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ON LITTLEWOOD'S PROBLEM FOR THE ASYMMETRIC OSCILLATOR

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In the papers [11, 12] Littlewood considered the equation

$$\ddot{x} + g(x) = f(t)\,,$$

with $g(x) \to \pm\infty$ as $x \to \pm\infty$, and he found some unexpected examples of unbounded solutions. His techniques were based on the mechanical interpretation of the equation as a forced oscillator. After these papers, the name of Littlewood was associated to the study of the boundedness problem for the forced oscillator. Many results on this problem have been obtained when f is periodic and g has superlinear growth at infinity (see [10] and the references therein). In these notes I shall present some recent results on this problem for a class of oscillators that have linear growth in g. These oscillators look very similar to the linear harmonic oscillator but they display interesting nonlinear dynamics. To compare the linear and nonlinear situations it will be interesting to start with the harmonic oscillator and look at it from a geometrical perspective.

1 The harmonic oscillator revisited

Consider the linear equation

$$\ddot{x} + \omega^2 x = f(t)\,, \tag{1.1}$$

where the frequency ω is a positive constant and the external force f is periodic (with period 2π). Let us review some elementary facts about resonance (unbounded/bounded solutions) and the Fredholm Alternative (periodic solutions):

(i) If $\omega \in \mathbb{N}$ and the Fourier coefficient $\hat{f_\omega}$ is different from zero then all solutions are unbounded. Here,

$$\hat{f_\omega} := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-i\omega t}dt\,.$$

(ii) If $\omega \in \mathbb{N}$ and $\hat{f_\omega} = 0$ then all solutions are 2π-periodic.

(iii) If $\omega \notin \mathbb{N}$ then all solutions are bounded

and there exists a unique 2π-periodic solution.

The previous results lead to a remarkable conclusion:

the solutions of (1.1) are bounded if and only if there exists at least one periodic solution.

Let us now forget these classical results and try to explore numerically the linear oscillator. Probably the easiest way is to draw on the screen of the computer the iterates of the Poincaré map. Given an initial state $s_0 = (x_0, v_0) \in \mathbf{R}^2$ we define the new state as

$$s_1 = (x_1, v_1)\,, \qquad x_1 = x(2\pi; s_0)\,, \qquad v_1 = \dot{x}(2\pi; s_0)\,,$$

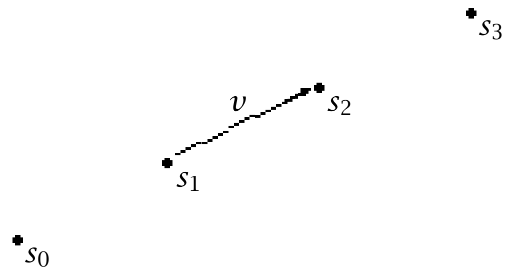
where $x(t; s_0)$ is the solution of (1.1) satisfying $x(0) = x_0$, $\dot{x}(0) = v_0$. The Poincaré map (also called the monodromy operator) is defined by

$$\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad s_0 \mapsto s_1\,.$$

It is easy to verify that $\mathcal{P}$ is an affine transformation that can be computed in terms of $\hat{f_\omega}$.

Let us see, in each of the cases (i), (ii) and (iii), the dynamics of $\mathcal{P}$ on the computer.

(i) $\omega \in \mathbb{N}$, $\hat{f}_\omega \neq 0$.



All orbits go to infinity and $\mathcal{P}$ is a translation. The translation vector v depends on $\hat{f}_\omega$. (The stepsize for numerical integration must be chosen small, otherwise the line on which the particle runs will appear slightly curved.)

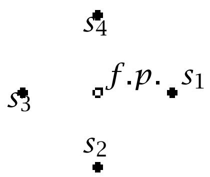
(ii) $\omega \in \mathbb{N}$, $\hat{f}_\omega = 0$.

$$s_0 = s_1 = s_2 = s_3 = \cdots$$

All orbits are fixed points and $\mathcal{P}$ is the identity. (After many iterations with the computer it will seem that the particle moves very slowly along a curve.)

