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A BAIRE CATEGORY APPROACH IN EXISTENCE THEORY OF DIFFERENTIAL EQUATIONS

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1 Introduction

The idea of using Baire category in differential equations has a long history dating back to Orlicz, Peixoto, Smale to mention only a few names. Usually one is given a complete metric space $\mathcal{M}$ of maps and for $f \in \mathcal{M}$ one considers the corresponding differential equation

$$x'(t) = f(t, x(t)). \tag{1.1}$$

In Banach spaces the question whether (1.1) has properties like existence, uniqueness, continuous dependence, existence of periodic solutions etc. is, in general, a difficult one. This has led mathematicians to rise, as a substitute, the question whether the property under consideration is perhaps generically satisfied in $\mathcal{M}$. Here Baire category enters naturally if we agree to say that a property is generic in $\mathcal{M}$ provided that it is satisfied by all f in a residual subset $\mathcal{M}_0$ of $\mathcal{M}$, i.e. with $\mathcal{M} \setminus \mathcal{M}_0$ of the first Baire category. It is worthwhile to point out, that knowing that a property of (1.1) is generic in $\mathcal{M}$ does not furnish, in general, any information whether a specific $f \in \mathcal{M}$ belongs or not to the generic set. In this note we discuss a method, based on the Baire category, by which the existence of solutions can be established.

2 Existence results for differential inclusions

This area of research was started more recently by De Blasi and Pianigiani [9, 10] in connection with existence problems for non convex valued differential inclusions of the form

$$x'(t) \in F(t, x(t)), \qquad x(0) = x_0 .$$

In this development a stimulus was offered by a paper by Cellina [3] in which it was shown that the solution set to the Cauchy problem

$$x'(t) \in \{-1, 1\}, \qquad x(0) = 0$$

is a residual subset of the solution set of

$$x'(t) \in [-1, 1], \qquad x(0) = 0,$$

if the latter is endowed with the metric of uniform convergence.

To give an idea of the method we consider two simple situations. If E is a Banach space, we denote by $\mathcal{K}(E))$ (resp. $\mathcal{K}_c(E))$ the space of all nonempty, bounded, closed (resp. nonempty, bounded, closed, convex) subsets of E and let $I = [0, 1]$. Consider the Cauchy problem

$$x'(t) \in \partial F(t, x(t)), \qquad x(0) = x_0 \tag{2.1}$$

and

$$x'(t) \in F(t, x(t)), \qquad x(0) = x_0, \tag{2.2}$$

where $\partial F(t, x(t))$ denotes the boundary of $F(t, x(t))$.

By a solution of (2.1) or (2.2) we mean a Lipschitzean function x satisfying (2.1) or (2.2) a.e. on I.

In E, an open ball with center x_0 and radius r is denoted by $B(x_0, r)$. For $A \subset E$ we denote by $intA$, $\overline{co}A$ respectively the interior of A and the closed convex hull of A.

THEOREM 2.1 *[9] Let E be a reflexive Banach space, $F : I \times B(x_0, r) \to \mathcal{K}_c(E)$ be Hausdorff continuous and bounded by a constant, say M. Assume, moreover, that for every $(t, x) \in I \times B(x_0, r)$ the set $F(t, x)$ has nonempty interior. Then the Cauchy problems (2.1) and (2.2) have solutions and the solution set $\mathcal{M}_{\partial F}$ of (2.1) is a residual subset of the solution set $\mathcal{M}_F$ of (2.2).*

PROOF. (Sketch) Set $\mathcal{M}_F = \{x \in C(I, E) : x \textit{ is a solution of (2.2)}\}$. $\mathcal{M}_F$ is nonempty and, under the metric of $C(I, E)$, is a complete metric space. For any $n \in \mathbb{N}$ set

$$\mathcal{M}_n = \{x \in \mathcal{M}_F : \int_I d(x'(t), \partial F(t, x(t)))dt < 1/n\},$$

where $d(x'(t), \partial F(t, x(t))) = \inf\{\|x'(t) - z\| : z \in F(t, x(t))\}$. The main step is to prove that each $\mathcal{M}_n$ is an open and dense subset of $\mathcal{M}_F$. Then, by the Baire category theorem, it follows that the set $\bigcap \mathcal{M}_n$ is a residual subset of $\mathcal{M}_F$. Since $\mathcal{M}_{\partial F} = \bigcap \mathcal{M}_n$ the statement follows. □

THEOREM 2.2 *[10] Let E be a reflexive, separable Banach space and let $F : I \times B(x_0, r) \to \mathcal{K}_c(E)$ be Hausdorff continuous and bounded by M. Moreover, assume that for every $(t, x) \in I \times B(x_0, r)$ the set $F(t, x)$ has nonempty interior. Then the solution set $\mathcal{M}_{extF}$ of the Cauchy problem*

$$x'(t) \in extF(t, x(t)), \qquad x(0) = x_0$$

is nonempty.

