some interesting connections between asymptotic and measure-theoretic densities:

(M1) $\mathcal{D}_\mu \subset D$ strictly,

where D denotes the class of the increasing sequences having asymptotic density;

(M2) $E \in \mathcal{D}_\mu \implies \mu(E) = \delta(E)$;

(M3) $\delta(E) = 1 \implies \mu^*(E) = 1$;

(M4) There exist *extremal* sequences $E \subset \mathbb{N}$, i.e. such that $\mu^*(E) = \mu^*(\mathbb{N} \setminus E) = 1$.

Moreover, it has to be remarked that measurability of a sequence E infers a more regular distribution of its elements than asymptotic density, arithmetic progressions being now involved.

As a consequence of this fact, we shall see that the conjecture (E,II), when density is substituted by the Banach-Buck measure, becomes a true sentence. In fact:

([Sk4]) *If* $f : \mathbb{N}_1 \to \mathbb{R}$ *is additive and* $f(u_{i+1}) - f(u_i) \to 0$ *as* $i \to \infty$ *along a sequence* $\{u_i\}$ *(* $i \in \mathbb{N}$*) having Banach-Buck measure 1, then* $f(n) = c \log n$*,* $n \geq 1$ *(*$c = f(2)/\log 2$*).*

The proof will use the Corollary in **4.a)**. In fact, according to the above property (M2), $\{u_i\}$ turns out to have asymptotic density 1 ; moreover, for fixed $\epsilon > 0$ there exist $A, B \in \mathcal{D}_0$ such that $A \mathring{\subset} \{u_i\} \mathring{\subset} B \subset \mathbb{N}_1$ and $\lambda(B) - \lambda(A) < \epsilon$. Writing $A =: \{a_r\}$ ($r \in \mathbb{N}$) with $a_r < a_{r+1}$, we get $a_{r+1} - a_r = O(1)$, whence also $u_{i+1} - u_i = O(1)$; then, by application of the Corollary the theorem is proved.

Obviously measurability of $\{u_i\}$ is a rather strong assumption; however we shall see in the next section how a further way of approaching the problem allows us to get the logarithmic form even in some situations of non-measurability of $\{u_i\}$.

4.c) A further approach to density of sequences involves topology. It will allow us to restrict the class of sequences of density 1 for which the Conjecture (E,II) is still an open problem.

In $\mathbb{N}$ we may define as a *neighbourhood* of the natural number a every arithmetical progression $\{a + kd\}$ $(k \in \mathbb{N})$. On this ground, we may consider "everywhere dense" and "nowhere dense" sequences in $\mathbb{N}$ and study possible connections between such topological density properties and the asymptotic densities 1 and 0 of sequences ([Sk3]), which are involved in the conjecture.

We just present here the properties which will be of use later:

(T1) Every $U = \{u_i\}$ having asymptotic density 1 is everywhere dense in $\mathbb{N}$;

(T2) If the sequence E is everywhere dense in $\mathbb{N}$ then its outer measure $\mu^*(E)$ is 1;

(T3) If $Z \subset \mathbb{N}$ has asymptotic density 0, then each of the following situations may occur: Z nowhere dense; Z neither nowhere nor everywhere dense; Z everywhere dense.

As an example of the last, and less natural, situation, let us consider

$$Z = \bigcup_{k=1}^{\infty} J_k \quad \text{where} \quad J_k = \{n \in \mathbb{N} : 2^k < n \leq 2^k + k\}.$$

Since every arithmetical progression meets both Z and $\mathbb{N} \setminus Z$ infinitely many times, every natural m is a boundary point of Z and of $\mathbb{N} \setminus Z$; then $\mathbb{N}$ is the closure of both Z and $\mathbb{N} \setminus Z$.

Moreover every such sequence Z with density 0 and everywhere dense in $\mathbb{N}$ is not measurable, since, according to (T2), both Z and $\mathbb{N} \setminus Z$ have outer measure 1.

This proves also that

There exist sequences with asymptotic density 1 which are not Banach-Buck measurable.

In this frame we can state some characterizations of the logarithm in the spirit of the Erdős conjecture and also derive an interesting information about sequences with density 0 which may be neglected when the condition $f(n+1) - f(n) \to 0$ has to be weakened. A result which is proved by resort to topological properties of the sequences involved in the limit condition, is the following one:

([Sk4]) *Let $U = \{u_i\} \subset \mathbb{N}$ be an increasing sequence with density 1 such that $\mu^*(\mathbb{N} \setminus U) < 1$. If $f : \mathbb{N}_1 \to \mathbb{R}$ is additive and $f(u_{i+1}) - f(u_i) \to 0$ for $i \to \infty$, then $f(n) = c \log n$, $n \geq 1$ ($c = f(2)/\log 2$).*

In fact, the assumptions imply that the sequence $\mathbb{N} \setminus U$ cannot be everywhere dense; then U has at least one inner point and therefore it contains an arithmetic progression; this implies $u_{i+1} - u_i = O(1)$ and the Corollary in 4.a) proves the theorem.