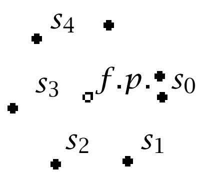
We shall distinguish two subcases for (iii).

(iii-1) $\omega \in \mathbb{Q}$, $\omega = p/q$ (the fraction is in reduced form with $q > 1$).



Most probably the reader is a very lucky person and will choose the point on the screen that corresponds to the initial condition of the periodic solution (the fixed point). An average person will find a periodic particle (of period q) that rotates around the fixed point. The Poincaré map is now a root of unity; that is, $\mathcal{P}^q = I$, and all solutions are periodic with period $2\pi q$ (subharmonic solutions of order q). After many iterates on the computer an ellipse will probably appear in the screen. This curve is invariant under $\mathcal{P}$ and it has appeared due to numerical effects. Notice that the orbit lies on the ellipse but it is not dense.

(iii-2) $\omega \notin \mathbb{Q}$.



It is possible that the reader will select again the point corresponding to the periodic solution. This means that she (or he) is really lucky. Typically, orbits rotate around the fixed point without periodicity. After many steps we should see an ellipse on the screen. This ellipse, $\mathcal{E}$, is an invariant curve of $\mathcal{P}$, $\mathcal{P}(\mathcal{E}) = \mathcal{E}$, and the orbit $\{s_n\}$ is dense in $\mathcal{E}$. In this case all solutions of (1.1) are quasi-periodic with frequencies $\omega_1 = 1$, $\omega_2 = \omega$. (A function $x : \mathbb{R} \to \mathbb{R}$ is said to be quasi-periodic with frequencies $\omega_1, \omega_2, \dots, \omega_n$ if there exists a continuous function $F : \mathbb{R}^n \to \mathbb{R}$, $F = F(\theta_1, \dots, \theta_n)$, that is 2π-periodic in each θ_i, and satisfies $x(t) = F(\omega_1 t, \dots, \omega_n t)$ for each t).

From the geometrical point of view we can sum up all the previous cases in two situations: either there exists a family of straight lines that are invariant under $\mathcal{P}$ and foliate the plane, case (i), or there exists a family of ellipses in the same conditions, cases (ii) and (iii). These ellipses are translates of $v^2/2 + \omega^2 x^2/2 = constant$, the energy levels of the autonomous oscillator.

2 The equation of an asymmetric oscillator

Let us consider a particle of mass $m = 1$ attached to two springs with Hooke's constants ω_1^2 and ω_2^2. The second spring is only in one side and has a stop at the origin. Under the action of an external forcing

$f = f(t)$, the equation of motion is

$$\ddot{x} + \begin{cases} (\omega_1^2 + \omega_2^2)x & \text{if } x \le 0 \\ \omega_1^2 x & \text{if } x \ge 0 \end{cases} = f(t)$$

or, equivalently,

$$\ddot{x} + ax^+ - bx^- = f(t)\,,$$

where $a = \omega_1^2$, $b = \omega_1^2 + \omega_2^2$. The above equation with $a \neq b$ will be referred to as the equation of the asymmetric oscillator. When $a = b$ we go back to the harmonic oscillator.

Piecewise linear oscillators appear in the engineering literature (see [5]) and, in particular, the equation of the asymmetric oscillator has been proposed by Lazer and McKenna in [9] as a simplified version of their model of suspension bridge.

3 The periodic Dancer-Fučik spectrum

In the seventies Fučik and Dancer studied the equation of an asymmetric oscillator by purely mathematical reasons. They were interested in nonlinear boundary value problems (Dirichlet, Neumann, Periodic, ...) and they found that the equation

$$\ddot{x} + ax^+ - bx^- = f(t)\,, \qquad a > 0\,, \quad b > 0 \tag{3.1}$$

was a paradigm for a large family of equations ("equations with jumping nonlinearities"). More information on this point of view can be found in the book by Fučik [7]. Next we shall review some of the results in [2, 3, 4] on the periodic problem for (3.1). They can be seen as a sort of Fredholm Alternative for the asymmetric oscillator.