PROOF. (Sketch) Let $\mathcal{M}_F$ be the solution set of the Cauchy problem (2.2) and set

$$\mathcal{M}' = cl\{x \in \mathcal{M}_F : x'(t) \in intF(t, x(t)) \quad a.e.\}.$$

Here cl denotes the closure in the topology of $C(I, E)$. $\mathcal{M}'$ is non empty, and equipped with the metric of $C(I, E)$ is a complete metric space. Now introduce the Choquet function[5, 10, 15] $d_F(t, x, z)$, which "measures" the distance of a point $z \in F(t, x)$ from the set of the extreme points of the convex set $F(t, x)$. We summarize below the main properties of d_F :

i. d_F is non negative and bounded on graph F;

ii. $d_F(t, x, z) = 0$ if and only if $z \in extF(t, x)$;

iii. if $\{x_n\} \subset \mathcal{M}'$ converges to x,
then limsup $\int_I d_F(t, x_n(t), x_n'(t))dt \leq \int_I d_F(t, x(t), x'(t))dt$.

Then we set

$$\mathcal{M}_n' = \{x \in \mathcal{M}' : \int_I d_F(t, x(t), x'(t))dt < 1/n\}$$

and we prove that each $\mathcal{M}_n'$ is an open and dense subset of $\mathcal{M}'$. Consequently $\bigcap \mathcal{M}_n'$ is a residual subset of $\mathcal{M}'$ and hence nonempty. Finally we show that

$$\bigcap \mathcal{M}_n' \subset \mathcal{M}_{extF} .$$

□

We point out that the right choice of the underlying metric space, in our case $\mathcal{M}'$, is fundamental. For example the above argument certainly fails if we replace $\mathcal{M}'$ by $\mathcal{M}_F$ for, if valid, it would imply that $\mathcal{M}_{extF}$ is dense in $\mathcal{M}_F$ which, in general, is not true by a well known counterexample due to Plis.

3 Existence results for Hamilton-Jacobi equations

In a recent paper Dacorogna and Marcellini [7], see also Bressan and Flores [2], have used the Baire method to establish existence of solutions to the Dirichlet problem for Hamilton-Jacobi equation of the form

$$\begin{cases} H(\nabla u(x)) = 0 & x \in \Omega \text{ a.e.} \\ u(x) = \varphi(x) & x \in \partial\Omega , \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is open. They consider both, the scalar and the vectorial case. After rewriting the above problem in the equivalent form

$$\begin{cases} \nabla u(x) \in C & x \in \Omega \text{ a.e.} \\ u(x) = \varphi(x) & x \in \partial\Omega, \end{cases}$$

where $C = \{z \in \mathbb{R}^n \mid H(z) = 0\}$, Dacorogna and Marcellini prove an existence theorem by using a Baire category approach very much in the spirit of Theorems 1 and 2 above. Their basic assumptions are that $\nabla\varphi(x)$ is compactly contained in the interior of $\overline{co}F(x, \varphi(x))$, $x \in \Omega$ and that $\varphi \in C^1(\Omega)$. In the sequel we discuss these results and show that, in the scalar case both assumptions can be relaxed. Details and generalizations can be found in [11]. Consider the following problem

$$\begin{cases} H(x, u(x), \nabla u(x)) = 0 & x \in \Omega \text{ a.e.} \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{3.1}$$

We associate with (3.1) the partial differential inclusion

$$\begin{cases} \nabla u(x)) \in F(x, u(x)) & x \in \Omega \text{ a.e.} \\ u(x) = \varphi(x) & x \in \partial\Omega, \end{cases} \tag{3.2}$$

where $F(x, y) = \{z \in \mathbb{R}^n : H(x, y, z) = 0\}$. Under suitable assumptions on H, the map $\overline{co}F$ turns out to be compact valued and continuous.

THEOREM 3.1 *[11] Let $F : \Omega \times \mathbb{R} \to \mathcal{K}(\mathbb{R}^n)$ be bounded by a constant $M \geq 0$ and assume that $\overline{co}F$ be continuous on $\Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $\varphi \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ be such that $\nabla\varphi(x) \in int\overline{co}F(x, \varphi(x))$, $x \in \Omega$ a.e..*
Then the Dirichlet problem

$$\begin{cases} \nabla u(x) \in ext\overline{co}F(x, u(x)) & x \in \Omega \text{ a.e.} \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases} \tag{3.3}$$

has a solution $u \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$.

PROOF. (Sketch) We introduce an appropriate space $\mathcal{M}$, given by

$$\mathcal{M} = cl\{u\in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega) : \nabla u(x) \in int\overline{co}F(x,u(x)), x\in\Omega \text{ a.e. and } u(x) = \varphi(x), x \in \partial\Omega\}.$$

Here, the closure is understood in the metric of $C(\overline{\Omega})$. Under this metric $\mathcal{M}$ turns out to be a nonempty complete metric space. For $n \in \mathbb{N}$, set

$$\mathcal{M}_n = \{u \in \mathcal{M} : \int_\Omega d_F(x,u(x),\nabla u(x))dx < 1/n\}.$$

Then by using the properties of the Choquet function d_F we show that $\mathcal{M}_n$ is open and dense in $\mathcal{M}$. A straightforward application of the Baire category theorem yields that the set $\mathcal{M}_0 = \bigcap \mathcal{M}_n$ is non empty. Further each $u \in \mathcal{M}_0$ is actually a solution of the Dirichlet problem (3.3). As F is assumed to have compact values it follows that each solution of (3.3), is also a solution of the initial Dirichlet problem (3.2) and hence of (3.1). □

The vectorial case is by far more difficult. A "particularly simple" vectorial problem, yet unsolved in its full generality, is the prescribed singular values problem. It has been recently studied by Cellina and Perrotta [4] in terms of potential wells and by Dacorogna and Marcellini [7, 8] by means of the Baire method which has proven to be effective, beside this, also in more general vectorial problems.

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