In conclusion, on the ground of the different points of view above mentioned, we can assert that the Conjecture (E, II) is still an open problem in the unique case when the increasing sequence U with asymptotic density 1, involved in the limit condition, has both the properties

i) $\mu^*(\mathbb{N} \setminus U) = 1$, then U is non measurable, since $\mu^*(U) = 1$;

ii) l.u.b. $(u_{i+1} - u_i) = +\infty$.

Of course sequences of this kind have also to be considered in order to construct a counterexample (if any) to disprove the conjecture.

5 Restricted additivity

According to the formulation of the problem by Erdős, we have assumed till now that the arithmetic function f is defined and additive all over $\mathbb{N}_1$ and satisfies the additional condition along a subsequence of $\mathbb{N}_1$. But if we frame the additivity relation $f(mn) = f(m) + f(n)$ in the context of functional equations, we may get some suggestions from recent branches in such theory, like that of “stability”, as we did in section **1**, or that of the equations “on a restricted domain”. Our aim is now to consider arithmetic functions f having a given subsequence $\{u_i\} \subset \mathbb{N}_1$ as domain, in order to formulate a suitable additional “if and only if” condition so that f can be the restriction of the function $c \log n$ or of $c \log n + O(1)$ over its domain.

Let us notice that the property $u_{i+1} - u_i = O(1)$ will play again an important role.

First, we shall report the more general result in this context, concerning the uniform approximation of $f(t)$ on its domain $W \subset \mathbb{N}_1$ by the function $c\log t$. The sequence W will be defined in connection with a given increasing sequence $U = \{u_i\} \subset \mathbb{N}_1$ satisfying the following properties:

(R1) $u_{i+1} - u_i = O(1)$

(R2) $u_i \in U_1 \Longrightarrow u_i^2 \in U_1$,

then, let us define

(R3) $U_{11} := \{u_i u_j \,:\, u_i, u_j \in U_1\}$

(R4) $W := U_1 \cup U_{11} = \{w_k\}, \quad w_k < w_{k+1}$.

It is easy to find examples of sequences U_1 fulfilling (R1) and (R2) such that $U_1 \cup U_{11}$ is strictly contained in $\mathbb{N}_1$.

The theorem reads as follows:

([Sk8]) *The arithmetical function $f : W \to \mathbb{R}$, with $W = U_1 \cup U_{11} = \{w_k\}$ such that (R1), (R2), (R3), (R4) are satisfied, has the form*

$$f(t) = c\log t + O(1), \qquad \textit{for } t \in W$$

if and only if the following conditions are both fulfilled:

(R5) $|f(u_i u_j) - f(u_i) - f(u_j)| \le \sigma$ *for some real* $\sigma \ge 0$ *and every* $(u_i, u_j) \in U_1 \times U_1$;

(R6) $\lim_{k\to\infty}\left(\liminf_{n\to\infty}\; 2^{-n}\left|f(w_{k+1}^{2^n}) - f(w_k^{2^n})\right|\right) = 0.$

In order to prove this result many tools have to be used: a preliminary lemma, which gives a stability result for arithmetic functions on a restricted domain and is proved by adapting the outline of the classical Hyers proof ([Hy]) through the definition of auxiliary functions and their suitable extensions; the central part of the proof consists in calculating $f(w_k)$ by a modified "direct" technique like that mentioned in **4.a)**. It has to be noticed that an important role is played by the condition $u_{i+1} - u_i = O(1)$.

Now, assuming in particular $\sigma = 0$ in the above statement we get the corollary:

([Sk7]) *The arithmetic function* $f : W \to \mathbb{R}$, *with* $W = U_1 \cup U_{11} = \{w_k\}$ *satisfying (R1), (R3), (R4), has the form*

$$f(t) = c \log t \qquad \textit{for } t \in W$$

if and only if the following conditions are both fulfilled

(R5)$_0$ $f(u_i u_j) = f(u_i) + f(u_j)$ *for every* $(u_i, u_j) \in U_1 \times U_1$;

(R6)$_0$ $f(w_{k+1}) - f(w_k) \to 0$ *for* $k \to \infty$.

In spite of likeness between this proposition and Conjecture (E,II), the underlying problem is not exactly the same.

Lastly, in case of $U_1 \cup U_{11} = \mathbb{N}_1$, this corollary becomes the classical theorem by Erdős.