In the space of parameters $(a, b) \in \mathbb{R}_+^2$ define the family of curves

$$C_n : \quad \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{n}\,, \qquad n = 1, 2, \ldots$$

These curves are similar to hyperbolae and the intersection of each of them with the diagonal $a = b$ is the point (n^2, n^2). These points are in the periodic spectrum of the harmonic oscillator.

The role played by the set $\{n^2 : \; n = 1, 2, \ldots\}$ in the periodic linear problem is played by $\Sigma = \bigcup_{n=1}^{\infty} C_n$ in the asymmetric oscillator. Namely,

(i) If $(a,b) \notin \Sigma$ then the equation (3.1) has a 2π-periodic solution for each $f \in L^1(\mathbb{R}/2\pi\mathbf{Z})$. Notice that this solution is not always unique.

(ii) If $(a,b) \in \Sigma$ then there exist functions $f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ such that (3.1) has no 2π-periodic solutions.

In [4] Dancer gave examples of piecewise constant functions f for which there are no periodic solutions. Later, in [9], Lazer and McKenna have found new examples of nonexistence for functions of the type $f(t) = \alpha\cos nt + \beta\sin nt$. Some sufficient conditions for existence of periodic solution when $(a,b) \in \Sigma$ were also given in [3]. They have been improved in a very recent paper by Fabry and Fonda [6].

4 Coexistence of unbounded and periodic motions

Our next task is to find under which conditions are there unbounded solutions of (3.1). The Massera's theorem applied to our equation says that the existence of a bounded solution (in the future or in the past) implies the existence of a 2π-periodic solution. In consequence, if $(a,b) \in \Sigma$ and f is such that there are no periodic solutions, then all solutions are unbounded. By analogy with the linear case we can formulate the following question:

> *are all solutions of (3.1) bounded when there exists a periodic solution?*

For some time I thought that the answer should be yes, but my colleague José Miguel Alonso had the opposite impression. My conjecture was wrong. In a paper recently appeared, [1], Alonso and I proved that for any couple $(a,b) \in \mathbb{R}^2_+$ with

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q}\,, \tag{4.1}$$

there exist functions $f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ for which "large solutions" are unbounded. By a large solution we mean a solution with sufficiently large initial energy. When (4.1) holds but $(a,b) \notin \Sigma$, the results of

Dancer imply the existence of a periodic solution. In consequence, coexistence of unbounded and periodic solutions can occur. This is a genuine nonlinear phenomenon. In a paper to appear [13], Liu Bin has considered again the case (4.1) and has found sufficient conditions of f for the boundedness of all solutions. The results in [13] and [1] are complementary. They provide conditions for the boundedness or unboundedness that are rather sharp when the function f is smooth. (The result of Liu requires $f \in C^5$). The precise conditions can be found in the original papers but it may be interesting to show how they apply in a concrete case.

EXAMPLE 1 $\ddot{x} + 4x^+ - x^- = \lambda + \cos 4t\,, \quad \lambda \in \mathbb{R}$.

The couple $(a, b) = (4, 1)$ is not in Σ and so we know the existence of a periodic solution for any λ. Also, the number $1/\sqrt{a} + 1/\sqrt{b} = 3/2$ is in $\mathbb{Q}$ and the results in [13] and [1] apply. They lead to the following assertions:

- If $|\lambda| < 1/45$ then all large solutions are unbounded
- If $|\lambda| > 1/45$ then all solutions are bounded.

The case $|\lambda| = 1/45$ cannot be decided with the results in these papers. The reader may demand how the number $1/45$ appears. In the next section I will try to justify it.

The previous example shows that the resonance set for the asymmetric oscillator is not Σ but a larger set that at least contains

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q}\,.$$

This is dense in the space of parameters and the next question is, of course, what happens in the complementary set. In a work in preparation I have proved that if

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \notin \mathbb{Q}$$

and $f \in C^4(\mathbb{R}/2\pi\mathbb{Z})$ is such that $\int_0^{2\pi} f(t)dt \neq 0$, then all solutions of (3.1) are bounded. Let us illustrate this result with another example.

EXAMPLE 2 $\ddot{x} + 5x^+ - x^- = \lambda + \cos 4t\,, \quad \lambda \in \mathbb{R}.$

The number $1/\sqrt{a} + 1/\sqrt{b} = 1 + 1/\sqrt{5}$ is not rational and so there is a periodic solution for any λ. When $\lambda \neq 0$ all solutions are bounded but for $\lambda = 0$ I do not know what happens.

5 The dynamics at infinity

We are now going to sketch the proofs of the results in [1] and [13]. As we shall see, these proofs give additional information on the behaviour of large solutions of the asymmetric oscillator.

The first step will be to introduce appropriate coordinates.

5.1 The asymmetric polar coordinates

Let us first consider the "homogeneous" equation

$$\ddot{x} + ax^+ - bx^- = 0\,. \tag{5.1}$$

In the phase space $(x, \dot{x})$ the orbits can be constructed by gluing two ellipses. These orbits are closed and have the minimal period

$$\tau = \frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}}\,.$$

This means that the origin is an isochronous centre.

The equation (5.1) is positively homogeneous and so the general solution can be expressed as

$$x(t) = \alpha C(t+\beta)\,, \qquad \alpha > 0\,, \quad \beta \in \mathbb{R}\,,$$

where $C(t)$ is the solution of (5.1) satisfying the initial conditions $C(0) = 1$, $\dot{C}(0) = 0$. This function can be determined explicitly by the formulas

$$C(-t) = C(t)\,, \qquad C(t+\tau) = C(t)\,,$$

$$C(t) = \begin{cases} \cos\sqrt{a}t\,, & 0 \le t \le \dfrac{\pi}{2\sqrt{a}} \\ -\sqrt{\dfrac{a}{b}}\sin\sqrt{b}\left(t - \dfrac{\pi}{2\sqrt{a}}\right), & \dfrac{\pi}{2\sqrt{a}} < t \le \dfrac{\tau}{2}\,. \end{cases}$$

The function $S(t) = \dot{C}(t)$ is also τ-periodic and the conservation of energy leads to the identity for S and C,

$$S(t)^2 + aC^+(t)^2 + bC^-(t)^2 = a\,, \qquad \forall t \in \mathbb{R}\,. \tag{5.2}$$

(It is convenient to think that the function C is an "asymmetric cosine". In fact, for $a = b = \omega^2$ one has $C(t) = \cos\omega t$, $S(t) = -\omega\sin\omega t$.)

Let $\omega = 2\pi/\tau$ be the frequency associated to (5.1). The functions $C(\theta/\omega)$ and $S(\theta/\omega)$ are 2π-periodic with respect to θ and allow us to define the change of variables

$$x = rC\left(\frac{\theta}{\omega}\right), \qquad y = rS\left(\frac{\theta}{\omega}\right); \qquad r > 0\,, \quad \theta \in \mathbf{S}^1\,.$$

The function C is of class C^2 while S is of class C^1. These facts, together with (5.2), imply that the mapping $(\theta, r) \to (x, y)$ is a C^1 diffeomorphism from $\mathbf{S}^1 \times (0, \infty)$ onto $\mathbb{R}^2 - \{0\}$. With this change (5.1) becomes

$$\dot{\theta} = \omega\,, \qquad \dot{r} = 0\,.$$

As a second step we shall apply the same change of variables to the forced equation and we shall study the equation for $r \to \infty$.

5.2 The Poincaré map at infinity

In the new variables θ, r the equation (3.1) becomes

$$\begin{cases} \dot{\theta} = \omega - \dfrac{\gamma}{2r} f(t) C\left(\dfrac{\theta}{\omega}\right) \\ \dot{r} = \dfrac{\gamma}{2\omega} f(t) S\left(\dfrac{\theta}{\omega}\right), \end{cases}$$

where γ is a positive constant. Let us consider the Poincaré map $\mathcal{P} : (\theta_0, r_0) \to (\theta_1, r_1)$, where

$$\theta_1 = \theta(2\pi; \theta_0, r_0)\,, \qquad r_1 = r(2\pi; \theta_0, r_0)\,.$$