6 Asymptotic additivity

We shall present here a last variant of the Erdős problem, referring to *asymptotically additive* functions, namely to the functions $f : \mathbb{N}_1 \to \mathbb{R}$ such that

$$f(mn) - f(m) - f(n) \to 0 \tag{6.1}$$

is satisfied as $m, n \to \infty$. It has to be noticed that when (6.1) is satisfied as the product mn goes to infinity, then it is equivalent to the equation $f(mn) = f(m) + f(n)$: this can be easily proved, for instance, by defining $\varphi(m,n) = f(mn) - f(m) - f(n)$ and writing $\varphi(p^k mn)$, for a fixed prime p, in two different ways, by use of $p^k mn = p^k.mn = p^k m.n$. By comparison one gets $\varphi(m,n) = \varphi(p^k m, n) + \varphi(p^k, m) - \varphi(p^k, mn) \to 0$ as $k \to \infty$, whence $\varphi(m,n) = 0$.

On the contrary, if (6.1) is satisfied as $m \to \infty$ and $n \to \infty$, it is not the same as additivity: this is shown, for instance, by $f(n) = \log(n+1)$.

Asymptotically additive functions are usefully studied in the setting of the theory of stability on a resticted domain, a very recent branch of research ([Sk5]). By a suitable choice of the unbounded domain in $\mathbb{N}$, we can prove the following proposition:

([Sk7]) *The function $f : \mathbb{N}_1 \to \mathbb{R}$ satisfies (6.1) as $m \to \infty$ and $n \to \infty$ if and only if there exists a completely additive function $A : \mathbb{N}_1 \to \mathbb{R}$ such that*

$$f(n) = A(n) + o(1) \qquad for \quad n \to \infty\,.$$

This allows us to close our talk with the following consequent result, which seems to be quite in harmony with the original Erdős propositions:

([Sk7]) *The function $f : \mathbb{N}_1 \to \mathbb{R}$ satisfies the asymptotic conditions*

$$\begin{aligned} f(mn) - f(m) - f(n) &\to 0 \qquad && \text{for } m \to \infty \text{ and } n \to \infty \\ f(n+1) - f(n) &\to 0 \qquad && \text{for } n \to \infty \end{aligned}$$

if and only if

$$f(n) = c \log n + o(1) \quad \text{as } n \to \infty\,, \quad c = \lim_{k\to\infty} f\left(2^{2^k}\right) / \log\left(2^{2^k}\right).$$

References

[Ba] S. BANACH, *Théorie des opérations linéaires*, Warszawa, 1932.

[Bu1] R.C. BUCK, *The measure theoretic approach to density*, Amer. J. of Math., **68** (1946) 560–580.

[Bu2] E.F. BUCK and R.C. BUCK, *A note on finitely additive measures*, Amer. J. of Math., **69** (1947) 413–420.

[Er1] P. ERDŐS, *On the distribution function of additive functions*, Ann. of Math., **47** (1946) 1–20.

[Er2] P. ERDŐS, *On the distribution function of additive arithmetical functions and some related problems*, Rend. Sem. Mat. Fis. Milano, **27** (1958) 45–49.

[Hy] D.H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA, **27** (1941) 222–224.

[Ma] E. MATE, *On a problem of P. Erdős*, Acta Sci. Math. (Szeged), **30** (1969) 301–304.

[Re] S.A. RENYI, *On a theorem of P. Erdős and its applications in information theory*, Mathematica, **1** (24) (1959) 341–344.

[Sk1] F. SKOF, *Un criterio di completa additività per le funzioni aritmetiche riguardante successioni di densità irregolare*, Accad. Naz. Lincei, Rend. Cl. Sci.Fis. Mat.Nat., **48** (1970) 1–4.

[Sk2] F. SKOF, *Proprietà di densità delle successioni e forma* $c \log n$ *delle funzioni aritmetiche additive*, Atti Accad. Sci. Torino, **111** (1976/77) 503–510.

[Sk3] F. SKOF, *Qualche osservazione sulle successioni di interi aventi densità 1 oppure 0*, Rend. Sem. Mat. Univ. Polit. Torino, **35** (1976/77) 391–395.

[Sk4] F. SKOF, *Densità, misura, equidistribuzione delle successioni di interi e proprietà delle funzioni aritmetiche additive*, Rend. di Mat., Roma (4) **10** (1977) 607–616.

[Sk5] F. SKOF, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano, **53** (1983) 113–129.

[Sk6] F. SKOF, *Sulle funzioni aritmetiche* $\delta-$*additive*, Atti Accad. Sci. Torino, **122** (1988) 297–306.

[Sk7] F. SKOF, *On existence of approximately additive extensions or quasi-extensions of* δ*-additive functions on unbounded restricted domains*, Commun. at the 34th ISFE, Wisla-Jawornik, Poland (1996).

[Sk8] F. SKOF, *Arithmetic functions close to the logarithm on some restricted domains*, Commun. at the 36th ISFE, Brno (1998).

[Wi1] E. WIRSING, *A characterization of* $\log n$ *as an additive arithmetic function*, Symp. Math. IV, 45–57, Academic Press, London/NewYork, 1970.

[Wi2] E. WIRSING, *Recent progress in analytic number theory*, Vol. 2, 231–280, Academic Press, London, 1981.