From the new system we can obtain the expansions

$$\begin{cases} \theta_1 = \theta_0 + 2\pi\omega + \frac{1}{r_0}\Phi(\theta_0) + o(\frac{1}{r_0}) \\ r_1 = r_0 - \Phi'(\theta_0) + o(1)\,, \end{cases}$$

with

$$\Phi(\theta) = -\frac{\gamma}{2}\int_0^{2\pi} f(t)\, C\left(\frac{\theta}{\omega} + t\right) dt\,.$$

The function Φ plays an important role in the theory of the asymmetric oscillator. It was already employed in [3] and [6] to study the existence of periodic solutions when $(a, b) \in \Sigma$.

We are now going to apply (4.1). In fact, we can find a fraction q/p in reduced form such that

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = 2\frac{q}{p}\,.$$

Then, $\tau = 2\pi q/p$ and $\omega = p/q$. The linearization of $\mathcal{P}$ at infinity is a rotation of the form

$$\mathcal{R}: \quad \theta_1 = \theta_0 + 2\pi\frac{p}{q}, \qquad r_1 = r_0\,.$$

This is a root of unity ($\mathcal{R}^q = I$) and it will be simpler to study $\mathcal{P}^q$ instead of $\mathcal{P}$. The expansion of $\mathcal{P}^q$ is

$$\begin{cases} \theta_q = \theta_0 + 2\pi p + \dfrac{q}{r_0}\mu(\theta_0) + o\left(\dfrac{1}{r_0}\right) \\ r_q = r_0 - \mu'(\theta_0) + o(1)\,, \end{cases} \tag{5.3}$$

where μ is a discrete average of Φ, namely

$$\mu(\theta) = \frac{1}{q}\sum_{k=0}^{q-1} \Phi\left(\theta + 2\pi\frac{p}{q}k\right)\,.$$

For example 1 this function can be computed, its exact value is

$$\mu(\theta) = -\frac{\gamma}{2}\left(-4\lambda + \frac{4}{45}\cos 3\theta\right)\,.$$

Our final step will be the study of the dynamics of (5.3).

5.3 The discrete mapping

The behaviour of the mapping (5.3) depends on the oscillatory properties of the function μ. We shall consider two cases:

- μ has a finite number of zeros $\theta_1, \dots, \theta_n$ with $\mu'(\theta_i) \neq 0$, $i = 1, \dots, n$

- $\mu(\theta) > 0$, $\forall \theta \in \mathbb{R}$ [or $\mu(\theta) < 0$, $\forall \theta \in \mathbb{R}$].

In the first situation one can find a large ball around the origin B and n narrow sectors K_i around $\theta = \theta_i$ such that $K_i \cap B$ is positively invariant [resp. negatively invariant] if $\mu'(\theta_i)$ is positive [resp. negative]. Moreover, the orbits in $K_i \cap B$ go to infinity in the future or in the past depending of the type of invariance of the sector. The rest of orbits outside the ball B travel between two consecutive sectors and so they are unbounded in the future and in the past.

When the function μ has a definite sign (second case) the map $\mathcal{P}^q$ has a twist at infinity. This means that the derivative

$$\frac{\partial \theta_q}{\partial r_0} = -\frac{1}{r_0^2}\mu(\theta_0) + o\left(\frac{1}{r_0^2}\right)$$

has a definite sign for large r_0. This fact suggests the use of Moser's Invariant Curve Theorem. Actually Liu uses a variant of this theorem that can be found in [15] and shows the existence of invariant curves surrounding infinity. This implies the boundedness of all orbits. As in the case of the harmonic oscillator, these curves produce quasi-periodic solutions with two frequencies. However, in contrast to the linear case, these invariant curves do not foliate the plane and isolated subharmonic solutions or Mather sets can appear (see [14] for more details).

To conclude we go back to example 1. In this case the function μ has 6 nondegenerate zeros when $|\lambda| < 1/45$ and does not vanish when $|\lambda| > 1/45$. The reader is invited to explore numerically the mapping (5.3) for different μ's and to compare the pictures with the linear situation that we discussed in the first section of these notes.

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